Wave Propagation in Strings with Continuous and Concentrated Loads. By Mr HAROLD JEFFREYS, St John's College.

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1·0. The normal modes of a stretched string, with equal masses attached at equal intervals along it, are well known. But the method of normal coordinates is not often a very convenient way of treating the motion of systems when started off by impulses or displacements at definite points, unless the number of degrees of freedom is very small, and the most useful in practice is the operational method of Heaviside or the equivalent method of Bromwich depending on the use of complex integrals. Apart from its intrinsic interest, the motion of a loaded string disturbed in this way serves to illustrate several other questions.

First, if the length, total mass, and tension of the string are kept constant, but the individual particles and the distance between them are made very small, we approach in the limit the problem of the uniform heavy string. Now the justification of the operational method is complete for systems with a finite number of degrees of freedom. Starting by defining σ^{-1} to mean the operation of integrating with regard to t from 0 to t, it can be shown without appeal to complex integration that any operator is intelligible if, when we regard σ temporarily as a complex variable, the operator is regular near $\sigma^{-1}=0$ or $\sigma=\infty$. It can also be shown that all the operators arising in the discussion of systems with a finite number of degrees of freedom satisfy this condition*. We may remark that σ is not defined explicitly: only powers of σ^{-1} occur in the operational solution. The operational solution can therefore be completely justified for light strings loaded at a finite number of points.

But when the operational method is applied to continuous systems the solution is found to involve such operators as $e^{-\sigma h/c}$, where h is positive and independent of t. This is not regular near $\sigma^{-1}=0$, and a new rule of interpretation is needed. The rule adopted is that

$$e^{-\sigma h/c}\phi(t)=\phi\left(t-\frac{h}{c}\right) \qquad\qquad (1).$$

This rule has been considered a symbolical form of Taylor's theorem; but it gives the correct answer when ϕ or some of its derivatives are discontinuous between t and $t-h/c$, when Taylor's theorem is untrue. If it is interpreted by means of complex integrals it is

* *Cambridge Math. Tracts* (in the press).

found to be equivalent, in ordinary cases, to Fourier's theorem. If we approach the continuous system as the limit of a finite one we may hope to find out how the exponential arises and obtain some light on the reason for its interpretation.

Second, the nature of the propagation of a disturbance in a string loaded discontinuously proves to be more complicated than in the continuous string, and dispersion occurs.

2·0. Consider a light string under tension, free to execute either transverse or longitudinal vibrations. Particles of mass ρl are attached to it at intervals l. If $y_0, y_1, y_2 \ldots$ are the displacements of the particles, the kinetic energy is given by

$$2T = \rho l (\dot{y}_0^2 + \dot{y}_1^2 + \dot{y}_2^2 + \ldots) \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and the potential energy by

$$2V = \rho c^2 l^{-1} \{(y_1 - y_0)^2 + (y_2 - y_1)^2 + \ldots\} \quad \ldots\ldots\ldots(2).$$

For transverse vibrations ρc^2 is the tension; for longitudinal ones it is the sum of Hooke's constant and the tension. The equations of motion are of the form

$$\ddot{y}_r = -\frac{c^2}{l^2}(2y_r - y_{r-1} - y_{r+1}) \quad \ldots\ldots\ldots\ldots\ldots(3).$$

If we make l approach zero while ρ and c remain constant, the system approaches the continuous string with line density ρ, and c becomes the velocity of propagation of waves. If further the quantity on the right of (3) tends to a definite limit we put

$$rl = x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

and the equation reduces to the ordinary form

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Suppose now that the system starts from rest, that the particle corresponding to y_m is kept fixed, and that y_0 is made to vary with the time in some prescribed manner. By our rules we replace the general equation of motion by

$$-\frac{\sigma^2 l^2}{c^2} y_r = 2y_r - y_{r-1} - y_{r+1} \quad \ldots\ldots\ldots\ldots\ldots(6),$$

subject to y_0 being given, so that the first equation of type (6) to arise involves $\sigma^2 y_1$. Also

$$y_m = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

In obtaining the operational solution of these equations we must treat σ as a constant and carry through the operations as for equations of finite differences. We find that if

$$y_r = Aa^r + Bb^r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

all the equations for $r=1$ to $m-1$ become identities provided that

$$a=e^{\lambda};\quad b=e^{-\lambda} \quad \text{.......................(9)},$$

where
$$\cosh\lambda = 1+\sigma^2 l^2/2c^2 \quad \text{..................(10)},$$

or
$$\sinh\tfrac{1}{2}\lambda=\frac{\sigma l}{2c} \quad \text{.......................(11)}.$$

The conditions for $r=0$ and $r=m$ determine A and B, and our operational solution is easily found to be

$$y_r=\frac{\sinh(m-r)\lambda}{\sinh m\lambda}y_0 \quad \text{..................(12)}.$$

Now $\sinh m\lambda/\sinh\lambda$ is a polynomial in $\sinh\frac{1}{2}\lambda$ of degree $2(m-1)$, the term of highest degree being $(2\sinh\frac{1}{2}\lambda)^{2(m-1)}$ or $(\sigma l/c)^{2(m-1)}$. As was to be expected, the operator is therefore a regular function of σ^{-1} near $\sigma^{-1}=0$, and its first term is $(\sigma l/c)^{-2r}$ or $(c/l\sigma)^{2r}$. It follows at once that the further a particle is from the disturbed end the more gradually it will begin to move.

If we take y_0 to be unity for all positive values of the time the first term in y_r is $\dfrac{1}{(2r)!}\left(\dfrac{ct}{l}\right)^{2r}$. If r is great we can approximate to this by Stirling's formula and find

$$y_r \doteqdot \left(\frac{1}{4\pi r}\right)^{\frac{1}{2}}\left(\frac{e}{2}\frac{ct}{rl}\right)^{2r}$$
$$=\left(\frac{l}{4\pi x}\right)^{\frac{1}{2}}\left(\frac{e}{2}\frac{ct}{x}\right)^{2x/l} \quad \text{.....................(13)}.$$

If then x is greater than $\frac{1}{2}ect$, y_r will tend to zero when l is made to tend to zero. Similar considerations will apply to the later terms in the expansion of the operator, and the fact that in a continuous string it takes a finite time for a disturbance at $x=0$ to produce any motion at all at a given distance from the end therefore becomes intelligible. This proposition is untrue for the string carrying discrete particles.

Still taking y_0 to be unity when t is positive, and using the complex integral interpretation, we have

$$y_r=\frac{1}{2\pi\iota}\int_L\frac{\sinh(m-r)\lambda}{\sinh m\lambda}e^{\gamma t}\frac{d\gamma}{\gamma} \quad \text{...........(14)},$$

where the integral is to be taken along a line parallel to the imaginary axis and on the positive side of it, and λ is defined now by

$$\sinh\tfrac{1}{2}\lambda=\frac{\gamma l}{2c} \quad \text{.........................(15)},$$

subject to λ and γ being real and positive at the same time. Then on the path of integration

$$|e^{\lambda}| > 1 \quad \text{......................(16),}$$

and we can expand in powers of $e^{-2m\lambda}$. The wave-expansion method that occurs so frequently in the operational treatment of continuous systems is therefore applicable to the string with discrete particles. The first term is

$$z_r = \frac{1}{2\pi\iota}\int_L e^{\gamma t - r\lambda}\frac{d\gamma}{\gamma} \quad \text{.................(17),}$$

and the rest follow at once. The first term gives the direct wave, the others the waves reflected at the ends. In operational form we can write

$$z_r = e^{-r\lambda} \quad \text{....................................(18)}$$

$$= \left\{\frac{\sigma l}{2c} + \left(\frac{\sigma^2 l^2}{4c^2} + 1\right)^{\frac{1}{2}}\right\}^{-2r} \quad \text{..............(19).}$$

We can evaluate the integral (17) by the method of steepest descents. For given values of t and r there is a saddle point where

$$t = r\frac{d\lambda}{d\gamma},$$

that is, $$\cosh\tfrac{1}{2}\lambda = rl/ct = x/ct = \xi \text{ say} \quad \text{...........(20).}$$

If $\xi > 1$, this makes λ, and therefore γ, real and positive. If $\xi < 1$, there are two saddle points on the imaginary axis, which both contribute to the integral.

If $x > ct$, γ at the saddle point is equal to $(2c/l)\sqrt{(\xi^2 - 1)}$. Near this point

$$\gamma t - r\lambda = \frac{2ct}{l}\left[(\xi^2 - 1)^{\frac{1}{2}} - \xi\cosh^{-1}\xi\right] + \frac{lt}{4c}\frac{(\xi^2 - 1)^{\frac{1}{2}}}{\xi^2}(\gamma - \gamma_0)^2 \quad (21),$$

and the line of steepest descent is there perpendicular to the real axis. We find by the usual formula

$$z_r \sim \frac{1}{2}\left(\frac{l}{\pi ct}\right)^{\frac{1}{2}}\frac{\xi}{(\xi^2 - 1)^{\frac{3}{4}}}\exp\left[-\frac{2ct}{l}\{\xi\cosh^{-1}\xi - (\xi^2 - 1)^{\frac{1}{2}}\}\right] \quad (22).$$

When ξ is great this approximates to (13).

If $x < ct$, λ at the saddle points will be $\pm i\mu$, where

$$\cos\tfrac{1}{2}\mu = \xi; \quad \gamma = \pm\frac{2c}{l}i\sin\tfrac{1}{2}\mu \quad \text{...........(23).}$$

But it is found that the line of steepest descent through $+i\mu$ is in the direction $\frac{1}{4}\pi$ and goes to $-\infty$; that through $-i\mu$ is in the direction $\frac{3}{4}\pi$ and also goes to $-\infty$. These lines are therefore not

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together equivalent to the line L, because they pass the origin, which is a pole, on the negative real side. A loop from $-\infty$ around the origin must be added. This makes a contribution unity, and we have

$$z_r \sim 1 - \left(\frac{l}{\pi ct}\right)^{\frac{1}{2}} \frac{\xi}{(1-\xi^2)^{\frac{3}{4}}} \cos\left[\frac{2ct}{l}\{(1-\xi^2) - \xi\cos^{-1}\xi\} + \tfrac{1}{4}\pi\right] \quad (24).$$

Considering now how (22) and (24) behave when l is made small, we see that in front of the point where $x = ct$ the displacement falls off rapidly, and that at all such points it tends to zero with l. Behind this point the displacement is unity except for a rapidly alternating portion, whose wave-length and amplitude both tend to zero with l. The solution for the continuous string is therefore zero displacement for $x > ct$ and unit displacement when $x < ct$. This corresponds to the function $H(t - x/c)$, which is the solution obtained by direct operational methods.

The part of the solution we have just estimated is

$$e^{-r\lambda} H(t) = \left\{\left(1 + \frac{\sigma^2 l^2}{4c^2}\right)^{\frac{1}{2}} + \frac{\sigma l}{2c}\right\}^{-2r} H(t) \quad \text{.........(25).}$$

The operator is regular near $\sigma^{-1} = 0$. If, however, we make l tend to zero, the operator tends (formally) to $\exp(-\sigma r l/c)$ or $\exp(-\sigma x/c)$. The rule that

$$\exp(-\sigma x/c)\, H(t) = H(t - x/c) \quad \text{............(26),}$$

is therefore justified provided that we define $\exp(-\sigma x/c)$ as meaning

$$\lim_{l \to 0} \left\{\left(1 + \frac{\sigma^2 l^2}{4c^2}\right)^{\frac{1}{2}} + \frac{\sigma l}{2c}\right\}^{-2x/l} \quad \text{...............(27).}$$

In this way the exponential operator is defined in terms of the fundamental concept of definite integration. Further, since any function of t can be built up out of the function $H(t)$, this result is immediately generalized to any function.

3·0. Consider next a string whose fixed ends correspond to $r = m$ and $r = -m'$. Initially y and $\dot{y}$ are zero for all particles, except that $y_0 = u$ and $\dot{y}_0 = v$. For positive values of r we still have

$$y_r = \frac{\sinh(m-r)\lambda}{\sinh m\lambda}\, y_0 \quad \text{..................(1),}$$

and for negative values, if $r = -r'$,

$$y_{-r'} = \frac{\sinh(m'-r')\lambda}{\sinh m'\lambda}\, y_0 \quad \text{..................(2).}$$

The subsidiary equation for y_0 is now

$$\left(2 + \frac{\sigma^2 l^2}{c^2}\right) y_0 - y_1 - y_{-1} = \frac{l^2}{c^2}(\sigma^2 u + \sigma v) \quad \text{.........(3),}$$

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whence, with the same meaning of λ as in the last section,

$$y_0 = \frac{l^2}{c^2}(\sigma^2 u + \sigma v)\frac{\sinh m\lambda \sinh m'\lambda}{\sinh(m+m')\lambda \sinh\lambda} \quad \text{.........(4)},$$

$$y_r = \frac{l^2}{c^2}(\sigma^2 u + \sigma v)\frac{\sinh m'\lambda \sinh(m-r)\lambda}{\sinh(m+m')\lambda \sinh\lambda} \quad \text{......(5)}.$$

If we expand y_r by the wave-expansion method, we find that the first wave is given by

$$z_r = \frac{l^2}{2c^2}(\sigma^2 u + \sigma v)\, e^{-r\lambda}/\sinh\lambda \quad \text{..............(6)}.$$

If the string is infinite in length, with the same masses and spacing of the particles, the reflected waves do not arise, and this expression is the complete solution.

3·1. The problem just solved operationally is not of immediate interest, but has important extensions. Suppose first that $y_{-r'}$ and $\dot{y}_{-r'}$ are initially equal to u and v for all values of r'. Then each initial disturbance makes its contribution to y_r and z_r, and we have

$$z_r = \frac{l^2}{c^2}\frac{\sigma^2 u + \sigma v}{2\sinh\lambda} e^{-r\lambda}(1 + e^{-\lambda} + e^{-2\lambda} + \ldots)$$

$$= \frac{l^2}{c^2}\frac{(\sigma^2 u + \sigma v)\, e^{-r\lambda}}{2\sinh\lambda\,(1 - e^{-\lambda})} \quad \text{...........................(7)}.$$

But by 2·0 (11) this can be written

$$z_r = \left(u + \frac{v}{\sigma}\right)\frac{e^{-(r-\frac{1}{2})\lambda}}{2\cosh\frac{1}{2}\lambda} \quad \text{.............................(8)}$$

$$= \frac{1}{2\pi\iota}\int_L \left(u + \frac{v}{\gamma}\right)\frac{e^{\frac{1}{2}\lambda}}{2\cosh\frac{1}{2}\lambda} e^{\gamma t - r\lambda}\frac{d\gamma}{\gamma} \quad \text{.........(9)},$$

λ being now redefined as in 2·0 (15), and the saddle points are as in 2·0. If $x > ct$, we can obtain the solution from 2·0 (22) by multiplying by the value of $\frac{1}{2}(u + v/\gamma)\, e^{\frac{1}{2}\lambda}\,\mathrm{sech}\,\frac{1}{2}\lambda$ at the saddle point

$$\gamma = \frac{2c}{l}(\xi^2 - 1)^{\frac{1}{2}} \quad \text{....................(10)}.$$

The factor required is found to be

$$\tfrac{1}{2}\left\{u + \frac{vl}{2c}\sqrt{(\xi^2 - 1)}\right\}\left\{1 + \frac{(\xi^2 - 1)^{\frac{1}{2}}}{\xi}\right\} \quad \text{........(11)},$$

and the disturbance when $x > ct$ is again insignificant if l is small. When $x < ct$ the contribution from the saddle points 2·0 (23) is again multiplied by a finite factor and therefore is comparable with its value when y_0 is constrained to be unity. But the contribution from the loop round the origin is altered in form, since the origin is now a double pole. We find that this portion is

$$\tfrac{1}{2}u + \tfrac{1}{2}v\{t - (r - \tfrac{1}{2})\,l/c\} \quad \text{..................(12)},$$

or practically

$$\tfrac{1}{2}u + \tfrac{1}{2}v\{t - x/c\} = \tfrac{1}{2}(u + v/\sigma)\,H(-x/c) \quad \text{.........(13)}.$$

By making l small and superposing motions due to assigned initial disturbances we can easily obtain the D'Alembert solution for a continuous string

$$y = \tfrac{1}{2}u(x+ct) + \tfrac{1}{2}u(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} v(x)\,dx \quad \text{...(14)},$$

where u and v are now functions of x.

3·2. Let us return now to the case where only particle number 0 is disturbed initially. Using 3·0 (6) we have

$$z_r = \frac{l}{2c}\,\frac{\sigma u + v}{\cosh \tfrac{1}{2}\lambda}\,e^{-r\lambda} \quad \text{.............................(15)}$$

$$= \frac{l}{2c}\,\frac{\sigma u + v}{\left\{\dfrac{\sigma l}{2c} + \left(\dfrac{\sigma^2 l^2}{4c^2} + 1\right)^{\frac{1}{2}}\right\}^{2r}\left(\dfrac{\sigma^2 l^2}{4c^2} + 1\right)^{\frac{1}{2}}} \quad \text{......(16)}.$$

Since the operational form of the Bessel functions is

$$J_n(kt) = \frac{\sigma/k}{\left\{\dfrac{\sigma}{k} + \left(\dfrac{\sigma^2}{k^2} + 1\right)^{\frac{1}{2}}\right\}^{n}\left(\dfrac{\sigma^2}{k^2} + 1\right)^{\frac{1}{2}}} \quad \text{.........(17)},$$

$z_r - z_{r-1}$ and $\dot{z}_r$ can be expressed explicitly in terms of Bessel functions of integral order and their derivatives, as has been found otherwise by Schrödinger*.

Consider the case where $x = rl < ct$, and interpret (15) as a complex integral. Near a saddle point we have

$$\gamma t - r\lambda = \frac{2ict}{l}\{\sqrt{(1-\xi^2)} - \xi\cos^{-1}\xi\} + \frac{1}{4}\,\frac{ilt}{c}\,\frac{(1-\xi^2)^{\frac{1}{2}}}{\xi^2}(\gamma - \gamma_0)^2 \quad (18),$$

and we find the approximation

$$z_r = \frac{lv}{2c} - \left(\frac{l}{\pi ct}\right)^{\frac{1}{2}}(1-\xi^2)^{-\frac{1}{4}}\left[u\sin\left\{\frac{2ct}{l}(\sqrt{(1-\xi^2)} - \xi\cos^{-1}\xi) + \tfrac{1}{4}\pi\right\}\right.$$

$$\left. + \frac{lv}{2c\sqrt{(1-\xi^2)}}\cos\left\{\frac{2ct}{l}(\sqrt{(1-\xi^2)} - \xi\cos^{-1}\xi) + \tfrac{1}{4}\pi\right\}\right] \quad \text{...(19)}.$$

* *Ann. d. Phys.* 44, 1914, 916–934.

When ξ is small, the argument of the periodic terms is nearly $\frac{2ct}{l}(1-\frac{1}{2}\pi\xi)$, the variable part of which is $-r\pi$. Thus the displacements of consecutive particles differ in phase by π. As ξ increases the difference of phase between consecutive particles diminishes, and tends to zero as $\xi \to 1$. The amplitude at the same time increases steadily. But when ξ is nearly unity the argument of the periodic term is nearly $\frac{4}{3}\sqrt{2}\,(1-\xi)^{\frac{3}{2}}\frac{ct}{l}$, and the difference of phase between consecutive particles is comparable with $(1-\xi)^{\frac{1}{2}}$. Also the actual displacements vary only slowly from one particle to the next. It can be shown that analogous relations hold for the velocities.

To estimate $\dot{z}_r$ and $z_r - z_{r-1}$ explicitly the easiest way is to return to the expressions for them as complex integrals. For the former we find

$$\dot{z}_r = \left(\frac{l}{\pi ct}\right)^{\frac{1}{2}}(1-\xi^2)^{-\frac{1}{4}}\left[v \sin\left\{\frac{2ct}{l}(\sqrt{(1-\xi^2)}-\xi\cos^{-1}\xi)+\tfrac{1}{4}\pi\right\}\right.$$
$$\left.+\frac{2c}{l}(1-\xi^2)^{\frac{1}{2}}\,u\cos\left\{\frac{2ct}{l}(\sqrt{(1-\xi^2)}-\xi\cos^{-1}\xi)+\tfrac{1}{4}\pi\right\}\right] \quad (20),$$

and the average kinetic energy of a given particle over a few oscillations is

$$\frac{1}{2}\rho l\,[\dot{z}_r^2]_{\text{mean}} = \frac{1}{4}\frac{\rho l^2}{\pi ct}(1-\xi^2)^{-\frac{1}{2}}\left\{v^2+\frac{4c^2}{l^2}(1-\xi^2)\,u^2\right\} \quad (21).$$

The average potential energy is the same. It can be verified by integration that the total energy is equal to the initial energy.

4·0. It follows easily from the foregoing discussion that the system has no property analogous to heat conduction. Suppose in fact that n consecutive particles are initially given velocities distributed in frequency according to the error law. If the mean square of these velocities is s^2, we can regard s as the initial velocity of agitation of the disturbed region and s^2 as proportional to the temperature. By 3·2 (20) the velocity of agitation s_r of a particle distant rl from the end of the disturbed region is given by

$$s_r^2 = \frac{l}{\pi ct}\Sigma\,[v\sin\{tf(\xi)+\tfrac{1}{4}\pi\}]^2(1-\xi^2)^{-\frac{1}{2}} \;\ldots\ldots\ldots(1),$$

and since by hypothesis the v's are distributed at random they are not correlated with the harmonic factors, and we have

$$s_r^2 = \frac{nl}{2\pi ct}(1-\xi^2)^{-\frac{1}{2}}\,s^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where ξ is to be given a value between rl/ct and $(r+n)l/ct$. If r/n is large we can ignore the difference and say that (2) gives the distribution of temperature after time t.

This expression bears no resemblance to the solution of the corresponding problem in heat-conduction. If in one-dimensional flow a short stretch of the region is heated and the remainder left at uniform temperature, the temperature at distance x from the heated place after time t is proportional to $t^{-\frac{1}{2}} \exp(-x^2/4h^2t)$, where h^2 is the thermometric conductivity. The factor $t^{-\frac{1}{2}}$ instead of t^{-1} alone is enough to indicate a fundamental difference: the extent of the region containing a given fraction of the energy originally supplied increases in proportion to t in our present problem, instead of $t^{\frac{1}{2}}$ as in the thermal case*.

5·0. In consequence of the failure to find any phenomenon resembling conduction in a string loaded with equal particles equally spaced, it is desirable to investigate whether it can occur when the particles are unequal and their masses are distributed at random. The restriction that they are equally spaced is retained. An operational solution can be obtained without much difficulty, apart from the writing down of large determinants and their expansion in series. Unfortunately, however, no satisfactory way of evaluating the resulting solution has been found. The nature of the solution can be seen more simply by considering a simpler system. Suppose that y_0 is given, that the string is infinite in the positive direction, and that all the particles have mass ρl except one, which has mass $\rho l(1+a_k)$. Let its displacement be y_k. Then the waves reaching it are given by

$$y_r = e^{-r\lambda} y_0 \quad \text{.........................(1)},$$

and the effect of the exceptional mass is to reflect part of the disturbance. We assume therefore

$$y_r = e^{-r\lambda} y_0 + A e^{-(k-r)\lambda} \qquad r \leqslant k \quad \text{..............(2)},$$

$$y_r = e^{-(r-k)\lambda} y_k \qquad r \geqslant k \quad \text{..............(3)}.$$

But y_r must reduce to y_k when $r=k$, and therefore

$$A = y_k - e^{-k\lambda} y_0 \quad \text{......................(4)}.$$

The equation of motion for this particle is

$$-y_{k-1} + \left\{2 + \frac{\sigma^2 l^2}{c^2}(1+a_k)\right\} y_k - y_{k+1} = 0 \qquad \text{......(5)},$$

whence, with
$$\frac{\sigma l}{2c} = \sinh \tfrac{1}{2}\lambda \quad \text{......................(6)},$$

* A general discussion of the related problem of thermal conduction in solids, with references, is given by M. Born, *Atomtheorie des festen Zustandes*, 1923, 708.

we find
$$y_k = \frac{e^{-k\lambda}}{1 + a_k \tanh \frac{1}{2}\lambda} y_0 \quad \text{.....................(7)},$$

and
$$y_r = e^{-r\lambda} y_0 - \frac{a_k \tanh \frac{1}{2}\lambda}{1 + a_k \tanh \frac{1}{2}\lambda} e^{-(2k-r)\lambda} y_0 \qquad r \leqslant k \text{ ......(8)},$$

$$y_r = \frac{1}{1 + a_k \tanh \frac{1}{2}\lambda} e^{-r\lambda} y_0 \qquad r \geqslant k \text{ ......(9)}.$$

Considering (9) now, we see that k does not appear explicitly. The effect of a particle of abnormal mass upon the transmitted wave is the same whichever it may be. Also, if y_r is found by the method of steepest descents, the situation of the saddle points is unaffected, and we can estimate the transmitted wave at once. If there is a pole at the origin, where λ is zero, its contribution is therefore unaffected. This was to be expected, because in the limit when the particles become indefinitely numerous this part of the solution reduces to the solution for a continuous string, which is obviously unaffected by increasing the mass of an indefinitely short piece by a finite fraction of itself. But the saddle points are at

$$\cosh \tfrac{1}{2}\lambda = \xi = rl/ct \quad \text{.....................(10)},$$

and we must introduce a factor $\left\{1 + i a_k \frac{\sqrt{(1-\xi^2)}}{\xi}\right\}^{-1}$. This factor is complex and therefore gives a shift of phase as well as a change of amplitude. The amplitude is multiplied by

$$\left\{1 + a_k^2 \frac{1-\xi^2}{\xi^2}\right\}^{-\frac{1}{2}} \quad \text{.....................(11)}.$$

When ξ is nearly unity, corresponding to the arrival of the first pulse, the amplitude is therefore hardly affected. But as time goes on and ξ decreases, this factor also decreases, and when ξ is small it approximates to ξ/a_k. The energy is at the same time multiplied by $(\xi/a_k)^2$.

This result shows that irregular distribution of mass must have a very marked effect on the propagation of the waves produced by a given initial disturbance. We have here considered such irregularity in its mildest form, where only one particle has a mass differing from the average, and we have found that it reflects the shortest waves almost completely. If many particles have abnormal masses, the small transmission factor will be raised to a high power. A certain fraction of the energy will therefore be concentrated near the region disturbed originally, and will not spread out according to the laws of dispersion. The longest waves, on the other hand, including those that travel fastest, are not much affected by the irregularity. We may say that the effect of irregularity will be to destroy the tail of the train of waves spreading out, and to retain its energy in the form of irregular movement near the origin of the disturbance. Irregularity in the coefficients of $(y_r - y_{r-1})^2$ in the potential energy has also been found to give internal reflexion of similar type.

The effect is very marked. If for instance $\sinh \frac{1}{2}\lambda = \frac{1}{2}i$, corresponding to $\lambda = \frac{1}{6}\pi i$ and a wave length $6l$, ξ is $\frac{1}{2}\sqrt{3}$ and the group velocity is $0{\cdot}866c$. But if the irregularity is as great as in an ordinary glass we may suppose a_k^2 on an average $> \frac{1}{100}$ and the transmission by a single particle is under $(1\frac{1}{300})^{-1}$. If then the atomic spacing is 10^{-8} cm. the energy is reduced as the wave travels 1 cm. to $e^{-3{\cdot}3 \times 10^5}$ of its initial value. Yet a long wave travels with velocity c without loss. Most of the energy supplied in such a case must therefore be regarded as converted into thermal agitation by internal reflexion.

Summary.

The motion of a light string loaded with equal masses at regular intervals has been discussed by operational methods. It is found that the system, though possessing only a finite number of degrees of freedom, shows most of the characteristic features of dispersion. The operator $e^{-\sigma x/c}$ that occurs in the discussion of uniform continuous strings is found to arise as the limit of an operator defined wholly in terms of definite integration, and its interpretation involves a theorem analogous to Taylor's theorem, but apparently more general. There is no phenomenon analogous to the conduction of heat. Any irregularity in the distribution of mass, however, produces strong internal reflexion of the shortest waves, and may provide a mechanism for conduction.