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ARTICLE

Large triangle packings and Tuza's conjecture in sparse random graphs

Patrick Bennett[1,†], Andrzej Dudek[1,*,‡] and Shira Zerbib[2]

[1]Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, USA, [2]Department of Mathematics, Iowa State University, Ames, IA 50011, USA
*Corresponding author. Email: andrzej.dudek@wmich.edu

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Abstract
The triangle packing number $\nu(G)$ of a graph G is the maximum size of a set of edge-disjoint triangles in G. Tuza conjectured that in any graph G there exists a set of at most $2\nu(G)$ edges intersecting every triangle in G. We show that Tuza's conjecture holds in the random graph $G = G(n, m)$, when $m \leqslant 0.2403n^{3/2}$ or $m \geqslant 2.1243n^{3/2}$. This is done by analysing a greedy algorithm for finding large triangle packings in random graphs.

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1. Introduction

Let G be a graph. The *triangle packing number* of G, denoted by $\nu(G)$, is the maximal size of a set of edge-disjoint triangles (*i.e.* copies of K_3). Let $G(n, m)$ be the Erdős–Rényi *random graph* that assigns equal probability to all graphs on a fixed set V of n vertices with exactly $m = m(n)$ edges. When we refer to an event occurring *with high probability* (w.h.p. for short), we mean that the probability of that event goes to 1 as n goes to infinity.

In this paper we consider a random greedy process that produces a triangle packing in the random graph $G(n, m)$. Our motivation is to investigate the likely value of $\nu(G(n, m))$. We will call our process the *online triangle packing process* since it reveals one edge of $G(n, m)$ at a time, and builds a triangle packing as the edges are revealed. In online triangle packing we start with an empty packing $M(0)$ in $G(n, 0)$. We reveal one edge at a time; if that edge forms a copy of the tripartite graph $K_{1,1,s}$ for some $s \geqslant 1$ that is edge-disjoint with $M(i)$, then we choose the maximal such s and add that copy of $K_{1,1,s}$ to the packing to form $M(i + 1)$. Note that the *unmatched graph* $U(i) = G(n, i) - M(i)$ is triangle-free by induction on i (here and below we identify a graph H with its edge set $E(H)$). Furthermore, observe that the triangle packing can be obtained from $M(i)$ by taking a triangle from each graph of $M(i)$.

The online triangle packing process is similar to three other, more well-studied processes that produce triangle-free graphs. In the *triangle-free process*, first introduced by Bollobás and Erdős (see [10]), one maintains a triangle-free subgraph $G_T(i) \subseteq G(n, i)$ by revealing one edge at a time,

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and adding that edge to $G_T(i)$ only if it does not create a triangle in $G_T(i)$. This process was originally motivated by the study of the Ramsey numbers $R(3, t)$, and several progressively better analyses of the process have repeatedly improved the best known lower bound on $R(3, t)$, until recently Bohman and Keevash [7] and independently Fiz Pontiveros, Griffiths and Morris [13] analysed the process in incredible detail and proved that $R(3, t) \geqslant (1/4 - o(1))t^2/\log t$.

Bollobás and Erdős introduced another process, now known as *random triangle removal*, where a triangle-free graph is created by 'working backwards' (see [8, 9]). In this process one starts with $G_R(0) = K_n$ and at each step removes the edges of one triangle chosen uniformly at random from all triangles in $G_R(i)$, stopping only when the graph becomes triangle-free. The triangles whose edges were removed form a triangle packing in K_n. Random triangle removal was also originally motivated by the study of $R(3, t)$, although it has not produced any good bounds on $R(3, t)$. Bollobás and Erdős also conjectured that the number of edges remaining at the end of this process (*i.e.* edges not in the triangle packing) is w.h.p. $\Theta(n^{3/2})$. The best known estimate (both upper and lower bound) on the number of edges remaining is $n^{3/2+o(1)}$ by Bohman, Frieze and Lubetzky [6].

Bollobás and Erdős introduced a third process they hoped could attack $R(3, t)$, now called the *reverse triangle-free process*, where we 'work backwards' in a different way. In this process we start with $G_{RT}(0) = K_n$ and at each step remove one edge that is in a triangle in $G_{RT}(i)$, stopping only when the graph becomes triangle-free. Erdős, Suen and Winkler [12] proved that the expected number of edges in the final graph is $(1 + o(1))\sqrt{\pi}\, n^{3/2}/4$. Makai [20] and independently Warnke [22] then proved that the final number of edges is concentrated about its expectation.

We analyse the online triangle packing process using methods similar to those used to analyse the triangle-free and random triangle removal processes. Specifically, we use the *dynamic concentration method* (also known as the *differential equation method*: see Wormald's survey [23]) to track a system of random variables using martingale concentration inequalities. Essentially, we define a 'good event' stipulating that all our random variables are what we expect them to be, and show that it is very unlikely to stray outside the good event.

In this paper we focus on the triangle packing process for sparse random graphs only. For dense graphs Frankl and Rödl proved the following.

Theorem 1.1. (Frankl and Rödl [14]). *Suppose $\varepsilon > 0$. Let $G = G(n, m)$ be a random graph of order n and size $m = cn^{3/2}$, where $c \geqslant (\log n)^2$. Then w.h.p.*

$$\nu(G) \geqslant \frac{1}{3}(1 - \varepsilon)cn^{3/2}.$$

Clearly this theorem is optimal in order, since it shows that almost all edges can be decomposed into edge-disjoint triangles. An unpublished result for Pippenger strengthened Theorem 1.1 by slightly decreasing the lower bound on c (see *e.g.* [2]).

In this paper we are interested in the case when $c < (\log n)^2$. Let $z = z(t)$, where $t \geqslant 0$, be a function satisfying the differential equation $z' = 2e^{-z^2} - 4z^2$ (this differential equation is discussed in detail in Section 2.2). Let $\zeta \approx 0.5930714217$ be the positive root of the equation $e^{-\zeta^2} - 2\zeta^2 = 0$. Define

$$L_\nu(c) := \frac{1}{3}\left[c - \frac{z(c)}{2} - 2\int_0^c [z(t)^2 - 1 + e^{-z(t)^2}]\, dt\right]. \tag{1.1}$$

Our main result is the following.

Theorem 1.2. *Let $G = G(n, m)$ be a random graph of order n and size $m = cn^{3/2}$.*

(i) *For an arbitrary small $\varepsilon > 0$, let*

$$n^{-(1/20)+\varepsilon} < c \leqslant \frac{1}{1000} \log\log n.$$

Then w.h.p.

$$\nu(G) \geqslant (1+o(1)) L_\nu(c) n^{3/2}.$$

Furthermore, if

$$\frac{\zeta}{6(1-\zeta^2)} \approx 0.1525 < c \leqslant \frac{1}{1000} \log\log n,$$

then w.h.p.

$$\nu(G) \geqslant (1+o(1)) n^{3/2} \left[c(1-2\zeta^2) - \frac{\zeta}{6} \right].$$

(ii) *Let $1 \ll c \leqslant (\log n)^2$. Then w.h.p.*

$$\nu(G) \geqslant (1+o(1)) n^{3/2} c(1-2\zeta^2) \geqslant (1+o(1)) 0.2965 c n^{3/2}.$$

Observe that the bound in part (ii) is only slightly worse than the best possible, as in Theorem 1.1. The proof of Theorem 1.2, presented in Section 2, employs the dynamic concentration method and is algorithmic.

We complement Theorem 1.2 with a straightforward result.

Theorem 1.3. *Let $G = G(n, m)$ be a random graph of order n and size $m = cn^{3/2}$. Let $t_\triangle = t_\triangle(G)$ denote the number of copies of K_3 in G.*

(i) *If $n^{-3/10}/\log n \leqslant c \leqslant 1$, then w.h.p.*

$$\nu(G) \geqslant (1+o(1)) \frac{4c^3}{3} n^{3/2} e^{-12c^2} = (1+o(1)) t_\triangle e^{-12c^2}.$$

(ii) *If $c = o(n^{-3/10})$, then w.h.p.*

$$\nu(G) = t_\triangle(G).$$

Since $\lim_{c\to 0} e^{-12c^2} = 1$, this theorem implies that when c is small enough, almost all triangles are edge-disjoint. Therefore the bound in Theorem 1.3 is very good for small c (even when c is a small constant). The proof is given in Section 3. It will also follow from the proof that the bound in Theorem 1.2 is always better than the one in Theorem 1.3 for $n^{-(1/20)+\varepsilon} < c \leqslant (1/1000) \log\log n$, given in Section 2.

As an application of our theorems we consider a well-known conjecture of Tuza [21] on triangle packings in graphs, in the special case of random graphs. For a given graph G, let $\tau(G)$ be the *triangle covering number* of G, that is, the minimal size of a set of edges intersecting all triangles. Trivially, $\nu(G) \leqslant \tau(G) \leqslant 3\nu(G)$ for any graph G. Tuza's conjecture asserts that the upper bound can be improved.

Conjecture 1. (Tuza [21]). For every graph G, $\tau(G) \leqslant 2\nu(G)$.

The conjecture is tight for the complete graphs of orders 4 and 5. Recently, Baron and Khan [3] showed (disproving a conjecture of Yuster [24]) that for any $\alpha > 0$ there are arbitrarily large

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graphs G of positive density satisfying $\tau(G) > (1 - o(1))|G|/2$ and $\nu(G) < (1 + \alpha)|G|/4$. Hence, in general, the multiplicative constant 2 in the Tuza's conjecture cannot be improved. The best known upper bound is due to Haxell [16], who proved that $\tau(G) \leqslant (66/23)\nu(G)$. For more related results, see *e.g.* [1, 17, 19]. Here we show that for random graphs the following holds.

Theorem 1.4. *There exist absolute constants* $0 < c_1 < c_2$ *such that if* $m \leqslant c_1 n^{3/2}$ *or* $m \geqslant c_2 n^{3/2}$, *then w.h.p. Tuza's conjecture holds for* $G = G(n, m)$.

The existence of one of these constants, c_1, was very recently also proved by Basit and Galvin [4]. The proof of Theorem 1.4 is given in Section 4, from which it will follow that one can take $c_1 := 0.2403$ and $c_2 := 2.1243$. So the gap is not too big but unfortunately we could not close it. (See Section 5 for some additional discussion.)

2. Finding a triangle packing through the random process

2.1 Outline of the algorithm

In the online triangle packing process we in fact find an edge-disjoint set of subgraphs of the form $K_{1,1,s}$, for $s \geqslant 1$ (*i.e.* a complete tripartite graph with two partition classes of size 1 and one partition class of size s).

Formally, we reveal one edge of $G(n, m)$ at each step, so at step i we have $G(n, i)$. We will partition the edges of $G(n, i)$ into a *matched graph* $M(i)$ and an *unmatched graph* $U(i)$. At step i we reveal a random edge e_i. If e_i creates a copy K of $K_{1,1,s}$, for some $s \geqslant 1$, with some other edges in $U(i)$, then we choose the maximal such s and form $M(i+1)$ by inserting all the edges of K into $M(i)$, and we form $U(i+1)$ by removing from $U(i)$ the edges of K. Note that e_i creates a new copy of $K_{1,1,s}$ with other edges in $U(i)$ precisely when the vertices in e_i have codegree s in U, where the *codegree* of vertices u, v in a graph H, denoted by $\mathrm{codeg}_H(u, v)$, is the number of vertices w such that both uw and vw are edges of H.

For a vertex v let $d_U(v, i) = \deg_{U(i)}(v)$ and $d_M(v, i) = \deg_{M(i)}(v)$ be the unmatched and matched degree at step i, respectively. Let $d_G(v, i) = \deg_{G(n,i)}(v) = d_U(v, i) + d_M(v, i)$. We will usually suppress the 'i'. Define the scaled time parameter

$$t = t(i) := \frac{i}{n^{3/2}},$$

where $0 \leqslant i \leqslant (1/1000)n^{3/2} \log\log n$. At each step we choose a random edge without replacement. Hence, at every step the probability of choosing any particular edge that has not been chosen yet is at least $1/\binom{n}{2} \geqslant 2/n^2$ and at most

$$\frac{1}{\binom{n}{2} - (1/1000)n^{3/2}\log\log n} = \frac{2}{n^2}(1 + \tilde{O}(n^{-1/2})),$$

where $a(n) \in \tilde{O}(b(n))$ if there exists $k \geqslant 0$ such that $a(n) \in O(b(n) \log^k b(n))$.

Our process is 'wasteful' because it might remove from $U(i)$ some $K_{1,1,s}$ with $s \geqslant 2$ instead of only removing a triangle, in which case $2(s-1)$ edges are 'wasted'. We will show that actually the process does not waste too many edges. Therefore taking triangles only instead of $K_{1,1,s}$ would not significantly improve the size of the triangle packing but the analysis of the process would be more involved (see Section 5 for additional discussion).

We make the following heuristic predictions that we will prove later. First, due to concentration of vertex degrees in $G(n, m)$ for large enough m, at any step i (ignoring steps near the beginning) we have for every vertex v that

$$d_U(v) + d_M(v) = deg_{G(n,i)}(v) = \frac{2i}{n}(1 + o(1)) = 2tn^{1/2}(1 + o(1)).$$

Now let us heuristically assume that $d_U(v) \approx z(t)n^{1/2}$ (and therefore $d_M(v) \approx (2t - z(t))n^{1/2}$) and the codegrees in $U(i)$ are distributed Poisson with expectation $n(zn^{-1/2})^2 = z^2$. Then the number of unmatched edges is approximately $\frac{1}{2}n^{3/2}z$. When the vertices of the new edge have codegree 0 (this happens with probability e^{-z^2}), no triangle is formed, so we gain one unmatched edge. Otherwise these vertices have codegree $r \geqslant 1$ (this happens with probability $(z^{2r}/r!)e^{-z^2}$) and we put a $K_{1,1,r}$ into the packing, so $2r$ previously unmatched edges become matched. Thus the expected one-step change in the number of unmatched edges, which we approximate using a derivative, should be about

$$\Delta\left(\frac{1}{2}z(t)n^{3/2}\right) \approx \left(\frac{1}{2}z'(t)n^{3/2}\right)\Delta t = \frac{1}{2}z' \approx 1 \cdot e^{-z^2} - \sum_{r\geqslant 1} 2r\frac{z^{2r}}{r!}e^{-z^2} = e^{-z^2} - 2z^2$$

since the change in t in one step is $\Delta t = n^{-3/2}$. Thus we assume z satisfies the differential equation $z' = 2e^{-z^2} - 4z^2$. Although this equation has no explicit solution, we can still derive several properties of z. Summarizing, at the end of the process (after $cn^{3/2}$ edges have been revealed) about $cn^{3/2} - (z/2)n^{3/2}$ edges are matched, and the unmatched edges create a triangle-free graph. In the most optimistic scenario this would imply that we have a triangle packing of size

$$\frac{1}{3}\left(cn^{3/2} - \frac{z}{2}n^{3/2}\right).$$

We will show that this is not far from being true.

2.2 Preliminaries

Let $z = z(t)$ for $t \geqslant 0$ be such that the following autonomous differential equation holds:

$$z' = 2e^{-z^2} - 4z^2.$$

Assume that $z(0) = 0$. Then z is an increasing function of t, and z approaches the smallest positive root of the equation $2e^{-x^2} - 4x^2 = 0$ (as t goes to infinity), which is about $\zeta \approx 0.5931$. Hence $0 \leqslant z \leqslant \zeta$. This also implies that $z'(t) \geqslant 0$.

Furthermore, note that

$$z'' = (2e^{-z^2} - 4z^2)' = (-4ze^{-z^2} - 8z)z' = -4(e^{-z^2} + 2)zz' \leqslant 0, \tag{2.1}$$

and consequently $0 \leqslant z' \leqslant z'(0) = 2$.

It is also not difficult to see that there exists an absolute constant $t_0 > 0$ such that

$$2t - 4t^3 \geqslant z(t) \quad \text{for } t \in [0, t_0]. \tag{2.2}$$

Indeed, consider the function $g(t) = 2t - 4t^3 - z(t)$. One can verify that

$$g'(t) = 2 - 12t^2 - z'(t), \quad g''(t) = -24t + 4(e^{-z(t)^2} + 2)z(t)z'(t) \quad \text{and}$$
$$g'''(t) = -24 + 4[-2e^{-z(t)^2}z(t)^2z'(t)^2 + (e^{-z(t)^2} + 2)z'(t)^2 + (e^{z(t)^2} + 2)z(t)z''(t)].$$

Thus, since $z(0) = z''(0) = 0$ and $z'(0) = 2$, we obtain $g(0) = g'(0) = g''(0) = 0$ and $g'''(0) = 24$. Since $g'''(t)$ is continuous (indeed it is differentiable, and we could calculate its derivative using the formulas above), the latter implies that there exists some absolute constant $t_0 > 0$ such that $g'''(t) \geqslant 0$ for every $t \in [0, t_0]$. Hence $g''(t)$ is increasing and so $g''(t) \geqslant g''(0) = 0$ for $t \in [0, t_0]$. Similarly, this implies that $g'(t) \geqslant 0$ and finally $g(t) \geqslant 0$.

For integers $r, s \geqslant 0$, let us define the following random variables for every step $i \geqslant 0$.

- $C_r(v) = C_r(v, i)$ is the set of vertices u such that $\text{codeg}_U(u, v) = r$.

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- $P_r(u,v) = P_r(u,v,i)$ is the set of vertices w such that w is a neighbour of exactly one of $\{u,v\}$, say w^*, and w has codegree r (in U) with the vertex in $\{u,v\} \setminus \{w^*\}$ which we call w^{**}.
- $Q_{r,s}(u,v) = Q_{r,s}(u,v,i)$ is the set of vertices w such that $\mathrm{codeg}_U(w,u) = r$ and $\mathrm{codeg}_U(w,v) = s$.

When it is convenient we will abuse notation by writing the name of a set when we mean the cardinality of that set.

We define now deterministic counterparts to the above random variables. If we assume that the unmatched graph is almost regular and the codegrees are almost independent Poisson variables, then we expect the above random variables to be close (after scaling by an appropriate power of n) to the following functions:

$$c_r = c_r(t) := \frac{e^{-z^2} z^{2r}}{r!}, \quad p_r = p_r(t) := \frac{2e^{-z^2} z^{2r+1}}{r!}, \quad q_{r,s} = q_{r,s}(t) := \frac{e^{-2z^2} z^{2r+2s}}{r!s!}.$$

Observe that when $r = s = 0$ we have $c_0 = e^{-z^2}$, $p_0 = 2e^{-z^2} z$ and $q_{0,0} = e^{-2z^2}$. Moreover, since for any $k \geqslant 0$ and $0 \leqslant x \leqslant 1$, we get $e^{-x^2} x^k \leqslant 1$, we obtain

$$c_r \leqslant \frac{1}{r!}, \quad p_r \leqslant \frac{1}{r!}, \quad q_{r,s} \leqslant \frac{1}{r!s!}.$$

Simple but tedious calculations (see Appendix A) show that the above functions satisfy the following differential equations, where c_r', p_r' and $q_{r,s}'$ denote derivatives of c_r, p_r and $q_{r,s}$ as functions of t:

$$c_r' = 2c_{r-1}p_0 + 8(r+1)c_{r+1}z - 2c_r(p_0 + 4rz), \tag{2.3}$$

$$p_r' = 4q_{r,0} + 2p_{r-1}p_0 + 8(r+1)p_{r+1}z - 2p_r[p_0 + (4r+2)z], \tag{2.4}$$

$$q_{r,s}' = 2(q_{r-1,s} + q_{r,s-1})p_0 + 8\big[(r+1)q_{r+1,s} + (s+1)q_{r,s+1}\big]z - 4q_{r,s}\big[p_0 + 2(r+s)z\big]. \tag{2.5}$$

These differential equations can be viewed as idealized one-step changes in the random variables $C_r(v)$, $P_r(u,v)$ and $Q_{r,s}(u,v)$. Each of these variables counts copies of some type of substructure, and these copies can be created or destroyed by the process when we add or remove edges. Equations (2.3)–(2.5) can be understood as expressing the one-step changes in the random variables in terms of these creations and deletions, on average. We will ultimately use these differential equations to argue that the random variables stay close to their deterministic counterparts.

Define an 'error function'

$$f(t) := \exp\left\{\frac{100\log n}{\log\log n} \cdot t\right\} n^{-1/5}$$

and observe that for $0 \leqslant t \leqslant (1/1000)\log\log n$ we have $n^{-1/5} \leqslant f(t) \leqslant n^{-1/10}$.

For a given step i, let $\mathscr{E}_i$ be the event such that in $G = G(n,i)$ we have the following.

(i) *No huge codegree.* For all $u, v \in V$, we have

$$\mathrm{codeg}_G(u,v) \leqslant \frac{3\log n}{\log\log n}.$$

(ii) *No dense set.* For every subset $S \subseteq V$ such that $|S| \leqslant 10n^{1/2}\log\log n$, we have

$$|G[S]| \leqslant n^{1/2}\log^2 n.$$

(iii) *No $K_{3,7}$ and not too many $K_{3,2}$.* For any $u, v \in V$, the number of vertices w such that there are two vertices x, y that are both connected to all of u, v, w (*i.e.* such that the induced graph of $G(n,i)$ on the set $\{x,y,u,v,w\}$ contains a copy of $K_{3,2}$ with partition classes $\{x,y\}$ and $\{u,v,w\}$) is at most $O(\log^3 n)$. Furthermore, $G(n,i)$ contains no $K_{3,7}$ subgraph.

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(iv) *Dynamic concentration.* For every $j \leqslant i$,
- $d_G(v,j) \in (2t \pm n^{-1/4}\log^2 n)n^{1/2}$,
- $d_U(v,j) \in (z \pm f)n^{1/2}$,
- $|C_r(v,j)| \in (c_r \pm (r+1)^{-3} f)n$,
- $|P_r(u,v,j)| \in (p_r \pm f)n^{1/2}$,
- $|Q_{r,s}(u,v,j)| \in (q_{r,s} \pm f)n$,

where $a \pm b$ denotes the interval $[a-b, a+b]$, and the functions z, f, c_r, p_r and $q_{r,s}$ are evaluated at the point $t(j)$.

It is easy to see that the first three conditions of the event $\mathscr{E}_i$ hold w.h.p. for every i under consideration. We use the asymptotic equivalence of the models $G(n,m)$ and $G(n,p)$ (where $p = m/\binom{n}{2}$) and the fact that the conditions (i)–(iii) are monotone graph properties (see [18]). Now, to see that (i) holds w.h.p., we calculate the expected number of pairs u, v with at least

$$r_{\max} := \frac{3\log n}{\log\log n}$$

common neighbours. At step i the number of edges we have added is at most $n^{3/2}(\log\log n)/1000$. Thus it is enough to show that (i) holds w.h.p. in $G(n,p)$ where $p \leqslant n^{-1/2}(\log\log n)/500$. Now the expected number of pairs of vertices in $G(n,p)$ with codegree at least $r_{\max}$ is at most

$$\begin{aligned} n^2\binom{n}{r_{\max}}p^{2r_{\max}} &\leqslant n^2\left(\frac{enp^2}{r_{\max}}\right)^{r_{\max}} \\ &\leqslant n^2\left(\frac{(\log\log n)^3}{\log n}\right)^{r_{\max}} \\ &\leqslant e^{2\log n}\left(\frac{(\log\log n)^3}{\log n}\right)^{r_{\max}} \\ &\leqslant e^{2\log n}\left(\frac{1}{(\log n)^{5/6}}\right)^{r_{\max}} \\ &= e^{-(\log n)/2} \\ &= o(1). \end{aligned}$$

To see that (ii) holds w.h.p., assume that $s \leqslant 10n^{1/2}\log\log n$ and set $L = n^{1/2}\log^2 n$. The expected number of subsets $S \subseteq V$ with $|S| = s$ that induce at least L edges is at most

$$\binom{n}{s}\binom{\binom{s}{2}}{L}p^L \leqslant \left(\frac{en}{s}\right)^s\left(\frac{es^2p}{2L}\right)^L. \tag{2.6}$$

Now

$$\left(\frac{en}{s}\right)^s \leqslant \left(\frac{en}{10n^{1/2}\log\log n}\right)^{10n^{1/2}\log\log n} \leqslant (n^{1/2})^{10n^{1/2}\log\log n} = e^{5n^{1/2}\log n\log\log n}$$

and

$$\left(\frac{es^2p}{2L}\right)^L \leqslant \left(\frac{(\log\log n)^3}{(\log n)^2}\right)^L \leqslant \left(\frac{1}{\log n}\right)^L = e^{-n^{1/2}\log^2 n\log\log n}.$$

Thus, (2.6) is at most $\exp\{-\Omega(n^{1/2}\log^2 n\log\log n)\}$, which is small enough to beat a union bound over all $s \leqslant 10n^{1/2}\log\log n$.

To see that (iii) holds w.h.p., first note that the expected number of copies of $K_{3,7}$ in $G(n,p)$ for $p \leqslant n^{-1/2}(\log\log n)/500$ is at most $n^{10}p^{21} = o(1)$, so by Markov's inequality w.h.p. there are no copies of $K_{3,7}$. Now, to address the copies of $K_{2,3}$, we fix u, v and bound the number of triples

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w, x, y such that all edges in $\{x, y\} \times \{u, v, w\}$ are present in $G(n, m)$. Since we already know that the 'no huge codegree' property (i) holds w.h.p., there are $O(\log^2 n)$ choices for x, y. But for each x, y we have again by (i) that there are $O(\log n)$ choices for w. Thus the number of triples x, y, w is at most $O(\log^3 n)$.

In Sections 2.3–2.6 we prove that (iv) also holds w.h.p.

2.3 Tracking $d_U(v, j)$

First observe that Chernoff's bound implies that w.h.p.

$$d_G(v, j) \in (2t \pm n^{-1/4} \log^2 n) n^{1/2}.$$

Moreover, in order to estimate $d_U(v, j)$ it suffices to track $d_M(v, j)$.

We define the natural filtration $\mathscr{F}_i$ to be the history of the process up to step i. In particular, conditioning on $\mathscr{F}_i$ tells us the current state of the process. Assuming we are in the event $\mathscr{E}_{i-1}$, we calculate the expected one-step change of the matched degree, conditional on $\mathscr{F}_{i-1}$, namely

$$\mathbb{E}[\Delta d_M(v, i) | \mathscr{F}_{i-1}] = \mathbb{E}[d_M(v, i) - d_M(v, i-1) | \mathscr{F}_{i-1}].$$

We have already revealed $i-1$ edges. Now we reveal a new edge e_i. Note that $d_M(v)$ is non-decreasing. If $e_i \subseteq N_U(v)$, where $N_U(v)$ is the set of vertices connected to v in the graph U, then $d_M(v)$ increases by 2. If e_i is the edge vu for some vertex u such that $\mathrm{codeg}_U(u, v) > 0$, then $d_M(v)$ increases by $1 + \mathrm{codeg}_U(u, v)$. Since at most $\tilde{O}(n^{1/2})$ edges within $N_U(v)$ have been chosen, we have

$$\begin{aligned}
&\mathbb{E}[\Delta d_M(v, i) | \mathscr{F}_{i-1}] \\
&= \left[2 \cdot \left(\binom{d_U(v, i-1)}{2} - \tilde{O}(n^{1/2})\right) + \sum_{r=1}^{r_{\max}} (1 + r) C_r(v, i-1)\right] \cdot \frac{2}{n^2}(1 + \tilde{O}(n^{-1/2})) \\
&= \left[d_U(v, i-1)^2 + \sum_{r=1}^{r_{\max}} (1 + r) C_r(v, i-1)\right] \cdot \frac{2}{n^2} + \tilde{O}(n^{-3/2}) \\
&\leqslant \left[((z + f) n^{1/2})^2 + \sum_{r=1}^{r_{\max}} (1 + r) \left(\frac{e^{-z^2} z^{2r}}{r!} + (r+1)^{-3} f\right) n\right] \cdot \frac{2}{n^2} + \tilde{O}(n^{-3/2}),
\end{aligned}$$

where the functions z and f are evaluated at point $t(i-1)$. Now

$$\begin{aligned}
\sum_{r=1}^{r_{\max}} (1 + r) \left(\frac{e^{-z^2} z^{2r}}{r!}\right) &= e^{-z^2} \left(\sum_{r=1}^{r_{\max}} \frac{z^{2r}}{r!} + z^2 \sum_{r=1}^{r_{\max}} \frac{z^{2(r-1)}}{(r-1)!}\right) \\
&= e^{-z^2} \left(\sum_{r=1}^{\infty} \frac{z^{2r}}{r!} + z^2 \sum_{r=1}^{\infty} \frac{z^{2(r-1)}}{(r-1)!}\right) + O(n^{-2}) \\
&= e^{-z^2} (e^{z^2} - 1 + z^2 e^{z^2}) + O(n^{-2}) \\
&= 1 - e^{-z^2} + z^2 + O(n^{-2}),
\end{aligned}$$

where the second equality uses the fact that for $r \geqslant r_{\max}$ we have

$$r! = \exp\{(1 + o(1)) r \log r\} \geqslant \exp\{(3 + o(1)) \log n\},$$

and so

$$\sum_{r=r_{\max}}^{\infty} \frac{z^{2r}}{r!} + z^2 \sum_{r=r_{\max}}^{\infty} \frac{z^{2(r-1)}}{(r-1)!} < n^{-3+o(1)} = O(n^{-2}).$$

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Also

$$\sum_{r=1}^{r_{\max}} (r+1)^{-2} \leqslant \frac{\pi^2}{6} - 1 \leqslant 1.$$

Thus, since $0 \leqslant z \leqslant \zeta$ and $f^2 = O(f)$, we get

$$\begin{aligned}
\mathbb{E}[\Delta d_M(v,i)|\mathscr{F}_{i-1}] &\leqslant [((z+f)n^{1/2})^2 + (1-e^{-z^2}+z^2)n + fn] \cdot \frac{2}{n^2} + \tilde{O}(n^{-3/2}) \\
&= [z^2 + 2fz + f^2 + 1 - e^{-z^2} + z^2 + f]2n^{-1} + \tilde{O}(n^{-3/2}) \\
&= [2 - 2e^{-z^2} + 4z^2 + O(f)]n^{-1} + \tilde{O}(n^{-3/2}) \\
&= [2 - 2e^{-z(t(i-1))^2} + 4z(t(i-1))^2 + O(f(t(i-1)))]n^{-1} + \tilde{O}(n^{-3/2}). \qquad (2.7)
\end{aligned}$$

Define variables

$$D^{\pm}(v) = D^{\pm}(v,i) := \begin{cases} d_M(v,i) - (2t(i) - z(t(i)) \pm f(t(i)))n^{1/2} & \text{if } \mathscr{E}_{i-1} \text{ holds,} \\ D^{\pm}(v, i-1) & \text{otherwise.} \end{cases}$$

We will show that the variables $D^+(v)$ are supermartingales. Symmetric calculations show that the $D^-(v)$ are submartingales. To do that, we first apply Taylor's theorem to approximate the change in the deterministic function by its derivative. Let $g(t) := 2t - z(t) + f(t)$ and $t(i) := i/n^{3/2}$. Then

$$(g\circ t)(i) - (g\circ t)(i-1) = (g\circ t)'(i-1) + \frac{(g\circ t)''(\omega)}{2} = g'(t(i-1))n^{-3/2} + \frac{(g\circ t)''(\omega)}{2},$$

where $\omega \in [i-1, i]$. But

$$(g\circ t)''(i) = (g'(t(i))n^{-3/2})' = g''(t(i))n^{-3} = (-z''(t) + f''(t))n^{-3}.$$

Furthermore, by (2.1) we get that $|z''(t)| \leqslant 24$. Also

$$f''(t) = \left(\frac{100 \log n}{\log\log n}\right)^2 \exp\left\{\frac{100\log n}{\log\log n}\cdot t\right\} n^{-1/5} = \left(\frac{100\log n}{\log\log n}\right)^2 f(t).$$

Thus $(g\circ t)''(\omega) = O(n^{-3})$ and

$$(g\circ t)(i) - (g\circ t)(i-1) = (2 - z'(t(i-1)) + f'(t(i-1)))n^{-3/2} + O(n^{-3}). \qquad (2.8)$$

Now, if we are in $\mathscr{E}_{i-1}$, then (2.7) and (2.8) for $t = t(i-1)$ imply

$$\begin{aligned}
\mathbb{E}[\Delta D^+(v,i)|\mathscr{F}_{i-1}] &\leqslant (-f'(t) + O(f(t)))n^{-1} + \tilde{O}(n^{-3/2}) \\
&= \left[-\left(\frac{100\log n}{\log\log n}\right) f(t) + O(f(t))\right] n^{-1} + \tilde{O}(n^{-3/2}) \\
&\leqslant 0,
\end{aligned}$$

showing that the sequence $D^+(v,i)$ is a supermartingale.

We now apply the following martingale inequality due to Freedman [15] to show that the probability of $D^+(v)$ becoming positive is small, and thus so is the probability that $d_M(v)$ is out of its bounds.

Lemma 2.1. (Freedman [15]). *Let $Y(i)$ be a supermartingale with $\Delta Y(i) \leqslant C$ for all i, and let*

$$V(i) := \sum_{k\leqslant i} \mathrm{Var}\,[\Delta Y(k)|\mathscr{F}_k].$$

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Then

$$\mathbb{P}[\exists i\colon V(i) \leqslant b, Y(i) - Y(0) \geqslant \lambda] \leqslant \exp\left(-\frac{\lambda^2}{2(b+C\lambda)}\right).$$

Observe that $|\Delta d_M(v,i)| = O(\log n) = \tilde{O}(1)$, since for any pair of vertices the codegree is $O(\log n)$. Moreover, due to (2.8), $|\Delta(2t(i) - z(t(i)) + f(t(i)))n^{1/2}| = O(1)$ trivially. The triangle inequality thus implies that $\Delta D^+(v,i) = O(\log n)$. Also, since the variable $d_M(v,i)$ is non-decreasing, we have

$$\mathbb{E}[|\Delta d_M(v,i)||\mathscr{F}_i] = \mathbb{E}[\Delta d_M(v,i)|\mathscr{F}_i] = O(n^{-1})$$

by (2.7). So the one-step variance is

$$\mathrm{Var}\,[\Delta D^+|\mathscr{F}_k] \leqslant \mathbb{E}[(\Delta D^+)^2|\mathscr{F}_k] \leqslant O(\log n)\cdot \mathbb{E}[|\Delta D^+||\mathscr{F}_k] = O(n^{-1}\log n).$$

Therefore, for Freedman's inequality we use

$$b = O(n^{-1}\log n)\cdot O(n^{3/2}\log\log n) = \tilde{O}(n^{1/2}).$$

The 'bad' event here is the event that we have $D^+(v,i) > 0$, and since $D^+(v,0) = -n^{3/10}$ we set $\lambda = n^{3/10}$. Then Lemma 2.1 yields that the failure probability is at most

$$\exp\left\{-\frac{n^{3/5}}{\tilde{O}(n^{1/2}) + \tilde{O}(1)\cdot n^{3/10}}\right\},$$

which is small enough to beat a union bound over all vertices.

Using symmetric calculations, one can apply Freedman's inequality to the supermartingale $-D^-(v,i)$ to show that the 'bad' event $D^-(v,i) < 0$ does not occur w.h.p.

2.4 Tracking $C_r(v)$

We would now like to estimate $\mathbb{E}[\Delta C_r(v,i)|\mathscr{F}_{i-1}]$. Since $C_r(v,i)$ counts the number of vertices u such that $\mathrm{codeg}_U(u,v) = r$, we are interested to know how these codegree functions can increase or decrease.

Note first that $\mathrm{codeg}_U(u,v)$ increases by at most 1 at any step. The only case in which $\mathrm{codeg}_U(u,v)$ increases at step i is if we choose an edge $e_i = xy$ such that $x = u$ (resp. $x = v$), y is connected to v (resp. u), and e_i does not create a triangle with other edges in U. In the event $\mathscr{E}_i$, the number of such edges e_i is $P_0(u,v) - \tilde{O}(1)$, where the $\tilde{O}(1)$ term accounts for the few edges that may already be in M (by condition (i) in the event $\mathscr{E}_i$).

On the other hand, $\mathrm{codeg}_U(u,v)$ can decrease by more than 1 in a single step, but we will argue that w.h.p. this does not happen often, and $\mathrm{codeg}_U(u,v)$ never decreases by more than 6. For example, a decrease of 2 occurs if the edge e_i has both vertices in the common neighbourhood of u and v (see Figure 1(a)). This happens with probability $\tilde{O}(n^{-2})$. Another way for $\mathrm{codeg}_U(u,v)$ to decrease by $b \geqslant 2$ is if the edge e_i has one vertex in $\{u,v\}$, and the other vertex w has b neighbours that are also neighbours of u and v (see Figure 1(b)). However, in the event $\mathscr{E}_i$ we never have $b \geqslant 7$ since the graph has no copy of $K_{7,3}$, and for any fixed u, v the number of vertices w that could play this role (for some $b \geqslant 2$) is at most $\tilde{O}(1)$. Altogether, the probability that at step i the unmatched codegree of u and v decreases by at least 2 is $\tilde{O}(n^{-2})$, and w.h.p. we never see $\mathrm{codeg}_U(u,v)$ decrease by more than 6 in any single step, for any vertices u, v.

Now we discuss the possibility that $\mathrm{codeg}_U(u,v)$ decreases by exactly 1. For any edge $e = xy$ in $U = U(i-1)$ let $K(e)$ be the set of edges e_i which, if chosen, would match the edge e, that is, e_i, e and some third unmatched edge form a triangle. Let

$$S(u,v) = \{uw, vw \mid w \in N_U(u) \cap N_U(v)\}$$

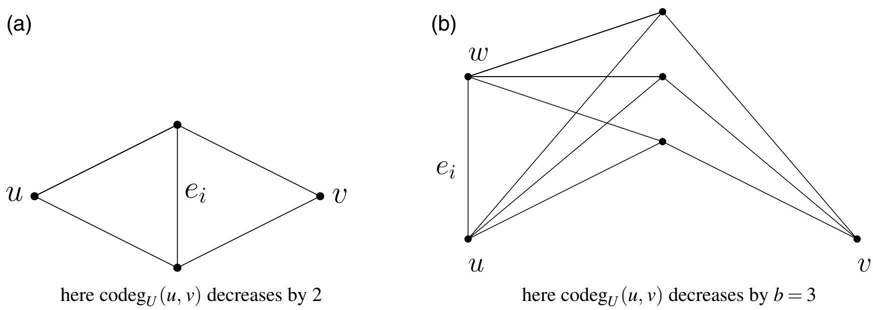

Figure 1. Rare ways for $\text{codeg}_U(u, v)$ to decrease.

be the set of edges that are in paths of two edges between u and v (so $|S(u, v)| = 2\text{codeg}_U(u, v)$). The number of edges e_i that, if chosen, would decrease $\text{codeg}_U(u, v)$ by 1 is

$$\bigcup_{e\in S(u,v)} K(e) = \sum_{e\in S(u,v)} |K(e)| - \tilde{O}(1),$$

where the $\tilde{O}(1)$ accounts for any edges that are in $K(e)$ for multiple edges e (see previous paragraph). Note also that for $e = xy$, $|K(e)| = d_U(x) + d_U(y) - \tilde{O}(1)$, so in the event $\mathscr{E}_i$ we have

$$|K(e)| \in 2(z \pm f)n^{1/2} + \tilde{O}(1).$$

Summarizing, we calculate $\mathbb{E}[\Delta C_r(v, i)|\mathscr{F}_{i-1}]$ by considering separately edges e_i that:

- increase $\text{codeg}_U(u, v)$ by 1 for some $u \in C_{r-1}(v)$,
- decrease $\text{codeg}_U(u, v)$ by 1 for some $u \in C_{r+1}(v)$,
- increase $\text{codeg}_U(u, v)$ for some $u \in C_r(v)$,
- decrease $\text{codeg}_U(u, v)$ for some $u \in C_r(v)$,
- decrease $\text{codeg}_U(u, v)$ by $b > 1$ for some $u \in C_{r+b}(v)$ (this is rare).

We get

$$\begin{aligned}
&\mathbb{E}[\Delta C_r(v, i)|\mathscr{F}_{i-1}]\\
&\quad= \left[\sum_{u\in C_{r-1}(v)} P_0(u, v) + \sum_{\substack{u\in C_{r+1}(v)\\ e\in S(u,v)}} K(e) - \sum_{u\in C_r(v)} \left(P_0(u, v) + \sum_{e\in S(u,v)} K(e)\right)\right] \cdot \frac{2}{n^2} + \tilde{O}(n^{-1})\\
&\quad\leqslant \Big[2(c_{r-1} + r^{-3}f) \cdot (p_0 + f) + 8(r + 1)(c_{r+1} + (r + 2)^{-3}f)(z + f)\\
&\qquad - 2(c_r - (r + 1)^{-3}f) \cdot [p_0 - f + 4r(z - f)]\Big]n^{-1/2} + \tilde{O}(n^{-1})\\
&\quad= \Big[2c_{r-1}p_0 + 8(r + 1)c_{r+1}z - 2c_r(p_0 + 4rz)\\
&\qquad + 16r(r + 1)^{-3}zf + O((r + 1)^{-3}f)\Big]n^{-1/2} + \tilde{O}(n^{-1}),
\end{aligned} \tag{2.9}$$

where all functions are evaluated at point $t(i - 1)$.

Define variables

$$C_r^{\pm}(v) = C_r^{\pm}(v, i) := \begin{cases} C_r(v, i) - (c_r(t(i)) \pm (r + 1)^{-3}f(t(i)))n & \text{if } \mathscr{E}_{i-1} \text{ holds,}\\ C_r^{\pm}(v, i - 1) & \text{otherwise.}\end{cases}$$

As in the previous section, we apply Taylor's theorem to approximate the change in the deterministic function by its derivative. Since

$$c_r''(t) = \frac{(4z^4 - 8rz^2 + 4r^2 - 2z^2 - 2r)z^{2r-2}e^{-z^2}}{r!},$$

we get $|c_r''(t(i-1))| = O(n^{-3})$ and

$$\Delta(c_r(t(i-1)) + (r+1)^{-3} f(t(i-1)))n = [c_r'(t(i-1)) + (r+1)^{-3} f'(t(i-1))]n^{-1/2} + O(n^{-2}).$$

Thus, by (2.3) and (2.9) for $t = t(i-1)$, we have

$$\begin{aligned}
&\mathbb{E}[\Delta C_r^+(v,i)|\mathscr{F}_{i-1}] \\
&\quad \leqslant [16r(r+1)^{-3} zf(t) + O((r+1)^{-3} f(t)) - (r+1)^{-3} f'(t)]n^{-1/2} + \tilde{O}(n^{-1}) \\
&\quad \leqslant \left[16rzf(t) + O(f(t)) - \left(\frac{100 \log n}{\log\log n}\right) f(t)\right] n^{-1/2}(r+1)^{-3} + \tilde{O}(n^{-1}) \\
&\quad \leqslant 0,
\end{aligned}$$

since $16rz < 100(\log n)/\log\log n$.

Now observe that $|\Delta C_r(v)| = \tilde{O}(n^{1/2})$. Indeed, if the new edge e_i has one vertex at v and the other at x, say, this only affects the codegree of v with the $\tilde{O}(n^{1/2})$ neighbours of x. On the other hand if e_i is not incident with v, then v loses at most two unmatched edges, say vx and vy, in which case only the codegree of v with the $\tilde{O}(n^{1/2})$ neighbours of x and y can be affected. Thus we also have $|\Delta C_r^+(v)| = \tilde{O}(n^{1/2})$, since the deterministic terms have much smaller one-step changes. Now we would like to bound $\mathbb{E}[|\Delta C_r(v)||\mathscr{F}_k]$, so we will re-examine (2.9). There are positive and negative contributions to $\Delta C_r(v)$, and of course (2.9) represents the expected positive contributions minus the expected negative contributions. Now by the triangle inequality $|\Delta C_r(v)|$ is at most the sum of the positive and negative contributions, and so

$$\begin{aligned}
&\mathbb{E}[|\Delta C_r(v)||\mathscr{F}_k] \\
&\quad \leqslant \left[\sum_{u \in C_{r-1}(v)} P_0(u,v) + \sum_{\substack{u \in C_{r+1}(v) \\ e \in S(u,v)}} K(e) + \sum_{u \in C_r(v)} \left(P_0(u,v) + \sum_{e \in S(u,v)} K(e)\right)\right] \cdot \frac{2}{n^2} + \tilde{O}(n^{-1}) \\
&\quad = O(n^{-1/2}),
\end{aligned} \tag{2.10}$$

since each term in (2.9) is $O(n^{-1/2})$. Thus

$$\mathbb{E}[|\Delta C_r^+(v)||\mathscr{F}_k] \leqslant \mathbb{E}[|\Delta C_r(v)||\mathscr{F}_k] + |\Delta(c_r(t) + (r+1)^{-3} f(t))|n = O(n^{-1/2}),$$

and hence the one-step variance is

$$\mathrm{Var}\,[\Delta C_r^+(v)|\mathscr{F}_k] \leqslant \mathbb{E}[(\Delta C_r^+(v))^2|\mathscr{F}_k] = \tilde{O}(n^{1/2}) \cdot \mathbb{E}[|\Delta C_r^+(v)||\mathscr{F}_k] = \tilde{O}(1).$$

The 'bad' event here is the event that $C_r^+(v,i) > 0$. Since $C_r^+(v,0) = -(r+1)^{-3} n^{4/5}$, we set $\lambda = (r+1)^{-3} n^{4/5} = \tilde{O}(n^{4/5})$. Then Lemma 2.1 yields that the failure probability is at most

$$\exp\left\{-\frac{\tilde{O}(n^{8/5})}{\tilde{O}(n^{3/2}) + \tilde{O}(n^{1/2}) \cdot \tilde{O}(n^{4/5})}\right\},$$

which is small enough to beat a union bound over all vertices as well as possible values of r.

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2.5 Tracking $P_r(u, v)$

Similarly, we calculate $\mathbb{E}[\Delta P_r(u, v, i)|\mathscr{F}_{i-1}]$. It is not difficult to see that

$$\begin{aligned}
&\mathbb{E}[\Delta P_r(u, v, i)|\mathscr{F}_{i-1}]\\
&= \Bigg[Q_{r,0}(u, v) + Q_{0,r}(u, v) + \sum_{w\in P_{r-1}(u,v)} P_0(w, w^{**}) + \sum_{\substack{w\in P_{r+1}(u,v)\\ e\in S(w,w^{**})}} (K(e) - \tilde{O}(1))\\
&\quad - \sum_{w\in P_r(u,v)} \Bigg(P_0(w, w^{**}) + \sum_{e\in S(w,w^{**})\cup\{ww*\}} (K(e) - \tilde{O}(1))\Bigg)\Bigg] \frac{2}{n^2}(1 + \tilde{O}(n^{-1/2}))\\
&\leqslant [4(q_{r,0} + f) + 2(p_{r-1} + f)(p_0 + f) + 8(r + 1)(p_{r+1} + f)(z + f)\\
&\quad - 2(p_r - f)[p_0 - f + 2(2r + 1)(z - f)]]n^{-1} + \tilde{O}(n^{-3/2})\\
&= [4q_{r,0} + 2p_{r-1}p_0 + 8(r + 1)p_{r+1}z - 2p_r[p_0 + (4r + 2)z]\\
&\quad + 16rzf(t) + O(f(t))]n^{-1} + \tilde{O}(n^{-3/2}). \qquad (2.11)
\end{aligned}$$

Define variables

$$P_r^{\pm}(u, v) = P_r^{\pm}(u, v, i) := \begin{cases} P_r(u, v, i) - (p_r(t(i)) \pm f(t(i)))n^{1/2} & \text{if } \mathscr{E}_{i-1} \text{ holds,}\\ P_r^{\pm}(u, v, i-1) & \text{otherwise.}\end{cases}$$

Note that by (2.4), (2.11) and Taylor's theorem, in the event $\mathscr{E}_{i-1}$ we have

$$\begin{aligned}
\mathbb{E}[\Delta P_r^{+}(u, v)|\mathscr{F}_i] &= \mathbb{E}[\Delta P_r(u, v)|\mathscr{F}_i] - (p_r'(t) + f'(t))n^{-1} + \tilde{O}(n^{-3/2})\\
&\leqslant [16rzf(t) + O(f(t)) - f'(t)]n^{-1} + \tilde{O}(n^{-3/2})\\
&\leqslant 0,
\end{aligned}$$

where $t = t(i-1)$. Now, since the codegrees are all $O(\log n)$, we have that at any step at most $O(\log n)$ edges become matched. Consider the effect on $P_r(u, v)$ by removing one edge e from G_U. If e is incident with u, say $e = ux$, then the removal of e can only affect vertices $w \in P_r(u, v)$ such that $w \in \{x\} \cup (N(x) \cap N(v))$ of which there are only $O(\log n)$. Similarly, if e is incident with v, then at most $O(\log n)$ vertices $w \in P_r(u, v)$ are affected. Finally, if e is not incident with u, v, then the only vertices $w \in P_r(u, v)$ that could be affected are the endpoints of e. Thus, since $O(\log n)$ edges are removed at any step and each one affects $O(\log n)$ vertices w, we always have $|\Delta P_r(u, v)| = O(\log^2 n)$. Also, $|\Delta P_r^{+}(u, v)| = O(\log^2 n)$, since the deterministic terms have much smaller one-step changes. We can also see that $\mathbb{E}[|\Delta P_r(u, v)||\mathscr{F}_k] = O(n^{-1})$ by an argument analogous to the one used to justify (2.10). Indeed, $\mathbb{E}[|\Delta P_r(u, v)||\mathscr{F}_k]$ is at most the sum of the absolute values of the terms in (2.11), all of which are $O(n^{-1})$. Thus

$$\mathbb{E}[|\Delta P_r^{+}(u, v)||\mathscr{F}_k] \leqslant \mathbb{E}[|\Delta P_r(u, v)||\mathscr{F}_k] + |\Delta(p_r(t) + f(t))n^{1/2}| = O(n^{-1})$$

and

$$\mathrm{Var}\,[\Delta P_r^{+}(u, v)|\mathscr{F}_k] \leqslant \mathbb{E}[(\Delta P_r^{+}(u, v))^2|\mathscr{F}_k] = O(\log n) \cdot \mathbb{E}[|\Delta P_r^{+}(u, v)||\mathscr{F}_k] = \tilde{O}(n^{-1}).$$

Therefore, using Lemma 2.1, our failure probability is at most

$$\exp\left\{-\frac{n^{3/5}}{\tilde{O}(n^{1/2}) + \tilde{O}(n^{3/10})}\right\},$$

which is small enough to beat a union bound over all pairs of vertices and values of r.

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2.6 Tracking $Q_{r,s}(u,v)$

Finally, we wish to calculate $\mathbb{E}[\Delta Q_{r,s}(u,v,i)|\mathscr{F}_{i-1}]$. Again it is not difficult to verify that

$$\begin{aligned}
&\mathbb{E}[\Delta Q_{r,s}(u,v,i)|\mathscr{F}_{i-1}] \\
&= \Bigg[\sum_{w\in Q_{r-1,s}(u,v)} P_0(u,w) + \sum_{w\in Q_{r,s-1}(u,v)} P_0(v,w) + \sum_{\substack{w\in Q_{r+1,s}(u,v)\\ e\in S(u,w)}} K(e) + \sum_{\substack{w\in Q_{r,s+1}(u,v)\\ e\in S(v,w)}} K(e) \\
&\quad - \sum_{w\in Q_{r,s}(u,v)} \Bigg(P_0(u,w) + P_0(v,w) + \sum_{e\in S(u,w)\cup S(v,w)} K(e)\Bigg)\Bigg]\frac{2}{n^2}(1+\tilde{O}(n^{-1/2})) \\
&\leqslant [2(q_{r-1,s}+f+q_{r,s-1}+f)(p_0+f) \\
&\quad + 8[(r+1)(q_{r+1,s}+f)+(s+1)(q_{r,s+1}+f)](z+f) \\
&\quad - 4(q_{r,s}-f)[p_0-f+2(r+s)(z-f)]]n^{-1/2} + \tilde{O}(n^{-1}) \\
&= [2(q_{r-1,s}+q_{r,s-1})p_0 + 8[(r+1)q_{r+1,s}+(s+1)q_{r,s+1}]z - 4q_{r,s}[p_0+2(r+s)z] \\
&\quad + 12(r+s)zf(t) + O(f)]n^{-1/2} + \tilde{O}(n^{-1}).
\end{aligned} \tag{2.12}$$

Define variables

$$Q^{\pm}_{r,s}(u,v) = Q^{\pm}_{r,s}(u,v,i) := \begin{cases} Q_{r,s}(u,v,i) - (q_{r,s}(t(i)) \pm f(t(i)))n & \text{if } \mathscr{E}_{i-1} \text{ holds,} \\ Q^{\pm}_{r,s}(v,i-1) & \text{otherwise.} \end{cases}$$

By (2.5) and (2.12), in the event $\mathscr{E}_{i-1}$ we have

$$\begin{aligned}
\mathbb{E}[\Delta Q^{+}_{r,s}(u,v)|\mathscr{F}_i] &= \mathbb{E}[\Delta Q_{r,s}(u,v)|\mathscr{F}_i] - (q'_{r,s}(t)+f'(t))n^{-1/2} + \tilde{O}(n^{-1}) \\
&\leqslant [12(r+s)zf(t) + O(f(t)) - f'(t)]n^{-1/2} + \tilde{O}(n^{-1}) \\
&\leqslant 0.
\end{aligned}$$

Let us consider the effect on $Q_{r,s}(u,v)$ by removing one edge e from G_U. If e is incident with u, say $e=ux$, then the only vertices $w\in Q_{r,s}(u,v)$ that could be affected are in the set $x\cup N(x)$ which has size $\tilde{O}(n^{1/2})$; and similarly if e is incident with v. If e is not incident with u,v then the only affected $w\in Q_{r,s}(u,v)$ would be the endpoints of e. Thus we have $|\Delta Q_{r,s}(u,v)| = \tilde{O}(n^{1/2})$, and also $|\Delta Q^{+}_{r,s}(u,v)| = \tilde{O}(n^{1/2})$ because the deterministic terms in $Q^{+}_{r,s}(u,v)$ have much smaller one-step changes. We can also see that $\mathbb{E}[|\Delta Q_{r,s}(u,v)||\mathscr{F}_k] = O(n^{-1/2})$ by another argument analogous to the one used to justify (2.10). Indeed, $\mathbb{E}[|\Delta Q_{r,s}(u,v)||\mathscr{F}_k]$ is at most the sum of the absolute values of the terms in (2.12), all of which are $O(n^{-1/2})$. Thus

$$\mathbb{E}[|\Delta Q^{+}_{r,s}(u,v)||\mathscr{F}_k] \leqslant \mathbb{E}[|\Delta Q_{r,s}(u,v)||\mathscr{F}_k] + |\Delta(q_{r,s}(t)+f(t))n| = O(n^{-1/2}),$$

and the one-step variance is

$$\text{Var}\,[\Delta Q^{+}_{r,s}(u,v)|\mathscr{F}_k] \leqslant \mathbb{E}[(\Delta Q^{+}_{r,s}(u,v))^2|\mathscr{F}_k] = \tilde{O}(n^{1/2})\cdot \mathbb{E}[|\Delta Q^{+}_{r,s}(u,v)||\mathscr{F}_k] = \tilde{O}(1).$$

Thus Lemma 2.1 yields that the failure probability is at most

$$\exp\left\{-\frac{n^{8/5}}{\tilde{O}(n^{3/2}) + \tilde{O}(n^{4/5}\cdot n^{1/2})}\right\},$$

which is again small enough to beat a union bound over all pairs of vertices and values of r,s.

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2.7 Proof of Theorem 1.2(i)

Let $r \geqslant 1$ and $i \geqslant 0$ be integers. Let $X_r(i)$ be an indicator random variable such that $X_r(i) = 1$ if the vertices of e_i have codegree r. We showed that w.h.p.

$$\Pr(X_r(i) = 1) \leqslant c_r(t(i)) + (r+1)^{-3} f(t(i)) \leqslant \frac{e^{-z(t(i))^2} z(t(i))^{2r}}{r!} + n^{-1/10} =: p_r(i).$$

Let $X_r'(i)$ be an indicator random variable such that $\Pr(X_r'(i) = 1) = p_r(i)$, and let the $X_r'(i)$ all be independent. Set $X_r = \sum_i X_r(i)$ and $X_r' = \sum_i X_r'(i)$, and observe that X_r' stochastically dominates X_r. Moreover,

$$\mathbb{E}(X_r') = \sum_{i=1}^{cn^{3/2}} \frac{e^{-z(t(i))^2} z(t(i))^{2r}}{r!} + cn^{7/5}.$$

Clearly $cn^{7/5} \leqslant \mathbb{E}(X_r') \leqslant cn^{3/2}$. Consequently, the general form of the Chernoff bound yields that

$$\Pr(X_r' \geqslant \mathbb{E}(X_r') + cn^{7/5}) \leqslant e^{-n^{\varepsilon}}$$

for some absolute constant $\varepsilon > 0$. Thus w.h.p. we have

$$X_r' \leqslant \sum_{i=1}^{cn^{3/2}} \frac{e^{-z(t(i))^2} z(t(i))^{2r}}{r!} + 2cn^{7/5}.$$

Recall that w.h.p. the codegree of two vertices is never larger than

$$r_{\max} = \frac{3 \log n}{\log \log n}$$

and $c = \tilde{O}(1)$. Consequently, the number of 'wasted' edges is at most

$$\begin{aligned}
\sum_{r=1}^{r_{\max}} 2(r-1) X_r &\leqslant \sum_{r=1}^{r_{\max}} 2(r-1) X_r' \\
&= 2 \sum_{r=1}^{r_{\max}} (r-1) \sum_{i=0}^{cn^{3/2}} \frac{e^{-z(t(i))^2} z(t(i))^{2r}}{r!} + \tilde{O}(cn^{7/5}) \\
&= 2 \sum_{i=0}^{cn^{3/2}} \sum_{r=1}^{r_{\max}} (r-1) \frac{e^{-z(t(i))^2} z(t(i))^{2r}}{r!} + \tilde{O}(cn^{7/5}).
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{r=1}^{\infty} (r-1) \frac{e^{-z^2} z^{2r}}{r!} &= e^{-z^2} \left(z^2 \sum_{r=1}^{\infty} \frac{z^{2(r-1)}}{(r-1)!} - \sum_{r=1}^{\infty} \frac{z^{2r}}{r!} \right) \\
&= e^{-z^2} [z^2 e^{z^2} - (e^{z^2} - 1)] \\
&= z^2 - 1 + e^{-z^2},
\end{aligned}$$

we get that w.h.p. we waste at most

$$2 \sum_{i=0}^{cn^{3/2}} [z(in^{-3/2})^2 - 1 + e^{-z(in^{-3/2})^2}] + \tilde{O}(cn^{7/5})$$

edges.

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Consider the function $g(t) := z(t)^2 - 1 + e^{-z(t)^2}$. Clearly $g'(t) = 2z(t)z'(t)(1 - e^{-z(t)^2})$. From the properties of z it follows that $g'(t)$ is positive, and hence $g(t)$ is increasing. Thus

$$
\begin{aligned}
2\sum_{i=0}^{cn^{3/2}} [z(in^{-3/2})^2 - 1 + e^{-z(in^{-3/2})^2}] &\leqslant 2\int_0^{cn^{3/2}+1} [z(\iota n^{-3/2})^2 - 1 + e^{-z(\iota n^{-3/2})^2}]\, d\iota \\
&= 2n^{3/2}\int_0^{c+n^{-3/2}} [z(t)^2 - 1 + e^{-z(t)^2}]\, dt \\
&= 2n^{3/2}\int_0^{c} [z(t)^2 - 1 + e^{-z(t)^2}]\, dt + O(1).
\end{aligned}
$$

Furthermore, since the number of unmatched edges is w.h.p. at most $(z(c)/2)n^{3/2} + n^{7/5}$, the number of matched edges is at least

$$cn^{3/2} - \frac{z(c)}{2}n^{3/2} - n^{7/5}.$$

Therefore the number of edge-disjoint triangles at the end of the online triangle packing process is w.h.p. at least

$$\frac{n^{3/2}}{3}\left[c - \frac{z(c)}{2} - 2\int_0^c [z(t)^2 - 1 + e^{-z(t)^2}]\, dt\right] - \tilde{O}(cn^{7/5}).$$

We now show that if $c \geqslant n^{-(1/20)+\varepsilon}$, then $\tilde{O}(cn^{7/5})$ is negligible. First we handle the case where $c = \Omega(1)$, in which case our claim will follow from the fact that the function

$$L_\nu(c) = \frac{1}{3}\left(c - \frac{z(c)}{2} - 2\int_0^c [z(t)^2 - 1 + e^{-z(t)^2}]\, dt\right)$$

is positive for all $c > 0$. Indeed, $L_\nu(0) = 0$ and we have

$$L_\nu'(c) = \frac{1}{3}\left(1 - \frac{z'(c)}{2} - 2[z(c)^2 - 1 + e^{-z(c)^2}]\right) = 1 - e^{-z(c)^2} > 0,$$

where we have used the differential equation $z' = 2e^{-z^2} - 4z^2$. This shows that for $c = \Omega(1)$ we have

$$\nu(G(n, cn^{3/2}) \geqslant (1 + o(1))L_\nu(c)n^{3/2}.$$

Now we handle the case where $n^{-(1/20)+\varepsilon} \leqslant c < t_0$, where t_0 is the constant obtained in (2.2). By (2.2) we obtain

$$c - \frac{z(c)}{2} \geqslant c - \frac{2c - 4c^3}{2} = 2c^3.$$

Since for any $x \geqslant 0$, $e^{-x} \leqslant 1 - x + x^2/2$, we have that $e^{-z(t)^2} \leqslant 1 - z(t)^2 + z(t)^4/2$. Hence $z(t)^2 - 1 + e^{-z(t)^2} \leqslant z(t)^4/2$. Thus, again by (2.2),

$$2\int_0^c [z(t)^2 - 1 + e^{-z(t)^2}]\, dt \leqslant \int_0^c z(t)^4\, dt \leqslant \int_0^c (2t)^4\, dt = \frac{16}{5}c^5.$$

Consequently

$$\frac{n^{3/2}}{3}\left[c - \frac{z(c)}{2} - 2\int_0^c [z(t)^2 - 1 + e^{-z(t)^2}]\, dt\right] \geqslant \frac{cn^{3/2}}{3}\left(2c^2 - \frac{16}{5}c^4\right) = \Omega(cn^{7/5+2\varepsilon}),$$

since by assumption

$$2c^2 - \frac{16}{5}c^4 = \Omega(n^{-(1/10)+2\varepsilon}),$$

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and $\Omega(cn^{7/5+2\varepsilon})$ is bigger than $\tilde{O}(cn^{7/5})$, as required.

The remaining part of the theorem follows immediately from the facts that $z(c) \leqslant \zeta$ and $z(t)^2 - 1 + e^{-z(t)^2}$ is increasing (as showed above). Thus

$$\frac{n^{3/2}}{3}\left[c - \frac{z(c)}{2} - 2\int_0^c [z(t)^2 - 1 + e^{-z(t)^2}]\,dt\right] \geqslant \frac{n^{3/2}}{3}\left[c(-2\zeta^2 + 3 - 2e^{-\zeta^2}) - \frac{\zeta}{2}\right]$$
$$= n^{3/2}\left[c(1 - 2\zeta^2) - \frac{\zeta}{6}\right],$$

since ζ satisfies $e^{-\zeta^2} - 2\zeta^2 = 0$.

2.8 Proof of Theorem 1.2(ii)

In the proof of Theorem 1.2(i) we assumed that the number of edges is at most $i_{\max} := (1/1000)n^{3/2}\log\log n$. If the number of edges is bigger than $i_{\max}$, then we do the so called *sprinkling*. First we run the process for the first $i_{\max}$ steps finding a packing M_1. Next we start the next round with $i_{\max}$ steps finding a new packing M_2. Here we make sure that we do not choose edges from the previous round. So we decrease the probability of choosing a new edge. If necessary we repeat the process again and again obtaining packings $M_1, \ldots, M_k$, where $k = O((\log n)^2)$. Recall that we reveal $G(n, m)$ one edge at a time by sampling edges without replacement, so the triangles in the packing M_i will all be edge-disjoint from the triangles in M_j for $i \neq j$. At any step of any round the probability of choosing any particular edge that has not been chosen yet will always be at least

$$\frac{1}{\binom{n}{2} - (\log n)^2 \cdot (1/1000)n^{3/2}\log\log n} = \frac{2}{n^2}(1 + \tilde{O}(n^{-1/2})).$$

Furthermore, it follows from the proof of Theorem 1.2(i) that in each round the failure probability is exponentially small in n. So after running at most $(\log n)^2$ rounds the failure probability is still $o(1)$, yielding the triangle packing of size $|M_1| + \cdots + |M_k|$.

3. Proof of Theorem 1.3

We will prove the theorem in the random graph $G(n, p)$ for suitable p, and show that this implies the theorem for $G(n, m)$.

First consider $G = G(n, p)$ with $p = o(n^{-4/5})$. This corresponds to $c = o(n^{-3/10})$ in $G(n, cn^{3/2})$, as in part (ii) of the theorem. Let X be the random variable that counts the number of copies of K_4 minus an edge in G. Clearly $\mathbb{E}(X) = O(n^4p^5) = o(1)$, so almost all triangles are edge-disjoint, yielding part (ii) of the theorem. Note that the graph property 'all triangles are edge-disjoint' is a monotone property (since if a graph H has this property then so does any subgraph of H), so it carries from $G(n, p)$ to $G(n, m)$.

To prove part (i) of the theorem, assume that

$$\frac{1}{(\log n)n^{4/5}} \leqslant p \leqslant \frac{2c}{n^{1/2}}.$$

Let Y be the random variable that counts the number of triangles in G that share no edge with any other triangle. Clearly the set of all such triangles is a triangle matching, and thus $\nu(G) \geqslant Y$. Let $Y_{u,v,w}$ be an indicator random variable which equals 1 if u, v, w induce a triangle and there is no vertex in $V(G) \setminus \{u, v, w\}$ that induces a triangle with two vertices in $\{u, v, w\}$. Clearly u, v, w induce a triangle with probability p^3. Now we first reveal edges incident to u and then edges incident to v while making sure that

$$(N(u) \setminus \{v, w\}) \cap (N(v) \setminus \{u, w\}) = \emptyset.$$

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This happens with probability $(1-p)^{|N(u)|-2}$. Next we reveal edges incident to w, making sure that

$$((N(u) \setminus \{v, w\}) \cup (N(v) \setminus \{u, w\})) \cap (N(w) \setminus \{u, v\}) = \emptyset.$$

The latter happens with probability $(1-p)^{|N(u)|+|N(v)|-4}$. So

$$\Pr(Y_{u,v,w} = 1) = p^3(1-p)^{2|N(u)|+|N(v)|-6}.$$

The Chernoff bound now implies that a.a.s. for every $v \in V(G)$ we have $\deg(v) = (1 + o(1))2cn^{1/2}$. Hence, for any choice of u, v, w,

$$\begin{aligned} &\Pr(Y_{u,v,w} = 1 \mid |N(u)|, |N(v)| = (1+o(1))2cn^{1/2}) \\ &= p^3(1-p)^{-(1+o(1))6cn^{1/2}} \\ &= (1+o(1))p^3 e^{-12c^2}. \end{aligned}$$

Thus

$$\mathbb{E}(Y) = \sum_{u,v,w} \mathbb{E}(Y_{u,v,w}) = (1+o(1))\binom{n}{3} p^3 e^{-12c^2}.$$

Subsequently the standard application of the Chebyshev inequality yields that w.h.p.

$$Y = (1+o(1))\binom{n}{3} p^3 e^{-12c^2}.$$

Note that the graph property $\nu(G) \geqslant s$ is monotone, so this result carries from $G(n,p)$ to $G(n,m)$, completing the proof of the theorem.

4. Proof of Theorem 1.4

It is easy to see that in every graph G one can always cover all the triangles using at most half of the edges. Indeed, let H be the largest bipartite subgraph of G. It is well known that $|E(H)| \geqslant \frac{1}{2}|E(G)|$ (see *e.g.* [11]). Now observe that $E(G) \setminus E(H)$ cover all triangles. Thus we always have

$$\tau(G(n,m)) \leqslant m/2. \tag{4.1}$$

Let $G = G(n,m)$ with $m = cn^{3/2}$ be a random graph. If $c \gg 1$, then (4.1) and Theorems 1.1 and 1.2 imply

$$\tau(G) \leqslant \frac{1}{2}cn^{3/2} \leqslant 2 \cdot 0.2965cn^{3/2} \leqslant 2\nu(G).$$

Now, if $c \geqslant 2.1243$, then Theorem 1.2 implies

$$\tau(G) \leqslant \frac{1}{2}cn^{3/2} \leqslant 2 \cdot n^{3/2}\left[c(1-2\zeta^2) - \frac{\zeta}{6}\right] \leqslant 2\nu(G).$$

On the other hand, for $c \leqslant 0.2403$ we can take one edge from each triangle obtaining a trivial cover set, implying

$$\tau(G) \leqslant t_\triangle \leqslant 2 \cdot t_\triangle e^{-12c^2} \leqslant 2\nu(G).$$

Therefore we can set $c_1 := 0.2403$ and $c_2 := 2.1243$ in the assumptions of Theorem 1.4. These constants can be slightly improved by using the general bound (i) in Theorem 1.2, where the function z can be found numerically.

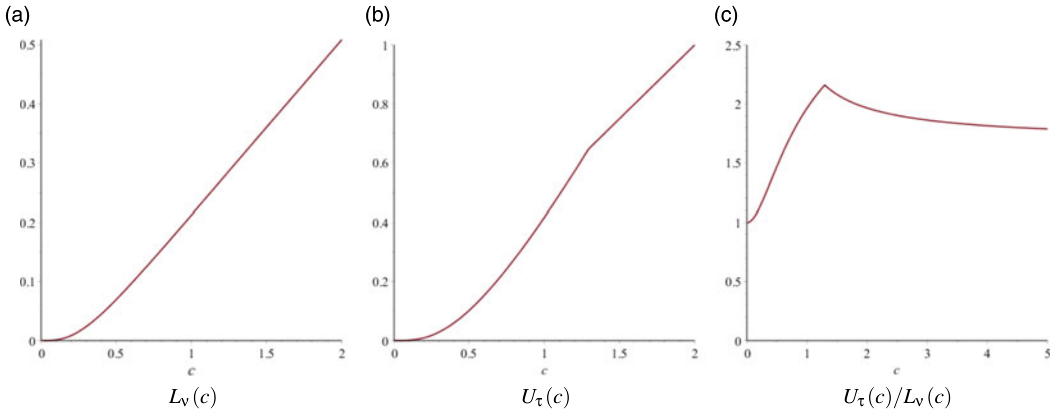

Figure 2. $U_\tau(c)$ versus $L_\nu(c)$ (where the latter was defined in (1.1)).

5. Concluding remarks

We note in passing that an upper bound on $\tau(G(n, m))$ can be obtained from the triangle-free process. This process accepts a set of edges forming a triangle-free subgraph of $G(n, m)$, so the rejected edges form a triangle cover. We will refer to Bohman's original triangle-free paper [5]. Recall that in this process one maintains a triangle-free subgraph $G_T(i) \subseteq G(n, i)$ by revealing one edge at a time, and adding that edge to $G_T(i)$ only if it does not create a triangle in $G_T(i)$. When we refer to Bohman's paper, to avoid confusion with our variable names we will replace his 'i' with '$\hat{i}$' and we will replace his 't' with '$\hat{t}$'. So the number of edges accepted by the process after $i = tn^{3/2}$ edges are proposed is $\hat{i} = \hat{t}n^{3/2}$.

Bohman proved that w.h.p. for all $\hat{i} \leqslant O(n^{3/2})$ the number $Q(\hat{i})$ of edges eligible to be inserted into the triangle-free graph (*i.e.* edges that would be accepted if proposed) is

$$Q(\hat{i}) = (1 + o(1))\binom{n}{2}e^{-4\hat{t}^2}.$$

Actually Bohman proved this for all $\hat{i}$ at most some constant times $n^{3/2}\log^{1/2} n$, but we will not fully use that here.

Heuristically, the number of edges the process proposes until it accepts the $(\hat{i} + 1)$st edge behaves like a geometric random variable with expectation $e^{4\hat{t}^2}$. Thus we derive the differential equation

$$\frac{d\hat{t}}{dt} = e^{-4\hat{t}^2}, \quad \hat{t}(0) = 0.$$

If the above heuristic analysis holds, then the number of edges rejected by the triangle-free process after $tn^{3/2}$ edges have been proposed should be $(1 + o(1))(t - \hat{t})n^{3/2}$, which would then be an upper bound on the triangle cover number. Also recall that the triangle cover number is always at most half the edges. To combine these two upper bounds (and we stress that only one is rigorously proven) on τ we let

$$U_\tau(c) := \min\{c/2, c - \hat{t}(c)\}.$$

Unfortunately, by itself such an improvement on the bound for τ would not be enough to show that Tuza's conjecture holds for all m. It would imply that Tuza's conjecture holds for $G(n, m)$ when $m \leqslant 1.0478n^{3/2}$, which is an improvement over the bound $m \leqslant 0.2403n^{3/2}$ given in Theorem 1.4 (see also Figure 2).

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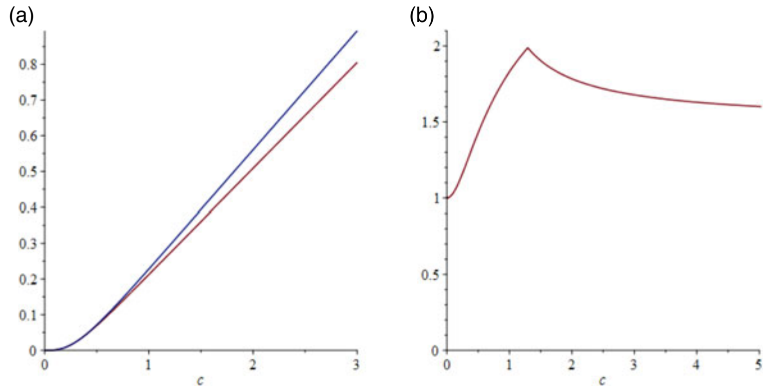

$L_\nu(c)$ (in red) and $L_\nu^*(c)$ (in blue) $U_\tau(c)/L_\nu^*(c)$

Figure 3. $U_\tau(c)$ versus $L_\nu^*(c)$.

Now we will describe how one might possibly improve the upper bound on the triangle packing number. In this paper we studied a random process that finds in $G(n, m)$ edge-disjoint subgraphs of the form $K_{1,1,s}$ for $s \geqslant 1$, instead of edge-disjoint triangles. It is easy to guess what we would get by considering a process where we take triangles only. Heuristically assume degrees in U are all close to $yn^{1/2}$ and that codegrees are Poisson with expectation y^2. Then the number of unmatched edges is $\frac{1}{2}yn^{3/2}$. Calculating the one-step change in the number of unmatched edges is easy: we gain one unmatched edge if e_i has endpoints with codegree 0 (this happens with probability e^{-y^2}), and otherwise we lose two unmatched edges which go into the constructed matching along with e_i. Using the expected one-step change as a derivative, we get the differential equation

$$\frac{1}{2}y' = 1 \cdot e^{-y^2} - 2 \cdot (1 - e^{-y^2}),$$

which is equivalent to $y' = 6e^{-y^2} - 4$. One can show again that $y(t)$ is an increasing function such that $y(t) \leqslant \upsilon$, where $\upsilon \approx 0.6367$. Since the number of matched edges is $cn^{3/2} - (y(c)/2)n^{3/2}$, we conclude that the number of edge-disjoint triangles (and hence our lower bound for ν) we would get is $(1 + o(1))L_\nu^*(c)n^{3/2}$, where

$$L_\nu^*(c) := \frac{1}{3}c - \frac{1}{6}y(c). \tag{5.1}$$

If our heuristic prediction above actually holds w.h.p. for this process, and if our heuristic analysis of the edges rejected by the triangle-free process also holds, then it would 'close the gap', implying Tuza's conjecture holds in $G = G(n, m)$ for any m (see Figure 3). In fact numerical calculations (see Figure 3(b)) would seem to show that that w.h.p. $\tau(G) \leqslant 1.9883 \cdot \nu(G)$. For $c \gg 1$ the bound (5.1) is also better, since in this case $L_\nu^*(c) = 1/3 + o(1)$ and this would imply that almost all edges can be decomposed into edge-disjoint triangles. We know that this is the case for $c = \Omega(\log^2 n)$ (see Theorem 1.1).

However, such a process is significantly more difficult to analyse than the one discussed in this paper. The reason is that when we choose an edge e_i at step i, we potentially create many copies of K_3 that share e_i. Since we would need to move only one such copy to the matched set, it is likely that we could choose a copy of K_3 sharing e_i uniformly at random. This part will make the analysis much more complicated.

While one is thinking of ways to produce large triangle matchings in random graphs, of course it is also natural to consider of the random triangle removal process on $G(n, m)$, where we take the graph $G(n, m)$ and then iteratively select a triangle uniformly at random and remove its edges until

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the graph is triangle-free. However, this process also seems difficult to analyse. For $m = \Theta(n^{3/2})$, if we choose a random triangle in $G(n, m)$ and remove its edges, the number of other triangles destroyed (*i.e.* the triangles that share an edge with the one that is removed) is not concentrated even for the very first step of the process, so the analysis of this process would not resemble the analysis of random triangle removal on the complete graph as in [6]. To analyse the process on $G(n, m)$ we would need to find a way to reveal a small number of edges of $G(n, m)$ at each step, in a manner that allows us to track how many triangles are remaining after we have removed a lot of them. However it is unclear how to do that.

Finally, let us mention one more problem that might be of some interest. The number of edges in the unmatched graph U seems to achieve a maximum of $\Theta(n^{3/2})$ edges, although we were only able to prove this in $G(n, m)$ for $m = O(n^{3/2} \log\log n)$. This is interesting because it is known that the final graph produced by the triangle-free process, as well as the final graph produced by random triangle removal process, also has $n^{3/2+o(1)}$ edges. It would be an interesting technical challenge to analyse the online triangle packing process in $G(n, m)$ for larger m. Ideally one would try for $m = \binom{n}{2}$ of course, but even $m = n^{3/2+\varepsilon}$ seems to be challenging. In particular, it would be interesting to know if the unmatched graph always has at most $\zeta n^{3/2}$ edges.

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Appendix A. The system of differential equations

Here we verify (2.3), (2.4) and (2.5). Recall that

$$z' = 2e^{-z^2} - 4z^2$$

and that

$$c_r = \frac{e^{-z^2} z^{2r}}{r!}, \quad p_r = \frac{2e^{-z^2} z^{2r+1}}{r!}, \quad q_{r,s} = \frac{e^{-2z^2} z^{2r+2s}}{r!s!}.$$

Equation (2.3) asserts that

$$c_r' = 2c_{r-1}p_0 + 8(r+1)c_{r+1}z - 2c_r(p_0 + 4rz).$$

On the one hand we have

$$\begin{aligned} c_r' &= \frac{2re^{-z^2} z^{2r-1} - 2e^{-z^2} z^{2r+1}}{r!}(2e^{-z^2} - 4z^2) \\ &= \frac{4re^{-2z^2} z^{2r-1} - 4e^{-2z^2} z^{2r+1} - 8re^{-z^2} z^{2r+1} + 8e^{-z^2} z^{2r+3}}{r!}, \end{aligned} \tag{A.1}$$

while on the other hand we have

$$\begin{aligned} &2c_{r-1}p_0 + 8(r+1)c_{r+1}z - 2c_r(p_0 + 4rz) \\ &\quad = 2\left(\frac{e^{-z^2} z^{2r-2}}{(r-1)!}\right)(2e^{-z^2} z) + 8(r+1)\left(\frac{e^{-z^2} z^{2r+2}}{(r+1)!}\right)z - 2\left(\frac{e^{-z^2} z^{2r}}{r!}\right)(2e^{-z^2} z + 4rz) \end{aligned}$$

which, after expanding and getting a common denominator, matches (A.1) and so (2.3) is verified.

Equation (2.4) asserts that

$$p_r' = 4q_{r,0} + 2p_{r-1}p_0 + 8(r+1)p_{r+1}z - 2p_r[p_0 + (4r+2)z].$$

On the one hand we have

$$\begin{aligned} p_r' &= \frac{2(2r+1)e^{-z^2} z^{2r} - 4e^{-z^2} z^{2r+2}}{r!}(2e^{-z^2} - 4z^2) \\ &= \frac{4(2r+1)e^{-2z^2} z^{2r} - 8e^{-2z^2} z^{2r+2} - 8(2r+1)e^{-z^2} z^{2r+2} + 16e^{-z^2} z^{2r+4}}{r!}, \end{aligned} \tag{A.2}$$

while on the other hand we have

$$\begin{aligned} &4q_{r,0} + 2p_{r-1}p_0 + 8(r+1)p_{r+1}z - 2p_r[p_0 + (4r+2)z] \\ &\quad = 4\left(\frac{e^{-2z^2} z^{2r}}{r!}\right) + 2\left(\frac{2e^{-z^2} z^{2r-1}}{(r-1)!}\right) \cdot 2e^{-z^2} z + 8(r+1)\left(\frac{2e^{-z^2} z^{2r+3}}{(r+1)!}\right)z \\ &\quad\quad - 2\left(2\frac{e^{-z^2} z^{2r+1}}{r!}\right)[2e^{-z^2} z + (4r+2)z] \end{aligned}$$

which, after expanding and getting a common denominator, matches (A.2) and so (2.4) is verified.

Equation (2.5) asserts that

$$q_{r,s}' = 2(q_{r-1,s} + q_{r,s-1})p_0 + 8\big[(r+1)q_{r+1,s} + (s+1)q_{r,s+1}\big]z - 4q_{r,s}\big[p_0 + 2(r+s)z\big].$$

On the one hand we have

$$\begin{aligned} q_{r,s}' &= \frac{(2r+2s)e^{-2z^2} z^{2r+2s-1} - 4e^{-2z^2} z^{2r+2s+1}}{r!s!}(2e^{-z^2} - 4z^2) \\ &= \frac{4(r+s)e^{-3z^2} z^{2r+2s-1} - 8e^{-3z^2} z^{2r+2s+1} - 8(r+s)e^{-2z^2} z^{2r+2s+1} + 16e^{-2z^2} z^{2r+2s+3}}{r!s!}, \end{aligned} \tag{A.3}$$

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while on the other hand we have

$$
\begin{aligned}
&2(q_{r-1,s}+q_{r,s-1})p_0+8\big[(r+1)q_{r+1,s}+(s+1)q_{r,s+1}\big]z-4q_{r,s}\big[p_0+2(r+s)z\big]\\
&\quad=2\left(\frac{e^{-2z^2}z^{2r+2s-2}}{(r-1)!s!}+\frac{e^{-2z^2}z^{2r+2s-2}}{r!(s-1)!}\right)2e^{-z^2}z\\
&\qquad+8\left[(r+1)\left(\frac{e^{-2z^2}z^{2r+2s+2}}{(r+1)!s!}\right)+(s+1)\left(\frac{e^{-2z^2}z^{2r+2s+2}}{r!(s+1)!}\right)\right]z\\
&\qquad-4\frac{e^{-2z^2}z^{2r+2s}}{r!s!}\big[2e^{-z^2}z+2(r+s)z\big],
\end{aligned}
$$

which matches (A.3) and so (2.5) is verified.

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