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# WELL-POSEDNESS OF SECOND-ORDER DEGENERATE DIFFERENTIAL EQUATIONS WITH FINITE DELAY

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*Abstract* We give necessary and sufficient conditions for the $L^p$-well-posedness of the second-order degenerate differential equations with finite delay

$$(Mu)''(t) + \alpha u'(t) = Au(t) + Fu_t + f(t) \quad (t \in [0, 2\pi])$$

with periodic boundary conditions $(Mu)(0) = (Mu)(2\pi)$, $(Mu)'(0) = (Mu)'(2\pi)$. Here $A$ and $M$ are closed operators on a complex Banach space $X$ satisfying $D(A) \subset D(M)$, $\alpha \in \mathbb{C}$ is fixed, $F$ is a bounded linear operator from $L^p([-2\pi, 0], X)$ into $X$, and $u_t$ is given by $u_t(s) = u(t + s)$ when $s \in [-2\pi, 0]$.

*Keywords:* degenerate differential equation with delay; $L^p$-well-posedness; Lebesgue–Bochner spaces; Fourier multipliers

2010 *Mathematics subject classification:* Primary 34G10; 47D06; 35G70; 34K13

## 1. Introduction

Recently, the first-order degenerate differential equations

$$(Mu)'(t) = Au(t) + f(t) \quad (0 \leqslant t \leqslant 2\pi) \tag{1.1}$$

with periodic boundary condition have been studied by Lizama and Ponce, where $A$ and $M$ are closed linear operators on a complex Banach space $X$ and $f$ is an $X$-valued function. Under suitable assumptions on the modified resolvent operator determined by (1.1), they gave necessary and sufficient conditions to ensure the well-posedness of (1.1) in Lebesgue–Bochner spaces $L^p(\mathbb{T}, X)$, periodic Besov spaces $B^s_{p,q}(\mathbb{T}, X)$ and periodic Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{T}, X)$ [**10**], where $\mathbb{T} := [0, 2\pi]$. The well-posedness of similar second-order degenerate differential equations

$$(Mu')'(t) = Au(t) + f(t) \quad (0 \leqslant t \leqslant 2\pi) \tag{1.2}$$

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with periodic boundary conditions have been studied in [**4**] by Bu, who gave necessary or sufficient conditions for the well-posedness of (1.2) in $L^p(\mathbb{T},X)$, $B^s_{p,q}(\mathbb{T},X)$ and $F^s_{p,q}(\mathbb{T},X)$ using operator-valued Fourier multiplier theorems of Arendt and Bu on $L^p(\mathbb{T},X)$ and $B^s_{p,q}(\mathbb{T},X)$ [**1**, **2**] and of Bu and Kim on $F^s_{p,q}(\mathbb{T},X)$ [**6**].

On the other hand, Lizama studied the first-order equations with finite delay

$$u'(t) = Au(t) + Fu_t + f(t) \quad (0 \leqslant t \leqslant 2\pi), \tag{1.3}$$

where $A$ is a closed linear operator on a complex Banach space $X$, $u_t(\cdot) = u(t+\cdot)$ is defined on $[-2\pi,0]$, $f \in L^p(\mathbb{T},X)$, and the delay operator $F\colon L^p([-2\pi,0],X) \to X$ is a bounded linear operator [**9**]. He obtained necessary and sufficient conditions for (1.3) to be $L^p$-well-posed using Fourier multiplier theorems on $L^p(\mathbb{T},X)$. Bu and Fang gave necessary and sufficient conditions for (1.3) to be well-posed in $B^s_{p,q}(\mathbb{T},X)$ and $F^s_{p,q}(\mathbb{T},X)$ under suitable assumptions on the Fourier transform of the delay operator $F$ [**5**]. We note that the problem of characterization of the well-posedness for evolution equations with periodic conditions has been studied extensively in recent years: see, for example, [**2**, **4**, **5**, **7**–**9**] and references therein.

The aim of this paper is to study the $L^p$-well-posedness of the second-order degenerate differential equations with finite delay

$$(Mu)''(t) + \alpha u'(t) = Au(t) + Fu_t + f(t) \quad (t \in [0,2\pi]), \tag{$P_2$}$$

where $A$, $M$ are closed operators on a complex Banach space $X$ satisfying $D(A) \subset D(M)$, and where $\alpha \in \mathbb{C}$ is fixed, the delay operator $F\colon L^p([-2\pi,0],X) \to X$ is a bounded linear operator, and $u_t$ is defined by $u_t(s) = u(t+s)$ for $s \in [-2\pi,0]$, where we identify a function defined on $[0,2\pi]$ with its $2\pi$-periodic extension on $\mathbb{R}$.

Let $1 \leqslant p < \infty$, let $\alpha \neq 0$ and let $f \in L^p(\mathbb{T},X)$; $u \in L^p(\mathbb{T},D(A))$ is called a strong $L^p$-solution of $(P_2)$ if $u \in W^{1,p}_{\mathrm{per}}(\mathbb{T},X)$, $Mu, (Mu)' \in W^{1,p}_{\mathrm{per}}(\mathbb{T},X)$, and $(P_2)$ is satisfied almost everywhere (a.e.) on $\mathbb{T}$, where $W^{1,p}_{\mathrm{per}}(\mathbb{T},X)$ is the first periodic Sobolev space (see the precise definition below). We say that $(P_2)$ is $L^p$-well-posed if, for each $f \in L^p(\mathbb{T},X)$, there exists a unique strong $L^p$-solution of $(P_2)$. Our main result gives a characterization of the well-posedness of $(P_2)$ in $L^p(\mathbb{T},X)$. Precisely, when $\alpha \neq 0$, $1 < p < \infty$ and the underlying Banach space is a UMD space, then $(P_2)$ is $L^p$-well-posed if and only if $\rho_{\alpha,M,F}(A) = \mathbb{Z}$ and the sets $\{k^2MN_k\colon k \in \mathbb{Z}\}$ and $\{kN_k\colon k \in \mathbb{Z}\}$ are Rademacher bounded, where $N_k = (-k^2M + \mathrm{i}\alpha k - A - B_k)^{-1}$ and $B_k$ is the bounded linear operator on $X$ given by $B_kx = F(e_kx)$, $e_k$ is the scalar function on $[0,2\pi]$ given by $e_k(t) = \mathrm{e}^{\mathrm{i}kt}$. Here $\rho_{\alpha,M,F}(A)$ denotes the modified $M$-resolvent set of $A$ (see the precise definition below).

When $\alpha = 0$, then $(P_2)$ has the following simpler form:

$$(Mu)''(t) = Au(t) + Fu_t + f(t) \quad (t \in [0,2\pi]). \tag{$P_2'$}$$

In this case, for $f \in L^p(\mathbb{T},X)$, the belonging of strong $L^p$-solutions of $(P_2')$ to $W^{1,p}_{\mathrm{per}}(\mathbb{T},X)$ is no longer needed. Precisely, $u \in L^p(\mathbb{T},D(A))$ is called a strong $L^p$-solution of $(P_2')$, if $Mu, (Mu)' \in W^{1,p}_{\mathrm{per}}(\mathbb{T},X)$ and $(P_2')$ is satisfied a.e. on $\mathbb{T}$. We say that $(P_2')$ is $L^p$-well-posed, if for each $f \in L^p(\mathbb{T},X)$, there exists a unique strong $L^p$-solution of $(P_2)$. We

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show that when $1 < p < \infty$ and $X$ is a UMD Banach space (where UMD stands for unconditional martingale difference), if the set $\{k(B_{k+1} - B_k) \colon k \in \mathbb{Z}\}$ is Rademacher bounded (R-bounded, for short), then $(P_2')$ is $L^p$-well-posed if and only if $\rho_{M,F}(A) = \mathbb{Z}$ and the sets $\{k^2 M N_k \colon k \in \mathbb{Z}\}$ and $\{N_k \colon k \in \mathbb{Z}\}$ are Rademacher bounded, where $N_k = (-k^2 M + \mathrm{i}\alpha k - A - B_k)^{-1}$. Here, $\rho_{M,F}(A)$ is the modified $M$-resolvent set of $A$ (see the precise definition below). At the end of this paper we give an application of our results to a concrete example.

Our results may be regarded as generalizations of the previous known results of Arendt and Bu [**1**] about the $L^p$-well-posedness of $(P_2)$ when $M = I_X$ is the identity operator on $X$ and $F = 0$. The main tools we will use in the study of the $L^p$-well-posedness of $(P_2)$ are operator-valued Fourier multiplier theorems obtained by Arendt and Bu on $L^p(\mathbb{T}, X)$ [**1**]. Indeed, we will transform the $L^p$-well-posedness of $(P_2)$ to an operator-valued Fourier multiplier problem in $L^p(\mathbb{T}, X)$.

## 2. Main results

Let $X$ and $Y$ be complex Banach spaces and let $\mathbb{T} := [0, 2\pi]$. We denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from $X$ to $Y$. If $X = Y$, we will simply denote it by $\mathcal{L}(X)$. For $1 \leqslant p < \infty$, we denote by $L^p(\mathbb{T}, X)$ the space of all equivalent classes of $X$-valued measurable functions $f$ defined on $\mathbb{T}$ satisfying

$$\|f\|_{L^p} := \left( \int_0^{2\pi} \|f(t)\|^p \frac{\mathrm{d}t}{2\pi} \right)^{1/p} < \infty.$$

For $f \in L^1(\mathbb{T}, X)$, we denote by

$$\hat{f}(k) := \frac{1}{2\pi} \int_0^{2\pi} e_{-k}(t) f(t) \, \mathrm{d}t$$

the $k$th Fourier coefficient of $f$, where $k \in \mathbb{Z}$ and $e_k(t) = \mathrm{e}^{\mathrm{i}kt}$ when $t \in \mathbb{T}$.

**Definition 2.1.** If $X$ and $Y$ are complex Banach spaces and $1 \leqslant p < \infty$, we say that $(M_k)_{k \in \mathbb{Z}} \subset \mathcal{L}(X, Y)$ is an $L^p$-Fourier multiplier if, for each $f \in L^p(\mathbb{T}, X)$, there exists $u \in L^p(\mathbb{T}, Y)$ such that $\hat{u}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$.

It follows easily from the closed graph theorem that when $(M_k)_{k \in \mathbb{Z}} \subset \mathcal{L}(X, Y)$ is an $L^p$-Fourier multiplier, then there exists a (unique) bounded linear operator $T \in \mathcal{L}(L^p(\mathbb{T}, X), L^p(\mathbb{T}, Y))$ satisfying $(Tf)^\wedge(k) = M_k \hat{f}(k)$ when $f \in L^p(\mathbb{T}, X)$ and $k \in \mathbb{Z}$. The operator-valued Fourier multiplier theorem on $L^p(\mathbb{T}, X)$ obtained in [**1**] involves the Rademacher boundedness for sets of bounded linear operators. Let $\gamma_j$ be the $j$th Rademacher function on $[0, 1]$ given by $\gamma_j(t) = \operatorname{sgn}(\sin(2^j t))$ when $j \geqslant 1$. For $x \in X$, we denote by $\gamma_j \otimes x$ the $X$-valued function $t \to r_j(t)x$ on $[0, 1]$.

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**Definition 2.2.** Let $X$ and $Y$ be complex Banach spaces. A set $\mathfrak{T} \subset \mathcal{L}(X,Y)$ is said to be R-bounded if there exists $C > 0$ such that

$$\Big\| \sum_{j=1}^{n} \gamma_j \otimes T_j x_j \Big\|_{L^1([0,1],Y)} \leqslant C \Big\| \sum_{j=1}^{n} \gamma_j \otimes x_j \Big\|_{L^1([0,1],X)}$$

for all $T_1, \dots, T_n \in \mathfrak{T}$, $x_1, \dots, x_n \in X$ and $n \in \mathbb{N}$.

The main tool in our study of the $L^p$-well-posedness of $(P_2)$ is the operator-valued $L^p$-Fourier multiplier theorem established in [**1**]. The following results will be fundamental in the proof of our main result of this section. For the notion of UMD Banach spaces, we refer readers to [**1**] and references therein.

**Proposition 2.3 (Arendt and Bu [1, Proposition 1.11]).** *Let $X$, $Y$ be complex Banach spaces, let $1 \leqslant p < \infty$ and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$ be an $L^p$-Fourier multiplier. The set $\{M_k : k \in \mathbb{Z}\}$ is then R-bounded.*

**Theorem 2.4 (Arendt and Bu [1, Theorem 1.3]).** *Let $X$, $Y$ be UMD Banach spaces and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$. If the sets $\{M_k : k \in \mathbb{Z}\}$ and $\{k(M_{k+1} - M_k) : k \in \mathbb{Z}\}$ are R-bounded, then $(M_k)_{k\in\mathbb{Z}}$ defines an $L^p$-Fourier multiplier whenever $1 < p < \infty$.*

For $1 \leqslant p < \infty$, we define the first-order periodic Sobolev spaces [**1**] by

$$W^{1,p}_{\text{per}}(\mathbb{T},X) := \{u \in L^p(\mathbb{T},X) : \text{there exists } v \in L^p(\mathbb{T},X), \text{ such that } \hat{v}(k) = \mathrm{i}k\hat{u}(k) \text{ for all } k \in \mathbb{Z}\}.$$

If $u \in L^p(\mathbb{T},X)$, then $u \in W^{1,p}_{\text{per}}(\mathbb{T},X)$ if and only if $u$ is differentiable a.e. on $\mathbb{T}$ and $u' \in L^p(\mathbb{T},X)$; in this case $u$ is actually continuous and $u(0) = u(2\pi)$ [**1**, Lemma 2.1].

We consider the following second-order degenerate differential equations with finite delay:

$$(Mu)''(t) + \alpha u'(t) = Au(t) + Fu_t + f(t), \quad t \in [0, 2\pi], \qquad (P_2)$$

where $A$, $M$ are closed operators on a complex Banach space $X$ satisfying $D(A) \subset D(M)$ and $\alpha \in \mathbb{C}$, $f \in L^p(\mathbb{T},X)$ is given, and the delay operator $F : L^p([-2\pi,0],X) \to X$ is a bounded linear operator. Here we identify a function defined on $\mathbb{T}$ with its $2\pi$-periodic extension on $\mathbb{R}$. Moreover, for fixed $t \in \mathbb{T}$, $u_t$ is an element of $L^p([-2\pi,0],X)$ given by $u_t(s) = u(t+s)$ when $s \in [-2\pi, 0]$.

Let $1 \leqslant p < \infty$, $\alpha \neq 0$; we define the solution space of $(P_2)$ in the $L^p$-well-posedness case by

$$S_p(A,M) := \{u \in L^p(\mathbb{T}, D(A)) : u, Mu, (Mu)' \in W^{1,p}_{\text{per}}(\mathbb{T},X),$$

where we consider $D(A)$ to be a Banach space equipped with its graph norm. We note that when $u \in S_p(A,M)$, then $\|Fu_t\| \leqslant \|F\| \, \|u\|_p$ for all $t \in \mathbb{T}$, which implies that $Fu_\cdot \in L^p(\mathbb{T},X)$. $S_p(A,M)$ is a Banach space with the norm

$$\|u\|_{S_p(A,M)} := \|u\|_{L^p} + \|u'\|_{L^p} + \|Au\|_{L^p} + \|Mu\|_{L^p} + \|(Mu)'\|_{L^p} + \|(Mu)''\|_{L^p}.$$

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It follows from [**1**, Lemma 2.1] that when $u \in S_p(A, M)$, then $u$, $Mu$ and $(Mu)'$ are $X$-valued continuous functions on $\mathbb{T}$, $u(0) = u(2\pi)$, $(Mu)(0) = (Mu)(2\pi)$ and $(Mu)'(0) = (Mu)'(2\pi)$.

**Definition 2.5.** Let $1 \leqslant p < \infty$, $\alpha \neq 0$ and $f \in L^p(\mathbb{T}, X)$; $u \in S_p(A, M)$ is called a strong $L^p$-solution of $(P_2)$ if $(P_2)$ is satisfied a.e. on $\mathbb{T}$. We say that $(P_2)$ is $L^p$-well-posed if, for each $f \in L^p(\mathbb{T}, X)$, there exists a unique strong $L^p$-solution of $(P_2)$.

If $(P_2)$ is $L^p$-well-posed, there exists a constant $C > 0$ such that for each $f \in L^p(\mathbb{T}, X)$, if $u \in S_p(A, M)$ is the unique strong $L^p$-solution of $(P_2)$, then

$$\|u\|_{S_p(A,M)} \leqslant C\|f\|_{L^p}. \tag{2.1}$$

This can be easily obtained by the closedness of the operators $A$, $M$ and the closed graph theorem.

Let $F \in \mathcal{L}(L^p([-2\pi, 0], X), X)$ and $k \in \mathbb{Z}$; we define the operator $B_k$ on $X$ by $B_k x = F(e_k x)$ for all $x \in X$. It is clear that $B_k$ is linear and $\|B_k\| \leqslant \|F\|$ as $\|e_k\|_{L^p} \leqslant 1$. A simple calculation shows that $(Fu.)^\wedge(k) = B_k \hat{u}(k)$. We define the modified $M$-resolvent set of $A$ associated with $(P_2)$ by

$$\rho_{\alpha,M,F}(A) := \{k \in \mathbb{Z}\colon -k^2M + \alpha\mathrm{i}k - A - B_k\colon D(A) \to X \\ \text{is bijective and } [-k^2M + \alpha\mathrm{i}k - A - B_k]^{-1} \in \mathcal{L}(X)\}. \tag{2.2}$$

We first give a necessary condition for the $L^p$-well-posedness of $(P_2)$.

**Theorem 2.6.** *Let $X$ be a complex Banach space, let $1 \leqslant p < \infty$, let $A$, $M$ be closed linear operators on $X$ satisfying $D(A) \subset D(M)$, let $\alpha \neq 0$ and let $F \in \mathcal{L}(L^p([-2\pi, 0], X), X)$ be a delay operator. Assume that $(P_2)$ is $L^p$-well-posed. Then $\rho_{\alpha,M,F}(A) = \mathbb{Z}$ and the sets $\{k^2MN_k\colon k \in \mathbb{Z}\}$ and $\{kN_k\colon k \in \mathbb{Z}\}$ are R-bounded, where $N_k = [-k^2M + \alpha\mathrm{i}k - A - B_k]^{-1}$ when $k \in \mathbb{Z}$.*

**Proof.** Let $k \in \mathbb{Z}$ and $y \in X$ be fixed. We define $f(t) = \mathrm{e}^{\mathrm{i}kt}y$ $(t \in \mathbb{T})$. Then $f \in L^p(\mathbb{T}, X)$, $\hat{f}(k) = y$ and $\hat{f}(n) = 0$ for $n \neq k$. Since $(P_2)$ is $L^p$-well-posed, there exists a unique $u \in S_p(A, M)$ satisfying

$$(Mu)''(t) + \alpha u'(t) = Au(t) + Fu_t + f(t)$$

a.e. on $\mathbb{T}$. We have $\hat{u}(n) \in D(A) \subset D(M)$ when $n \in \mathbb{Z}$ by [**1**, Lemma 3.1]. Taking Fourier transforms on both sides, we obtain

$$[-n^2M + \alpha\mathrm{i}n - A - B_n]\hat{u}(n) = \begin{cases} 0, & n \neq k, \\ y, & n = k. \end{cases} \tag{2.3}$$

We have shown that $-k^2M + \alpha\mathrm{i}k - A - B_k$ is surjective. To show that $-k^2M + \alpha\mathrm{i}k - A - B_k$ is also injective, we take $x \in D(A)$ such that $[-k^2M + \alpha\mathrm{i}k - A - B_k]x = 0$. Let $u(t) = \mathrm{e}^{\mathrm{i}kt}x$ when $t \in \mathbb{T}$, then clearly $u \in S_p(A, M)$ and $(P_2)$ holds a.e. on $\mathbb{T}$ when $f = 0$. Thus $u$ is a strong $L^p$-solution of $(P_2)$ when $f = 0$. This implies that $x = 0$ by the uniqueness assumption. We have shown that $-k^2M+\alpha\mathrm{i}k-A-B_k$ is injective. Therefore, $-k^2M+\alpha\mathrm{i}k-A-B_k$ is bijective from $D(A)$ onto $X$.

Next we show that $[-k^2M+\alpha\mathrm{i}k-A-B_k]^{-1} \in \mathcal{L}(X)$. For $f(t) = \mathrm{e}^{\mathrm{i}kt}y$, we let $u \in S_p(A, M)$ be the unique strong $L^p$-solution of $(P_2)$. Then

$$\hat{u}(n) = \begin{cases} 0, & n \neq k, \\ [-k^2M+\alpha\mathrm{i}k-A-B_k]^{-1}y, & n = k, \end{cases}$$

by (2.3). Consequently, $u(t) = \mathrm{e}^{\mathrm{i}kt}[-k^2M+\alpha\mathrm{i}k-A-B_k]^{-1}y$. By (2.1), there exists a constant $C > 0$ independent from $y$ and $k$ such that

$$\|u\|_{L^p} + \|u'\|_{L^p} + \|Au\|_{L^p} + \|Mu\|_{L^p} + \|(Mu)'\|_{L^p} + \|(Mu)''\|_{L^p} \leqslant C\|f\|_{L^p}.$$

This implies that $\|N_k y\| \leqslant C\|y\|$ for all $y \in X$. Therefore, $\|N_k\| \leqslant C$. Thus $\rho_{\alpha,M,F}(A) = \mathbb{Z}$.

Now we are going to show that if $M_k = -k^2M[-k^2M+\alpha\mathrm{i}k-A-B_k]^{-1}$, $S_k = kN_k$ when $k \in \mathbb{Z}$, then $(M_k)_{k\in\mathbb{Z}}$ and $(S_k)_{k\in\mathbb{Z}}$ define $L^p$-Fourier multipliers. Let $f \in L^p(\mathbb{T}, X)$, then there exists a $u \in S_p(A, M)$ that is a strong $L^p$-solution of $(P_2)$ by assumption. Taking Fourier transforms on both sides of $(P_2)$, we get that $\hat{u}(k) \in D(A)$ and

$$[-k^2M+\alpha\mathrm{i}k-A-B_k]\hat{u}(k) = \hat{f}(k)$$

when $k \in \mathbb{Z}$. Since $-k^2M+\alpha\mathrm{i}k-A-B_k$ is invertible, this implies that $\hat{u}(k) = N_k\hat{f}(k)$ when $k \in \mathbb{Z}$. It follows from $u \in S_p(A, M)$ that $u, Mu, (Mu)' \in W^{1,p}_{\mathrm{per}}(\mathbb{T}, X)$. We have $[(Mu)'']^\wedge(k) = -k^2M\hat{u}(k)$ when $k \in \mathbb{Z}$, and thus

$$[(Mu)'']^\wedge(k) = -k^2M\hat{u}(k) = M_k\hat{f}(k)$$

and

$$(u')^\wedge(k) = \mathrm{i}k\hat{u}(k) = \mathrm{i}kN_k\hat{f}(k) = \mathrm{i}S_k\hat{f}(k)$$

when $k \in \mathbb{Z}$. We conclude that $(M_k)_{k\in\mathbb{Z}}$ and $(S_k)_{k\in\mathbb{Z}}$ define $L^p$-Fourier multipliers as $(Mu)'', u' \in L^p(\mathbb{T}, X)$. It follows from Proposition 2.3 that the sets $\{M_k : k \in \mathbb{Z}\}$ and $\{S_k : k \in \mathbb{Z}\}$ are R-bounded. This finishes the proof. □

In the proof of our main result of this section, we will use the following result.

**Proposition 2.7.** *Let $A$ and $M$ be closed linear operators on a UMD Banach space $X$ satisfying $D(A) \subset D(M)$, $1 < p < \infty$ and $\alpha \neq 0$. Let $F\colon L^p([-2\pi, 0], X) \to X$ be a bounded linear operator. Suppose that $\rho_{\alpha,M,F}(A) = \mathbb{Z}$ and the sets $\{k^2MN_k : k \in \mathbb{Z}\}$ and $\{kN_k : k \in \mathbb{Z}\}$ are R-bounded, where $N_k = (-k^2M+\alpha\mathrm{i}k-A-B_k)^{-1}$. Then $(k^2MN_k)_{k\in\mathbb{Z}}$, $(kN_k)_{k\in\mathbb{Z}}$ and $(B_kN_k)_{k\in\mathbb{Z}}$ are $L^p$-Fourier multipliers.*

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**Proof.** Let $M_k = k^2 M N_k$, $S_k = k N_k$, $H_k = B_k N_k$. Then the sets $\{M_k \colon k \in \mathbb{Z}\}$ and $\{S_k \colon k \in \mathbb{Z}\}$ are R-bounded by assumption. It follows from [**9**, Proposition 3.2] that $\{B_k \colon k \in \mathbb{Z}\}$ is R-bounded. It is sufficient to show that the sets $\{k(M_{k+1} - M_k) \colon k \in \mathbb{Z}\}$, $\{k(S_{k+1} - S_k) \colon k \in \mathbb{Z}\}$ and $\{k(H_{k+1} - H_k) \colon k \in \mathbb{Z}\}$ are R-bounded by Theorem 2.4. We observe that

$$
\begin{aligned}
N_{k+1} - N_k &= N_{k+1}(N_k^{-1} - N_{k+1}^{-1}) N_k \\
&= N_{k+1}[(2k+1)M - \alpha \mathrm{i} + B_{k+1} - B_k] N_k \\
&= (2k+1) N_{k+1} M N_k + N_{k+1}(B_{k+1} - B_k - \alpha \mathrm{i}) N_k. \qquad (2.4)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&k(M_{k+1} - M_k) \\
&\quad = k[(k+1)^2 M N_{k+1} - k^2 M N_k] \\
&\quad = k^3 M (N_{k+1} - N_k) + k(2k+1) M N_{k+1} \\
&\quad = k^3 (2k+1) M N_{k+1} M N_k + k^3 M N_{k+1}(B_{k+1} - B_k - \alpha \mathrm{i}) N_k + k(2k+1) M N_{k+1} \\
&\quad = \frac{k(2k+1)}{(k+1)^2} M_{k+1} M_k + \frac{k^2}{(k+1)^2} M_{k+1}(B_{k+1} - B_k - \alpha \mathrm{i}) S_k + \frac{k(2k+1)}{(k+1)^2} M_{k+1}, \qquad (2.5)
\end{aligned}
$$

$$
\begin{aligned}
&k(S_{k+1} - S_k) \\
&\quad = k[(k+1) N_{k+1} - k N_k] \\
&\quad = k^2 (N_{k+1} - N_k) + k N_{k+1} \\
&\quad = k^2 (2k+1) N_{k+1} M N_k + k^2 N_{k+1}(B_{k+1} - B_k - \alpha \mathrm{i}) N_k + k N_{k+1} \\
&\quad = \frac{2k+1}{k+1} S_{k+1} M_k + \frac{k}{k+1} S_{k+1}(B_{k+1} - B_k - \alpha \mathrm{i}) S_k + \frac{k}{k+1} S_{k+1} \qquad (2.6)
\end{aligned}
$$

and

$$
k(H_{k+1} - H_k) = k(B_{k+1} N_{k+1} - B_k N_k) = \frac{k}{k+1} B_{k+1} S_{k+1} - B_k S_k \qquad (2.7)
$$

when $k \neq -1$. It follows from (2.5)–(2.7) that the sets $\{k(M_{k+1} - M_k) \colon k \in \mathbb{Z}\}$, $\{k(S_{k+1} - S_k) \colon k \in \mathbb{Z}\}$ and $\{k(H_{k+1} - H_k) \colon k \in \mathbb{Z}\}$ are R-bounded since the product of two R-bounded sets is still R-bounded. This completes the proof. □

Next we give a necessary and sufficient condition for the $L^p$-well-posedness of $(P_2)$ that is the main result of this section.

**Theorem 2.8.** *Let $X$ be a UMD Banach space, let $1 < p < \infty$ and let $A$, $M$ be closed linear operators on $X$ satisfying $D(A) \subset D(M)$ and $\alpha \neq 0$. Let $F \colon L^p([-2\pi, 0], X) \to X$ be a bounded linear operator. Then $(P_2)$ is $L^p$-well-posed if and only if $\rho_{\alpha,M,F}(A) = \mathbb{Z}$ and the sets $\{k^2 M N_k \colon k \in \mathbb{Z}\}$ and $\{k N_k \colon k \in \mathbb{Z}\}$ are R-bounded, where $N_k = [-k^2 M + \alpha \mathrm{i} k - A - B_k]^{-1}$ when $k \in \mathbb{Z}$.*

**Proof.** If $(P_2)$ is $L^p$-well-posed, then $\rho_{\alpha,M,F}(A) = \mathbb{Z}$ and the sets $\{k^2MN_k\colon k \in \mathbb{Z}\}$ and $\{kN_k\colon k \in \mathbb{Z}\}$ are R-bounded by Theorem 2.6. Conversely, we assume that $\rho_{\alpha,M,F}(A) = \mathbb{Z}$ and the sets $\{k^2MN_k\colon k \in \mathbb{Z}\}$ and $\{kN_k\colon k \in \mathbb{Z}\}$ are R-bounded. Let $M_k = k^2MN_k$, $S_k = kN_k$ and $H_k = B_kN_k$ when $k \in \mathbb{Z}$. It follows from Proposition 2.7 that $(M_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$ and $(H_k)_{k\in\mathbb{Z}}$ are $L^p$-Fourier multipliers. Then, for all $f \in L^p(\mathbb{T}, X)$, there exists $u \in L^p(\mathbb{T}, X)$ satisfying

$$\hat{u}(k) = -M_k\hat{f}(k) \tag{2.8}$$

when $k \in \mathbb{Z}$. Since $((1/k)I_X)_{k\in\mathbb{Z}}$ is an $L^p$-Fourier multiplier by Theorem 2.4, it follows that $(N_k)_{k\in\mathbb{Z}}$ is also an $L^p$-Fourier multiplier as the product of two $L^p$-Fourier multipliers is still an $L^p$-Fourier multiplier. There exists $v \in L^p(\mathbb{T}, X)$ such that $\hat{v}(k) = N_k\hat{f}(k)$ for all $k \in \mathbb{Z}$. This implies that $\hat{v}(k) \in D(A) \subset D(M)$ and

$$(-k^2M + \alpha\mathrm{i}k - A - B_k)\hat{v}(k) = \hat{f}(k) \tag{2.9}$$

when $k \in \mathbb{Z}$. On the other hand, since $((\mathrm{i}/k)I_X)_{k\in\mathbb{Z}}$ and $(M_k)_{k\in\mathbb{Z}}$ are $L^p$-Fourier multipliers, we deduce that $(\mathrm{i}kMN_k)_{k\in\mathbb{Z}}$ is also an $L^p$-Fourier multiplier. Thus there exists $h \in L^p(\mathbb{T}, X)$ such that

$$\hat{h}(k) = \mathrm{i}kMN_k\hat{f}(k) = \mathrm{i}kM\hat{v}(k).$$

Thus $v(t) \in D(M)$ for almost all $t \in \mathbb{T}$ and $Mv \in W^{1,p}_{\text{per}}(\mathbb{T}, X)$ by [**1**, Lemmas 2.1 and 3.1]. In view of (2.8), we obtain

$$\hat{u}(k) = -k^2MN_k\hat{f}(k) = -k^2M\hat{v}(k) = \mathrm{i}k[(Mv)']^\wedge(k)$$

when $k \in \mathbb{Z}$, which implies that $(Mv)' \in W^{1,p}_{\text{per}}(\mathbb{T}, X)$ by [**1**, Lemmas 2.1 and 3.1]. We note that $(Fv.)^\wedge(k) = B_k\hat{v}(k) = B_kN_k\hat{f}(k) = H_k\hat{f}(k)$. Hence $Fv. \in L^p(\mathbb{T}, X)$ since $(H_k)_{k\in\mathbb{Z}}$ is an $L^p$-Fourier multiplier. On the other hand, $\mathrm{i}k\hat{v}(k) = \mathrm{i}kN_k\hat{f}(k) = \mathrm{i}S_k\hat{f}(k)$ when $k \in \mathbb{Z}$. It follows from the fact that $(S_k)_{k\in\mathbb{Z}}$ defines an $L^p$-Fourier multiplier that $v \in W^{1,p}_{\text{per}}(\mathbb{T}, X)$ by [**1**, Lemma 2.1]. We have $\hat{v}(k) \in D(A)$ and $A\hat{v}(k) = \hat{u}(k) + \alpha\mathrm{i}k\hat{v}(k) - B_k\hat{v}(k) - \hat{f}(k) = [u + \alpha v' - Fv. - f]^\wedge(k)$ by (2.9). By [**1**, Lemma 3.1], we conclude that $v \in L^p(\mathbb{T}, D(A))$ as $u + \alpha v' - Fv. - f \in L^p(\mathbb{T}, X)$. We have shown that $v \in S_p(A, M)$ and thus

$$(Mv)''(t) + \alpha v'(t) = Av(t) + Fv_t + f(t)$$

a.e. on $[0, 2\pi]$ by the uniqueness of Fourier coefficients [**1**, p. 314]. Therefore, $v$ is a strong $L^p$-solution of $(P_2)$. This shows the existence.

To show the uniqueness, we let $v \in S_p(A, M)$ satisfy

$$(Mv)''(t) + \alpha v'(t) = Av(t) + Fv_t$$

a.e. on $\mathbb{T}$. Taking Fourier transforms on both sides, we have $\hat{v}(k) \in D(A)$ and $-k^2M\hat{v}(k) + \alpha\mathrm{i}k\hat{v}(k) = A\hat{v}(k) + B_k\hat{v}(k)$ when $k \in \mathbb{Z}$. It follows that

$$[-k^2M + \alpha\mathrm{i}k - A - B_k]\hat{v}(k) = 0$$

for all $k \in \mathbb{Z}$. Since $\rho_{\alpha,M,F}(A) = \mathbb{Z}$, this implies that $\hat{v}(k) = 0$ for all $k \in \mathbb{Z}$ and thus $v = 0$. Therefore, $(P_2)$ is $L^p$-well-posed. The proof is complete. $\square$

When $\alpha = 0$, the problem $(P_2)$ has the simpler form

$$(Mu)''(t) = Au(t) + Fu_t + f(t), \quad t \in [0, 2\pi], \tag{$P_2'$}$$

without the term $\alpha u'$. In this case, it is more natural to consider the solution space as

$$S_p'(A, M) = \{u \in L^p(\mathbb{T}, D(A)) \colon Mu, (Mu)' \in W^{1,p}_{\text{per}}(\mathbb{T}, X)\}.$$

Here again we consider $D(A)$ to be a Banach space equipped with its graph norm. It follows from [**1**, Lemma 2.1] that when $u \in S_p'(A, M)$, then $Mu$ and $(Mu)'$ are $X$-valued continuous functions on $\mathbb{T}$, $(Mu)(0) = (Mu)(2\pi)$ and $(Mu)'(0) = (Mu)'(2\pi)$.

For $f \in L^p(\mathbb{T}, X)$, $u \in S_p'(A, M)$ is called a strong $L^p$-solution of $(P_2')$ if $(P_2')$ is satisfied a.e. on $\mathbb{T}$. We say that $(P_2')$ is $L^p$-well-posed if, for each $f \in L^p(\mathbb{T}, X)$, there exists a unique strong $L^p$-solution of $(P_2')$.

We define the modified $M$-resolvent set of $A$ associated with $(P_2')$ by

$$\rho_{M,F}(A) := \{k \in \mathbb{Z} \colon k^2M + A + B_k \colon D(A) \to X \\ \text{is bijective and } [k^2M + A + B_k]^{-1} \in \mathcal{L}(X)\}.$$

A similar argument to that used in the proofs of Theorems 2.6 and 2.8 gives the following results. We notice that in our characterization of the $L^p$-well-posedness of $(P_2')$, an extra condition on the delay operator $F$ is needed.

**Theorem 2.9.** *Let $X$ be a complex Banach space, let $1 \leqslant p < \infty$, let $A$, $M$ be closed linear operators on $X$ satisfying $D(A) \subset D(M)$, and let $F \in \mathcal{L}(L^p([-2\pi, 0], X), X)$ be a delay operator. Assume that $(P_2')$ is $L^p$-well-posed. Then $\rho_{M,F}(A) = \mathbb{Z}$ and the sets $\{k^2MN_k \colon k \in \mathbb{Z}\}$ and $\{N_k \colon k \in \mathbb{Z}\}$ are R-bounded, where $N_k = [k^2M + A + B_k]^{-1}$ when $k \in \mathbb{Z}$.*

**Theorem 2.10.** *Let $X$ be a UMD Banach space, let $1 < p < \infty$ and let $A$, $M$ be closed linear operators on $X$ satisfying $D(A) \subset D(M)$. Let $F \colon L^p([-2\pi, 0], X) \to X$ be a bounded linear operator. We assume that the set $\{k(B_{k+1} - B_k) \colon k \in \mathbb{Z}\}$ is R-bounded. Then $(P_2')$ is $L^p$-well-posed if and only if $\rho_{M,F}(A) = \mathbb{Z}$ and the sets $\{k^2MN_k \colon k \in \mathbb{Z}\}$ and $\{N_k \colon k \in \mathbb{Z}\}$ are R-bounded, where $N_k = [k^2M + A + B_k]^{-1}$ when $k \in \mathbb{Z}$.*

**Example 2.11.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, let $m$ be a bounded measurable function on $\Omega$ such that $m(x) > 0$ a.e. on $\Omega$, and let $f$ be a given function on $[0, 2\pi] \times \Omega$. We let $X$ be the Hilbert space $H^{-1}(\Omega)$ and $\alpha \neq 0$, $1 < p < \infty$. We consider the following second-order problem with periodic boundary conditions:

$$\left.\begin{aligned} \frac{\partial^2(m(x)u)}{\partial t^2} + \alpha\frac{\partial u}{\partial t} = -\Delta u + Fu_t + f(t, x), \quad (t, x) \in [0, 2\pi] \times \Omega, \\ u(t, x) = 0, \quad (t, x) \in [0, 2\pi] \times \partial\Omega, \\ u(0, x) = u(2\pi, x), \quad x \in \Omega, \\ \frac{\partial}{\partial t}u(0, x) = \frac{\partial}{\partial t}u(2\pi, x), \quad x \in \Omega, \end{aligned}\right\} \tag{2.10}$$

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where $u_t(s,x) = u(t+s,x)$ when $t \in [0, 2\pi]$ and $s \in [-2\pi, 0]$, $\alpha \in \mathbb{C}$ is fixed, and the delay operator $F\colon L^p([-2\pi, 0], X) \to X$ is a bounded linear operator. We remark that the last two boundary conditions are equivalent to

$$m(x)u(0,x) = m(x)u(2\pi,x), \qquad \frac{\partial}{\partial t}(m(x)u(0,x)) = \frac{\partial}{\partial t}(m(x)u(2\pi,x))$$

when $x \in \Omega$.

If $M$ is the multiplication operator by $m$ on $H^{-1}(\Omega)$, then, by [**7**, § 3.7] (see also references therein), there exists a constant $c > 0$ such that

$$\big\|M(zM-\Delta)^{-1}\big\| \leqslant \frac{c}{1+|z|}$$

when $\mathrm{Re}(z) \geqslant -c(1+|\mathrm{Im}(z)|)$. This implies in particular that

$$\|M(k^2M-\Delta)^{-1}\| \leqslant \frac{c}{1+|k|^2}$$

whenever $k \in \mathbb{Z}$. If we assume, furthermore, that $m^{-1}$ is regular enough that the multiplication operator by the function $m^{-1}$ is bounded on $H^{-1}(\Omega)$, then there exists a constant $c' > 0$ such that

$$\|(k^2M-\Delta)^{-1}\| \leqslant \frac{c'}{1+|k|^2} \tag{2.11}$$

when $k \in \mathbb{Z}$. We assume that $\alpha \neq 0$ and $\rho_{\alpha,M,F}(-\Delta) = \mathbb{Z}$. Now the identity

$$k^2M - \alpha \mathrm{i}k - \Delta + B_k = [I + (k^2M-\Delta)^{-1}(B_k - \alpha \mathrm{i}k)](k^2M-\Delta)$$

implies that

$$(k^2M - \alpha \mathrm{i}k - \Delta + B_k)^{-1} = (k^2M-\Delta)^{-1}[I + (k^2M-\Delta)^{-1}(B_k - \alpha \mathrm{i}k)]^{-1}.$$

The set $\{B_k\colon k \in \mathbb{Z}\}$ is R-bounded [**9**, Proposition 3.2], and therefore norm bounded, and this together with (2.11) shows that

$$\lim_{|k|\to+\infty} \|(k^2M-\Delta)^{-1}(B_k - \alpha \mathrm{i}k)\| = 0.$$

Therefore, by (2.11),

$$\sup_{k\in\mathbb{Z}} \|k(k^2M - \alpha \mathrm{i}k - \Delta + B_k)^{-1}\| < \infty$$

and

$$\sup_{k\in\mathbb{Z}} \|k^2M(k^2M - \alpha \mathrm{i}k - \Delta + B_k)^{-1}\| < \infty.$$

We deduce from Theorem 2.8 that the above periodic problem is $L^p$-well-posed when taking $X = H^{-1}(\Omega)$. Here we have used the fact that when $H$ is a Hilbert space, every norm bounded subset of $\mathcal{L}(H)$ is actually R-bounded [**1**].

A similar argument shows that when $\alpha = 0$, $\rho_{M,F}(-\Delta) = \mathbb{Z}$ and $\{k(B_{k+1} - B_k)\colon k \in \mathbb{Z}\}$ is bounded, the above periodic problem is $L^p$-well-posed whenever $1 < p < \infty$ by Theorem 2.10.

**Example 2.12.** Let $H$ be a complex Hilbert space, let $\alpha \neq 0$, let $1 < p < \infty$, let $A$, $M$ be closed linear operators on $H$ satisfying $D(A) \subset D(M)$, and let $F \in \mathcal{L}(L^p([-2\pi, 0], H), H)$ be a delay operator. Let $P$ be a densely defined positive self-adjoint operator on $H$ with $P \geqslant \delta > 0$. Let $M = P - \varepsilon$ with $\varepsilon < \delta$, and let $A = \sum_{i=0}^{k} a_i P^i$ with $a_i \geqslant 0$, $a_k > 0$. Then there exists a constant $c > 0$ such that

$$\|M(zM + A)^{-1}\| \leqslant \frac{c}{1 + |z|} \tag{2.12}$$

whenever $\operatorname{Re} z \geqslant -c(1 + |\operatorname{Im} z|)$ by [**7**, p. 73]. This implies in particular that

$$\sup_{k\in\mathbb{Z}} \|k^2 M(k^2 M + A)^{-1}\| < \infty.$$

If we assume that $\rho_{\alpha,M,F} = \mathbb{Z}$ and $0 \in \rho(M)$. Then the argument used in Example 2.11 shows that $(P_2)$ is $L^p$-well-posed by Theorem 2.8.

A similar argument shows that when $\alpha = 0$, $\rho_{M,F}(A) = \mathbb{Z}$, $0 \in \rho(M)$ and

$$\{k(B_{k+1} - B_k)\colon k \in \mathbb{Z}\}$$

is bounded, then $(P_2')$ is $L^p$-well-posed by Theorem 2.10.

**Example 2.13.** We consider the following problem:

$$\frac{\partial^2}{\partial t^2}\left(1 - \frac{\partial^2}{\partial x^2}\right)u(t,x) + \alpha u'(t,x) = \frac{\partial^4}{\partial x^4}u(t,x) + Fu_t(\cdot,x) + f(t,x), \quad (t,x) \in (0,2\pi) \times \Omega,$$
$$u(t,0) = u(t,1) = \frac{\partial^2}{\partial x^2}u(t,0) = \frac{\partial^2}{\partial x^2}u(t,1) = 0, \quad t \in [0,2\pi],$$
$$u(0,x) = u(2\pi,x), \quad \left(1 - \frac{\partial^2}{\partial x^2}\right)u(0,x) = \left(1 - \frac{\partial^2}{\partial x^2}\right)u(2\pi,x), \quad x \in \Omega,$$
$$\frac{\partial}{\partial t}\left(1 - \frac{\partial^2}{\partial x^2}\right)u(0,x) = \frac{\partial}{\partial t}\left(1 - \frac{\partial^2}{\partial x^2}\right)u(2\pi,x), \quad x \in \Omega,$$

where $\Omega = (0,1)$, $\alpha \neq 0$, $F \in \mathcal{L}(L^p([-2\pi,0]; L^2(\Omega)), L^2(\Omega))$ and $u_t(s,x) := u(t+s,x)$ when $t \in [0,2\pi]$ and $s \in [-2\pi,0]$. Let $X = L^2(\Omega)$ and let $P = -\partial^2/\partial x^2$ with domain $D(P) = H^2(\Omega) \cap H_0^1(\Omega)$, i.e. $P$ is the Laplacian on $L^2(\Omega)$ with Dirichlet boundary conditions. Then $P$ is positive self-adjoint on $X$. Let $M = P + I_X$ and $A = P^2$. It is clear that $-P$ generates a contraction semigroup on $L^2(\Omega)$ [**3**, Example 3.4.7], and hence $1 \in \rho(-P)$, or equivalently $M = I_X + P$ has a bounded inverse, i.e. $0 \in \rho(M)$. The abstract results obtained in Example 2.12 can then be applied: if $\rho_{\alpha,M,F}(P^2) = \mathbb{Z}$, then the above problem is $L^p$-well-posed for all $1 < p < \infty$.

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The abstract results obtained in Example 2.12 can be also applied to the following problem:

$$\frac{\partial^2}{\partial t^2}\left(1-\frac{\partial^2}{\partial x^2}\right)u(t,x)=\frac{\partial^4}{\partial x^4}u(t,x)+Fu_t(\cdot,x)+f(t,x),\quad (t,x)\in(0,2\pi)\times\Omega,$$
$$u(t,0)=u(t,1)=\frac{\partial^2}{\partial x^2}u(t,0)=\frac{\partial^2}{\partial x^2}u(t,1)=0,\quad t\in[0,2\pi],$$
$$u(0,x)=u(2\pi,x),\quad \left(1-\frac{\partial^2}{\partial x^2}\right)u(0,x)=\left(1-\frac{\partial^2}{\partial x^2}\right)u(2\pi,x),\quad x\in\Omega,$$
$$\frac{\partial}{\partial t}\left(1-\frac{\partial^2}{\partial x^2}\right)u(0,x)=\frac{\partial}{\partial t}\left(1-\frac{\partial^2}{\partial x^2}\right)u(2\pi,x),\quad x\in\Omega,$$

where $\Omega=(0,1)$, $F\in\mathcal{L}(L^p([-2\pi,0];L^2(\Omega)),L^2(\Omega))$ and $u_t(s,x):=u(t+s,x)$ when $t\in[0,2\pi]$ and $s\in[-2\pi,0]$. If $\rho_{M,F}(P^2)=\mathbb{Z}$ and $\{k(B_{k+1}-B_k)\colon k\in\mathbb{Z}\}$ is bounded, then the above problem is $L^p$-well-posed for all $1<p<\infty$, where $-P$ is the Laplacian with Dirichlet boundary conditions on $L^2(\Omega)$.

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## References

1. W. Arendt and S. Bu, The operator-valued Marcinkiewicz multiplier theorem and maximal regularity, *Math. Z.* **240** (2002), 311–343.
2. W. Arendt and S. Bu, Operator-valued Fourier multipliers on periodic Besov spaces and applications, *Proc. Edinb. Math. Soc.* **47** (2004), 15–33.
3. W. Arendt, C. Batty, M. Hieber and F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems* (Birkhäuser, 2001).
4. S. Bu, Well-posedness of second order degenerate differential equations in vector-valued function spaces, *Studia Math.* **214**(1) (2013), 1–16.
5. S. Bu and Y. Fang, Periodic solutions of delay equations in Besov spaces and Triebel–Lizorkin spaces, *Taiwanese J. Math.* **13**(3) (2009), 1063–1076.
6. S. Bu and J. Kim, Operator-valued Fourier multipliers on periodic Triebel spaces, *Acta Math. Sinica* **17** (2004), 15–25.
7. A. Favini and A. Yagi, *Degenerate differential equations in Banach spaces*, Pure and Applied Mathematics, Volume 215 (Marcel Dekker, New York, 1999).
8. V. Keyantuo and C. Lizama, Fourier multipliers and integro-differential equations in Banach spaces, *J. Lond. Math. Soc.* **69**(3) (2004), 737–750.
9. C. Lizama, Fourier multipliers and periodic solutions of delay equations in Banach spaces, *J. Math. Analysis Applic.* **324** (2006), 921–933.
10. C. Lizama and R. Ponce, Periodic solutions of degenerate differential equations in vector-valued function spaces, *Studia Math.* **202**(1) (2011), 49–63.