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Relaxation for an optimal design problem with linear growth and perimeter penalization

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This paper is devoted to the relaxation and integral representation in the space of functions of bounded variation for an integral energy arising from optimal design problems. The presence of a perimeter penalization is also considered in order to avoid non-existence of admissible solutions and, in addition, this leads to an interaction in the limit energy. More general models have also been taken into account.

1. Introduction

The optimal design problem, devoted to finding the minimal energy configurations of a mixture of two conductive materials, has been widely studied since the pioneering work of Kohn and Strang [28–30]. It is well known that, given a container Ω and prescribing only the volume fraction of the material where it is expected to have a certain conductivity, an optimal configuration might not exist. To overcome this difficulty, Ambrosio and Buttazzo in [6] imposed a perimeter penalization and studied the minimization problem

$$\min\left\{\int_E(\alpha|Du|^2+g_1(x,u))\,\mathrm{d}x+\int_{\Omega\setminus E}(\beta|Du|^2+g_2(x,u))\,\mathrm{d}x + \sigma P(E,\Omega)\colon E\subset\Omega,\ u\in H^1_0(\Omega)\right\},$$

finding the solution (u,E) and describing the regularity properties of the optimal set E.

In this paper we consider the minimization of a similar functional, where the energy density $|\cdot|^2$ has been replaced by the more general W_i, $i=1,2$, without any convexity assumptions and with linear growth, and since the lower-order terms $g_1(x,u)$ and $g_2(x,u)$ do not play any role in the asymptotics, we omit them in our

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subsequent analysis. The case of W_i, $i = 1, 2$, not convex with superlinear growth has been studied in the context of thin films in [16].

Thus, given Ω, a bounded open subset of $\mathbb{R}^N$, we assume that $W_i\colon \mathbb{R}^{d\times N} \to \mathbb{R}$ are continuous functions such that there exist positive constants α and β for which

$$\alpha|\xi| \leqslant W_i(\xi) \leqslant \beta(1+|\xi|) \quad \text{for every } \xi \in \mathbb{R}^{d\times N},\ i = 1, 2. \tag{1.1}$$

We consider the following optimal design problem

$$\inf_{\substack{u\in W^{1,1}(\Omega;\mathbb{R}^d)\\ \chi_E\in \mathrm{BV}(\Omega;\{0,1\})}} \left\{ \int_\Omega (\chi_E W_1(\nabla u) + (1-\chi_E) W_2)(\nabla u)\,\mathrm{d}x + P(E;\Omega)\colon u = u_0 \text{ on } \partial\Omega \right\}, \tag{1.2}$$

where χ_E is the characteristic function of $E \subset \Omega$, which has finite perimeter (see (2.2)).

Note that by (2.2) and the definition of total variation, $P(E;\Omega) = |D\chi_E|(\Omega)$ and we are led to the subsequent minimum problem

$$\inf_{\substack{u\in W^{1,1}(\Omega;\mathbb{R}^d)\\ \chi_E\in \mathrm{BV}(\Omega;\{0,1\})}} \left\{ \int_\Omega (\chi_E W_1 + (1-\chi_E) W_2)(\nabla u)\,\mathrm{d}x + |D\chi_E|(\Omega)\colon u = u_0 \text{ on } \partial\Omega \right\}.$$

The lack of convexity of the energy requires a relaxation procedure. To this end, we start by localizing our energy. As a first step, we introduce the functional $F_{\mathcal{OD}}\colon L^1(\Omega;\{0,1\}) \times L^1(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0,+\infty]$ defined by

$$F_{\mathcal{OD}}(\chi, u; A) := \begin{cases} \displaystyle\int_A (\chi_E W_1(\nabla u) + (1-\chi_E) W_2(\nabla u))\,\mathrm{d}x + |D\chi_E|(A) & \\ \qquad\qquad \text{in } \mathrm{BV}(A;\{0,1\}) \times W^{1,1}(A;\mathbb{R}^d), & \\ +\infty \qquad\qquad \text{otherwise.} & \end{cases} \tag{1.3}$$

We then consider the relaxed localized energy of (1.3) given by

$$\begin{aligned} \mathcal{F}_{\mathcal{OD}}(\chi, u; A) := \inf \biggl\{ \liminf_{n\to\infty} &\int_A (\chi_n W_1(\nabla u_n) + (1-\chi_n) W_2(\nabla u_n))\,\mathrm{d}x \\ &+ |D\chi_n|(A)\colon \{u_n\} \subset W^{1,1}(A;\mathbb{R}^d), \{\chi_n\} \subset \mathrm{BV}(A;\{0,1\}), \\ &u_n \to u \text{ in } L^1(A;\mathbb{R}^d) \text{ and } \chi_n \overset{*}{\rightharpoonup} \chi \text{ in } \mathrm{BV}(A;\{0,1\}) \biggr\}. \end{aligned}$$

Let $V\colon \{0,1\} \times \mathbb{R}^{d\times N} \to (0,+\infty)$ be given by

$$V(q,z) := qW_1(z) + (1-q)W_2(z) \tag{1.4}$$

and let $\bar{F}_{\mathcal{OD}}\colon \mathrm{BV}(\Omega;\{0,1\}) \times \mathrm{BV}(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0,+\infty]$ be defined as

$$\begin{aligned} \bar{F}_{\mathcal{OD}}(\chi, u; A) := \int_A QV(\chi, \nabla u)\,\mathrm{d}x + \int_A QV^\infty\left(\chi, \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\right)\mathrm{d}|D^c u| \\ + \int_{J_{(\chi,u)}\cap A} K_2(\chi^+, \chi^-, u^+, u^-, \nu)\,\mathrm{d}\mathcal{H}^{N-1}, \end{aligned} \tag{1.5}$$

where QV is the quasi-convex envelope of V given in (3.2), QV^∞ is the recession function of QV, namely,

$$QV^\infty(q,z) := \lim_{t\to\infty} \frac{QV(q,tz)}{t} \tag{1.6}$$

and

$$K_2(a,b,c,d,\nu) := \inf\left\{ \int_{Q_\nu} QV^\infty(\chi(x),\nabla u(x))\,\mathrm{d}x + |D\chi|(Q_\nu)\colon (\chi,u)\in\mathcal{A}_2(a,b,c,d,\nu)\right\}, \tag{1.7}$$

where

$$\begin{aligned}&\mathcal{A}_2(a,b,c,d,\nu)\\ &\quad := \{(\chi,u)\in \mathrm{BV}(Q_\nu;\{0,1\})\times W^{1,1}(Q_\nu;\mathbb{R}^d)\colon\\ &\qquad (\chi(y),u(y)) = (a,c) \text{ if } y\cdot\nu = \tfrac{1}{2},\ (\chi(y),u(y)) = (b,d)\\ &\qquad\quad \text{if } y\cdot\nu = -\tfrac{1}{2},\ (\chi,u) \text{ are 1-periodic in } \nu_1,\dots,\nu_{N-1} \text{ directions}\}\end{aligned} \tag{1.8}$$

for $(a,b,c,d,\nu)\in\{0,1\}\times\{0,1\}\times\mathbb{R}^d\times\mathbb{R}^d\times S^{N-1}$, with $\{\nu_1,\nu_2,\dots,\nu_{N-1},\nu\}$ an orthonormal basis of $\mathbb{R}^N$, and Q_ν the unit cube, centred at the origin, with one direction parallel to ν.

In § 6 we obtain the following integral representation.

THEOREM 1.1. *Let $\Omega\subset\mathbb{R}^N$ be a bounded open set and let $W_i\colon \mathbb{R}^d\times N\to[0,+\infty)$, $i=1,2$, be continuous functions satisfying (1.1). Let $\bar{F}_{\mathcal{OD}}$ be the functional defined in (1.5). Then, for every $(\chi,u)\in L^1(\Omega;\{0,1\})\times L^1(\Omega;\mathbb{R}^d)$,*

$$\mathcal{F}_{\mathcal{OD}}(\chi,u;A) = \begin{cases}\bar{F}_{\mathcal{OD}}(\chi,u;A) & \text{if } (\chi,u)\in \mathrm{BV}(\Omega;\{0,1\})\times\mathrm{BV}(\Omega;\mathbb{R}^d),\\ +\infty & \text{otherwise.}\end{cases}$$

This result will be achieved as a particular case of a more general theorem dealing with special functions of bounded variation that are piecewise constants.

In fact, we provide an integral representation for the relaxation of the functional $F\colon L^1(\Omega;\mathbb{R}^m)\times L^1(\Omega;\mathbb{R}^d)\times\mathcal{A}(\Omega)\to[0,+\infty]$ defined by

$$F(v,u;A) := \begin{cases}\displaystyle\int_A f(v,\nabla u)\,\mathrm{d}x + \int_{A\cap J_v} g(v^+,v^-,\nu_v)\,\mathrm{d}\mathcal{H}^{N-1} & \\ \qquad\qquad\qquad \text{in } \mathrm{SBV}_0(A;\mathbb{R}^m)\times W^{1,1}(A;\mathbb{R}^d), & \\ +\infty \qquad\qquad \text{otherwise,} & \end{cases} \tag{1.9}$$

where $\mathrm{SBV}_0(A;\mathbb{R}^m)$ is defined in (2.4) and $f\colon\mathbb{R}^m\times\mathbb{R}^{d\times N}\to[0,+\infty[$ and $g\colon\mathbb{R}^m\times\mathbb{R}^m\times S^{N-1}\to[0,+\infty[$ satisfy the following hypotheses:

(F_1) f is continuous;

(F_2) there exist $0<\beta'\leqslant\beta$ such that

$$\beta'|z|\leqslant f(q,z)\leqslant\beta(1+|z|)$$

for every $(q,z)\in\mathbb{R}^m\times\mathbb{R}^{d\times N}$;

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(F$_3$) there exists $L > 0$ such that

$$|f(q_1, z) - f(q_2, z)| \leqslant L|q_1 - q_2|(1 + |z|)$$

for every $q_1, q_2 \in \mathbb{R}^m$ and $z \in \mathbb{R}^{d\times N}$;

(F$_4$) there exist $\alpha \in (0, 1)$ and $C, L > 0$ such that

$$t|z| > L \implies \left| f^\infty(q, z) - \frac{f(q, tz)}{t} \right| \leqslant C\frac{|z|^{1-\alpha}}{t^\alpha} \quad \text{for every } (q, z) \in \mathbb{R}^m \times \mathbb{R}^{d\times N},\ t \in \mathbb{R},$$

with f^∞ the recession function of f with respect to the last variable, defined as

$$f^\infty(q, z) := \limsup_{t\to\infty} \frac{f(q, tz)}{t} \tag{1.10}$$

for every $(q, z) \in \mathbb{R}^m \times \mathbb{R}^{d\times N}$;

(G$_1$) g is continuous;

(G$_2$) there exists a constant $C > 0$ such that

$$\frac{1}{C}(1 + |\lambda - \theta|) \leqslant g(\lambda, \theta, \nu) \leqslant C(1 + |\lambda - \theta|)$$

for every $(\lambda, \theta, \nu) \in \mathbb{R}^m \times \mathbb{R}^m \times S^{N-1}$;

(G$_3$) $g(\lambda, \theta, \nu) = g(\theta, \lambda, -\nu)$ for every $(\lambda, \theta, \nu) \in \mathbb{R}^m \times \mathbb{R}^m \times S^{N-1}$.

The relaxed localized energy of (1.9) is given by

$$\begin{aligned}\mathcal{F}(v, u; A) &:= \inf\bigg\{ \liminf_{n\to\infty} \left(\int_A f(v_n, \nabla u_n)\,\mathrm{d}x + \int_{J_{v_n}\cap A} g(v_n{}^+, v_n{}^-, \nu_{v_n})\,\mathrm{d}\mathcal{H}^{N-1} \right) : \\ &\qquad \{u_n\} \subset W^{1,1}(A; \mathbb{R}^d),\ \{v_n\} \subset \mathrm{SBV}_0(A; \mathbb{R}^m), \\ &\qquad u_n \to u \text{ in } L^1(A; \mathbb{R}^d) \text{ and } v_n \to v \text{ in } L^1(A; \mathbb{R}^m) \bigg\}.\end{aligned} \tag{1.11}$$

Let $\bar{F}_0 \colon \mathrm{SBV}_0(\Omega; \mathbb{R}^m) \times \mathrm{BV}(\Omega; \mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0, +\infty]$ be given by

$$\begin{aligned}\bar{F}_0(v, u; A) &:= \int_A Qf(v, \nabla u)\,\mathrm{d}x + \int_A Qf^\infty\left(v, \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\right)\mathrm{d}|D^c u| \\ &\quad + \int_{J_{(v,u)}\cap A} K_3(v^+, v^-, u^+, u^-, \nu)\,\mathrm{d}\mathcal{H}^{N-1},\end{aligned} \tag{1.12}$$

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where Qf is the quasi-convex envelope of f given in (3.2), Qf^∞ is the recession function of Qf and $K_3\colon \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1} \to [0, +\infty[$ is defined as

$$\begin{aligned} &K_3(a,b,c,d,\nu) \\ &\quad := \inf\biggl\{ \int_{Q_\nu} Qf^\infty(v(x), \nabla u(x))\,\mathrm{d}x \\ &\qquad\qquad + \int_{J_v \cap Q_\nu} g(v^+(x), v^-(x), \nu(x))\,\mathrm{d}\mathcal{H}^{N-1}\colon (v,u) \in \mathcal{A}_3(a,b,c,d,\nu)\biggr\}, \end{aligned} \tag{1.13}$$

where

$$\begin{aligned} &\mathcal{A}_3(a,b,c,d,\nu) \\ &\quad := \biggl\{(v,u) \in (\mathrm{SBV}_0(Q_\nu;\mathbb{R}^m) \cap L^\infty(Q_\nu;\mathbb{R}^m)) \times W^{1,1}(Q_\nu;\mathbb{R}^d)\colon \\ &\qquad (v(y),u(y)) = (a,c) \text{ if } y\cdot\nu = \tfrac{1}{2},\ (v(y),u(y)) = (b,d) \\ &\qquad\quad \text{if } y\cdot\nu = -\tfrac{1}{2},\ (v,u) \text{ are } 1-\text{periodic in } \nu_1,\dots,\nu_{N-1} \text{ directions}\biggr\} \end{aligned} \tag{1.14}$$

with $\{\nu_1, \nu_2, \dots, \nu_{N-1}, \nu\}$ an orthonormal basis of $\mathbb{R}^N$.

In the following theorem we present the main result.

THEOREM 1.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, let $f\colon \mathbb{R}^m \times \mathbb{R}^{d\times N} \to [0,+\infty[$ be a function satisfying (F_1)–(F_4) and let $g\colon \mathbb{R}^m \times \mathbb{R}^m \times S^{N-1} \to [0,+\infty[$ be a function satisfying (G_1)–(G_3). Let F be the functional defined in (1.9). Then, for every $(v,u) \in L^1(\Omega;\mathbb{R}^m) \times L^1(\Omega;\mathbb{R}^d)$,*

$$\mathcal{F}(v,u;\Omega) = \begin{cases} \bar{F}_0(v,u;\Omega) & \text{if } (v,u) \in \mathrm{SBV}_0(\Omega;\mathbb{R}^m) \times \mathrm{BV}(\Omega;\mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases}$$

The paper is organized as follows. Preliminary results dealing with functions of bounded variation, perimeters and special functions of bounded variation that are piecewise constant are covered in § 2. The properties of the energy densities and several auxiliary results involved in the proofs of representation theorems 1.1 and 1.2 are discussed in § 3. The proof of the lower bound for $\mathcal{F}$ in (1.11) is presented in § 4, while § 5 contains the upper bound and the proof of theorem 1.2. The applications to optimal design problems as in [6] and the comparison with previous related relaxation results as in [25] (such as theorem 1.1) are discussed in § 6.

2. Preliminaries

We give a brief survey of functions of bounded variation and sets of finite perimeter.

In the following, $\Omega \subset \mathbb{R}^N$ is an open bounded set and we denote by $\mathcal{A}(\Omega)$ the family of all open subsets of Ω. The N-dimensional Lebesgue measure is denoted by $\mathcal{L}^N$, while $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure. The unit cube in $\mathbb{R}^N$, $(-\frac{1}{2},\frac{1}{2})^N$, is denoted by Q and we set $Q(x_0,\varepsilon) := x_0 + \varepsilon Q$ for $\varepsilon > 0$.

For every $\nu \in S^{N-1}$ we define $Q_\nu := R_\nu(Q)$, where R_ν is a rotation such that $R_\nu(e_N) = \nu$. The constant C may vary from line to line.

We denote by $\mathcal{M}(\Omega)$ the space of all signed Radon measures in Ω with bounded total variation. By the Riesz representation theorem, $\mathcal{M}(\Omega)$ can be identified with the dual of the separable space $\mathcal{C}_0(\Omega)$ of continuous functions on Ω vanishing on the boundary $\partial\Omega$. If $\lambda \in \mathcal{M}(\Omega)$ and $\mu \in \mathcal{M}(\Omega)$ is a non-negative Radon measure, we denote by $\mathrm{d}\lambda/\mathrm{d}\mu$ the Radon–Nikodým derivative of λ with respect to μ.

The following version of the Besicovitch differentiation theorem was proven by Ambrosio and Dal Maso [7, proposition 2.2].

THEOREM 2.1. *If λ and μ are Radon measures in Ω, $\mu \geqslant 0$, then there exists a Borel measure set $E \subset \Omega$ such that $\mu(E) = 0$ and, for every $x \in \operatorname{supp}\mu - E$,*

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}(x) := \lim_{\varepsilon\to 0^+} \frac{\lambda(x+\varepsilon C)}{\mu(x+\varepsilon C)}$$

exists and is finite whenever C is a bounded convex open set containing the origin.

We recall that the exceptional set E does not depend on C. An immediate corollary is the generalization of the Lebesgue–Besicovitch differentiation theorem given below.

THEOREM 2.2. *If μ is a non-negative Radon measure and if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N, \mu)$, then*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\mu(x+\varepsilon C)} \int_{x+\varepsilon C} |f(y) - f(x)|\,\mathrm{d}\mu(y) = 0$$

for μ-almost everywhere (a.e.), $x \in \mathbb{R}^N$ and for every bounded convex open set C containing the origin.

DEFINITION 2.3. A function $w \in L^1(\Omega;\mathbb{R}^d)$ is said to be of *bounded variation*, and we write $w \in \mathrm{BV}(\Omega;\mathbb{R}^d)$, if all its first distributional derivatives $D_j w_i$ belong to $\mathcal{M}(\Omega)$ for $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant N$.

The matrix-valued measure whose entries are $D_j w_i$ is denoted by Dw and $|Dw|$ stands for its total variation. We observe that if $w \in \mathrm{BV}(\Omega;\mathbb{R}^d)$, then $w \mapsto |Dw|(\Omega)$ is lower semi-continuous in $\mathrm{BV}(\Omega;\mathbb{R}^d)$ with respect to the $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$ topology.

We briefly recall some facts about functions of bounded variation. For more details we refer the reader to [9, 21, 27, 32].

DEFINITION 2.4. Let $w, w_n \in \mathrm{BV}(\Omega;\mathbb{R}^d)$. The sequence $\{w_n\}$ strictly converges in $\mathrm{BV}(\Omega;\mathbb{R}^d)$ to w if $\{w_n\}$ converges to w in $L^1(\Omega;\mathbb{R}^d)$ and $\{|Dw_n|(\Omega)\}$ converges to $|Dw|(\Omega)$ as $n \to \infty$.

DEFINITION 2.5. Given $w \in \mathrm{BV}(\Omega;\mathbb{R}^d)$ the *approximate upper limit* and the *approximate lower limit* of each component w^i, $i = 1, \dots, d$, are defined by

$$(w^i)^+(x) := \inf\left\{t \in \mathbb{R}\colon \lim_{\varepsilon\to 0^+} \frac{\mathcal{L}^N(\{y \in \Omega \cap Q(x,\varepsilon)\colon w^i(y) > t\})}{\varepsilon^N} = 0\right\}$$

and

$$(w^i)^-(x) := \sup\left\{t \in \mathbb{R}\colon \lim_{\varepsilon\to 0^+} \frac{\mathcal{L}^N(\{y \in \Omega \cap Q(x,\varepsilon)\colon w^i(y) < t\})}{\varepsilon^N} = 0\right\},$$

respectively. The *jump set* of w is given by

$$J_w := \bigcup_{i=1}^{d} \{x \in \Omega \colon (w^i)^-(x) < (w^i)^+(x)\}.$$

It can be shown that J_w and the complement of the set of Lebesgue points of w differ at most by a set of $\mathcal{H}^{N-1}$ measure 0. Moreover, J_w is $(N-1)$ rectifiable, i.e. there are C^1 hypersurfaces Γ_i such that $\mathcal{H}^{N-1}(J_w \setminus \bigcup_{i=1}^{\infty} \Gamma_i) = 0$.

Proposition 2.6. *If $w \in \mathrm{BV}(\Omega; \mathbb{R}^d)$, then the following hold.*

(i) *For $\mathcal{L}^N$-a.e. $x \in \Omega$,*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left\{ \frac{1}{\varepsilon^N} \int_{Q(x,\varepsilon)} |w(y) - w(x) - \nabla w(x)(y-x)|^{N/(N-1)} \, \mathrm{d}y \right\}^{(N-1)/N} = 0. \tag{2.1}$$

(ii) *For $\mathcal{H}^{N-1}$-a.e. $x \in J_w$ there exist $w^+(x), w^-(x) \in \mathbb{R}^d$ and $\nu(x) \in S^{N-1}$ normal to J_w at x, such that*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{Q_\nu^+(x,\varepsilon)} |w(y) - w^+(x)| \, \mathrm{d}y = 0,$$
$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{Q_\nu^-(x,\varepsilon)} |w(y) - w^-(x)| \, \mathrm{d}y = 0,$$

where $Q_\nu^+(x,\varepsilon) := \{y \in Q_\nu(x,\varepsilon) \colon \langle y-x, \nu \rangle > 0\}$ and $Q_\nu^-(x,\varepsilon) := \{y \in Q_\nu(x,\varepsilon) \colon \langle y-x, \nu \rangle < 0\}$.

(iii) *For $\mathcal{H}^{N-1}$-a.e. $x \in \Omega \setminus J_w$,*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{Q(x,\varepsilon)} |w(y) - w(x)| \, \mathrm{d}y = 0.$$

We observe that in the vector-valued case, in general $(w^i)^\pm \neq (w^\pm)^i$. In the following w^+ and w^- denote the vectors introduced in (ii), above.

Choosing a normal $\nu_w(x)$ to J_w at x, we denote the *jump* of w across J_w by $[w] := w^+ - w^-$. The distributional derivative of $w \in \mathrm{BV}(\Omega; \mathbb{R}^d)$ admits the decomposition

$$Dw = \nabla w \mathcal{L}^N \lfloor \Omega + ([w] \otimes \nu_w) \mathcal{H}^{N-1} \lfloor J_w + D^c w,$$

where ∇w represents the density of the absolutely continuous part of the Radon measure Dw with respect to the Lebesgue measure. The *Hausdorff*, or *jump*, *part* of Dw is represented by $([w] \otimes \nu_w)\mathcal{H}^{N-1} \lfloor J_w$ and $D^c w$ is the *Cantor part* of Dw. The measure $D^c w$ is singular with respect to the Lebesgue measure and it is diffuse, i.e. every Borel set $B \subset \Omega$ with $\mathcal{H}^{N-1}(B) < \infty$ has Cantor measure 0.

The following result, which will be exploited later, can be found in [25, lemma 2.6].

Lemma 2.7. *Let $w \in \mathrm{BV}(\Omega; \mathbb{R}^d)$. Then, for $\mathcal{H}^{N-1}$-a.e. x in J_w,*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^{N-1}} \int_{J_w \cap Q_{\nu(x)}(x,\varepsilon)} |w^+(y) - w^-(y)| \, \mathrm{d}\mathcal{H}^{N-1} = |w^+(x) - w^-(x)|.$$

In the following we give some preliminary notions related to sets of finite perimeter. For a detailed treatment we refer the reader to [9].

Definition 2.8. Let E be an $\mathcal{L}^N$-measurable subset of $\mathbb{R}^N$. For any open set $\Omega \subset \mathbb{R}^N$ the perimeter of E in Ω, denoted by $P(E;\Omega)$, is the variation of χ_E in Ω, i.e.

$$P(E;\Omega) := \sup\left\{ \int_E \operatorname{div}\varphi \,\mathrm{d}x \colon \varphi \in C_c^1(\Omega;\mathbb{R}^d),\ \|\varphi\|_{L^\infty} \leqslant 1 \right\}. \tag{2.2}$$

We say that E is a set of finite perimeter in Ω if $P(E;\Omega) < +\infty$.

Recalling that if $\mathcal{L}^N(E\cap\Omega)$ is finite, then $\chi_E \in L^1(\Omega)$, by [9, proposition 3.6], and it follows that E has finite perimeter in Ω if and only if $\chi_E \in \mathrm{BV}(\Omega)$ and $P(E;\Omega)$ coincides with $|D\chi_E|(\Omega)$, the total variation in Ω of the distributional derivative of χ_E. Moreover, a generalized Gauss–Green formula holds:

$$\int_E \operatorname{div}\varphi \,\mathrm{d}x = \int_\Omega \langle \nu_E, \varphi\rangle \,\mathrm{d}|D\chi_E| \quad \forall \varphi \in C_c^1(\Omega;\mathbb{R}^d),$$

where $D\chi_E = \nu_E|D\chi_E|$ is the polar decomposition of $D\chi_E$.

We also recall that, when dealing with sets of finite measure, a sequence of sets $\{E_n\}$ converges to E in measure in Ω if $\mathcal{L}^N(\Omega\cap(E_n\Delta E))$ converges to 0 as $n\to\infty$, where Δ stands for the symmetric difference. Analogously, the local convergence in measure corresponds to the above convergence in measure for any open set $A \subset\subset \Omega$. These convergences are equivalent to $L^1(\Omega)$ and $L^1_{\mathrm{loc}}(\Omega)$ convergences of the characteristic functions. We also recall that the local convergence in measure in Ω is equivalent to the convergence in measure in domains Ω with finite measure.

Denoting by $\mathcal{P}(\Omega)$ the family of all sets with finite perimeters in Ω, we recall the Fleming–Rishel formula (see [22, (4.59)]): for every $\Phi \in W^{1,1}(\Omega)$ the set $\{t \in \mathbb{R}\colon \{\Phi > t\} \notin \mathcal{P}(\Omega)\}$ is negligible in $\mathbb{R}$ and

$$\int_\Omega h|\nabla\Phi| \,\mathrm{d}x = \int_{-\infty}^{+\infty} \int_{\partial^*\{\Phi>t\}} h \,\mathrm{d}\mathcal{H}^{N-1} \,\mathrm{d}t \tag{2.3}$$

for every bounded Borel function $h\colon \Omega\to\mathbb{R}$, where $\partial^*\{\Phi>t\}$ denotes the essential boundary of $\{\Phi>t\}$ (see [9, definition 3.60]).

At this point we deal with functions of bounded variation whose Cantor part is null.

Definition 2.9. A function $v \in \mathrm{BV}(\Omega;\mathbb{R}^m)$ is said to be a special function of bounded variation, and we write $v \in \mathrm{SBV}(\Omega;\mathbb{R}^m)$, if $D^c v = \underline{0}$, i.e.

$$Dv = \nabla v \mathcal{L}^N \lfloor \Omega + ([v]\otimes \nu_v)\mathcal{H}^{N-1}\lfloor J_v.$$

The space $\mathrm{SBV}_0(\Omega;\mathbb{R}^m)$ is defined by

$$\mathrm{SBV}_0(\Omega;\mathbb{R}^m) := \{v \in \mathrm{SBV}(\Omega;\mathbb{R}^m)\colon \nabla v = 0 \text{ and } \mathcal{H}^{N-1}(J_v) < +\infty\}. \tag{2.4}$$

Clearly, any characteristic function of a set of finite perimeter is in $\mathrm{SBV}_0(\Omega)$.

We recall that a sequence of sets $\{E_i\}$ is a Borel partition of a Borel set $B \in \mathcal{B}(\mathbb{R}^N)$ if and only if

$$E_i \in \mathcal{B}(\mathbb{R}^N) \text{ for every } i,\ E_i \cap E_j = \emptyset \text{ for every } i \neq j \text{ and } \bigcup_{i=1}^\infty E_i = B.$$

The above requirements could be weakened by requiring that $|E_i \cap E_j| = 0$ for $i \neq j$ and $|B \Delta \bigcup_{i=1}^{\infty} E_i| = 0$. Such a sequence $\{E_i\}$ is said to be a Caccioppoli partition if and only if each E_i is a set of finite perimeter.

The following result, the proof of which can be found in [18], expresses the relation between Caccioppoli partitions and SBV_0 functions.

LEMMA 2.10. *If $v \in \mathrm{SBV}_0(\Omega; \mathbb{R}^m)$, then there exists a Borel partition $\{E_i\}$ of Ω and a sequence $\{v_i\} \subset \mathbb{R}^m$ such that*

$$v = \sum_{i=1}^{\infty} v_i \chi_{E_i} \quad \text{a.e. } x \in \Omega,$$

$$\mathcal{H}^{N-1}(J_v \cap \Omega) = \frac{1}{2} \sum_{i=1}^{\infty} \mathcal{H}^{N-1}(\partial^* E_i \cap \Omega) = \frac{1}{2} \sum_{i \neq j = 1}^{\infty} \mathcal{H}^{N-1}(\partial^* E_i \cap \partial^* E_j \cap \Omega),$$

$$(v^+, v^-, \nu_v) \equiv (v^i, v^j, \nu_i) \quad \text{a.e. } x \in \partial^* E_i \cap \partial E_j^* \cap \Omega,$$

where ν_i is the unit normal to $\partial^* E_i \cap \partial E_j^*$,

In the following we identify $(v, u) \in \mathrm{SBV}_0(\Omega; \mathbb{R}^m) \times \mathrm{BV}(\Omega; \mathbb{R}^d)$ with their precise representatives $(\tilde{v}, \tilde{u})$. (See [9, definition 3.63 and corollary 3.80] for the definition.)

REMARK 2.11. Since we have that $\mathrm{SBV}_0(\Omega; \mathbb{R}^m) \subset \mathrm{BV}(\Omega; \mathbb{R}^m)$, it follows that $(v, u) \in \mathrm{BV}(\Omega; \mathbb{R}^{m+d})$ for every $(v, u) \in \mathrm{SBV}_0(\Omega; \mathbb{R}^m) \times \mathrm{BV}(\Omega; \mathbb{R}^d)$. Thus, (v, u) is $|D^c(v, u)|$ measurable and since $D^c(v, u) = (\underline{0}, D^c u)$, we may say that v is $|D^c u|$ measurable.

The following compactness result for bounded sequences in $\mathrm{SBV}(\Omega; \mathbb{R}^m)$ is due to Ambrosio (see [2, 4]).

THEOREM 2.12. *Let $\Phi \colon [0, +\infty) \to [0, +\infty)$ and $\Theta \colon (0, +\infty] \to (0, +\infty]$ be two functions, convex and concave, respectively, such that*

$$\lim_{t \to \infty} \frac{\Phi(t)}{t} = +\infty, \quad \Phi \text{ is non-decreasing},$$

$$\Theta(+\infty) = \lim_{t \to \infty} \Theta(t), \quad \lim_{t \to 0^+} \frac{\Theta(t)}{t} = +\infty, \quad \Theta \text{ is non-decreasing.}$$

Let $\{v_n\}$ be a sequence of functions in $\mathrm{SBV}(\Omega; \mathbb{R}^m)$ such that

$$\sup_n \left\{ \int_\Omega \Phi(|\nabla v_n|) \, \mathrm{d}x + \int_{J_{v_n}} \Theta(|[v_n]|) \, \mathrm{d}\mathcal{H}^{N-1} + \int_\Omega |v_n| \, \mathrm{d}x \right\} < +\infty.$$

There then exists a subsequence $\{v_{n_k}\}$ converging in $L^1(\Omega; \mathbb{R}^m)$ to a function $v \in \mathrm{SBV}(\Omega; \mathbb{R}^m)$ and

$$\nabla v_{n_k} \rightharpoonup \nabla v \quad \text{in } L^1(\Omega; \mathbb{R}^{N \times m}), \qquad [v_{n_k}] \otimes \nu_{v_{n_k}} \mathcal{H}^{N-1} \lfloor J_{v_{n_k}} \overset{*}{\rightharpoonup} [v] \otimes \nu_v \mathcal{H}^{N-1} \lfloor J_v,$$

$$\int_{J_v \cap \Omega} \Theta(|[v]|) \, \mathrm{d}\mathcal{H}^{N-1} \leqslant \liminf_{n \to +\infty} \int_{J_{v_n} \cap \Omega} \Theta(|[v_n]|) \, \mathrm{d}\mathcal{H}^{N-1}.$$

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3. Auxiliary results

This section is mainly devoted to describing the properties of the energy densities involved in the integral representation of the relaxed functionals (1.5) and (1.12).

Recall that a Borel function $f\colon \mathbb{R}^m \times \mathbb{R}^{d\times N} \to [-\infty, +\infty]$ is said to be quasi-convex if

$$f(q,z) \leqslant \frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega f(q, z + \nabla\varphi(y))\,\mathrm{d}y \tag{3.1}$$

for every open bounded set $\Omega \subset \mathbb{R}^N$ with $\mathcal{L}^N(\partial\Omega) = 0$ for every $(q,z) \in \mathbb{R}^m \times \mathbb{R}^{d\times N}$ and every $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^d)$ whenever the right-hand side of (3.1) exists as a Lebesgue integral.

The quasi-convex envelope of $f\colon \mathbb{R}^m \times \mathbb{R}^{d\times N} \to [0, +\infty]$ is the largest quasi-convex function below f and it is denoted by Qf. If f is Borel and locally bounded from below then it can be shown that

$$Qf(q,z) = \inf\left\{ \int_Q f(q, z + \nabla\varphi)\,\mathrm{d}x \colon \varphi \in W_0^{1,\infty}(Q;\mathbb{R}^d) \right\} \tag{3.2}$$

for every $(q,z) \in \mathbb{R}^m \times \mathbb{R}^{d\times N}$.

The following result guarantees that the properties of f are inherited by Qf. Since the proof develops along the lines of [31, proposition 2.2], in turn inspired by [19], we omit it.

Proposition 3.1. *Let $f\colon \mathbb{R}^m \times \mathbb{R}^{d\times N} \to [0, +\infty)$ be a function satisfying (F_1)–(F_3) and let $Qf\colon \mathbb{R}^m \times \mathbb{R}^{d\times N} \to [0, +\infty)$ be its quasi-convexification, as in (3.2). Then Qf satisfies (F_1)–(F_3).*

Remark 3.2. Let $f\colon \mathbb{R}^m \times \mathbb{R}^{d\times N} \to [0, +\infty)$ be a function satisfying (F_1)–(F_4) with f^∞ as in (1.10).

(i) Recall that the recession function $f^\infty(q,\cdot)$ is positively 1-homogeneous for every $q \in \mathbb{R}^m$.

(ii) We observe that if f satisfies the growth condition (F_2), then we have that $\beta'|z| \leqslant f^\infty(q,z) \leqslant \beta|z|$ holds. Moreover, if f satisfies (F_3), then f^∞ satisfies

$$|f^\infty(q,z) - f^\infty(q',z)| \leqslant L|q-q'|\,|z|,$$

where L is the constant appearing in (F_3).

(iii) As showed in [25, remark 2.2(ii)], if a function $f\colon \mathbb{R}^m \times \mathbb{R}^{d\times N} \to [0, +\infty)$ is quasi-convex in the last variable and such that $f(q,z) \leqslant c(1 + |z|)$ for some $c > 0$, then its recession function $f^\infty(q,\cdot)$ is also quasi-convex.

(iv) A proof entirely similar to [10, proposition 3.4] (see also [31, proposition 2.6]) ensures that for every $(q,z) \in \mathbb{R}^m \times \mathbb{R}^{d\times N}$, $Q(f^\infty)(q,z) = (Qf)^\infty(q,z)$, and hence we will adopt the notation Qf^∞. In particular, if f satisfies (F_1)–(F_3), proposition 3.1 guarantees that Qf^∞ is continuous in both variables. Furthermore, for every $q \in \mathbb{R}^m$, $Qf^\infty(q,\cdot)$ is Lipschitz continuous in the last variable.

(v) $(Qf)^\infty$ satisfies the analogous condition to (F_4). We also observe, as emphasized in [25], that (F_4) is equivalent to saying that there exist $C > 0$ and $\alpha \in (0,1)$ such that

$$|f^\infty(q,z) - f(q,z)| \leqslant C(1 + |z|^{1-\alpha})$$

for every $(q,z) \in \mathbb{R}^m \times \mathbb{R}^{d\times N}$.

An argument entirely similar to [31, proposition 2.7] ensures that there exist $\alpha \in (0,1)$ and $C' > 0$ such that

$$|(Qf)^\infty(q,z) - Qf(q,z)| \leqslant C'(1 + |z|^{1-\alpha})$$

for every $(q,z) \in \mathbb{R}^m \times \mathbb{R}^{d\times N}$.

The following proposition, whose proof can be obtained by arguing exactly as in [12, p. 132], establishes the properties of the density K_3.

Proposition 3.3. *Let* $f\colon \mathbb{R}^m \times \mathbb{R}^{d\times N} \to [0,+\infty)$ *and let* $g\colon \mathbb{R}^m \times \mathbb{R}^m \times S^{N-1} \to (0,+\infty)$. *Let* K_3 *be the function defined in (1.13). If* (F_1)*–*(F_4) *and* (G_1)*–*(G_3) *hold, then so do the following.*

(a) $|K_3(a,b,c,d,\nu) - K_3(a',b',c',d',\nu)| \leqslant C(|a-a'| + |b-b'| + |c-c'| + |d-d'|)$ *for every* $(a,b,c,d,\nu), (a',b',c',d',\nu) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1}$.

(b) $\nu \mapsto K_3(a,b,c,d,\nu)$ *is upper semi-continuous for every* $(a,b,c,d) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^d$.

(c) K_3 *is upper semi-continuous in* $\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1}$.

(d) $K_3(a,b,c,d,\nu) \leqslant C(|a-b| + |c-d| + 1)$ *for every* $\nu \in S^{N-1}$. *More precisely, from the growth conditions* (F_2), (G_2) *and the definition of* K_3 *we have* $K_3(a,a,c,d,\nu) \leqslant C(|c-d|)$, $K_3(a,b,c,c,\nu) \leqslant C(1+|a-b|)$.

A Borel measurable function $g\colon \mathbb{R}^m \times \mathbb{R}^m \times S^{N-1} \to \mathbb{R}$ is BV-elliptic (see [3,9,14]) if, for all $(a,b,\nu) \in \mathbb{R}^m \times \mathbb{R}^m \times S^{N-1}$ and for any finite subset T of $\mathbb{R}^m$,

$$\int_{J_w \cap Q_\nu} g(w^+, w^-, \nu_w)\,\mathrm{d}\mathcal{H}^{N-1} \geqslant g(a,b,\nu) \tag{3.3}$$

for all $w \in \mathrm{BV}(Q_\nu;T)$ such that $w = v_0$ on ∂Q_ν, where

$$v_0 := \begin{cases} a & \text{if } x\cdot\nu > 0, \\ b & \text{if } x\cdot\nu \leqslant 0. \end{cases} \tag{3.4}$$

We are now in a position to provide some approximation results that allow us to reobtain the relaxed functionals and the related energy densities in terms of suitable relaxation procedures. To this end, we start by stating a result very similar to [12, proposition 3.5], which allows us to obtain K_3.

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PROPOSITION 3.4. *Let* $f\colon \mathbb{R}^m \times \mathbb{R}^{d\times N} \to [0,+\infty)$ *and* $g\colon \mathbb{R}^m \times \mathbb{R}^m \times S^{N-1} \to (0,+\infty)$ *be functions such that* (F_1)–(F_4) *and* (G_1)–(G_3) *hold, respectively. Let* K_3 *be the function defined in (1.13) and let* (v_0, u_0) *be given by*

$$v_0(x) := \begin{cases} a & \text{if } x\cdot\nu > 0, \\ b & \text{if } x\cdot\nu < 0, \end{cases} \qquad u_0(x) := \begin{cases} c & \text{if } x\cdot\nu > 0, \\ d & \text{if } x\cdot\nu < 0. \end{cases} \tag{3.5}$$

Then

$$\begin{aligned} K_3(a,b,c,d,\nu) = \inf_{(v_n,u_n)} \Big\{ \liminf_{n\to\infty} \Big(&\int_{Q_\nu} Qf^\infty(v_n(x), \nabla u_n(x))\,\mathrm{d}x \\ &+ \int_{Q_\nu\cap J_{v_n}} g(v_n^+(x), v_n^-(x), \nu_n(x))\,\mathrm{d}\mathcal{H}^{N-1}\Big) : \\ &(v_n,u_n) \in \mathrm{SBV}_0(Q_\nu;\mathbb{R}^m)\times W^{1,1}(Q_\nu;\mathbb{R}^d), \\ &(v_n,u_n)\to(v_0,u_0) \text{ in } L^1(Q_\nu;\mathbb{R}^{m+d}) \Big\} \\ =: k_3^*(a,b,c,d,\nu). \end{aligned}$$

REMARK 3.5.

(i) It is worthwhile observing that the above result ensures a sharper result than the one that is stated; namely, the same type of arguments in [12, proposition 3.5] allow us to obtain $K_3(a,b,c,d,\nu)$ as a relaxation procedure but with test sequences in $\mathcal{A}_3(a,b,c,d,\nu)$ converging to (v_0,u_0) in (3.5).

(ii) Notice that in (1.14), by virtue of the growth conditions on Qf^∞ (see remark 3.2), we can replace the space $W^{1,1}(Q_\nu;\mathbb{R}^d)$ with $W^{1,\infty}(Q_\nu;\mathbb{R}^d)$.

(iii) Under assumptions (G_1)–(G_3) the function K_3 in (1.13) can be obtained by taking test functions v either in $\mathrm{BV}(\Omega;T)$ for every $T\subset\mathbb{R}^m$ with $\mathrm{card}(T)$ finite, or in $\mathrm{SBV}_0(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)$. This is easy to verify by virtue of lemma 2.10. Namely, one can approximate functions v in $\mathrm{SBV}_0(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)$ by sequences $\{v_n\}$ in $\mathrm{BV}(\Omega;T_n)$ with $T_n\subset\mathbb{R}^m$ and $\mathrm{card}(T_n)$ finite. Moreover, $(v_n^+, v_n^-, \nu_{v_n}) \to (v^+, v^-, \nu_v)$ pointwise and we can apply the reverse of Fatou's lemma to obtain the equivalence between the two possible definitions of K_3.

(iv) Observe that the properties of K_3 and the assumptions on f and g allow us to replace the set $\mathrm{SBV}_0(Q;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)$ by $\mathrm{SBV}_0(\Omega;\mathbb{R}^m)$ in the definition of $\mathcal{A}_3$ (see (1.14)).

By the proposition below, in (1.11) we can replace f by its quasi-convexification Qf. We will omit the proof, which is quite standard, and exploits the relaxation results in the Sobolev spaces (see [19, theorem 9.8]).

PROPOSITION 3.6. *Let* $\Omega\subset\mathbb{R}^N$ *be a bounded open set, let* f *and* g *be as in theorem 4.1, let* Qf *be as in (3.2) and let* $\mathcal{F}$ *be given by (1.11). Then, for every*

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$A \in \mathcal{A}(\Omega)$ *and for every* $(v,u) \in \mathrm{SBV}_0(A;\mathbb{R}^m) \times \mathrm{BV}(A;\mathbb{R}^d)$,

$$\mathcal{F}(v,u;A) = \inf\left\{ \liminf_{n\to\infty} \int_A Qf(v_n, \nabla u_n)\,\mathrm{d}x + \int_{A\cap J_{v_n}} g(v_n^+, v_n^-, \nu_n)\,\mathrm{d}\mathcal{H}^{N-1} : \{(v_n,u_n)\} \subset \mathrm{SBV}_0(A;\mathbb{R}^m) \times W^{1,1}(A;\mathbb{R}^d), (v_n,u_n) \to (v,u) \text{ in } L^1(A;\mathbb{R}^m) \times L^1(A;\mathbb{R}^d) \right\}.$$

The following result is analogous to [24, proposition 2.4] and is devoted to replacing the test functions in (1.11) with smooth ones. We will omit the proof and just observe that (i) follows the same arguments as those in [1] with an application of Morse's measure covering theorem (see [23, theorem 1.147]).

Proposition 3.7. *Let* $f : \mathbb{R}^m \times \mathbb{R}^{d\times N} \to [0,+\infty]$ *be a function satisfying* (F_1)–(F_3) *and let* Qf *be given by (3.2).*

(i) *Let* B *be a ball in* $\mathbb{R}^N$. *If*

$$\bar{F}_0(v,u;B) \leqslant \liminf_{n\to\infty} \left(\int_B Qf(v_n, \nabla u_n)\,\mathrm{d}x + \int_{J_{v_n}\cap B} g(v_n^+, v_n^-, \nu_{v_n})\,\mathrm{d}\mathcal{H}^{N-1} \right) \tag{3.6}$$

holds for every $(v_n,u_n),(v,u) \in \mathrm{SBV}_0(\Omega;\mathbb{R}^m) \times W^{1,1}(\Omega;\mathbb{R}^d)$ *such that we have* $(v_n,u_n) \to (v,u)$ *in* $L^1(\Omega;\mathbb{R}^m) \times L^1(\Omega;\mathbb{R}^d)$, *then it holds for all open bounded sets* $\Omega \subset \mathbb{R}^N$.

(ii) *For every* $(v,u) \in L^1(\Omega;\mathbb{R}^m) \times L^1(\Omega;\mathbb{R}^d)$, $\{(v_n,u_n)\} \subset \mathrm{SBV}_0(\Omega;\mathbb{R}^m) \times W^{1,1}(\Omega;\mathbb{R}^d)$ *such that* $(v_n,u_n) \to (v,u)$ *in* $L^1(\Omega;\mathbb{R}^m) \times L^1(\Omega;\mathbb{R}^d)$, *there exists* $\{(\tilde{v}_n,\tilde{u}_n)\} \subset C_0^\infty(\mathbb{R}^N;\mathbb{R}^m) \times C_0^\infty(\mathbb{R}^N;\mathbb{R}^d)$ *such that* $(\tilde{v}_n,\tilde{u}_n) \to (v,u)$ *strictly in* $\mathrm{BV}(\Omega;\mathbb{R}^m) \times \mathrm{BV}(\Omega;\mathbb{R}^d)$ *and*

$$\liminf_{n\to\infty} \int_\Omega Qf(\tilde{v}_n, \nabla\tilde{u}_n)\,\mathrm{d}x = \liminf_{n\to\infty} \int_\Omega Qf(v_n, \nabla u_n)\,\mathrm{d}x.$$

In order to achieve the integral representation in (1.2) for the jump part, we need to modify $\{(v_n,u_n)\}$ to match the boundary in such a way that the new sequences will be in $\mathcal{A}_3(v^+(x), v^-(x), u^+(x_0), u^-(x_0), \nu(x_0))$, given in (1.14), and such that the energy doesn't increase. This is achieved in the next lemma that, for the sake of simplicity, is stated in the unit cube $Q \subset \mathbb{R}^N$ and with the normal to the jump set $\nu = e_N$. The proof relies on the techniques of [15, lemma 3.5], [25, lemma 3.1] and [5, lemma 4.4].

Lemma 3.8. *Let* $Q := [0,1]^N$ *and*

$$v_0(y) := \begin{cases} a & \text{if } x_N > 0, \\ b & \text{if } x_N < 0, \end{cases} \qquad u_0(y) := \begin{cases} c & \text{if } x_N > 0, \\ d & \text{if } x_N < 0. \end{cases}$$

Let $\{v_n\} \subset \mathrm{SBV}_0(Q;\mathbb{R}^m)$ *and* $\{u_n\} \subset W^{1,1}(Q;\mathbb{R}^d)$ *such that* $v_n \to v_0$ *in* $L^1(Q;\mathbb{R}^m)$ *and* $u_n \to u_0$ *in* $L^1(Q;\mathbb{R}^d)$.

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If ρ is a mollifier, $\rho_n := n^N \rho(nx)$, then there exists $\{(\zeta_n, \xi_n)\} \in \mathcal{A}_3(a, b, c, d, e_N)$ such that

$$\begin{aligned} \zeta_n &= v_0 \text{ on } \partial Q, & \zeta_n &\to v_0 \text{ in } L^1(Q; \mathbb{R}^m), \\ \xi_n &= \rho_{i(n)} * u_0 \text{ on } \partial Q, & \xi_n &\to u_0 \text{ in } L^1(Q; \mathbb{R}^d) \end{aligned}$$

and

$$\begin{aligned} \limsup_{n\to\infty} &\left(\int_Q Qf(\zeta_n, \nabla \xi_n)\,\mathrm{d}x + \int_{J_{\zeta_n}\cap Q} g(\zeta_n^+, \zeta_n^-, \nu_{\zeta_n})\,\mathrm{d}\mathcal{H}^{N-1} \right) \\ &\leqslant \liminf_{n\to\infty} \left(\int_Q Qf(v_n, \nabla u_n)\,\mathrm{d}x + \int_{J_{v_n}\cap Q} g(v_n^+, v_n^-, \nu_{v_n})\,\mathrm{d}\mathcal{H}^{N-1} \right). \end{aligned}$$

Proof. Without loss of generality, we may assume that

$$\begin{aligned} \liminf_{n\to\infty} &\left(\int_Q Qf(v_n, \nabla u_n)\,\mathrm{d}x + \int_{J_{v_n}\cap Q} g(v_n^+, v_n^-, \nu_{v_n})\,\mathrm{d}\mathcal{H}^{N-1} \right) \\ &= \lim_{n\to\infty} \left(\int_Q Qf(v_n, \nabla u_n)\,\mathrm{d}x + \int_{J_{v_n}\cap Q} g(v_n^+, v_n^-, \nu_{v_n})\,\mathrm{d}\mathcal{H}^{N-1} \right) < +\infty. \end{aligned}$$

The proof is divided into two steps.

STEP 1. First we claim that for every $\varepsilon > 0$, denoting $\|(v_0, u_0)\|_\infty$ by M_0, there exist sequences $\{\bar{u}_n\} \subset W^{1,1}(Q; \mathbb{R}^d) \cap L^\infty(Q; \mathbb{R}^d)$ and $\{\bar{v}_n\} \subset \mathrm{SBV}_0(Q; \mathbb{R}^m) \cap L^\infty(Q; \mathbb{R}^m)$ and a constant $C > 0$ such that $\|\bar{u}_n\|_\infty, \|\bar{v}_n\|_\infty \leqslant C$ for every n and

$$\begin{aligned} \liminf_{n\to\infty} &\left(\int_Q Qf(\bar{v}_n, \nabla \bar{u}_n)\,\mathrm{d}x + \int_{J_{\bar{v}_n}\cap Q} g(\bar{v}_n^+, \bar{v}_n^-, \nu_{\bar{v}_n})\,\mathrm{d}\mathcal{H}^{N-1} \right) \\ &\leqslant \lim_{n\to\infty} \left(\int_Q Qf(v_n, \nabla u_n)\,\mathrm{d}x + \int_{J_{v_n}\cap Q} g(v_n^+, v_n^-, \nu_{v_n})\,\mathrm{d}\mathcal{H}^{N-1} \right) + \varepsilon. \end{aligned} \tag{3.7}$$

To achieve the claim we can apply a truncation argument as in [15, lemma 3.5] (see also [12, lemma 3.7]). For $a_i \in \mathbb{R}$ (to be determined later) depending on ε and M_0, we define $\phi_i \in W_0^{1,\infty}(\mathbb{R}^{m+d}; \mathbb{R}^{m+d})$ such that

$$\phi_i(x) = \begin{cases} x, & |x| < a_i, \\ 0, & |x| \geqslant a_{i+1}, \end{cases} \tag{3.8}$$

$\|\nabla \phi_i\|_\infty \leqslant 1$, with $x \in \mathbb{R}^{m+d}$, $x \equiv (x_1, x_2)$, $x_1 \in \mathbb{R}^m$ and $x_2 \in \mathbb{R}^d$.

For any $n \in \mathbb{N}$ and for any i as above, let $(v_n^i, u_n^i) \in \mathrm{SBV}_0(Q; \mathbb{R}^m) \times W^{1,1}(Q; \mathbb{R}^d) \cap L^\infty(Q; \mathbb{R}^{m+d})$ be given by

$$(v_n^i, u_n^i) := \phi_i(v_n, u_n).$$

Considering the bulk part of the energy F in (1.9) and exploiting proposition 3.6 and the growth conditions on f and Qf, we have

$$\begin{aligned}\int_Q Qf(v_n^i,\nabla u_n^i)\,\mathrm{d}x &= \int_{Q\cap\{|(v_n,u_n)|\leqslant a_i\}} Qf(v_n,\nabla u_n)\,\mathrm{d}x\\ &\quad+\int_{Q\cap\{|(v_n,u_n)|>a_{i+1}\}} Qf(0,0)\,\mathrm{d}x\\ &\quad+\int_{Q\cap\{a_i<|(v_n,u_n)|\leqslant a_{i+1}\}} Qf(v_n^i,\nabla u_n^i)\,\mathrm{d}x\\ &\leqslant \int_Q Qf(v_n,\nabla u_n)\,\mathrm{d}x + C|Q\cap\{|(v_n,u_n)|>a_{i+1}\}|\\ &\quad+C_1\int_{A\cap\{a_i<|(v_n,u_n)|\leqslant a_{i+1}\}}(1+|\nabla u_n|)\,\mathrm{d}x.\end{aligned}$$

Concerning the surface term of the energy in (1.9), given that $((v_n^i)^{\pm},(u_n^i)^{\pm}) = \phi_i(v_n^{\pm},u_n^{\pm})$, without loss of generality one can assume that $|(v_n^-,u_n^-)|\leqslant|(v_n^+,u_n^+)|$ $\mathcal{H}^{N-1}$-a.e. on $J_{(v_n,u_n)}$ so we have that

$$\begin{aligned}&\int_{Q\cap J_{v_n^i}} g((v_n^i)^+,(v_n^i)^-,\nu_{v_n^i})\,\mathrm{d}\mathcal{H}^{N-1}\\ &\quad\leqslant\int_{J_{v_n}\setminus\{a_{i+1}\leqslant|(v_n^-,u_n^-)|\}\cap Q} g(\phi_i((v_n^i)^+,(u_n^i)^+),\phi_i((v_n^i)^-,(u_n^i)^-),\nu_{(v_n^i,u_n^i)})\,\mathrm{d}\mathcal{H}^{N-1}.\end{aligned}$$

Arguing as in [15, lemma 3.5] (see also [15, remark 3.6]) and exploiting the growth conditions on g, we can estimate $(1/k)\sum_{i=1}^k F(v_n^i,u_n^i;Q)$ for any fixed $k\in\mathbb{N}$ and for every $n\in\mathbb{N}$, with k independent on n. Then

$$\begin{aligned}&\frac{1}{k}\sum_{i=1}^k F(v_n^i,u_n^i;Q)\\ &\quad\leqslant F(v_n,u_n;Q)\\ &\qquad+\frac{1}{k}\sum_{i=2}^k\biggl(C|Q\cap\{|(v_n,u_n)|>a_{i+1}\}|+C_4\int_{J_2^i\cap Q}(1+|v_n^-|)\,\mathrm{d}\mathcal{H}^{N-1}\biggr)\\ &\qquad+\frac{1}{k}\biggl(c_2\int_Q(1+|\nabla u_n|)\,\mathrm{d}x+3C_4\int_{J_{v_n}\cap Q}(1+|v_n^+-v_n^-|)\,\mathrm{d}\mathcal{H}^{N-1}\biggr),\end{aligned}$$

where $J_2^i := \{|v_n^-|\leqslant a_i,\ |v_n^+|\geqslant a_{i+1}\}$. By the growth conditions, there exists a constant C such that

$$c_2\int_Q(1+|\nabla u_n|)\,\mathrm{d}x+3c_4\int_{J_{v_n}\cap Q}(1+|v_n^+-v_n^-|)\,\mathrm{d}\mathcal{H}^{N-1}\leqslant C$$

for every $n\in\mathbb{N}$. Choose $k\in\mathbb{N}$ such that $c/k\leqslant\varepsilon/3$. Moreover,

$$C\geqslant\int_{J_2^i\cap Q}|v_n^+-v_n^-|\,\mathrm{d}\mathcal{H}^{N-1}\geqslant\int_{J_2^i\cap Q}(|v_n^+|-|v_n^-|)\,\mathrm{d}\mathcal{H}^{N-1}\geqslant(a_{i+1}-a_i)\mathcal{H}^{N-1}(J_2^i\cap Q),$$

whence

$$\int_{J_2^i \cap Q} (1 + |v_n^-|) \, d\mathcal{H}^{N-1} \leqslant C \frac{1 + a_i}{a_{i+1} - a_i}.$$

The sequence $\{a_i\}$ can be chosen recursively as follows:

$$C_2 |Q \cap \{|(v_n, u_n)| > a_i\}| \leqslant \frac{\varepsilon}{3} \quad \text{for every } n \in \mathbb{N}, \; a_{i+1} \geqslant M_0,$$
$$c_4 C \frac{1 + a_i}{a_{i+1} - a_i} \leqslant \frac{\varepsilon}{3} \quad \text{for every } i \in \mathbb{N}.$$

This is possible since $\{(v_n, u_n)\}$ is bounded in L^1. We thus obtain

$$\frac{1}{k} \sum_{j=1}^{k} F(v_n^{i_j}, u_n^{i_j}; Q) \leqslant F(v_n, u_n; Q) + \varepsilon.$$

Therefore, for every $n \in \mathbb{N}$ there exists $i(n) \in \{1, \dots, k\}$ such that

$$F(v_n^{i_n}, u_n^{i_n}; Q) \leqslant F(v_n, u_n; Q) + \varepsilon.$$

It suffices to define $\bar{v}_n := v_n^{i_n}$ and $\bar{u}_n := u_n^{i_n}$ to achieve (3.7) and observe that $\{\bar{u}_n\}$ and $\{\bar{v}_n\}$ are bounded in L^∞, by construction.

STEP 2. This step is devoted to the construction of sequences $\{\xi_n\}$ and $\{\zeta_n\}$ as in the statement of lemma 3.8. Let $\bar{v}_n$ and $\bar{u}_n$ be as in step 1. Define

$$w_n(x) := (\rho_n * u_0)(x) = \int_{B(x,1/n)} \rho_n(x - y) u_0(y) \, dy.$$

As ρ is a mollifier we have, for each tangential direction $i = 1, \dots, N-1$, $w_n(x+e_i) = w_n(x)$ and so

$$w_n(y) = \begin{cases} c & \text{if } x_N > \dfrac{1}{n}, \\ d & \text{if } x_N < -\dfrac{1}{n}, \end{cases} \qquad \|\nabla w_n\|_\infty = O(n), \quad w_n \in \mathcal{A}_1(c, d, e_N),$$

where

$$\mathcal{A}_1(c, d, e_N) := \{u \in W^{1,1}(Q_\nu; \mathbb{R}^d) \colon u(y) = c \text{ if } y \cdot \nu = \tfrac{1}{2}, \; u(y) = d \text{ if } y \cdot \nu = -\tfrac{1}{2} \\ \text{with } u \text{ 1-periodic in } \nu_1, \dots, \nu_{N-1} \text{directions}\}.$$

Let

$$\alpha_n := \sqrt{\|\bar{u}_n - w_n\|_{L^1(Q;\mathbb{R}^d)} + \|\bar{v}_n - v_0\|_{L^1(Q)}},$$
$$k_n := n[1 + \|\bar{u}_n\|_{W^{1,1}(Q;\mathbb{R}^d)} + \|w_n\|_{W^{1,1}(Q;\mathbb{R}^d)} + \|\bar{v}_n\|_{\mathrm{BV}(Q)} + \|v_0\|_{\mathrm{BV}(Q)} + \mathcal{H}^{N-1}(J_{\bar{v}_n})]$$

and $s_n := \alpha_n / k_n$, where $[k]$ denotes the largest integer less than or equal to k. Since $\alpha_n \to 0^+$, we may assume that $0 \leqslant \alpha_n < 1$ and set $Q_0 := (1 - \alpha_n)Q$, $Q_i := (1 - \alpha_n + i s_n)Q$, $i = 1, \dots, k_n$.

Consider a family of cut-off functions $\varphi_i \in C_0^\infty(Q_i)$, $0 \leqslant \varphi_i \leqslant 1$, $\varphi_i = 1$ in Q_{i-1}, $\|\nabla\varphi_i\|_\infty = O(1/s_n)$ for $i = 1, \dots, k_n$ and define

$$u_n^{(i)}(x) := (1 - \varphi_i(x))w_n(x) + \varphi_i(x)\bar{u}_n(x).$$

Since $u_n^{(i)} = w_n$ on ∂Q, we have that $u_n^{(i)} \in \mathcal{A}_1(c, d, e_N)$. Clearly,

$$\nabla u_n^{(i)} = \nabla\bar{u}_n \text{ in } Q_{i-1}, \qquad \nabla u_n^{(i)} = \nabla w_n \text{ in } Q \setminus Q_i$$

and in $Q_i \setminus Q_{i-1}$,

$$\nabla u_n^{(i)} = \nabla w_n + \varphi_i(\nabla\bar{u}_n - \nabla w_n) + (\bar{u}_n - w_n) \otimes \nabla\varphi_i.$$

For $0 < t < 1$ define

$$v_{n,i}^t(x) := \begin{cases} v_0(x) & \text{if } \varphi_i(x) < t, \\ \bar{v}_n(x) & \text{if } \varphi_i(x) \geqslant t. \end{cases}$$

Clearly, $\lim_{n\to\infty} \|v_{n,i}^t - v_0\|_{L^1(Q)} = 0$ as $n \to \infty$, independently on i and t. For every n and i, by the Fleming–Rishel formula (2.3), it is possible to find $t_{n,i} \in \,]0, 1[$ such that

$$\{x \in Q \colon \varphi_i(x) < t_{n,i}\} \in \mathcal{P}(Q),$$
$$\mathcal{H}^{N-1}(J_{v_0} \cap \{x \in Q \colon \varphi_i(x) = t_{n,i}\}) = \mathcal{H}^{N-1}(J_{\bar{v}_n} \cap \{x \in Q \colon \varphi_i(x) = t_{n,i}\}) = 0,$$

where $\mathcal{P}(Q)$ denotes the family of sets with finite perimeter in Q. Let

$$v_{n,i}^{t_{n,i}} := \begin{cases} v_0(x) & \text{in } Q \cap \{x \in Q \colon \varphi_i(x) < t_{n,i}\}, \\ \bar{v}_n(x) & \text{in } Q \cap \{x \in Q \colon \varphi_i(x) \geqslant t_{n,i}\}. \end{cases}$$

Clearly, $\lim_{n\to\infty} \|v_{n,i}^{t_{n,i}} - v_0\|_{L^1(Q)} = 0$, $\{v_{n,i}^{t_{n,i}}\} \subset \mathrm{SBV}_0(Q; \mathbb{R}^m) \cap L^\infty(Q; \mathbb{R}^m)$ and, from step 1, it is uniformly bounded on n, i and t.

We have

$$\begin{aligned}
&\int_Q Qf(v_{n,i}^{t_{n,i}}, \nabla u_n^{(i)}) \, \mathrm{d}x + \int_{J_{v_{n,i}^{t_{n,i}}} \cap Q} g((v_{n,i}^{t_{n,i}})^+, (v_{n,i}^{t_{n,i}})^-, \nu_{v_{n,i}^{t_{n,i}}}) \, \mathrm{d}\mathcal{H}^{N-1} \\
&\quad \leqslant \int_Q Qf(\bar{v}_n, \nabla\bar{u}_n) \, \mathrm{d}x \\
&\qquad + C \int_{Q_i \setminus Q_{i-1}} \Big(1 + |\bar{u}_n(x) - w_n(x)| \frac{1}{s_n} + |\nabla\bar{u}_n(x)| + |\nabla w_n(x)|\Big) \, \mathrm{d}x \\
&\qquad + C \int_{Q \setminus Q_i} (1 + |\nabla w_n(x)|) \, \mathrm{d}x + \int_{Q \cap \{\varphi_i > t_{n,i}\}_1} g(\bar{v}_n^+, \bar{v}_n^-, \nu_{\bar{v}_n}) \, \mathrm{d}\mathcal{H}^{N-1} \\
&\qquad + |Dv_{n,i}^{t_{n,i}}|(Q \cap \{\varphi_i > t_{n,i}\}_0) + \mathcal{H}^{N-1}((Q \cap \{\varphi_i > t_{n,i}\}_0)) \\
&\qquad + |Dv_{n,i}^{t_{n,i}}|(\partial^*\{\varphi_i < t_{n,i}\}) + \mathcal{H}^{N-1}(\partial^*\{\varphi_i < t_{n,i}\}) \\
&\quad \leqslant \int_Q Qf(\bar{v}_n, \nabla\bar{u}_n) \, \mathrm{d}x + I_1 + \int_{Q \cap J_{\bar{v}_n}} g(\bar{v}_n^+, \bar{v}_n^-, \nu_{\bar{v}_n}) \, \mathrm{d}\mathcal{H}^{N-1} \\
&\qquad + C|Dv_0|(Q \setminus Q_i \colon \{\varphi_i > t_{n,i}\}_0) + \frac{C}{s_n} \int_{Q_i \setminus Q_{i-1}} |\bar{v}_n - v_0| \, \mathrm{d}x + \frac{1}{s_n} O(s_n),
\end{aligned}$$

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where

$$\{\varphi_i > t_{n,i}\}_1 := \left\{x \in Q\colon \frac{|\{x \in Q\colon \varphi_i > t_{n,i}\} \cap B_\rho(x)|}{|B_\rho(x)|} = 1\right\},$$
$$\{\varphi_i > t_{n,i}\}_0 := \left\{x \in Q\colon \frac{|\{x \in Q\colon \varphi_i > t_{n,i}\} \cap B_\rho(x)|}{|B_\rho(x)|} = 0\right\},$$
$$I_1 := C\int_{Q_i\setminus Q_{i-1}} \left(1 + |\bar{u}_n(x) - w_n(x)|\frac{1}{s_n} + |\nabla \bar{u}_n(x)| + |\nabla w_n(x)|\right)\mathrm{d}x + C\int_{Q\setminus Q_i}(1 + |\nabla w_n(x)|)\,\mathrm{d}x$$

and we have used (2.3) in the last two terms of the above estimate.

Averaging over all layers $Q_i \setminus Q_{i-1}$ one obtains

$$\begin{aligned}
&\frac{1}{k_n}\sum_{i=1}^{k_n}\left(\int_Q Qf(v_{n,i}^{t_{n,i}}, \nabla u_n^{(i)})\,\mathrm{d}x + \int_{Q\cap J_{v_{n,i}^{t_{n,i}}}} g((v_{n,i}^{t_{n,i}})^+, (v_{n,i}^{t_{n_i}})^-, \nu_{v_{n_i}^{t_{n,i}}})\,\mathrm{d}\mathcal{H}^{N-1}\right)\\
&\quad\leqslant \int_Q Qf(\bar{v}_n, \bar{u}_n)\,\mathrm{d}x + \int_{Q\cap J_{v_n}} g(\bar{v}_n^+, \bar{v}_n^-, \nu_{\bar{v}_n})\,\mathrm{d}\mathcal{H}^{N-1}\\
&\qquad + \frac{C}{k_n}\int_Q(1 + |\nabla\bar{u}_n| + |\nabla\bar{v}_n|)\,\mathrm{d}x + \frac{C}{k_n}\int_Q|\bar{u}_n - w_n|\frac{1}{s_n}\,\mathrm{d}x\\
&\qquad + C\int_{Q\setminus Q_0}(1 + |\nabla w_n|)\,\mathrm{d}x + C|Dv_0|(Q\setminus Q_0)\\
&\qquad + \frac{C}{s_n k_n}\int_{Q\setminus Q_0}|\bar{v}_n - v_0|\,\mathrm{d}x + \frac{C}{k_n}\\
&\quad\leqslant \int_Q Qf(\bar{v}_n, \nabla\bar{u}_n)\,\mathrm{d}x + \int_{Q\cap J_{v_n}} g(\bar{v}_n^+, \bar{v}_n^-, \nu_{\bar{v}_n})\,\mathrm{d}\mathcal{H}^{N-1}\\
&\qquad + \frac{C}{k_n}\int_Q(1 + |\nabla\bar{u}_n| + |\nabla\bar{v}_n|)\,\mathrm{d}x + \frac{C}{\alpha_n}\|\bar{u}_n - w_n\|_{L^1} + C\int_{Q\setminus Q_0}(1 + |\nabla w_n|)\,\mathrm{d}x\\
&\qquad + C|Dv_0|(Q\setminus Q_0) + \frac{C}{\alpha_n}\|\bar{v}_n - v_0\|_{L^1(Q)} + \frac{C}{k_n}.
\end{aligned}$$

Since $|Q \setminus Q_0| = O(\alpha_n)$ and $\nabla w_n(x) = 0$ if $|x_N| > 1/N$, we estimate

$$\int_{Q\setminus Q_0}(1+|\nabla w_n|)\,\mathrm{d}x \leqslant O(\alpha_n) + \mathcal{H}^{N-1}(Q\setminus Q_0\cap\{x_N = 0\})\int_{-1/n}^{1/n} O(n)\,\mathrm{d}x_N = O(\alpha_n).$$

The same argument exploited above in order to estimate $\int_{Q\setminus Q_0}\mathrm{d}x$ applies to estimate $|Dv_0|(Q\setminus Q_0)$ since v_0 is a jump function across $x_N = 0$, namely, $|Dv_0|(Q \setminus Q_0) = C\mathcal{H}^{N-1}(Q\setminus Q_0 \cap \{x_N = 0\})$, where we also recall that $Q_0 = \alpha_n Q$.

Setting

$$\varepsilon_n := O\left(\frac{1}{n}\right) + C\sqrt{\|\bar{u}_n - w_n\|_{L^1(Q;\mathbb{R}^d)} + \|\bar{v}_n - v_0\|_{L^1(Q)}} + O(\alpha_n)$$

we have that $\varepsilon_n \to 0^+$ and

$$\frac{1}{k_n}\sum_{i=1}^{k_n}\bigg(\int_Q Qf(v_{n,i}^{t_{n,i}},\nabla u_n^{(i)})\,\mathrm{d}x+\int_{Q\cap J_{v_{n_i}^{t_{n,i}}}} g((v_{n,i}^{t_{n,i}})^+,(v_{n,i}^{t_{n,i}})^-,\nu_{v_{n,i}^{t_{n,i}}})\,\mathrm{d}\mathcal{H}^{N-1}\bigg) \leqslant \int_Q Qf(\bar v_n,\nabla\bar u_n)\,\mathrm{d}x+\int_{Q\cap J_{\bar v_n}} g(\bar v_n^+,\bar v_n^-,\nu_{\bar v_n})\,\mathrm{d}\mathcal{H}^{N-1}+\varepsilon_n$$

and so there exists an index $i(n)\in\{1,\dots,k_n\}$ for which

$$\int_Q Qf(v_{n,i(n)}^{t_{n,i(n)}},\nabla u_n^{i(n)})\,\mathrm{d}x+\int_{Q\cap J_{v_{n_i}^{t_{n,i}}}} g((v_{n,i}^{t_{n,i}})^+,(v_{n,i}^{t_{n,i}})^-,\nu_{v_{n,i}^{t_{n,i}}})\,\mathrm{d}\mathcal{H}^{N-1} \leqslant \int_Q Qf(\bar v_n,\nabla\bar u_n)\,\mathrm{d}x+\int_{Q\cap J_{\bar v_n}} g(\bar v_n^+.\bar v_n^-,\nu_{\bar v_n})\,\mathrm{d}\mathcal{H}^{N-1}+\varepsilon_n.$$

It suffices to define $\xi_n := u_n^{i(n)}$ and $\zeta_n := v_{n,i(n)}^{t_{n,i(n)}}$ to get

$$\limsup_{n\to\infty}\bigg(\int_Q Qf(\zeta_n,\nabla\xi_n)\,\mathrm{d}x+\int_{J_{\zeta_n}\cap Q} g(\zeta_n^+,\zeta_n^-,\nu_{\zeta_n})\,\mathrm{d}\mathcal{H}^{N-1}\bigg) \leqslant \liminf_{n\to\infty}\bigg(\int_Q Qf(\bar v_n,\nabla\bar u_n)\,\mathrm{d}x+\int_{J_{\bar v_n}\cap Q} g(\bar v_n^+,\bar v_n^-,\nu_{\bar v_n})\,\mathrm{d}\mathcal{H}^{N-1}\bigg),$$

which concludes the proof. □

REMARK 3.9.
(i) Observe that arguing as in the first step of lemma 3.8, we have that for every $u\in \mathrm{BV}(\Omega;\mathbb{R}^d)$ and $v\in \mathrm{SBV}_0(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)$,

$$\begin{aligned}\mathcal{F}(v,u;A) &= \inf\bigg\{\liminf_{n\to\infty}\bigg(\int_A f(v_n,\nabla u_n)\,\mathrm{d}x+\int_{J_{v_n}\cap A} g({v_n}^+,{v_n}^-,\nu_{v_n})\,\mathrm{d}\mathcal{H}^{N-1}\bigg): \\ &\qquad \{v_n\}\subset \mathrm{SBV}_0(A;\mathbb{R}^m)\cap L^\infty(A;\mathbb{R}^m),\ \{u_n\}\subset W^{1,1}(A;\mathbb{R}^d), \\ &\qquad (v_n,u_n)\to(v,u) \text{ in } L^1(A;\mathbb{R}^{m+d}),\ \sup_n\|v_n\|_\infty<+\infty\bigg\}.\end{aligned}$$

(ii) Similarly, if $u\in \mathrm{BV}(\Omega;\mathbb{R}^d)\cap L^\infty(\Omega;\mathbb{R}^d)$, then

$$\begin{aligned}\mathcal{F}(v,u;A) &= \inf\bigg\{\liminf_{n\to\infty}\bigg(\int_A f(v_n,\nabla u_n)\,\mathrm{d}x+\int_{J_{v_n}\cap A} g({v_n}^+,{v_n}^-,\nu_{v_n})\,\mathrm{d}\mathcal{H}^{N-1}\bigg): \\ &\qquad \{v_n\}\subset \mathrm{SBV}_0(A;\mathbb{R}^m)\cap L^\infty(A;\mathbb{R}^m),\ \{u_n\}\subset W^{1,1}(A;\mathbb{R}^d)\cap L^\infty(A;\mathbb{R}^d), \\ &\qquad (v_n,u_n)\to(v,u) \text{ in } L^1(A;\mathbb{R}^{m+d}),\ \sup_n\|(v_n,u_n)\|_\infty<+\infty\bigg\}.\end{aligned}$$

(iii) Notice that an argument entirely similar to [14, lemmas 13 and 14] allows us to say that for every $(v,u)\in \mathrm{SBV}_0(\Omega;\mathbb{R}^m)\times \mathrm{BV}(\Omega;\mathbb{R}^d)$, we have that

$$\mathcal{F}(v,u;A)=\lim_{j\to\infty}\mathcal{F}(\phi_j(v,u);A),$$

where ϕ_j are the functions defined in (3.8).

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We conclude this section with a result that will be exploited later on.

Lemma 3.10. *Let X be a function space. For any $F\colon \mathbb{R}\times X\to[0,\infty]$,*

$$\limsup_{\varepsilon\to 0^+}\inf_{u\in X}F(\varepsilon,u)\leqslant\inf_{u\in X}\limsup_{\varepsilon\to 0^+}F(\varepsilon,u).$$

Proof. For any $\tilde{u}\in X$,

$$\inf_{u\in X}F(\varepsilon,u)\leqslant F(\varepsilon,\tilde{u}).$$

Thus,

$$\limsup_{\varepsilon\to 0^+}\inf_{u\in X}F(\varepsilon,u)\leqslant\limsup_{\varepsilon\to 0^+}F(\varepsilon,\tilde{u})$$

for every $\tilde{u}\in X$. Applying the infimum in the previous inequality, one obtains

$$\inf_{\tilde{u}\in X}\limsup_{\varepsilon\to 0^+}\inf_{u\in X}F(\varepsilon,u)\leqslant\inf_{\tilde{u}\in X}\limsup_{\varepsilon\to 0^+}F(\varepsilon,\tilde{u}).$$

Hence,

$$\limsup_{\varepsilon\to 0^+}\inf_{u\in X}F(\varepsilon,u)\leqslant\inf_{u\in X}\limsup_{\varepsilon\to 0^+}F(\varepsilon,u).$$

□

4. Lower bound

This section is devoted to the proof of the lower bound inequality for theorem 1.2. Recall that $\mathcal{F}$ and $\bar{F}_0$ are the functionals introduced in (1.11) and (1.12).

Theorem 4.1. *Let $\Omega\subset\mathbb{R}^N$ be a bounded open set, let $f\colon\mathbb{R}^m\times\mathbb{R}^d\to[0,+\infty)$ satisfy (F_1)–(F_4) and let $g\colon\mathbb{R}^m\times\mathbb{R}^m\times S^{N-1}\to[0,+\infty)$ satisfy (G_1)–(G_3). Then, for every $(v,u)\in\mathrm{SBV}_0(\Omega;\mathbb{R}^m)\times\mathrm{BV}(\Omega;\mathbb{R}^d)$ and for every sequence $\{(v_n,u_n)\}\subset\mathrm{SBV}_0(\Omega;\mathbb{R}^m)\times W^{1,1}(\Omega;\mathbb{R}^d)$ such that $(v_n,u_n)\to(v,u)$ in $L^1(\Omega;\mathbb{R}^m)\times L^1(\Omega;\mathbb{R}^d)$,*

$$\bar{F}_0(v,u;\Omega)\leqslant\liminf_{n\to\infty}F(v_n,u_n;\Omega),\tag{4.1}$$

where $\bar{F}_0$ is given by (1.12).

Proof. Let $(v,u)\in\mathrm{SBV}_0(\Omega;\mathbb{R}^m)\times\mathrm{BV}(\Omega;\mathbb{R}^d)$. Without loss of generality, we may assume that for every $\{(v_n,u_n)\}\subset\mathrm{SBV}_0(\Omega;\mathbb{R}^m)\times\mathrm{BV}(\Omega;\mathbb{R}^d)$ converging to (v,u) in $L^1(\Omega;\mathbb{R}^m)\times L^1(\Omega;\mathbb{R}^d)$,

$$\begin{aligned}\liminf_{n\to\infty}&\left(\int_\Omega f(v_n,\nabla u_n)\,\mathrm{d}x+\int_{J_{v_n}\cap\Omega}g(v_n^+,v_n^-,\nu_{v_n})\,\mathrm{d}\mathcal{H}^{N-1}\right)\\&=\lim_{n\to\infty}\left(\int_\Omega f(v_n,\nabla u_n)\,\mathrm{d}x+\int_{J_{v_n}\cap\Omega}g(v_n^+,v_n^-,\nu_{v_n})\,\mathrm{d}\mathcal{H}^{N-1}\right)<+\infty.\end{aligned}$$

For every Borel set $B\subset\Omega$ define

$$\mu_n(B):=\int_B f(v_n,\nabla u_n)\,\mathrm{d}x+\int_{J_{v_n}\cap B}g(v_n^+,v_n^-,\nu_{v_n})\,\mathrm{d}\mathcal{H}^{N-1}.$$

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Since $\{\mu_n\}$ is a sequence of non-negative Radon measures uniformly bounded in the space of measures, we can extract a subsequence, still denoted by $\{\mu_n\}$, weakly $*$ converging in the sense of measures to some Radon measure μ. Using the Radon–Nikodým theorem we can decompose μ as the sum of four mutually singular non-negative measures, namely

$$\mu = \mu_a \mathcal{L}^N + \mu_c |D^c u| + \mu_j \mathcal{H}^{N-1} \lfloor J_{(v,u)} + \mu_s, \tag{4.2}$$

where we are considering (v,u) as a unique field in $\mathrm{BV}(\Omega; \mathbb{R}^{m+d})$ and have exploited the fact that $D^c(v,u) = (\underline{0}, D^c u)$ (see remark 2.11). By the Besicovitch derivation theorem

$$\mu_a(x_0) = \lim_{\varepsilon \to 0^+} \frac{\mu(B(x_0,\varepsilon))}{\mathcal{L}^N(B(x_0,\varepsilon))} < +\infty \qquad \text{for } \mathcal{L}^N\text{-a.e. } x_0 \in \Omega, \tag{4.3 a}$$

$$\mu_j(x_0) = \lim_{\varepsilon \to 0^+} \frac{\mu(Q_\nu(x_0,\varepsilon))}{\mathcal{H}^{N-1}(Q_\nu(x_0,\varepsilon) \cap J_{(v,u)})} < +\infty \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x_0 \in J_{(v,u)} \cap \Omega, \tag{4.3 b}$$

$$\mu_c(x_0) = \lim_{\varepsilon \to 0^+} \frac{\mu(Q(x_0,\varepsilon))}{|Du|(Q(x_0,\varepsilon))} < +\infty \qquad \text{for } |D^c u|\text{-a.e. } x_0 \in \Omega. \tag{4.3 c}$$

We claim that

$$\mu_a(x_0) \geqslant Qf(v(x_0), \nabla u(x_0)) \qquad \text{for } \mathcal{L}^N\text{-a.e. } x_0 \in \Omega, \tag{4.4}$$

$$\mu_j(x_0) \geqslant K_3(v^+(x_0), v^-(x_0), u^+(x_0), u^-(x_0), \nu_{(v,u)}) \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x_0 \in J_{(v,u)} \cap \Omega, \tag{4.5}$$

$$\mu_c(x_0) \geqslant (Qf)^\infty\left(v(x_0), \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}(x_0)\right) \quad \text{for } |D^c u|\text{-a.e. } x_0 \in \Omega, \tag{4.6}$$

where Qf is the density introduced in (3.2), Qf^∞ is its recession function as in (1.10) and K_3 is given by (1.13). If (4.4)–(4.6) hold, then (4.1) follows immediately. Indeed, since $\mu_n \overset{*}{\rightharpoonup} \mu$ in the sense of measures,

$$\begin{aligned}
&\liminf_{n\to\infty} \left(\int_\Omega f(v_n, \nabla u_n)\,\mathrm{d}x + \int_{J_{v_n} \cap \Omega} g(v_n^+, v_n^-, \nu_{v_n})\,\mathrm{d}\mathcal{H}^{N-1} \right) \\
&\quad \geqslant \liminf_{n\to\infty} \mu_n(\Omega) \\
&\quad \geqslant \mu(\Omega) \\
&\quad \geqslant \int_\Omega \mu_a \;\mathrm{d}x + \int_{J_{(v,u)}} \mu_j \,\mathrm{d}\mathcal{H}^{N-1} + \int_\Omega \mu_c \,\mathrm{d}|D^c u| \\
&\quad \geqslant \int_\Omega Qf(v(x), \nabla u(x))\,\mathrm{d}x + \int_{J_u \cap \Omega} K_3(v^+(x), v^-(x), u^+(x), u^-(x), \nu_{(v,u)})\,\mathrm{d}\mathcal{H}^{N-1} \\
&\qquad\quad + \int_\Omega (Qf)^\infty\left(v(x), \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}(x)\right) \mathrm{d}|D^c u|,
\end{aligned}$$

where we have used the fact that μ_s is non-negative.

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We prove (4.4)–(4.6) using the blow-up method introduced in [24].

STEP 1. Let $x_0 \in \Omega$ be a Lebesgue point for ∇u and v such that $x_0 \notin J_{(v,u)}$ and (2.1) applied to u and (4.3 a) hold.

We observe that

$$\liminf_{n\to\infty}\left(\int_\Omega f(v_n,\nabla u_n)\,\mathrm{d}x+\int_{J_{v_n}\cap\Omega} g(v_n^+,v_n^-,\nu_{v_n})\,\mathrm{d}\mathcal{H}^{N-1}\right)$$
$$\geqslant \liminf_{n\to\infty}\int_\Omega f(v_n,\nabla u_n)\,\mathrm{d}x$$
$$\geqslant \liminf_{n\to\infty}\int_\Omega Qf(v_n,\nabla u_n)\,\mathrm{d}x.$$

Note that, by proposition 3.1, Qf satisfies (F_1)–(F_3). By proposition 3.7, we may assume that $\{(v_n,u_n)\}\subset C_0^\infty(\mathbb{R}^N;\mathbb{R}^m)\times C_0^\infty(\mathbb{R}^N;\mathbb{R}^d)$ and applying [25, (2.10) in theorem 2.19] to the functional

$$G\colon (v,u)\in W^{1,1}(\Omega;\mathbb{R}^{m+d})\to\int_\Omega Qf(v,\nabla u)\,\mathrm{d}x$$

we obtain (4.4).

STEP 2. Now we prove (4.5).

Recall that $J_{(v,u)}=J_v\cup J_u$ and $\nu_{(v,u)}=\nu_v$ for every $(v,u)\in \mathrm{SBV}_0(\Omega;\mathbb{R}^m)\times W^{1,1}(\Omega;\mathbb{R}^d)$. By lemma 2.7, proposition 2.6(ii) and theorem 2.1, we may fix $x_0\in J_{(v,u)}\cap\Omega$ such that

$$\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon^{N-1}}\int_{J_{(v,u)}\cap Q_\nu(x_0,\varepsilon)}(|v^+(x)-v^-(x_0)|+|u^+(x)-u^-(x_0)|)\,\mathrm{d}\mathcal{H}^{N-1}$$
$$=|v^+(x_0)-v^-(x_0)|+|u^+(x_0)-u^-(x_0)|, \tag{4.7}$$

$$\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon^{N}}\int_{\{x\in Q_\nu(x_0,\varepsilon)\colon (x-x_0)\cdot\nu(x)>0\}}|v(x)-v^+(x_0)|^{N/(N-1)}\,\mathrm{d}x$$
$$+\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon^{N}}\int_{\{x\in Q_\nu(x_0,\varepsilon)\colon (x-x_0)\cdot\nu(x)>0\}}|u(x)-u^+(x_0)|^{N/(N-1)}\,\mathrm{d}x=0, \tag{4.8}$$

$$\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon^{N}}\int_{\{x\in Q_\nu(x_0,\varepsilon)\colon (x-x_0)\cdot\nu(x)<0\}}|v(x)-v^-(x_0)|^{N/(N-1)}\,\mathrm{d}x$$
$$+\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon^{N}}\int_{\{x\in Q_\nu(x_0,\varepsilon)\colon (x-x_0)\cdot\nu(x)<0\}}|u(x)-u^-(x_0)|^{N/(N-1)}\,\mathrm{d}x=0, \tag{4.9}$$

$$\mu_j(x_0)=\lim_{\varepsilon\to0^+}\frac{\mu(x_0+\varepsilon Q_{\nu(x_0)})}{\mathcal{H}^{N-1}\lfloor J_{(v,u)}(x_0+\varepsilon Q_{\nu(x_0)})}\text{ exists and is finite.} \tag{4.10}$$

For simplicity of notation we write $Q:=Q_{\nu(x_0)}$. Then, by (4.10),

$$\mu_j(x_0)=\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon^{N-1}}\int_{x_0+\varepsilon Q}\mathrm{d}\mu(x). \tag{4.11}$$

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Without loss of generality, we may choose $\varepsilon > 0$ such that $\mu(\partial(x_0+\varepsilon Q)) = 0$. Since $Qf \leqslant f$, we have

$$
\begin{aligned}
&\mu_j(x_0)\\
&\quad\geqslant \lim_{\varepsilon\to 0^+}\lim_{n\to\infty}\frac{1}{\varepsilon^{N-1}}\left(\int_{x_0+\varepsilon Q} Qf(v_n(x),\nabla u_n(x))\,\mathrm{d}x + \int_{J_{v_n}} g(v_n^+,v_n^-,\nu_{v_n})\,\mathrm{d}\mathcal{H}^{N-1}\right)\\
&\quad= \lim_{\varepsilon\to 0^+}\lim_{n\to\infty}\varepsilon\int_Q Qf(v_n(x_0+\varepsilon y),\nabla u_n(x_0+\varepsilon y))\,\mathrm{d}y\\
&\qquad\quad + \int_{Q\cap J(v_n,u_n)-x_0/\varepsilon} g(v_n^+(x_0+\varepsilon y), v_n^-(x_0+\varepsilon y), \nu_{(v_n,u_n)}(x_0+\varepsilon y))\,\mathrm{d}\mathcal{H}^{N-1}(y).
\end{aligned}
$$

Define

$$
\left.\begin{aligned}
v_{n,\varepsilon}(y) := v_n(x_0+\varepsilon y),\qquad u_{n,\varepsilon}(y) := u_n(x_0+\varepsilon y),\\
\nu_{n,\varepsilon}(y) := \nu_{(v_n,u_n)}(x_0+\varepsilon y)
\end{aligned}\right\} \tag{4.12}
$$

and

$$
v_0(y) := \begin{cases} v^+(x_0) & \text{if } y\nu(x_0) > 0,\\ v^-(x_0) & \text{if } y\nu(x_0) < 0,\end{cases}\qquad u_0(y) := \begin{cases} u^+(x_0) & \text{if } y\nu(x_0) > 0,\\ u^-(x_0) & \text{if } y\nu(x_0) < 0.\end{cases} \tag{4.13}
$$

Since $(v_n,u_n)\to(v,u)$ in $L^1(\Omega;\mathbb{R}^{m+d})$, by (4.8) and (4.9) one obtains

$$
\begin{aligned}
&\lim_{\varepsilon\to 0^+}\lim_{n\to\infty}\int_Q |v_{n,\varepsilon}(y)-v_0(y)|\,\mathrm{d}y\\
&\qquad= \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^N}\left(\int_{\{x\in x_0+\varepsilon\partial Q\colon (x-x_0)\nu(x_0)>0\}} |v(x)-v^+(x_0)|\,\mathrm{d}x\right.\\
&\qquad\qquad\qquad\left. + \int_{\{x\in x_0+\varepsilon\partial Q\colon (x-x_0)\nu(x_0)<0\}} |v(x)-v^-(x_0)|\,\mathrm{d}x\right) = 0
\end{aligned} \tag{4.14}
$$

and

$$
\begin{aligned}
&\lim_{\varepsilon\to 0^+}\lim_{n\to\infty}\int_Q |u_{n,\varepsilon}(y)-u_0(y)|\,\mathrm{d}y\\
&\qquad= \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^N}\left(\int_{\{x\in x_0+\varepsilon\partial Q\colon (x-x_0)\nu(x_0)>0\}} |u(x)-u^+(x_0)|\,\mathrm{d}x\right.\\
&\qquad\qquad\qquad\left. + \int_{\{x\in x_0+\varepsilon\partial Q\colon (x-x_0)\nu(x_0)<0\}} |u(x)-u^-(x_0)|\,\mathrm{d}x\right) = 0.
\end{aligned} \tag{4.15}
$$

Thus,

$$
\begin{aligned}
\mu_j(x_0) \geqslant \lim_{\varepsilon\to 0^+}\lim_{n\to\infty}&\left(\int_Q Qf^\infty\Big(v_{n,\varepsilon}(y),\nabla u_{n,\varepsilon}(y)\Big)\,\mathrm{d}y\right.\\
&\quad + \int_{Q\cap J(v_{n,\varepsilon},u_{n_\varepsilon})} g(v_{n,\varepsilon}^+, v_{n,\varepsilon}^-, \nu_{v_{n,\varepsilon}})\,\mathrm{d}\mathcal{H}^{N-1}(y)\\
&\quad\left. + \int_Q\left(\varepsilon Qf\Big(v_{n,\varepsilon}(y),\frac{1}{\varepsilon}\nabla u_{n,\varepsilon}(y)\Big) - Qf^\infty(v_{n,\varepsilon},\nabla u_{n,\varepsilon})\right)\mathrm{d}y\right).
\end{aligned}
$$

Exploiting remark 3.2(v), we can argue as in the estimates [25, (3.3)–(3.5)], thus obtaining

$$\mu_j(x_0) \geqslant \liminf_{\varepsilon\to 0^+} \liminf_{n\to\infty} \biggl(\int_Q Qf^\infty(v_{n,\varepsilon}(y), \nabla u_{n,\varepsilon}(y))\,\mathrm{d}y + \int_{Q\cap J(v_{n,\varepsilon},u_{n,\varepsilon})} g(v^+_{n,\varepsilon}, v^-_{n,\varepsilon}, \nu_{v_{n,\varepsilon}})\,\mathrm{d}\mathcal{H}^{N-1}(y)\biggr).$$

Since $(v_{n,\varepsilon}, u_{n,\varepsilon}) \to (v_0, u_0)$ in $L^1(Q;\mathbb{R}^{m+d})$ as $n\to\infty$ and $\varepsilon \to 0^+$, by a standard diagonalization argument, as in [12, theorem 4.1, steps 2 and 3], we obtain a sequence $(\bar v_k, \bar u_k)$ converging to (v_0, u_0) in $L^1(Q;\mathbb{R}^{m+d})$ as $k\to\infty$ such that

$$\mu_j(x_0) \geqslant \lim_{k\to\infty} \biggl(\int_Q Qf^\infty(\bar v_k(y), \nabla \bar u_k(y))\,\mathrm{d}y + \int_{Q\cap J_{(v_k,w_k)}} g(\bar v^+_k, \bar v^-_k, \nu_{\bar v_k})\,\mathrm{d}\mathcal{H}^{N-1}(y)\biggr).$$

Applying lemma 3.8 with Qf replaced by Qf^∞ and using remark 3.2(v), we may find $\{(\zeta_k, \xi_k)\} \in \mathcal{A}_3(v^+(x_0), v^-(x_0), u^+(x_0), u^-(x_0), \nu(x_0))$ such that

$$\begin{aligned}\mu_j(x_0) &\geqslant \lim_{k\to\infty} \biggl(\int_Q Qf^\infty(\zeta_k, \nabla\xi_k)\,\mathrm{d}x + \int_{Q\cap J_{(\zeta_k,\xi_k)}} g(\zeta^+_k, \zeta^-_k, \nu_{\zeta_k})\,\mathrm{d}\mathcal{H}^{N-1}\biggr)\\ &\geqslant K_3(v^+(x_0), v^-(x_0), u^+(x_0), u^-(x_0), \nu(x_0)).\end{aligned}$$

Step 3. Here we show (4.6).

Let $(v, u) \in \mathrm{SBV}_0(\Omega;\mathbb{R}^m) \times \mathrm{BV}(\Omega;\mathbb{R}^d)$ and note, as already emphasized in remark 2.11, that $|D^c(v, u)| = |D^c u|$. For $|D^c u|$-a.e. $x_0 \in \Omega$, we have

$$\lim_{\varepsilon\to 0^+} \frac{|D(v,u)|(Q(x_0,\varepsilon))}{|D^c(v,u)|(Q(x_0,\varepsilon))} = \lim_{\varepsilon\to 0^+} \frac{|D(v,u)|(Q(x_0,\varepsilon))}{|D^c u|(Q(x_0,\varepsilon))} = 1.$$

And so, by [25, theorems 2.4(iii) and 2.11] and by theorem 2.1, for $|D^c u|$-a.e. $x_0 \in \Omega$ we have

$$\mu_c(x_0) = \lim_{\varepsilon\to 0^+} \frac{\mu(Q(x_0,\varepsilon))}{|Du|(Q(x_0,\varepsilon))},$$

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \int_{Q(x_0,\varepsilon)} (|u(x) - u(x_0)| + |v(x) - v(x_0)|)\,\mathrm{d}x = 0$$

for $\mathcal{H}^{N-1}$-a.e. $x_0 \in \Omega \setminus J_{(v,u)}$,

$$A(x_0) = \lim_{\varepsilon\to 0^+} \frac{(D(v,u))(Q(x_0,\varepsilon))}{|D(v,u)|(Q(x_0,\varepsilon))}, \quad \|A(x_0)\| = 1, \quad A(x_0) = a\otimes\nu$$

with $a \in \mathbb{R}^d$ and $\nu \in S^{N-1}$,

$$\lim_{\varepsilon\to 0^+} \frac{|D(v,u)|(Q(x_0,\varepsilon))}{\varepsilon^{N-1}} = \lim_{\varepsilon\to 0^+} \frac{|Du|(Q(x_0,\varepsilon))}{\varepsilon^{N-1}} = 0,$$

$$\lim_{\varepsilon\to 0^+} \frac{|D(v,u)|(Q(x_0,\varepsilon))}{\varepsilon^{N}} = \lim_{\varepsilon\to 0^+} \frac{|Du|(Q(x_0,\varepsilon))}{\varepsilon^{N}} = \infty.$$

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Arguing as in the end of step 1, by proposition 3.7(ii) we may assume that $\{(\tilde{v}_n, \tilde{u}_n)\} \subset C_0^\infty(\mathbb{R}^N; \mathbb{R}^{m+d})$. Applying [25, (2.12) in theorem 2.19] to the functional $G\colon (v,u) \in W^{1,1}(\Omega; \mathbb{R}^{m+d}) \to \int_\Omega Qf(v, \nabla u)\,\mathrm{d}x$, we obtain, for $|D^c(v,u)|$-a.e. $x_0 \in \Omega$,

$$\mu_c(x_0) \geqslant (Qf)^\infty\biggl(v(x_0), \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}(x_0)\biggr),$$

which concludes the proof. □

5. Upper bound

This section is devoted to proving that $\mathcal{F} \leqslant \bar{F}_0$.

THEOREM 5.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, let $f\colon \mathbb{R}^d \times \mathbb{R}^m \to [0, +\infty)$ be a function satisfying (F_1)–(F_4) and let $g\colon \mathbb{R}^m \times \mathbb{R}^m \times S^{N-1} \to [0, +\infty[$ be a function satisfying (G_1)–(G_3).*

Then, for every $(v,u) \in \mathrm{SBV}_0(\Omega; \mathbb{R}^m) \times \mathrm{BV}(\Omega; \mathbb{R}^d)$ and for every $A \in \mathcal{A}(\Omega)$, there exist sequences $\{v_n\} \subset \mathrm{SBV}_0(\Omega; \mathbb{R}^m)$, $\{u_n\} \subset W^{1,1}(\Omega; \mathbb{R}^d)$ such that $v_n \to v$ in $L^1(\Omega; \mathbb{R}^m)$, $u_n \to u$ in $L^1(\Omega; \mathbb{R}^d)$ and

$$\liminf_{n\to\infty} F(v_n, u_n; A) \leqslant \bar{F}_0(v, u; A).$$

Before proving the upper bound we recall our strategy, which was first proposed in [8] and further developed in [25]. Namely, first we will show that $\mathcal{F}(v,u;\cdot)$ is a variational functional with respect to the L^1 topology and that

$$\mathcal{F}(v,u;\cdot) \leqslant \mathcal{L}^N + |Dv| + |Du| + \mathcal{H}^{N-1}\lfloor J_v.$$

Next, using Besicovitch's differentiation theorem, a blow-up argument will provide an upper bound estimate in terms of $\bar{F}_0$, first for bulk and Cantor parts, then also for the jump part, when the target functions (v,u) are bounded. Finally, the same approximation as in [8, theorem 4.9] will give the estimate for every $(v,u) \in \mathrm{SBV}_0(\Omega; \mathbb{R}^m) \times \mathrm{BV}(\Omega; \mathbb{R}^d)$.

We recall that $\mathcal{F}(v,u;\cdot)$ is said to be a variational functional with respect to the L^1 topology if the following hold.

(i) $\mathcal{F}(\cdot,\cdot;A)$ is local, i.e. $\mathcal{F}(v,u;A) = \mathcal{F}(v',u';A)$ for every $v, v' \in \mathrm{SBV}_0(A; \mathbb{R}^m)$, $u, u' \in \mathrm{BV}(A; \mathbb{R}^d)$ satisfying $u = u'$ and $v = v'$ a.e. in A.

(ii) $\mathcal{F}(\cdot,\cdot;A)$ is sequentially lower semi-continuous, i.e. if $v_n, v \in \mathrm{BV}(A; \mathbb{R}^m)$, $u_n, u \in \mathrm{BV}(A; \mathbb{R}^d)$ and $v_n \to v$ in $L^1(A; \mathbb{R}^m)$, $u_n \to u$ in $L^1(A; \mathbb{R}^d)$, then $\mathcal{F}(v,u;A) \leqslant \liminf_{n\to\infty} \mathcal{F}(v_n, u_n; A)$.

(iii) $\mathcal{F}(\cdot,\cdot;A)$ is the trace on $\{A \subset \Omega\colon A \text{ is open}\}$ of a Borel measure on $\mathcal{B}(\Omega)$, the family of all Borel subsets of Ω.

Since the lower semi-continuity and the locality of $\mathcal{F}(\cdot,\cdot;A)$ follow from its definition, it remains to prove (iii). This is the target of the following lemma, where (iii) will be obtained via a refinement of De Giorgi–Letta criterion (see [20, corollary 5.2]).

LEMMA 5.2. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary and let f and g be as in theorem 5.1. For every $(v,u) \in \mathrm{SBV}_0(\Omega;\mathbb{R}^m) \times \mathrm{BV}(\Omega;\mathbb{R}^d)$ the set function $\mathcal{F}(v,u;\cdot)$ in (1.11) is the trace of a Radon measure absolutely continuous with respect to $\mathcal{L}^N + |Dv| + |Du| + \mathcal{H}^{N-1}\lfloor J_v$.*

Proof. An argument very similar to [13, lemma 2.6 and remark 2.7] and [10, lemma 4.7] entails

$$\mathcal{F}(v,u;A) \leqslant C(\mathcal{L}^N(A) + |Dv|(A) + |Du|(A) + \mathcal{H}^{N-1}\lfloor J_v(A)).$$

By [20, corollary 5.2], to obtain (iii) it suffices to prove that

$$\mathcal{F}(v,u;A) \leqslant \mathcal{F}(v,u;B) + \mathcal{F}(v,u;A \setminus \bar{U})$$

for all $A, U, B \in \mathcal{A}(\Omega)$ with $U \subset\subset B \subset\subset A$, $u \in \mathrm{BV}(\Omega;\mathbb{R}^d)$ and $v \in \mathrm{SBV}_0(\Omega;\mathbb{R}^m)$.

We start by assuming that $v \in \mathrm{SBV}_0(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$.

Fix $\eta > 0$ and find $\{w_n\} \subset W^{1,1}((A \setminus \bar{U});\mathbb{R}^d)$, $\{v_n\} \subset \mathrm{SBV}_0(A \setminus \bar{U};\mathbb{R}^m) \cap L^\infty(A \setminus \bar{U};\mathbb{R}^m)$ (see remark 3.9) such that $w_n \to u$ in $L^1((A \setminus \bar{U});\mathbb{R}^d)$, $v_n \to v$ in $L^1((A \setminus \bar{U});\mathbb{R}^m)$ and

$$\limsup_{n\to\infty} \left(\int_{A\setminus\bar{U}} f(v_n, \nabla w_n)\,\mathrm{d}x + \int_{A\setminus\bar{U}\cap J_{v_n}} g(v_n^+, v_n^-, \nu_{v_n})\,\mathrm{d}\mathcal{H}^{N-1} \right) \leqslant \mathcal{F}(v,u;A\setminus\bar{U}) + \eta. \quad (5.1)$$

Extract a subsequence still denoted by n such that the above upper limit is a limit.

Let B_0 be an open subset of Ω with Lipschitz boundary such that $U \subset\subset B_0 \subset\subset B$. There then exist $\{u_n\} \subset W^{1,1}(B_0;\mathbb{R}^d)$ and $\{\bar{v}_n\} \subset \mathrm{SBV}_0(B_0;\mathbb{R}^m) \cap L^\infty(B_0;\mathbb{R}^m)$ (see remark 3.9(i)) such that $u_n \to u$ in $L^1(B_0;\mathbb{R}^d)$, $\bar{v}_n \to v$ in $L^1(B_0;\mathbb{R}^m)$ and

$$\mathcal{F}(v,u;B_0) = \lim_{n\to\infty} \left(\int_{B_0} f(\bar{v}_n, \nabla u_n)\,\mathrm{d}x + \int_{J_{\bar{v}_n}\cap B_0} g(\bar{v}_n^+, \bar{v}_n^-, \nu_{\bar{v}_n})\,\mathrm{d}\mathcal{H}^{N-1} \right). \quad (5.2)$$

For every $(\bar{v}, w) \in \mathrm{SBV}_0(A;\mathbb{R}^m) \cap L^\infty(A;\mathbb{R}^m) \times W^{1,1}(A;\mathbb{R}^d)$ consider

$$\mathcal{G}_n(\bar{v}, w; A) := \int_A (1 + |\nabla w|)\,\mathrm{d}x + (1 + [\bar{v}])\mathcal{H}^{N-1}\lfloor(J_{\bar{v}} \cap A).$$

Due to the coercivity condition (1.1), up to a subsequence, not relabelled, $\nu_n := \mathcal{G}_n(v_n, w_n;\cdot) + \mathcal{G}_n(\bar{v}_n, u_n;\cdot)$ restricted to $B_0 \setminus \bar{U}$ converges in the sense of distributions to some Radon measure ν, defined on $B_0 \setminus \bar{U}$. Analogously, for every $w \in \mathrm{SBV}_0(A;\mathbb{R}^m) \cap L^\infty(A;\mathbb{R}^m)$ we can define a sequence of measures

$$\mathcal{H}_n(w;E) := \int_{J_w \cap E} \mathrm{d}\mathcal{H}^{N-1}.$$

For every $t > 0$, let $B_t := \{x \in B_0 \mid \mathrm{dist}(x, \partial B_0) > t\}$. Define, for $0 < \delta < \eta$, the subsets $L_\delta := B_{\eta-2\delta} \setminus \bar{B}_{\eta+\delta}$. Consider a smooth cut-off function $\varphi_\delta \in C_0^\infty(B_{\eta-\delta};[0,1])$ such that $\varphi_\delta(x) = 1$ on B_η. As the thickness of the strip is of order δ, we have an upper bound of the form $\|\nabla\varphi_\delta\|_{L^\infty(B_{\eta-\delta})} \leqslant C/\delta$.

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Define $\bar{w}_n(x) := \varphi_\delta(x)u_n(x) + (1-\varphi_\delta(x))w_n(x)$. Clearly, $\{\bar{w}_n\}$ converges to u in $L^1(A)$ as $n \to \infty$ and

$$\nabla \bar{w}_n = \varphi_\delta \nabla u_n + (1-\varphi_\delta)\nabla w_n + \nabla\varphi_\delta \otimes (u_n - w_n).$$

Arguing as in [5, lemma 4.4], we may consider a sharp transition for the SBV_0 functions. Namely, let $\{v_n\}$ and $\{\bar{v}_n\}$ be as above. Then for every $0 < t < 1$ we may define $\tilde{v}_n^t$ such that $\tilde{v}_n^t \to v$ in $L^1(A)$ as $n \to \infty$ and

$$\tilde{v}_n^t(x) := \begin{cases} v_n(x) & \text{in } \{x\colon \varphi_\delta(x) < t\}, \\ \bar{v}_n(x) & \text{in } \{x\colon \varphi_\delta(x) \geqslant t\}. \end{cases}$$

Clearly, $\tilde{v}_n^t(x) \in \{v_n(x), \bar{v}_n(x)\}$ almost everywhere in A and since we have that $\mathcal{H}^{N-1}(J_{v_n}), \mathcal{H}^{N-1}(J_{\bar{v}_n}) < +\infty$ for all but at most countable $t \in\,]0,1[$, it results that

$$\mathcal{H}^{N-1}(J_{v_n} \cap \{x \in A\colon \varphi_\delta(x) = t\}) = \mathcal{H}^{N-1}(J_{\bar{v}_n} \cap \{x \in A\colon \varphi_\delta(x) = t\}) = 0.$$

Moreover, using the coarea formula (2.3) and the mean value theorem it is possible to find a t for which the integral over the level set is comparable to the double integral with t varying between 0 and 1. Thus, we have

$$\int_{\partial^*\{\varphi_\delta < t\}} \mathrm{d}\mathcal{H}^{N-1} \leqslant \frac{C}{\delta}\mathcal{L}^N(B_{\eta-\delta} \setminus B_\eta) \leqslant C.$$

An analogous reasoning provides for the same t that

$$\int_{\partial^*\{\varphi_\delta < t\}} |[\tilde{v}_n^t]| \, \mathrm{d}\mathcal{H}^{N-1} \leqslant \frac{C}{\delta}\int_{B_{\eta-\delta}\setminus B_\eta} |v_n(x) - \bar{v}_n(x)| \, \mathrm{d}x. \tag{5.3}$$

Thus, as for the $\{\mathcal{G}_n\}$ above, we may extract a bounded subsequence, not relabelled, from the sequence of measures $\mathcal{H}_n(\tilde{v}_n^t, \cdot)$ that is restricted to $B_0 \setminus \bar{U} \cap \partial^*\{\varphi_\delta < t\}$, converging in the sense of distributions to some Radon measure ν_1 and defined on $B_0 \setminus \bar{U}$.

By (1.1) we have the estimate

$$\begin{aligned}
&\int_A f(\tilde{v}_n^t, \nabla \bar{w}_n)\, \mathrm{d}x + \int_{A\cap J_{\tilde{v}_n^t}} g((\tilde{v}_n^t)^+, (\tilde{v}_n^t)^-, \nu_{\tilde{v}_n}^t)\, \mathrm{d}\mathcal{H}^{N-1} \\
&\quad\leqslant \int_{B_\eta} f(\bar{v}_n, \nabla u_n)\, \mathrm{d}x + \int_{J_{\bar{v}_n}\cap B_\eta} g(\bar{v}_n^+, \bar{v}_n^-, \nu_{\bar{v}_n})\, \mathrm{d}\mathcal{H}^{N-1} \\
&\qquad + \int_{(A\setminus \bar{B}_{\eta-\delta})} f(v_n, \nabla w_n)\, \mathrm{d}x + \int_{J_{v_n}\cap(A\setminus \bar{B}_{\eta-\delta})} g(v_n^+, v_n^-, \nu_{v_n})\, \mathrm{d}\mathcal{H}^{N-1} \\
&\qquad + C(\mathcal{G}_n(v_n, w_n; L_\delta) + \mathcal{G}_n(\bar{v}_n, u_n; L_\delta)) + \frac{1}{\delta}\int_{L_\delta} |w_n - u_n|\, \mathrm{d}x \\
&\qquad + \int_{\partial^*\{\varphi_\delta < t\}} |[\tilde{v}_n^t]|\, \mathrm{d}\mathcal{H}^{N-1} + \mathcal{H}_n(\tilde{v}_n^t; L_\delta \cap \partial^*\{\varphi_\delta < t\})
\end{aligned}$$

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$$\begin{aligned}
&\leqslant \int_{B_0} f(\bar{v}_n, \nabla u_n)\,\mathrm{d}x + \int_{J_{\bar{v}_n}\cap B_0} g(\bar{v}_n^+, \bar{v}_n^-, \nu_{\bar{v}_n})\,\mathrm{d}\mathcal{H}^{N-1} \\
&\quad + \int_{(A\setminus\bar{U})} f(v_n, \nabla w_n)\,\mathrm{d}x + \int_{J_{v_n}\cap(A\setminus\bar{U})} g(v_n^+, v_n^-, \nu_{v_n})\,\mathrm{d}\mathcal{H}^{N-1} \\
&\quad + C(\mathcal{G}_n(v_n, w_n, L_\delta) + \mathcal{G}_n(\bar{v}_n, u_n, L_\delta)) + \frac{1}{\delta}\int_{L_\delta} |w_n - u_n|\,\mathrm{d}x \\
&\quad + \int_{\partial^*\{\varphi_\delta<t\}} |[\tilde{v}_n^t]|\,\mathrm{d}\mathcal{H}^{N-1} + \mathcal{H}_n(\tilde{v}_n^t; L_\delta \cap \partial^*\{\varphi_\delta < t\}).
\end{aligned}$$

Passing to the limit as $n \to \infty$ and applying (5.1)–(5.3) and the L^1 convergence of $\{v_n\}$ and $\{\bar{v}_n\}$ to v, we have that

$$\begin{aligned}
\mathcal{F}(v, u; A) &\leqslant \mathcal{F}(v, u; B_0) + \mathcal{F}(v, u; A \setminus \bar{U}) + \eta + C\nu(\bar{L}_\delta) + C\nu_1(\bar{L}_\delta) \\
&\quad + \limsup_{n\to\infty} \int_{\partial^*\{\varphi_\delta<t\}} |[\tilde{v}_n^t]|\,\mathrm{d}\mathcal{H}^{N-1} \\
&\leqslant \mathcal{F}(v, u; B) + \mathcal{F}(v, u; A \setminus \bar{U}) + \eta + C\nu(\bar{L}_\delta) + C\nu_1(\bar{L}_\delta).
\end{aligned}$$

Letting δ go to 0, we obtain

$$\mathcal{F}(v, u; A) \leqslant \mathcal{F}(v, u; B) + \mathcal{F}(v, u; (A \setminus \bar{U})) + \eta + C\nu(\partial B_\eta) + C\nu_1(\partial B_\eta).$$

It suffices to choose a subsequence $\{\eta_i\}$ such that $\eta_i \to 0^+$ and $\nu(\partial B_{\eta_i}) = \nu_1(\partial B_{\eta_i}) = 0$ to conclude the proof of subadditivity for the case $v \in \mathrm{SBV}_0 \cap L^\infty$.

In the general case, by virtue of remark 3.9, we can argue as in the last part of [14, theorem 10]. □

Proof of theorem 5.1. We assume first that $(v, u) \in (\mathrm{SBV}_0(\Omega; \mathbb{R}^m) \times \mathrm{BV}(\Omega; \mathbb{R}^d)) \cap L^\infty(\Omega; \mathbb{R}^{m+d})$.

Step 1. In order to prove the upper bound, we start by recalling that by proposition 3.6 we can replace Qf by f in (1.11). First we deal with the bulk part.

Since the $\mathcal{F}(v, u; \cdot)$ is a measure absolutely continuous with respect to $\mathcal{L}^N + |Du| + (1 + [v])\mathcal{H}^{N-1}\lfloor J_v$, we claim that

$$\frac{\mathrm{d}\mathcal{F}(v, u; \cdot)}{\mathrm{d}\mathcal{L}^N}(x_0) \leqslant Qf(v(x_0), \nabla u(x_0))$$

for $\mathcal{L}^N$-a.e. $x_0 \in \Omega$, where x_0 is a Lebesgue point of v and u such that

$$\left.\begin{aligned}
\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\biggl\{\frac{1}{\varepsilon^N}\int_{B(x_0,\varepsilon)} |u(x) - u(x_0) - \nabla u(x_0)(x - x_0)|^{N/(N-1)}\,\mathrm{d}x\biggr\}^{(N-1)/N} &= 0, \\
\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\biggl\{\frac{1}{\varepsilon^N}\int_{B(x_0,\varepsilon)} |v(x) - v(x_0)|^{N/(N-1)}\,\mathrm{d}x\biggr\}^{(N-1)/N} &= 0, \\
\mu_a(x_0) = \lim_{\varepsilon\to 0^+} \frac{\mu(B(x_0, \varepsilon))}{\mathcal{L}^N(B(x_0, \varepsilon))} &< \infty.
\end{aligned}\right\} \tag{5.4}$$

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Let $U := (v,u)$. By (5.4) and theorems 2.1 and 2.2, for $\mathcal{L}^N$-a.e. $x_0 \in \Omega$ we have

$$\left.\begin{aligned}
&\lim_{\varepsilon\to 0^+} \frac{1}{\mathcal{L}^N(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)} |U(x)-U(x_0)|(1+|\nabla U(x)|)\,\mathrm{d}x = 0,\\
&\lim_{\varepsilon\to 0^+} \frac{|D_sU|(B(x_0,\varepsilon))}{\mathcal{L}^N(B(x_0,\varepsilon))} = 0,\\
&\lim_{\varepsilon\to 0^+} \frac{|DU|(B(x_0,\varepsilon))}{\mathcal{L}^N(B(x_0,\varepsilon))} \quad \text{exists and is finite},\\
&\lim_{\varepsilon\to 0^+} \frac{1}{\mathcal{L}^N(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)} Qf(v(x_0),\nabla u(x))\,\mathrm{d}x = Qf(v(x_0),\nabla u(x_0)),\\
&\qquad\qquad \frac{\mathrm{d}\mathcal{F}(v,u;\cdot)}{\mathrm{d}\mathcal{L}^N}(x_0) \quad \text{exists and is finite.}
\end{aligned}\right\} \tag{5.5}$$

We observe that the assumptions imposed on f and proposition 3.1 allow us to apply, for every $v \in \mathrm{SBV}_0(\Omega;\mathbb{R}^m)$, the global method (see [13, theorem 4.1.4]) to the functional $u \in W^{1,1}(\Omega;\mathbb{R}^d)\times\mathcal{A}(\Omega) \to G(u;A) := \int_A Qf(v(x),\nabla u(x))\,\mathrm{d}x$, thus obtaining an integral representation for the relaxed functional

$$\mathcal{G}(u;A) = \inf\Big\{\liminf_{n\to\infty} G(u_n;A) \colon u_n \to u \text{ in } L^1(A;\mathbb{R}^d)\Big\} \tag{5.6}$$

for every $(u,A) \in \mathrm{BV}(\Omega;\mathbb{R}^d)\times\mathcal{A}(\Omega)$.

Recall that the growth condition (G_2) and the lower semi-continuity with respect to the L^1-topology of the functional $v \in \mathrm{SBV}_0(\Omega;\mathbb{R}^m) \mapsto (1+[v])\mathcal{H}^{N-1}\lfloor(J_v\cap A)$ entail

$$\mathcal{F}(v,u;A) \leqslant \mathcal{G}(u;A) + (1+[v])\mathcal{H}^{N-1}\lfloor(J_v\cap A). \tag{5.7}$$

Differentiating with respect to $\mathcal{L}^N$ at x_0 and exploiting (5.4) and (5.5), we obtain that

$$\frac{\mathrm{d}\mathcal{F}((v,u);\cdot)}{\mathrm{d}\mathcal{L}^N}(x_0) \leqslant f_0(x_0,\nabla u(x_0)),$$

where for every $x_0 \in \Omega$ and $\xi \in \mathbb{R}^d$, $f_0(x_0,\xi)$ is given as in [13, (4.1.5)], namely,

$$f_0(x_0,\xi) := \limsup_{\varepsilon\to 0^+} \inf_{\substack{z\in W^{1,1}(Q;\mathbb{R}^d)\\ z(y)=\xi y \text{ on } \partial Q}} \Big\{\int_Q Qf(v(x_0+\varepsilon y),\nabla z(y))\,\mathrm{d}y\Big\}. \tag{5.8}$$

To conclude the proof, we claim that $f_0(x_0,\xi) \leqslant Qf(v(x_0),\xi)$ for every $x_0 \in \Omega$ satisfying (5.4) and (5.5) and $\xi \in \mathbb{R}^d$.

By virtue of lemma 3.10 we have that

$$\begin{aligned}
\limsup_{\varepsilon\to 0^+} &\inf_{\substack{z\in W^{1,1}(Q;\mathbb{R}^d)\\ z(y)=\xi y \text{ on } \partial Q}} \Big\{\int_Q Qf(v(x_0+\varepsilon y),\nabla z(y))\,\mathrm{d}y\Big\}\\
&\leqslant \inf_{\substack{z\in W^{1,1}(Q;\mathbb{R}^d)\\ z(y)=\xi y \text{ on } \partial Q}} \Big\{\limsup_{\varepsilon\to 0^+}\int_Q Qf(v(x_0+\varepsilon y),\nabla z(y))\,\mathrm{d}y\Big\}.
\end{aligned}$$

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Computing the lim sup on the right-hand side, we have

$$\limsup_{\varepsilon\to 0^+}\int_Q Qf(v(x_0+\varepsilon y),\nabla z(y))\,\mathrm{d}y = \limsup_{\varepsilon\to 0^+}\left(\int_Q Qf(v(x_0+\varepsilon y),\nabla z(y))\,\mathrm{d}y - \int_Q Qf(v(x_0),\nabla z(y))\,\mathrm{d}y\right) + \int_Q Qf(v(x_0),\nabla z(y))\,\mathrm{d}y.$$

Since x_0 is a Lebesgue point for v and recalling that $v\in \mathrm{SBV}_0(Q;\mathbb{R}^m)\cap L^\infty(Q;\mathbb{R}^m)$, by the Lebesgue dominated convergence theorem and (F_3) applied to Qf (see proposition 3.1), we have that

$$\limsup_{\varepsilon\to 0^+}\left(\int_Q Qf(v(x_0+\varepsilon y),\nabla z(y))\,\mathrm{d}y - \int_Q Qf(v(x_0),\nabla z(y))\,\mathrm{d}y\right) \leqslant \limsup_{\varepsilon\to 0^+}\int_Q L|v(x_0+\varepsilon y)-v(x_0)|(1+|\nabla z(y)|)\,\mathrm{d}y = 0.$$

Hence,

$$\limsup_{\varepsilon\to 0^+}\int_Q Qf(v(x_0+\varepsilon y),\nabla z(y))\,\mathrm{d}y = \int_Q Qf(v(x_0),\nabla z(y))\,\mathrm{d}y.$$

By the quasi-convexity of $Qf(v(x_0),\cdot)$ and (5.8), one obtains

$$f_0(x_0,\xi)\leqslant Qf(v(x_0),\xi),$$

which concludes the proof on replacing ξ by $\nabla u(x_0)$.

Step 2. We prove the upper bound for the Cantor part.

By the Radon–Nikodým theorem, we can write

$$|DU| = |D^c u| + \sigma, \tag{5.9}$$

where $U := (v,u)\in(\mathrm{SBV}_0(\Omega;\mathbb{R}^m)\times \mathrm{BV}(\Omega;\mathbb{R}^d))\cap L^\infty(\Omega;\mathbb{R}^{m+d})$ and where σ and $|D^c u|$ are mutually singular Radon measures.

Observe that $U\equiv(v,u)$ is $|D^c u|$-measurable, Dv is singular with respect to $|D^c u|$ and, by theorems 2.1, 2.2 and [25, theorem 2.11] for $|D^c u|$-a.e. $x\in B(x_0,\varepsilon)$,

$$\left.\begin{aligned}
&\lim_{\varepsilon\to 0^+}\frac{\mu(B(x_0,\varepsilon))}{|D^c u|(B(x_0,\varepsilon))} = 0,\\
&\lim_{\varepsilon\to 0^+}\frac{|Du|(B(x_0,\varepsilon))}{|D^c u|(B(x_0,\varepsilon))}\quad \text{exists and is finite},\\
&\lim_{\varepsilon\to 0^+}\frac{\varepsilon^N}{|D^c u|(B(x_0,\varepsilon))} = 0,\\
&\lim_{\varepsilon\to 0^+}\frac{1}{\mathcal{L}^N(B(x_0,\varepsilon))}\int_{B(x_0,\varepsilon)}(|u(x)-u(x_0)|+|v(x)-v(x_0)|)\,\mathrm{d}x = 0.
\end{aligned}\right\} \tag{5.10}$$

Moreover,

$$A(x) := \lim_{\varepsilon\to 0^+}\frac{D^c u(B(x,\varepsilon))}{|D^c u|(B(x,\varepsilon))},\qquad \lim_{\varepsilon\to 0^+}\frac{D^c U(B(x,\varepsilon))}{|D^c U|(B(x,\varepsilon))} =: D(x) \tag{5.11}$$

exist and they are rank 1 matrices of norm 1. In particular,

$$A(x) = a_u(x) \otimes \nu_u(x), \tag{5.12}$$

where $(a_u(x), \nu_u(x)) \in \mathbb{R}^d \times S^{N-1}$. By theorem 2.2 we have

$$\lim_{\varepsilon\to 0^+} \frac{1}{|D^c u|(B(x_0,\varepsilon))} \int_{B(x_0,\varepsilon)} f^\infty(v(x_0), A(x))\,\mathrm{d}|D^c u| = f^\infty(v(x_0), A(x_0)).$$

As in step 1, we recall that via the global method (see [13, theorem 4.1.4]) we can obtain an integral representation for the functional $\mathcal{G}(u; A)$ in (5.6) for every $(v, u) \in \mathrm{BV}(\Omega; \mathbb{R}^{m+d})$. Moreover, by proposition 3.6, we can replace f by Qf in (1.11) and (5.7) holds.

Differentiating with respect to $|D^c u|$ at x_0 and exploiting (5.9) and (5.10), we deduce

$$\frac{\mathrm{d}\mathcal{F}((v,u);\cdot)}{\mathrm{d}|D^c u|}(x_0) \leqslant h(x_0, a_u, \nu_u),$$

where $\nu_u(x)$ agrees with the unit vector that, together with a_u, satisfies (5.12) for $|D^c u|$-a.e. $x \in \Omega \setminus J_u$ and where $h(x_0, a, \nu)$ is given as in [13, (4.1.7)], namely,

$$h(x_0,a,\nu) := \limsup_{k\to\infty} \limsup_{\varepsilon\to 0^+} \inf_{\substack{z\in W^{1,1}(Q_\nu^{(k)};\mathbb{R}^d)\\ z(y)=a(\nu y) \text{ on } \partial Q_\nu^{(k)}}} \left\{ \frac{1}{k^{N-1}} \int_{Q_\nu^{(k)}} Qf^\infty(v(x_0+\varepsilon y), \nabla z(y))\,\mathrm{d}y \right\}, \tag{5.13}$$

where $a \in \mathbb{R}^d$, $\nu \in S^{N-1}$, $Q_\nu^{(k)} := R_\nu((-\frac{1}{2}k, \frac{1}{2}k)^{N-1} \times (-\frac{1}{2}, \frac{1}{2}))$ and R_ν is a rotation such that $R_\nu(e_N) = \nu$.

We also recall that, by remark 3.2(iv), $Q(f^\infty) = (Qf)^\infty = Qf^\infty$.

To conclude the proof, it is enough to show that

$$h(x_0, a, \nu) \leqslant Qf^\infty(v(x_0), a \otimes \nu).$$

By lemma 3.10,

$$h(x_0,a,\nu) \leqslant \limsup_{k\to\infty} \inf_{\substack{z\in W^{1,1}(Q_\nu^{(k)};\mathbb{R}^d)\\ z(y)=a(\nu y) \text{ on } \partial Q_\nu^{(k)}}} \left\{ \limsup_{\varepsilon\to 0^+} \frac{1}{k^{N-1}} \int_{Q_\nu^{(k)}} Qf^\infty(v(x_0+\varepsilon y), \nabla z(y))\,\mathrm{d}y \right\}. \tag{5.14}$$

In order to compute

$$\limsup_{\varepsilon\to 0^+} \frac{1}{k^{N-1}} \int_{Q_\nu^{(k)}} Qf^\infty(v(x_0+\varepsilon y), \nabla z(y))\,\mathrm{d}y,$$

we add and subtract inside the integral $Qf^\infty(v(x_0), \nabla z(y))$. Then, as in step 1, exploiting the fact that x_0 is a Lebesgue point for $v \in \mathrm{SBV}_0(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega; \mathbb{R}^m)$ and that Qf^∞ satisfies (F_3) (see remark 3.2 where (F_3) has been deduced for f^∞

and proposition 3.1), via Lebesgue's dominated convergence theorem we conclude that

$$\limsup_{\varepsilon\to 0^+} \frac{1}{k^{N-1}} \int_{Q_\nu^{(k)}} Qf^\infty((v(x_0+\varepsilon y), \nabla z(y))\,\mathrm{d}y) = \frac{1}{k^{N-1}} \int_{Q_\nu^{(k)}} Qf^\infty(v(x_0), \nabla z(y))\,\mathrm{d}y.$$

Finally, the quasi-convexity of Qf^∞ (deduced via remark 3.2 and proposition 3.1) provides

$$Qf^\infty(v(x_0), a\otimes\nu) = \inf_{\substack{z\in W^{1,1}(Q_\nu^{(k)};\mathbb{R}^d)\\ z(y)=a(\nu\cdot y) \text{ on } \partial Q_\nu^{(k)}}} \left\{\frac{1}{k^{N-1}} \int_{Q_\nu^{(k)}} Qf^\infty(v(x_0), \nabla z(y))\,\mathrm{d}y\right\},$$

which, together with (5.14) concludes the proof of the upper bound for the Cantor part when $(v,u) \in (\mathrm{SBV}_0(\Omega;\mathbb{R}^m)\times \mathrm{BV}(\Omega;\mathbb{R}^d)) \cap L^\infty(\Omega;\mathbb{R}^{m+d})$.

Step 3. We now prove the upper bound for the jump. Namely, we claim that

$$\mathcal{F}(U;J_U) \equiv \mathcal{F}(v,u,J_{(v,u)}) \leqslant \int_{J_U} K_3(v^+,v^-,u^+,u^-,\nu)\,\mathrm{d}\mathcal{H}^{N-1} \tag{5.15}$$

for every $U\equiv(v,u)\in(\mathrm{SBV}_0(\Omega;\mathbb{R}^m)\times \mathrm{BV}(\Omega;\mathbb{R}^d)) \cap L^\infty(\Omega;\mathbb{R}^{m+d})$.

The proof is divided into three parts according to the assumptions on the limit function U.

Case 1. $U(x) := (a,c)\chi_E(x) + (b,d)(1-\chi_E(x))$ with $P(E,\Omega)<\infty$.

Case 2. $U(x) := \sum_{i=1}^\infty (a_i,c_i)\chi_{E_i}(x)$, where $\{E_i\}_{i=1}^\infty$ forms a partition of Ω into sets of finite perimeter and $(a_i,c_i)\in\mathbb{R}^m\times\mathbb{R}^d$.

Case 3. $U\in(\mathrm{SBV}_0(\Omega;\mathbb{R}^m)\times \mathrm{BV}(\Omega;\mathbb{R}^d)) \cap L^\infty(\Omega;\mathbb{R}^{m+d})$.

Proof of case 1. We start by proving that for every open set $A\subset\Omega$,

$$\mathcal{F}(U;A)\equiv\mathcal{F}(v,u;A) \leqslant \int_A Qf(v(x),0)\,\mathrm{d}x + \int_{J_U\cap A} K_3(a,b,c,d,\nu)\,\mathrm{d}\mathcal{H}^{N-1}.$$

(a) Assume first that

$$v(x) := \begin{cases} a & \text{if } x\cdot\nu>0,\\ b & \text{if } x\cdot\nu<0,\end{cases} \quad\text{and}\quad u(x) := \begin{cases} c & \text{if } x\cdot\nu>0,\\ d & \text{if } x\cdot\nu<0.\end{cases}$$

We start with the case in which $A = a+\lambda Q$ is an open cube with two faces orthogonal to ν. For simplicity we also assume that $\nu = e_N$ and Q_ν will be denoted simply by Q. Our proof develops as in [26, proposition 4.1 and lemma 4.2] (see also [12, proposition 5.1]) and we will thus present only the main steps.

Suppose first that $a=0$ and $\lambda=1$. By proposition 3.4 (see also remark 3.5), there exists $(v_n,u_n)\in\mathcal{A}_3(a,b,c,d,\nu)$ such that $(v_n,u_n)\to(v,u)$ in $L^1(Q;\mathbb{R}^{m+d})$

and

$$K_3(a,b,c,d,\nu) = \lim_{n\to\infty}\left(\int_Q Qf^\infty(v_n(x),\nabla u_n(x))\,\mathrm{d}x + \int_{J_{v_n}\cap Q} g(v_n^+(x),v_n^-(x),\nu_n(x))\,\mathrm{d}\mathcal{H}^{N-1}\right). \tag{5.16}$$

We denote by Q' the set $\{x\in Q\colon x_N=0\}$. For $k\in\mathbb{N}$ we label the elements of

$$(\mathbb{Z}\cap[-k,k])^{N-1}\times\{0\}$$

by $\{a_i\}_{i=1}^{(2k+1)^{N-1}}$ and we observe that

$$(2k+1)\bar{Q}' = \bigcup_{i=1}^{(2k+1)^{N-1}} (a_i+\bar{Q}')$$

with

$$(a_i+Q')\cap(a_j+Q')=\emptyset \quad \text{for } i\neq j.$$

We define

$$z_{n,k}(x) := \begin{cases} a & \text{if } x_N > \dfrac{1}{2(2k+1)},\\ v_n((2k+1)x) & \text{if } |x_N| < \dfrac{1}{2(2k+1)},\\ b & \text{if } x_N < -\dfrac{1}{2(2k+1)} \end{cases}$$

and

$$w_{n,k}(x) := \begin{cases} c & \text{if } x_N > \dfrac{1}{2(2k+1)},\\ u_n((2k+1)x) & \text{if } |x_N| < \dfrac{1}{2(2k+1)},\\ d & \text{if } x_N < -\dfrac{1}{2(2k+1)}. \end{cases}$$

By the periodicity of the functions v_n and u_n, it is easily seen that

$$\lim_{n\to\infty}\lim_{k\to\infty}\|z_{n,k}-v\|_{L^1(Q;\mathbb{R}^m)}=0, \qquad \lim_{n\to\infty}\lim_{k\to\infty}\|w_{n,k}-u\|_{L^1(Q;\mathbb{R}^d)}=0.$$

Thus, by a standard diagonalization argument, we have

$$\mathcal{F}(v,u;Q) \leqslant \limsup_{n\to\infty}\limsup_{k\to\infty}\left(\int_Q Qf(z_{n,k}(x),\nabla w_{n,k}(x))\,\mathrm{d}x + \int_{Q\cap J_{z_{n,k}}} g(z_{n,k}^+(x),z_{n,k}^-(x),\nu_{n,k}(x))\,\mathrm{d}\mathcal{H}^{N-1}\right).$$

Arguing as in [12, proposition 5.1], for the bulk part we have

$$\limsup_{k\to\infty}\int_Q Qf(z_{n,k}(x),\nabla w_{n,k}(x))\,\mathrm{d}x = \int_Q Qf(v(y),0)\,\mathrm{d}y + \int_Q Qf^\infty(v_n(y),\nabla u_n(y))\,\mathrm{d}y,$$

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and for the surface term

$$\int_{Q\cap J_{z_{n,k}}} g(z_{n,k}^+(x), z_{n,k}^-(x), \nu_{n,k}(x))\,\mathrm{d}\mathcal{H}^{N-1} \leqslant \int_{Q\cap J_{v_n}} g(v_n^+(y), v_n^-(y), \nu_n(y))\,\mathrm{d}\mathcal{H}^{N-1}(y).$$

Putting together the estimates for bulk and surface terms and exploiting (5.16), we obtain that

$$\begin{aligned}\mathcal{F}(v,u;Q) &\leqslant \limsup_{n\to\infty}\biggl(\int_Q Qf(v,0)\,\mathrm{d}x + \int_Q Qf^\infty(v_n(y),\nabla u_n(y))\,\mathrm{d}y \\ &\qquad + \int_{Q\cap J_{v_n}} g(v_n^+(y), v_n^-(y), \nu_n(y))\,\mathrm{d}\mathcal{H}^{N-1}\biggr) \\ &= \int_Q Qf(v(x),0)\,\mathrm{d}x + K_3(a,b,c,d,e_N) \\ &= \frac{Qf(a,0)+Qf(b,0)}{2} + K_3(a,b,c,d,e_N).\end{aligned}$$

In order to consider sets $A = a + \lambda Q$ with $a \in \mathbb{R}^N$ and $\lambda > 0$, we define

$$(Qf)_\lambda(b,B) := Qf\biggl(b, \frac{B}{\lambda}\biggr), \qquad g_\lambda(\xi,\zeta,\nu) := \frac{1}{\lambda} g(\xi,\zeta,\nu)$$

and, for every $E \subset \Omega$,

$$\begin{aligned}\mathcal{F}_\lambda(v,u;E) := \inf_{\{(v_n,u_n)\}}\biggl\{\liminf_{n\to\infty}\biggl(\int_E (Qf)_\lambda(v_n(x),\nabla u_n(x))\,\mathrm{d}x \\ + \int_{E\cap J_{v_n}} g_\lambda(v_n^+(x), v_n^-(x), \nu_n(x))\,\mathrm{d}\mathcal{H}^{N-1}\biggr) : \\ (v_n,u_n) \in \mathrm{SBV}_0(E;\mathbb{R}^m)\times W^{1,1}(E;\mathbb{R}^d), \\ (v_n,u_n) \to (v,u) \text{ in } L^1(E;\mathbb{R}^{m+d})\biggr\}.\end{aligned}$$

It is easily seen that for every $(v,u) \in L^1(\Omega;\mathbb{R}^{m+d})$ we have

$$\mathcal{F}(v,u;A) = \lambda^N \mathcal{F}_\lambda(v_\lambda,u_\lambda;Q),$$

where

$$v_\lambda(x) := v\biggl(\frac{x-a}{\lambda}\biggr), \qquad u_\lambda(x) := u\biggl(\frac{x-a}{\lambda}\biggr).$$

Since $Qf_\lambda^\infty = (1/\lambda)Qf^\infty$, by the definition of K_3 for f_λ and g_λ we have that $(K_3)_\lambda(a,b,c,d,\nu) = (1/\lambda)K_3(a,b,c,d,\nu)$.

By the definition of u_λ and v_λ we have that

$$v_\lambda = \begin{cases} a & \text{if } x_N > 0, \\ b & \text{if } x_N < 0, \end{cases} \qquad u_\lambda = \begin{cases} c & \text{if } x_N > 0, \\ d & \text{if } x_N < 0. \end{cases}$$

So by the previous case, it results that

$$\mathcal{F}(v,u;A)\lambda^N = \mathcal{F}_\lambda(v_\lambda,u_\lambda;Q) \leqslant \lambda^N\left(\frac{Qf_\lambda(a,0)+Qf_\lambda(b,0)}{2}+(K_3)_\lambda(a,b,c,d,e_N)\right).$$

(b) Now let $U := (v,u)$ as in (a) and let A be any open set. The proof of this step is identical to [25, § 5, step 3, case 1(b)]. Indeed, it is enough to apply the same strategy but replacing u and K in [25] by U and K_3, respectively, herein, thereby obtaining

$$\mathcal{F}(v,u;A) \leqslant \int_A Qf(v(x),0)\,\mathrm{d}x + \int_{J_U\cap A} K_3(a,b,c,d,\nu)\,\mathrm{d}\mathcal{H}^{N-1}. \tag{5.17}$$

(c) Now suppose that U has a polygonal interface, i.e. $U=(a,c)\chi_E+(b,d)(1-\chi_E)$, where E is a polyhedral set, i.e. E is a bounded strongly Lipschitz domain and $\partial E = H_1\cup H_2\cup\cdots\cup H_M$ are closed subsets of hyperplanes of type $\{x\in\mathbb{R}^N : x\cdot\nu_i=\alpha_i\}$.

The details of the proof are omitted since they are very similar to [25, § 5, step 3, case 1(c)]. We just observe that, given an open set A contained in Ω, the argument relies on an inductive procedure on $I := \{i\in\{1,\dots,M\} : \mathcal{H}^{N-1}(H_i\cap A)>0\}$ starting from the case $I=0$ when $u\in W^{1,1}(A;\mathbb{R}^d)$ and $v\in \mathrm{SBV}_0(A;\mathbb{R}^m)\cap L^\infty(A;\mathbb{R}^m)$, for which it suffices to consider $u_n=u$ and $v_n=v$ with (5.17) reducing to

$$\mathcal{F}(v,u;A) \leqslant \int_A Qf(v(x),0)\,\mathrm{d}x.$$

The case $\operatorname{card} I = 1$ was studied in part (b), where E is a large cube so that $J_U\cap\Omega$ reduces to the flat interface $\{x\in\Omega : x\cdot\nu=0\}$.

The induction step (which first assumes that (5.17) is true if $\operatorname{card} I = k$, $k\leqslant M-1$, and then proves that it is still true if $\operatorname{card} I = k$) then develops exactly as in [12, proposition 5.1, step 2(c)], the only difference being that the slicing method used to connect the sequence across the interfaces relies on the same techniques as lemma 3.8 but referred to more general open sets than cubes (see also [25, § 5, step 3, case 1(c)]). Thus, one can conclude that

$$\mathcal{F}(v,u;A) \leqslant \int_A Qf(v(x),0)\,\mathrm{d}x + \int_{J_U\cap A} K_3(a,b,c,d,\nu)\,\mathrm{d}\mathcal{H}^{N-1}.$$

(d) If E is an arbitrary set of finite perimeter, the step develops in strong analogy with [25, § 5, step 3, case 1(f)]. Essentially, exploiting proposition 3.3(b), the approximation via polyhedral sets with finite perimeter as in [11, lemma 3.1] and applying Lebesgue's monotone convergence theorem gives

$$\mathcal{F}(v,u;A) \leqslant \int_A Qf(v(x),0)\,\mathrm{d}x + \int_{A\cap J_U} K_3(a,b,c,d,\nu)\,\mathrm{d}\mathcal{H}^{N-1}.$$

This last inequality, together with lemma 5.2, yields

$$\mathcal{F}(v,u;J_{(v,u)}) \leqslant \int_{J_{(v,u)}} K_3(a,b,c,d,\nu)\,\mathrm{d}\mathcal{H}^{N-1},$$

which gives (5.15) when $U\equiv(v,u)=(a,c)\chi_E+(b,d)(1-\chi_E)$ is the characteristic function of a set of finite perimeter. □

Proof of case 2. Arguing as in [25, § 5, step 3, case 2] and referring to [8, proposition 4.8, step 1], we clearly obtain for every $(v,u) \in \mathrm{BV}(\Omega;T) \times \mathrm{BV}(\Omega;T)$, with T a finite subset of $\mathbb{R}^d$,

$$\mathcal{F}(v,u;A) = \mathcal{F}(v,u;A \cap J_{(v,u)}) \leqslant \int_{J_{(v,u)}} K_3(v^+,v^-,u^+,u^-,\nu_{v,u}(x))\,\mathrm{d}\mathcal{H}^{N-1}(x).$$

□

Proof of case 3. For $U \equiv (v,u) \in (\mathrm{SBV}_0(\Omega;\mathbb{R}^m) \times \mathrm{BV}(\Omega;\mathbb{R}^d)) \cap L^\infty(\Omega;\mathbb{R}^{m+d})$ the proof develops analogously to [8, proposition 4.8, step 2] and we add some details for the reader's convenience.

First we observe that the jump set $J_U \equiv J_{(v,u)}$ can be decomposed as $(J_u \setminus J_v) \cup (J_v \setminus J_u) \cup (J_u \cap J_v)$, recalling that these sets are mutually disjoint and that the tangent hyperplanes to J_u and J_v coincide up to a set of $\mathcal{H}^{N-1}$ measure 0.

Let $A \in \mathcal{A}(\Omega)$ such that $A \supset J_U$. We assume that $U(x) \in [0,1]^{m+d}$ for a.e. $x \in A$. For every $h \in \mathbb{N}$, $h \geqslant 2$, it is possible to define a set

$$B_h := A \setminus J_U \cup \left\{ x \in J_U \colon |U^+(x) - U^-(x)| \leqslant \frac{1}{4(m+d)h} \right\}$$

and define the sequence $\{U_h\} \equiv \{(v_h,u_h)\}$ according to [8, proposition 4.8, step 2]. Observe that $J_{v_h} \subset J_v$. Then, by step 2, we have that

$$\begin{aligned}\mathcal{F}(v,u,;A) &\leqslant \liminf_{h\to\infty} \mathcal{F}(v_h,u_h;A) \\ &= \liminf_{h\to\infty} \biggl(\int_A Qf(v_h,0)\,\mathrm{d}x \\ &\qquad + \int_A Qf^\infty\Bigl(v_h, \frac{\mathrm{d}D^c u_h}{\mathrm{d}|D^c u_h|}\Bigr)\mathrm{d}|D^c u_h| \\ &\qquad + \int_{A\cap(J_{u_h}\cup J_{v_h})} K_3(v_h^+,v_h^-,u_h^+,u_h^-,\nu_{v_h,u_h})\,\mathrm{d}\mathcal{H}^{N-1} \biggr).\end{aligned} \tag{5.18}$$

We restrict our attention to the surface integral. Clearly,

$$\begin{aligned}&\int_{A\cap(J_{u_h}\cup J_{v_h})} K_3(v_h^+,v_h^-,u_h^+,u_h^-,\nu_{v_h,u_h})\,\mathrm{d}\mathcal{H}^{N-1} \\ &\quad = \int_{A\cap(J_{u_h}\cup J_{v_h})\cap B_h} K_3(v_h^+,v_h^-,u_h^+,u_h^-,\nu_{v_h,u_h})\,\mathrm{d}\mathcal{H}^{N-1} \\ &\qquad + \int_{A\cap(J_{u_h}\cup J_{v_h})\cap(A\setminus B_h)} K_3(v_h^+,v_h^-,u_h^+,u_h^-,\nu_{v_h,u_h})\,\mathrm{d}\mathcal{H}^{N-1}.\end{aligned}$$

By the decomposition of the jump set $J_{(v_h,u_h)}$, proposition 3.3(d), the fact that $J_{v_h} \subset J_v$ and the same type of estimates as in [8, p. 300] entail (with the constant

C varying from place to place)

$$\begin{aligned}\int_{A\cap(J_{u_h}\cup J_{v_h})\cap B_h} & K_3(v_h^+,v_h^-,u_h^+,u_h^-,\nu_{v_h,u_h})\,\mathrm{d}\mathcal{H}^{N-1}\\ &= \int_{A\cap(J_{u_h}\setminus J_{v_h})\cap B_h} K_3(v_h^+,v_h^-,u_h^+,u_h^-,\nu_{v_h,u_h})\,\mathrm{d}\mathcal{H}^{N-1}\\ &\quad + \int_{A\cap(J_{v_h}\setminus J_{u_h})\cap B_h} K_3(v_h^+,v_h^-,u_h^+,u_h^-,\nu_{v_h,u_h})\,\mathrm{d}\mathcal{H}^{N-1}\\ &\quad + \int_{A\cap J_{u_h}\cap J_{v_h}\cap B_h} K_3(v_h^+,v_h^-,u_h^+,u_h^-,\nu_{v_h,u_h})\,\mathrm{d}\mathcal{H}^{N-1}\\ &\leqslant C\int_{A\cap(J_{u_h}\setminus J_{v_h})\cap B_h} |u_h^+-u_h^-|\,\mathrm{d}\mathcal{H}^{N-1}\\ &\quad + C\int_{A\cap(J_{v_h}\setminus J_{u_h})\cap B_h} (|v_h^+-v_h^-|+1)\,\mathrm{d}\mathcal{H}^{N-1}\\ &\quad + C\int_{A\cap J_{u_h}\cap J_{v_h}\cap B_h} (|v_h^+-v_h^-|+|u_h^+-u_h^-|+1)\,\mathrm{d}\mathcal{H}^{N-1}\\ &\leqslant 2C(m+d)|Du|(A\cap B_h)+C(m+d)|Dv|(A\cap B_h)+C\mathcal{H}^{N-1}(J_v\cap B_h\cap A).\end{aligned} \tag{5.19}$$

Moreover, by proposition 3.3(c), (d) and the reverse of Fatou's lemma, we have

$$\begin{aligned}\int_{(J_{v_h}\cup J_{u_h})\cap(A\setminus B_h)} & K_3(v_h^+,v_h^-,u_h^+,u_h^-,\nu_{(v_h,u_h)})\,\mathrm{d}\mathcal{H}^{N-1}\\ &\leqslant \int_{A\cap(J_v\cup J_u)} K_3(v^+,v^-,u^+,u^-,\nu_{(v,u)})\,\mathrm{d}\mathcal{H}^{N-1}.\end{aligned}$$

Clearly, taking the limit as $h\to\infty$, from the above inequality and (5.19) we may conclude that

$$\begin{aligned}\mathcal{F}(v,u;A) &\leqslant \int_{A\cap(J_v\cup J_u)} K_3(v^+,v^-,u^+,u^-,\nu_{(v,u)})\,\mathrm{d}\mathcal{H}^{N-1}\\ &\quad + C(|Du|(A\setminus(J_v\cup J_u))+|Dv|(A\setminus(J_u\cup J_v)))+\int_A Qf(v,0)\,\mathrm{d}x,\end{aligned}$$

where we have exploited the fact that the Cantor term in (5.18) is 0 from the construction of the u_h and $\liminf_{h\to\infty}\mathcal{H}^{N-1}(J_v\cap B_h\cap A)=\mathcal{H}^{N-1}(J_v\cap(A\setminus(J_u\cup J_v)))=0$. Now, since $\mathcal{F}(v,u;\cdot)$ is a Radon measure, the above inequality holds for every Borel set B and in particular for the set $B=A\cap(J_v\cup J_u)$, and this gives

$$\mathcal{F}(v,u;J_v\cap J_u)\leqslant \int_{J_v\cap J_u} K_3(v^+,v^-,u^+,u^-,\nu_{(v,u)})\,\mathrm{d}\mathcal{H}^{N-1}.$$

This concludes the proof of step 2 when $(v,u)\in \mathrm{SBV}_0(\Omega;\mathbb{R}^m)\times \mathrm{BV}(\Omega;\mathbb{R}^d)\cap L^\infty(\Omega;\mathbb{R}^{m+d})$.

The general case $(v,u)\in \mathrm{SBV}_0(\Omega;\mathbb{R}^m)\times \mathrm{BV}(\Omega;\mathbb{R}^d)$ follows from remark 3.9(iii) (see [25, § 5, step 4] and [8, theorem 4.9]). □

Proof of theorem 1.2. It follows from theorems 4.1 and 5.1 □

REMARK 5.3. We observe that, as can be easily conjectured from the proofs of theorem 4.1, step 2, and theorem 5.1, step 3, case 3(i) and (ii), K_3 admits the following equivalent representations.

- On $J_u \setminus J_v$, $K_3(a,a,c,d,\nu) = Qf^\infty(a,(c-d)\otimes\nu)$, where Qf^∞ represents the recession function of the quasi-convexification of f, as in remark 3.2. In fact, one inequality is trivial by definition 1.13, while the other can be obtained through proposition 3.4 by invoking the quasi-convexity and the growth properties of $Qf^\infty(a,\cdot)$ (see remark 3.2) and analogous arguments to the ones leading to [9, (5.84)].

- On $J_v \setminus J_u$, $K_3(a,b,c,c,\nu) = \mathcal{R}g(a,b,\nu)$, where $\mathcal{R}g$ represents the BV-elliptic envelope of g; namely, the greatest BV-elliptic function less than or equal to g, which under the assumptions (G_1)–(G_3) admits the representation

$$\mathcal{R}g(a,b,\nu) = \inf\biggl\{\int_{J_w\cap Q_\nu} g(w^+,w^-,\nu)\,\mathrm{d}\mathcal{H}^{N-1}\colon \\ w\in \mathrm{SBV}_0(Q_\nu;\mathbb{R}^m)\cap L^\infty(Q_\nu;\mathbb{R}^m),\ w=v_0 \text{ on } \partial Q_\nu\biggr\}, \tag{5.20}$$

  as in [14,15,17], where v_0 is defined as in (3.4). This is a consequence of (1.13) and (5.20).

We observe that the above characterizations of K_3 could be deduced directly, thereby reproducing the proofs of the lower bound and the upper bound for theorem 1.2 for the jump part on the sets $J_u \setminus J_v$ and $J_v \setminus J_u$, respectively.

6. Applications

This section is devoted to the proof of theorem 1.1, which is very similar to that of theorem 1.2. In particular, we replace lemma 3.8 and proposition 3.3 by lemma 6.1 and proposition 6.2, respectively. However, keeping in mind the application that we describe in more detail in remark 6.4, we state the proof with more generality but, in order to prove theorem 1.1, we consider $m=1$ and $T=\{0,1\}$.

Let $T\subset\mathbb{R}^m$ be a finite set and let

$$V\colon T\times\mathbb{R}^{d\times N}\to(0,+\infty) \quad\text{and}\quad g\colon T\times T\times S^{N-1}\to[0,+\infty[\tag{6.1}$$

satisfy (F_1)–(F_4) and (G_1)–(G_3), respectively. Denote by $\mathcal{A}_{fr}$ the set defined in (1.8), where the range $\{0,1\}$ is replaced by T.

For simplicity we will consider $\nu=e_N$ and consequently $Q_\nu = Q = [0,1]^N$.

LEMMA 6.1. *Let $T\subset\mathbb{R}^m$ be a finite set and let*

$$v_0(y) := \begin{cases} a & \text{if } x_N>0,\\ b & \text{if } x_N<0,\end{cases} \qquad u_0(y) := \begin{cases} c & \text{if } x_N>0,\\ d & \text{if } x_N<0.\end{cases}$$

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Let $\{v_n\} \subset \mathrm{BV}(\Omega;T)$ *and* $\{u_n\} \subset W^{1,1}(Q;\mathbb{R}^d)$ *be such that* $v_n \to v_0$ *is in* $L^1(Q;\mathbb{R}^m)$ *and* $u_n \to u_0$ *is in* $L^1(Q;\mathbb{R}^d)$*.*

If ρ *is a mollifier,* $\rho_n := n^N\rho(nx)$*, then there exists a sequence of functions* $\{(\zeta_n,\xi_n)\} \in \mathcal{A}_{fr}(a,b,c,d,e_N)$*, such that*

$$\zeta_n = v_0 \text{ on } \partial Q, \qquad \zeta_n \to v_0 \text{ in } L^1(Q;\mathbb{R}^m), \tag{6.2}$$

$$\xi_n = \rho_{i(n)} * u_0 \text{ on } \partial Q, \quad \xi_n \to u_0 \text{ in } L^1(Q;\mathbb{R}^d) \tag{6.3}$$

and

$$\begin{aligned}\limsup_{n\to\infty}\biggl(\int_Q QV(\zeta_n,\nabla\xi_n)\,\mathrm{d}x + \int_{J_{\zeta_n}\cap Q} g(\zeta_n^+,\zeta_n^-,\nu_{\zeta_n})\,\mathrm{d}\mathcal{H}^{N-1}\biggr)\\ \leqslant \liminf_{n\to\infty}\biggl(\int_Q QV(v_n,\nabla u_n)\,\mathrm{d}x + \int_{J_{v_n}\cap Q} g(v_n^+,v_n^-,\nu_{v_n})\,\mathrm{d}\mathcal{H}^{N-1}\biggr),\end{aligned} \tag{6.4}$$

where QV *represents the quasi-convex envelope of* V *as in (3.2).*

We omit the proof since it is entirely similar to that of lemma 3.8. We just observe that there is no need for the first step where a truncation argument for v was built, since in the present context we deal with functions with finite range.

The following result, which contains the properties satisfied by K_2 in (1.7), is analogous to proposition 3.3 and it is stated for the reader's convenience.

PROPOSITION 6.2. *Let* V *be as in (1.4). Let* K_2 *be the function introduced in (1.7). The following properties hold.*

(a) $|K_2(a,b,c,d,\nu) - K_2(a',b',c',d',\nu)| \leqslant C(|a-a'|+|b-b'|+|c-c'|+|d-d'|)$ *for every* (a,b,c,d,ν)*,* $(a',b',c',d',\nu) \in \{0,1\}\times\{0,1\}\times\mathbb{R}^d\times\mathbb{R}^d\times S^{N-1}$*.*

(b) $\nu \mapsto K_2(a,b,c,d,\nu)$ *is upper semi-continuous for every* $(a,b,c,d) \in \{0,1\}\times\{0,1\}\times\mathbb{R}^d\times\mathbb{R}^d$*.*

(c) K_2 *is upper semi-continuous in* $\{0,1\}\times\{0,1\}\times\mathbb{R}^d\times\mathbb{R}^d\times S^{N-1}$*.*

(d) $K_2(a,b,c,d,\nu) \leqslant C(|a-b|+|c-d|)$ *for every* $\nu \in S^{N-1}$*.*

Proof of theorem 1.1. The arguments develop as in theorem 1.2, essentially replacing f by V in (1.4), v by χ, the surface integral by $|D\chi|$ and using the blow-up argument introduced in [24]; thus, we will present just the main differences.

(i) Lower bound: let $(\chi,u) \in \mathrm{BV}(\Omega;\{0,1\})\times\mathrm{BV}(\Omega;\mathbb{R}^d)$. Without loss of generality we may assume that for every $\{(\chi_n,u_n)\} \subset \mathrm{BV}(\Omega;\{0,1\})\times\mathrm{BV}(\Omega;\mathbb{R}^d)$ converging to (χ,u) in $L^1(\Omega;\{0,1\})\times L^1(\Omega;\mathbb{R}^d)$, $\liminf_{n\to\infty}(\int_\Omega V(\chi_n,\nabla u_n)\,\mathrm{d}x + |D\chi_n|(\Omega))$ is indeed a limit. For every Borel set $B \subset \Omega$ define

$$\mu_n(B) := \int_{\mathrm{BV}}(\chi_n,\nabla u_n)\,\mathrm{d}x + |D\chi_n|(B).$$

The sequence $\{\mu_n\}$ behaves as in theorem 1.2 and its weak $*$ limit (up to a not relabelled subsequence) μ can be decomposed as in (4.2), where, as in the remainder of the proof, $J_{(v,u)}$ has been replaced by $J_{(\chi,u)}$. Moreover, we emphasize that we

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have been considering (χ, u) as a unique field in $\mathrm{BV}(\Omega;\mathbb{R}^{1+d})$ and we have been exploiting the fact that $D^c(\chi,u) = (0, D^c u)$ (see remark 2.11). By the Besicovitch derivation theorem, we deduce (4.3).

We claim that

$$\mu_a(x_0) \geqslant QV(\chi(x_0), \nabla u(x_0)) \qquad \text{for} \mathcal{L}^N\text{-a.e. } x_0 \in \Omega, \tag{6.5}$$

$$\mu_j(x_0) \geqslant K_2(\chi^+(x_0), \chi^-(x_0), u^+(x_0), u^-(x_0), \nu_{(\chi,u)}) \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x_0 \in J_{(\chi,u)} \cap \Omega, \tag{6.6}$$

$$\mu_c(x_0) \geqslant (QV)^\infty\left(\chi(x_0), \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}(x_0)\right) \quad \text{for } |D^c u|\text{-a.e. } x_0 \in \Omega. \tag{6.7}$$

If (6.5)–(6.7) hold then the lower bound inequality for theorem 1.1 follows.

Step 1. Observing that, by proposition 3.1, QV satisfies (F_1)–(F_3), the proof of (6.5) develops as in step 1 of theorem 1.2, just applying [25, (2.10) in theorem 2.19], to the functional $G\colon (\chi,u) \in W^{1,1}(\Omega;\mathbb{R}^{1+d}) \to \int_\Omega QV(\chi, \nabla u)\,\mathrm{d}x$.

Step 2. The proof of (6.6) is very similar to the one of (4.5). Recall that $J_{(\chi,u)} = J_\chi \cup J_u$ and $\nu_{(\chi,u)} = \nu_\chi$ for every $(\chi,u) \in \mathrm{BV}(\Omega;\{0,1\}) \times W^{1,1}(\Omega;\mathbb{R}^d)$. The same arguments as those of step 2 in theorem 1.2 allow us to fix $x_0 \in J_{(\chi,u)} \cap \Omega$ such that (4.7)–(4.11) hold.

Recall that we denote $Q_{\nu(x_0)}$ by Q and we may choose $\varepsilon > 0$ such that $\mu(\partial(x_0 + \varepsilon Q)) = 0$. We then have

$$\begin{aligned}\mu_j(x_0) &\geqslant \lim_{\varepsilon\to 0^+}\lim_{n\to\infty} \frac{1}{\varepsilon^{N-1}}\left(\int_{x_0+\varepsilon Q} QV(\chi_n(x), \nabla u_n(x))\,\mathrm{d}x + |D\chi_n|(x_0+\varepsilon Q)\right)\\ &= \lim_{\varepsilon\to 0^+}\lim_{n\to\infty}\biggl(\varepsilon\int_Q QV(\chi_n(x_0+\varepsilon y), \nabla u_n(x_0+\varepsilon y))\,\mathrm{d}y\\ &\qquad + |D\chi_n(x_0+\varepsilon y)|\left(Q \cap J(\chi_n, u_n) - \frac{x_0}{\varepsilon}\right)\biggr).\end{aligned}$$

Define $\chi_{n,\varepsilon}$, $u_{n,\varepsilon}$, $\nu_{n,\varepsilon}$ and χ_0, u_0 according to (4.12) and (4.13). Since $(\chi_n, u_n) \to (\chi, u)$ in $L^1(\Omega;\mathbb{R}^{1+d})$, we obtain (4.14) and (4.15) with $v_{n,\varepsilon}$ and v_0 replaced by $\chi_{n,\varepsilon}$ and χ_0, respectively.

Thus,

$$\begin{aligned}\mu_j(x_0) \geqslant \lim_{\varepsilon\to 0^+}\lim_{n\to\infty}\biggl(&\int_Q QV^\infty(\chi_{n,\varepsilon}(y), \nabla u_{n,\varepsilon}(y))\,\mathrm{d}y\\ &+ |D\chi_{n,\varepsilon}|(Q) + \int_Q \varepsilon QV\left(\chi_{n,\varepsilon}(y), \frac{1}{\varepsilon}\nabla u_{n,\varepsilon}(y)\right)\\ &- QV^\infty(\chi_{n,\varepsilon}, \nabla u_{n,\varepsilon})\,\mathrm{d}y\biggr).\end{aligned}$$

By remark 3.2(v) we can argue as in the estimates [25, (3.3)–(3.5)], thereby obtaining

$$\mu_j(x_0) \geqslant \liminf_{\varepsilon\to 0^+}\liminf_{n\to\infty}\left(\int_Q QV^\infty(\chi_{n,\varepsilon}(y), \nabla u_{n,\varepsilon}(y))\,\mathrm{d}y + |D\chi_{n,\varepsilon}|(Q)\right).$$

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Applying lemma 6.1 with QV replaced by QV^∞, $T \subset \mathbb{R}^m$ replaced by $\{0,1\}$, the surface integral replaced by the total variation, K_{fr} and $\mathcal{A}_{fr}$ replaced by K_2 and $\mathcal{A}_2$, respectively, and using remark 3.2 we may find

$$\{(\zeta_k, \xi_k)\} \in \mathcal{A}_2(\chi^+(x_0), \chi^-(x_0), u^+(x_0), u^-(x_0), \nu(x_0))$$

such that

$$\begin{aligned}\mu_j(x_0) &\geqslant \lim_{k\to\infty}\left(\int_Q QV^\infty(\zeta_k, \nabla\xi_k)\,\mathrm{d}x + |D\zeta_k|(Q)\right)\\ &\geqslant K_2(\chi^+(x_0), \chi^-(x_0), u^+(x_0), u^-(x_0), \nu(x_0)).\end{aligned}$$

Step 3. The proof of (6.7) identically follows step 3 in the proof of theorem 4.1; namely, by applying [25, (2.12) in theorem 2.19] to the functional G introduced in step 1 here. This concludes the proof.

(ii) Upper bound: the proof of the upper bound develops in three steps in the same way as the proof of theorem 5.1. Furthermore, proposition 3.6 can be readapted by replacing Qf by QV and the surface integral by $|D\chi|$.

Step 1. For $\mathcal{L}^N$-a.e. $x_0 \in \Omega$, x_0 is a Lebesgue point for $U \equiv (\chi, u)$ such that (5.4) and (5.5) hold for QV. In analogy with theorem 5.1 step 1, we apply for every $\chi \in \mathrm{BV}(\Omega; \{0,1\})$ the global method [13, theorem 4.1.4] to the functional $G\colon (u, A) \in W^{1,1}(\Omega; \mathbb{R}^m) \times \mathcal{A}(\Omega) \to \int_\Omega QV(\chi, \nabla u)\,\mathrm{d}x$ to obtain an integral representation for the functional (5.6) for every $(u, A) \in \mathrm{BV}(\Omega; \mathbb{R}^m) \times \mathcal{A}(\Omega)$. Moreover, we can write

$$\mathcal{F}_{\mathcal{OD}}(\chi, u; A) \leqslant \mathcal{G}(u; A) + |D\chi|(A).$$

Differentiating with respect to $\mathcal{L}^N$ we obtain

$$\frac{\mathrm{d}\mathcal{F}_{\mathcal{OD}}(\chi, u; \cdot)}{\mathrm{d}\mathcal{L}^N} \leqslant V_0(x_0, \nabla u(x_0)),$$

where V_0 is the co-respective of f_0 in (5.8) where Qf has been replaced by QV. Arguing as in the last part of theorem 5.1, step 1 and applying lemma 3.10, we deduce that $V_0(x_0, \xi_0) \leqslant QV(\chi(x_0), \xi_0)$ and this leads to the conclusion when $u \in \mathrm{BV}(\Omega; \mathbb{R}^d) \cap L^\infty(\Omega; \mathbb{R}^d)$.

Step 2. The same type of arguments as those in step 1 apply to the proof of the upper bound for the Cantor part. The Radon–Nikodým theorem implies (5.9) for every $U \equiv (\chi, u) \in \mathrm{BV}(\Omega; \{0,1\}) \times (\mathrm{BV}(\Omega; \mathbb{R}^d) \cap L^\infty(\Omega; \mathbb{R}^d))$, with $|D^c u|$ and σ mutually singular. Moreover, (5.10)–(5.12) hold, the global method [13, theorem 4.1.4] applies to (5.6) and a differentiation with respect to $|D^c u|$ at x_0 provides

$$\frac{\mathrm{d}\mathcal{F}_{\mathcal{OD}}(\chi, u; \cdot)}{\mathrm{d}|D^c u|}(x_0) \leqslant h(x_0, a_u, \nu_u),$$

where $h(x, a, \nu)$ is given by (5.13). Remark 3.2 applied to QV^∞, lemma 3.10 and the same techniques employed in the last part of theorem 5.1, step 2 entail

$$h(x_0, a, \nu) \leqslant QV^\infty(\chi(x_0), a \otimes \nu),$$

which concludes the proof of the Cantor part for

$$(\chi, u) \in \mathrm{BV}(\Omega; \{0,1\}) \times (\mathrm{BV}(\Omega; \mathbb{R}^d) \cap L^\infty(\Omega; \mathbb{R}^d)).$$

STEP 3. We claim that

$$\mathcal{F}_{\mathcal{OD}}(U;J_U) \leqslant \int_{J_U} K_2(\chi^+,\chi^-,u^+,u^-,\nu_{\chi,u})\,\mathrm{d}\mathcal{H}^{N-1} \tag{6.8}$$

for every $(\chi,u) \in \mathrm{BV}(\Omega;\{0,1\}) \times (\mathrm{BV}(\Omega;\mathbb{R}^d) \cap L^\infty(\Omega;\mathbb{R}^d))$. The proof of (6.8) is divided into three parts according to the assumptions on the limit functions u.

CASE 1. $U(x) := (1,c)\chi_E(x) + (0,d)(1-\chi_E(x))$, with $P(E,\Omega) < +\infty$.

CASE 2. $u(x) = \sum_{i=1}^\infty c_i\chi_{E_i}(x)$, where $\{E_i\}_{i=1}^\infty$ forms a partition of Ω into sets of finite perimeter and $c_i \in \mathbb{R}^d$.

CASE 3. $u(x) \in \mathrm{BV}(\Omega;\mathbb{R}^d) \cap L^\infty(\Omega;\mathbb{R}^d)$.

Concerning case 1, we first consider the unit open cube $Q \subset \mathbb{R}^N$ and make the same assumptions on the target function U as those in theorem 5.1, step 3, case 1. We can then invoke an argument analogous to proposition 3.4 without invoking any truncation arguments such as those in remark 3.5. This guarantees that there exist $(\chi_n,u_n) \in \mathcal{A}_2(1,0,c,d,e_N)$ such that $(\chi_n,u_n) \to (\chi,u)$ in $L^1(Q;\mathbb{R}^{1+d})$ and

$$K_2(1,0,c,d,e_N) = \lim_{n\to\infty}\biggl(\int_Q QV^\infty(\chi_n(x),\nabla u_n(x))\,\mathrm{d}x + |D\chi_n|(Q)\biggr). \tag{6.9}$$

The proof then develops exactly as that of theorem 5.1 but taking into account that the sequence $z_{n,k}$ therein is built by replacing a, b and v_n by 1, 0 and χ_n, respectively, thus leading to

$$\mathcal{F}_{OD}(\chi,u;Q) \leqslant \frac{QV(1,0)+QV(0,0)}{2} + K_2(1,0,c,d,e_N).$$

With regard to a more general set A than Q, like that in theorem 5.1, step 3, case 1, we achieve the following representation:

$$\mathcal{F}_{\mathcal{OD}}(\chi,u;A) \leqslant \int_A QV(\chi(x),0)\,\mathrm{d}x + \int_{J_U} K_2(1,0,c,d,\nu)\,\mathrm{d}\mathcal{H}^{N-1}.$$

Then the strategy follows (b), (c) and (d) in theorem 5.1, step 3, case 1, and hence we obtain

$$\mathcal{F}_{\mathcal{OD}}(\chi,u;J_{\chi,u}) \leqslant \int_{J_{\chi,u}} K_2(1,0,c,d,\nu)\,\mathrm{d}\mathcal{H}^{N-1}.$$

Turning to case 2 and case 3, by the properties of K_2 in proposition 6.2, the proof develops in the same way as in [8, proposition 4.8, cases 2 and 3]. This concludes the proof of the upper bound when $(\chi,u) \in \mathrm{BV}(\Omega;\{0,1\}) \times (\mathrm{BV}(\Omega;\mathbb{R}^d) \cap L^\infty(\Omega;\mathbb{R}^d))$.

The general case, since $\chi \in \mathrm{BV}(\Omega;\{0,1\})$ and can be fixed, is identical to [25, § 5, step 4], where the truncation procedure involves just u.

Putting (i) and (ii) together, we achieve the desired result. □

REMARK 6.3. We observe that, as in remark 5.3, K_2 admits the following equivalent representations.

(i) On $J_u \setminus J_\chi$, $K_2(a,a,c,d,\nu) = QV^\infty(a,(c-d)\otimes\nu)$ with QV^∞ as in (1.6).

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(ii) On $J_\chi \setminus J_u$, $K_2(a,b,c,c,\nu) = |(a-b)\otimes\nu|$, i.e.

$$\int_{J_\chi} K_2(\chi^+,\chi^-,u^+,u^+,\nu)\,\mathrm{d}\mathcal{H}^{N-1} = |D\chi|(\Omega).$$

(iii) Note that

$$K_2(a,b,c,d,\nu) \geqslant \inf\left\{\int_{Q_\nu}(QV^\infty(w(x),\nabla u(x)) + |\nabla w(x)|)\,\mathrm{d}x \colon (w,u)\in\mathcal{A}(a,b,c,d,\nu)\right\},$$

where this latter density is the density $K(a,b,c,d,\nu)$ first introduced in [25] (see also [9, (5.83)]) and

$$\begin{aligned}\mathcal{A}(a,b,c,d,\nu) := \{(w,u)\in W^{1,1}(Q_\nu;\mathbb{R}^{1+d})\colon & \\ (w(y),u(y)) = (a,c) \text{ if } y\cdot\nu = \tfrac{1}{2}, & \\ (w(y),u(y)) = (b,d) \text{ if } y\cdot\nu = -\tfrac{1}{2}, & \\ (w,u) \text{ are 1-periodic in } \nu_1,\dots,\nu_{N-1} \text{ directions}\}. &\end{aligned}$$

On the other hand, if W_i, $i=1,2$, in (1.1) are proportional (as in the model presented in [6]), i.e. $W_2 = \alpha W_1$, $\alpha > 1$, taking V as in (1.4), since for every $q\in[0,1]$ $QV^\infty(q,z) = qQW_1^\infty(z) + \alpha(1-q)QW_1^\infty(z)$, then we claim that K_2 is equal to K of [25]. Indeed, without loss of generality, assuming W_1 is quasi-convex and positively 1-homogeneous, it is enough to observe that, for every $(w,u)\in\mathcal{A}(a,b,c,d,\nu)$,

$$\begin{aligned}K(1,0,c,d,\nu) &\geqslant \int_{Q_\nu}(w(x)W_1(\nabla u(x)) + \alpha(1-w(x))W_1(\nabla u(x)) + |\nabla w(x)|)\,\mathrm{d}x \\ &\geqslant \int_{Q_\nu}(W_1(\nabla u(x))+1)\,\mathrm{d}x,\end{aligned}$$

where we have used the fact that $\alpha + (1-\alpha)w \geqslant 1$ and

$$\int_{Q_\nu}|\nabla w|\,\mathrm{d}x \geqslant \left|\int_{Q_\nu}\nabla w\right|\mathrm{d}x = \left|\int_{\partial Q_\nu} w\otimes\nu(x)\,\mathrm{d}\mathcal{H}^{N-1}\right| = 1.$$

Taking a sequence of characteristic functions $\{\chi_\varepsilon\}$, admissible for $\mathcal{A}_2(1,0,c,d,\nu)$ in (1.8), such that their value is 1 in a strip of the cube orthogonal to ν and of thickness $1-\varepsilon$, we have

$$\begin{aligned}&\int_{Q_\nu} W_1(\nabla u(x))\,\mathrm{d}x + 1 \\ &\quad= \lim_{\varepsilon\to 0^+}\int_{Q_\nu}(\chi_\varepsilon W_1(\nabla u(x)) + \alpha(1-\chi_\varepsilon)W_1(\nabla u(x)))\,\mathrm{d}x + |D\chi_\varepsilon|(Q_\nu) \\ &\quad\geqslant K_2(1,0,c,d,\nu)\end{aligned}$$

and this proves our claim. Observe also that if $\alpha\in(0,1)$, then the result remains true; it is enough to express W_1 in terms of W_2.

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As emphasized in [6, remark 2.4], one can consider mixtures of more than two conductive materials, and hence we observe that theorem 1.1 can be extended with minor changes to these models, thereby leading to (6.12) in the remark below.

REMARK 6.4. Let T be a finite subset of $\mathbb{R}^m$. Theorem 1.1 also applies to energies of the type $F_{fr}\colon L^1(\Omega;T)\times L^1(\Omega;\mathbb{R}^d)\times\mathcal{A}(\Omega)\to[0,+\infty]$ defined by

$$F_{fr}(v,u;A):=\begin{cases}\displaystyle\int_A V(v,\nabla u)\,\mathrm{d}x+\int_{J_v}\bigcap A g(v^+,v^-,\nu_v)\,\mathrm{d}\mathcal{H}^{N-1} & \\ \qquad\qquad\text{in } \mathrm{BV}(A;T)\times W^{1,1}(A;\mathbb{R}^d), & \\ +\infty \qquad\qquad \text{otherwise.}\end{cases} \tag{6.10}$$

Indeed, consider the relaxed localized energy of (6.10) given by

$$\begin{aligned}\mathcal{F}_{fr}(v,u;A) &:=\inf\biggl\{\liminf_{n\to\infty}\int_A V(v_n,\nabla u_n)\,\mathrm{d}x+\int_{J_{v_n}\cap A} g(v_n^+,v_n^-,\nu_{v_n})\,\mathrm{d}\mathcal{H}^{N-1}\colon\\ &\qquad\{(v_n,u_n)\}\subset\mathrm{BV}(A;T)\times W^{1,1}(A;\mathbb{R}^d),\\ &\qquad(v_n,u_n)\to(v,u)\text{ in } L^1(A;T)\times L^1(A;\mathbb{R}^d)\biggr\},\end{aligned}$$

with V and g as in (6.1) satisfying (F_1)–(F_4) and (G_1)–(G_3), respectively.

Moreover, define $\bar{F}_{fr}\colon\mathrm{BV}(A;T)\times\mathrm{BV}(A;\mathbb{R}^d)\times\mathcal{A}(\Omega)\to[0,+\infty]$ as

$$\begin{aligned}\bar{F}_{fr}(v,u;A) &:=\int_A QV(v,\nabla u)\,\mathrm{d}x+\int_A QV^\infty\biggl(v,\frac{\mathrm{d}D^cu}{\mathrm{d}|D^cu|}\biggr)\,\mathrm{d}|D^cu|\\ &\qquad+\int_{J_{(v,u)}\bigcap A} K_{fr}(v^+,v^-,u^+,u^-,\nu)\,\mathrm{d}\mathcal{H}^{N-1},\end{aligned}$$

where QV is the quasi-convex envelope of V given in (3.2), QV^∞ is the recession function of QV, introduced in (1.6), and

$$\begin{aligned}K_{fr}(a,b,c,d,\nu) &:=\inf\biggl\{\int_{Q_\nu} QV^\infty(v,\nabla u(x))\,\mathrm{d}x\\ &\qquad+\int_{Q_\nu} g(v^+,v^-,\nu_v)\,\mathrm{d}\mathcal{H}^{N-1}\colon(v,u)\in\mathcal{A}_{fr}(a,b,c,d,\nu)\biggr\},\end{aligned} \tag{6.11}$$

where $\mathcal{A}_{fr}$ is the set defined in (1.8) with $\{0,1\}$ replaced by the finite set $T\subset\mathbb{R}^m$. Thus, we are led to the following representation: for every $(v,u)\in L^1(\Omega;T)\times L^1(\Omega;\mathbb{R}^d)$,

$$\mathcal{F}_{fr}(v,u;A)=\begin{cases}\bar{F}_{fr}(v,u;A) & \text{if } (v,u)\in\mathrm{BV}(A;T)\times\mathrm{BV}(A;\mathbb{R}^d),\\ +\infty & \text{otherwise.}\end{cases} \tag{6.12}$$

REMARK 6.5. In general we cannot expect $K_3=K_{fr}$ since in (6.11) the function g is defined in $T\times T\times S^{N-1}$ with $T\subset\mathbb{R}^d$ and $\mathrm{card}(T)$ finite, while in (1.13), g is defined in $\mathbb{R}^d\times\mathbb{R}^d\times S^{N-1}$. In particular, we recall that in $J_v\setminus J_u$, K_3 coincides with $\mathcal{R}g$, the SBV-elliptic envelope of g as in [14], while K_{fr} in (6.11) is given by the BV-elliptic envelope introduced by Ambrosio and Braides (see [9, definition 5.13]). Analogously, it is easily seen that K_2 coincides with $|D\chi|$ in $J_\chi \setminus J_u$.

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References

1 E. Acerbi and N. Fusco. Semicontinuity problems in the calculus of variations. *Arch. Ration. Mech. Analysis* **86** (1984), 125–145.

2 L. Ambrosio. A compactness theorem for a special class of functions of bounded variation. *Boll. UMI* B **3** (1989), 857–881.

3 L. Ambrosio. Existence theory for a new class of variational problems. *Arch. Ration. Mech. Analysis* **111** (1990), 291–322.

4 L. Ambrosio. A new proof of the SBV compactness theorem. *Calc. Var.* **3** (1995), 127–137.

5 L. Ambrosio and A. Braides. Functionals defined on partitions in sets of finite perimeter. I. Integral representation and Γ-convergence. *J. Math. Pures Appl.* **69** (1990), 285–306.

6 L. Ambrosio and G. Buttazzo. An optimal design problem with perimeter penalization. *Calc. Var. PDEs* **1** (1993), 55–69.

7 L. Ambrosio and G. Dal Maso. On the relaxation in $\mathrm{BV}(\Omega;\mathbb{R}^m)$ of quasi-convex integrals. *J. Funct. Analysis* **109** (1992), 76–97.

8 L. Ambrosio, S. Mortola and V. M. Tortorelli. Functionals with linear growth defined on vector valued BV functions. J. Math. Pures Appl. **70** (1991), 269–323.

9 L. Ambrosio, N. Fusco and and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs, (New York: Clarendon, 2000).

10 J.-F. Babadjian, E. Zappale and H. Zorgati. Dimensional reduction for energies with linear growth involving the bending moment. *J. Math. Pures Appl.* **90** (2008), 530–549.

11 S. Baldo. Minimal interface criterion for phase transitions in mixtures of Cahn–Hilliard fluids. *Annales Inst. H. Poincaré Analyse Non Linéaire* **7** (1990), 67–90.

12 A. C. Barroso, G. Bouchitté, G. Buttazzo and I. Fonseca. Relaxation of bulk and interfacial energies. *Arch. Ration. Mech. Analysis* **135** (1996), 107–173.

13 G. Bouchitté, I. Fonseca and L. Mascarenhas. A global method for relaxation. *Arch. Ration. Mech. Analysis* **144** (1998), 51–98.

14 G. Bouchitté, I. Fonseca, G. Leoni and L. Mascarenhas. A global method for relaxation in $W^{1,p}$ and SBV_p. *Arch. Ration. Mech. Analysis* **165** (2002), 187–242.

15 A. Braides, A. Defranceschi and E. Vitali. Homogenization of free discontinuity problems. *Arch. Ration. Mech. Analysis* **135** (1996), 297–356.

16 G. Carita and E. Zappale. 3D–2D dimensional reduction for a nonlinear optimal design problem with perimeter penalization. *C. R. Math.* **350** (2012), 1011–1016.

17 R. Choksi and I. Fonseca. Bulk and interfacial energy densities for structured deformations of continua. *Arch. Ration. Mech. Analysis* **138** (1997), 37–103.

18 G. Congedo and I. Tamanini. On the existence of solutions to a problem in multidimensional segmentation. *Annales Inst. H. Poincaré Analyse Non Linéaire* **8** (1991), 175–195.

19 B. Dacorogna. *Direct methods in the calculus of variations*, 2nd edn. Applied Mathematical Sciences, vol. 78 (Springer, 2008).

20 G. Dal Maso, I. Fonseca and G. Leoni. Nonlocal character of the reduced theory of thin films with higher order perturbations. *Adv. Calc. Var.* **3** (2010), 287–319.

21 L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. (Boca Raton, FL: CRC Press, 1992).

22 H. Federer. *Geometric measure theory* (Springer, 1969).

23 I. Fonseca and G. Leoni. *Modern methods in the calculus of variations: L^p spaces* (Springer, 2007).

24 I. Fonseca and S. Müller. Quasi-convex integrands and lower semicontinuity in L^1. *SIAM J. Math. Analysis* **23** (1992), 1081–1098.

25 I. Fonseca and S. Müller. Relaxation of quasiconvex functionals in $\mathrm{BV}(\Omega, \mathbb{R}^d)$ for integrands $f(x, u, \nabla u)$. *Arch. Ration. Mech. Analysis* **123** (1993), 1–49.

26 I. Fonseca and P. Ribka. Relaxation of multiple integrals in the space $\mathrm{BV}(\Omega; \mathbb{R}^d)$. *Proc. R. Soc. Edinb.* A **121** (1992), 321–348.

27 E. Giusti. *Minimal surfaces and functions of bounded variation*. Monographs in Mathematics, vol. 80 (Birkhäuser, 1984).

28 R. V. Kohn and G. Strang. Optimal design and relaxation of variational problems. I. *Commun. Pure Appl. Math.* **39** (1986), 113–137.

29 R. V. Kohn and G. Strang. Optimal design and relaxation of variational problems. II. *Commun. Pure Appl. Math.* **39** (1986), 139–182.

30 R. V. Kohn and G. Strang. Optimal design and relaxation of variational problems. III. *Commun. Pure Appl. Math.* **39** (1986), 353–377.

31 A. M. Ribeiro and E. Zappale. Relaxation of certain integral functionals depending on strain and chemical composition. *Chin. Annals Math.* **34** (2013), 491–514.

32 W. P. Ziemer. *Weakly differentiable functions: Sobolev spaces and functions of bounded variation*. Graduate Texts in Mathematics, vol. 120, p. 308 (Springer, 1989).

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