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ASYMPTOTIC TRACTS OF HARMONIC FUNCTIONS II

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An *asymptotic tract* of a real function u harmonic and non-constant in $\mathbb{C}$ is a component of the set $\{z: u(z) \neq c\}$, for some real number c; a *quasi-tract* $T(\neq \mathbb{C})$ is an unbounded simply-connected domain in $\mathbb{C}$ such that there exists a function u that is positive, unbounded and harmonic in T such that, for each point $\zeta \in \partial T \cap \mathbb{C}$,

$$\lim_{z \to \zeta} u(z) = 0;$$

and a $\mathcal{T}$-tract is an unbounded simply-connected domain T in $\mathbb{C}$ whose every prime end that contains ∞ in its impression is of the first kind.

The authors study the growth of a harmonic function in one of its asymptotic tracts, and the question of whether a quasi-tract is an asymptotic tract. The branching of either type of tract is also taken into consideration.

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1. Introduction

This paper continues a study, begun in [1], of the asymptotic tracts of functions harmonic in $\mathbb{C}$ (entire harmonic functions).

Definition 1.1. An asymptotic tract (or tract) of a real function $u(z)$ harmonic and non-constant in $\mathbb{C}$ is a component of the set $\{z: u(z) \neq c\}$ for some real number c.

It was shown in [1] that each tract T is necessarily simply-connected and unbounded, and that u is necessarily unbounded in each tract T; in addition, ∞ is an accessible boundary point (in $\mathbb{C}$) of each tract T. The local mapping properties of analytic functions show that the set $\{z: u(z) \neq c\}$ consists of a finite or countable number of curves which are locally analytic, except at the zeros of $\hat{f}'(z)$ (where $\hat{f}$ is any analytic completion of u) — where the set $\{z: u(z) = c\}$ branches. Observe that the angle between the "branches" must be equal to $2\pi/n$ for some $n \geqq 1$. The growth of u in tracts was studied in detail in [1] and [2]; and the geometry of level curves of u corresponding to non-constant functions harmonic in $\mathbb{C}$ was considered by Flatto, Newman and Shapiro with particularly beautiful results in [6].

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In Sections 2 and 3 below we shall be dealing with both these questions and also with the relationship between tracts and quasi-tracts and $\mathcal{T}$-tracts, which we now define.

Definition 1.2. An unbounded simply-connected domain T in $\mathbb{C}$, $T \neq \mathbb{C}$, is called a *quasi-tract* if it has the following property:

(P_1) there exists a function $u(z)$ positive, unbounded and harmonic in T such that, for each point $\zeta \in \partial T \cap \mathbb{C}$,

$$\lim_{z \to \zeta} u(z) = 0.$$

Remark. We will frequently say that "T supports u".

Also, an Inversen-type construction (see, for example, [**11**; p. 26]) shows that there exists a path Γ in T on which $u(z) \to +\infty$. Since Γ cannot tend to any finite point of ∂T (since $u(z)=0$ on ∂T), it follows that ∞ must be an accessible boundary point of any quasi-tract T.

Definition 1.3. An unbounded simply-connected domain T in $\mathbb{C}$, $T \neq \mathbb{C}$, is called a $\mathcal{T}$-tract ("$\mathcal{T}$" stands for "topological") if it has the following property:

(P_2) Every prime end that contains ∞ in its impression is of the first kind.

(A good reference on the various types of prime ends is [**5**; p. 180].)

The two principal areas of interest in our work are:

Question 1.1. Given (the geometry) of an asymptotic tract T, what can be said about the function $u(z)$ supported by T?

Question 1.2. Given a quasi-tract T, how can we decide when it is an asymptotic tract?

It was shown in [**3**] that if the plane consists of the union of (the closures of) k asymptotic tracts for a single harmonic function u, then $u(z)$ is necessarily a polynomial whose degree n satisfies the bounds $n+1 \leqq k \leqq 2n$. In fact, Question 1.1 was also tackled earlier in [**1**] and [**2**]; we shall tackle Question 1.2 in particular in Section 3 below.

If T is an asymptotic tract, its complement consists of a number of unbounded simply-connected domains (also asymptotic tracts) and their boundaries. Since T is simply-connected, it cannot have any "bounded holes" in itself. However, $\mathbb{C} - T$ (which can be regarded as "unbounded holes in T") may consist of a single tract, or of a finite number of tracts, or it may have the power of the continuum; this phenomenon, which we shall discuss in Section 4, of $\operatorname{Int}(\mathbb{C} - T)$ "breaking up" is called "branching". To be precise, we give the following definition:

Definition 1.4. An unbounded simply-connected domain T in $\mathbb{C}$ is said to be *branched of order* n_T (possibly $n_T = +\infty$) if it has the following property:

There exists a family $\mathcal{F}_T$ *of* n_T non-homotopic (in T) and disjoint (except for the

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end-point z_T) Jordan curves in T connecting some fixed point in T, z_T say, to ∞; in addition, any Jordan curve in T joining z_T to ∞ is homotopic (in T) to one of the elements of $\mathscr{F}_T$.

If $n_T=1$, we say that T is unbranched; if $n_T<+\infty$, we say that T is finitely branched; if $n_T=+\infty$, we say that T is infinitely branched. So far as we are aware, this question of branching has not previously been studied. In Section 5 we study the mapping properties of $\hat{f}$ (where $\hat{f}$ is any analytic completion of u) and show that $\hat{f}$ has a rather simple form when the tract has branching of finite order.

The results in Section 6 show that [**2**; Theorem 1] can be improved when the tract T is finitely branched.

The authors wish to thank Professor Maurice Heins for a suggestion that simplified the proof of Theorem 2.1, and the referee for his many exceedingly helpful suggestions and comments.

2. Asymptotic tracts

Rather loosely, an asymptotic tract can be (both topologically and analytically) "rather badly—but not too badly-behaved"! In this Section we shall study what can be said topologically about a tract T. After a couple of definitions we shall begin with two interesting domains.

Definition 2.1. A set S in $\mathbb{C}$ is said to be *connected at infinity* (in the sense of Arakelian [**7**; page 11]) if, for each neighbourhood N of ∞, there exists a neighbourhood M of ∞ (with $M\subset N$) such that each point $z\in M\cap S$ can be joined to ∞ by a Jordan curve Γ entirely contained in $(N\cap S)\cup\{\infty\}$. We shall say that a point $z\in S$ is *connected to infinity* in S if z can be joined to ∞ by a Jordan curve that lies in $(N\cap S)\cup\{\infty\}$.

Remark. It follows from the definition of local-connectedness (see, for example, [**12**; page 84]) that a set S is connected at ∞ if and only if $S\cup\{\infty\}$ is locally-connected at ∞. Hence results involving connectedness at ∞ can be restated in terms of local-connectedness at ∞.

Example 2.1. Let

$$D_1=\{z\colon z=x+iy, x>0, 0<y<(x+1)^{-1}\}-\{z\colon z=x+iy, x\geqq 1, y^{-1}\in\mathbb{N}, y^{-1}\geqq 3\},$$

and

$$D_2=D_1\cup\{z\colon z=x+iy, x>0, 1/2<y<3/4\}.$$

Both D_1 and D_2 are unbounded simply-connected domains in $\mathbb{C}$. Neither domain can be an asymptotic tract, since the level curve corresponding to ∂D_1 or ∂D_2 "piles up" on the segment $[+1,+\infty)$ of the positive real axis (technically, neither ∂D_1 nor ∂D_2 is locally-connected). We remark that D_2 can be made "more satisfying" by replacing each of the segments $\{z\colon z=x+iy, x\geqq 1, y^{-1}\in\mathbb{N}, y^{-1}\geqq 3\}$ by a suitably-chosen thin unbounded continuum bounded by a single analytic curve.

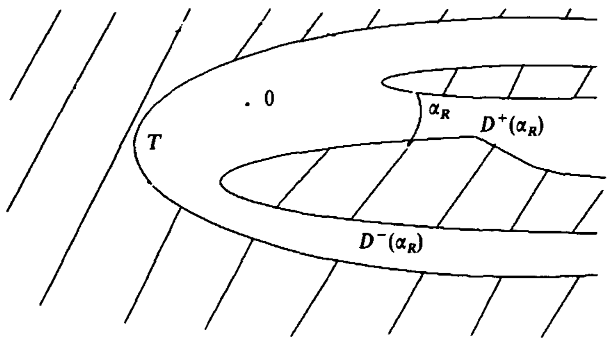

FIGURE 1

The domain D_1 does not contain a path going to ∞ so it cannot be a quasi-tract, but D_2 does contain such a path. To see that D_2 is a quasi-tract just map D_2 onto $\{w: |w|<1\}$ by the function $w=f(z)$ so that one of the paths going to ∞ in D_2 corresponds to a path tending to $w=1$. The function

$$u(z)=\mathrm{Re}\left(\frac{1+f(z)}{1-f(z)}\right)$$

shows that D_2 has the property (P_1) and hence is a quasi-tract. However neither domain is connected at ∞. This question of how a tract can "approach" ∞ leads us to the following result.

Theorem 2.1. *Let $u(z)$ be an entire non-constant harmonic function. Let T be a component of $\{z: u(z)\neq c\}$. Then T is connected at ∞ (in the sense of Arakelian).*

Proof. It suffices to prove the theorem when T is a component of $\{z: u(z)>0\}$. We may also assume that $O\in T$.

Since $T\neq\mathbb{C}$, it follows that for all sufficiently large R the set $\{z: |z|=R\}\cap T$ consists of at least one and at most a finite number of open arcs. Let α_R be a component of $\{z: |z|=R\}\cap T$. Since T is simply-connected, it follows that $T-\alpha_R$ consists of two simply-connected domains in $\mathbb{C}$. Since T is unbounded, at least one of the two domains in $T-\alpha_R$ is unbounded.

Let $D^+(\alpha_R)$ be that component of $T-\alpha_R$ that contains the intersection of $\{z: |z|>R\}$ with all neighbourhoods of points of α_R; we call this the *outer domain* associated with α_R. The *inner domain* $D^-(\alpha_R)$ associated with α_R is defined in an analogous fashion.

Let $U=\{z: |z|>R\}\cap T$ be any neighbourhood (in T) of ∞. We need to show that there exists a neighbourhood $V=\{z: |z|>R_0\}\cap T$ of ∞ such that any point z_0 in V can be connected to ∞ by a curve lying entirely in U. Consider the various outer domains $D^+(\alpha_R^i)$, $i=1,2,\ldots,N$, corresponding to the finite number of arcs contained in $T\cap\{z: |z|=R\}$. Each domain $D^+(\alpha_R^i)$ is either bounded or unbounded. Let R_0 be chosen so

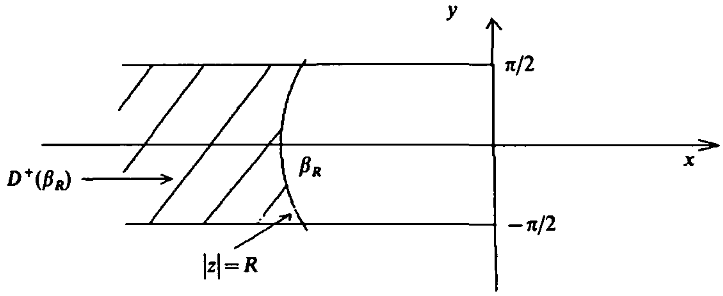

FIGURE 2

that all the bounded outer domains are contained in $\{z:|z|<R_0\}$. Now consider any point $z_0 \in V \equiv \{z:|z|>R_0\} \cap T$. It must lie in one of the unbounded outer domains, call it $D^+(\beta_R)$.

Denote the restriction of $u(z)$ to $D^+(\beta_R)$ by $U(z)$. On β_R we have $0<U(z) \leqq M$ for some M and $U(z)=0$ on $\partial D^+(\beta_R)-\beta_R$.

If $U(z)$ is unbounded in $D^+(\beta_R)$ then the usual Iversen-type argument shows that there exists a path Γ in $D^+(\beta_R)$ on which $U(z) \to +\infty$. Hence ∞ is an accessible boundary point of $D^+(\beta_R)$, and it follows that z_0 can be connected to ∞ by a path lying in $D^+(\beta_R)$. Thus the definition of connected at ∞ is satisfied in this case.

Now suppose that $U(z)$ is bounded in $D^+(\beta_R)$. This can happen; for example, consider the domain $\{z:|z|>R, \operatorname{Re} z<0, |\operatorname{Im} z|<\pi/2\}$ and the entire harmonic function $e^x \cos y$.

Given any point $z_0 \in D^+(\beta_R) \cap V$, we need to show that z_0 can be joined to ∞ by a path lying in $D^+(\beta_R)$ (in U). We will do this by constructing a function $\hat{U}(z)$ that is harmonic and positive in $D^+(\beta_R)$, tends continuously to 0 as z approaches $\partial D^+(\beta_R)$, and is unbounded in $D^+(\beta_R)$.

Let z_0 be a fixed point of $D^+(\beta_R)$ and let $\{r_n\}_{n=1}^{\infty}$ be an increasing sequence of real numbers which satisfy $r_1>|z_1|$ and $\lim_{n\to\infty} r_n=\infty$. Set Ω_n equal to the component of $D^+(\beta_R) \cap \{z:|z|<r_n\}$ which contains z_1. It is clear that Ω_n lies in Ω_{n+1} and $D^+(\beta_R)=\bigcup_{n=1}^{\infty} \Omega_n$. It is easy to see that the Dirichlet problem with boundary values

$$u(\zeta)=\begin{cases} 0, & \zeta \in \partial D^+(\beta_R) \cap \Omega_n, \\ 1, & \zeta \in \partial \Omega_n - \partial D^+(\beta_R), \end{cases}$$

has a solution $u_n(z)$ that is positive and harmonic in Ω_n. Now normalise the functions $u_n(z)$ by setting

$$\tilde{u}_n(z)=u_n(z)/u_n(z_0).$$

We now need the following [9; Theorem 3.3, page 64]:

Theorem A. *Let Ω be a domain, $a \in \Omega$, and let Φ denote the class of functions u that are harmonic on Ω, positive and satisfy the normalisation $u(a)=1$. Set*

$$\lambda(z)=\inf_{u\in\Phi} u(z), \quad \mu(z)=\sup_{u\in\Phi} u(z), \quad z\in\Omega.$$

Then λ is continuous and positive on Ω, and μ is a continuous finite-valued function on Ω.

With the use of Theorem A it is straightforward to show that $\{\tilde{u}_n(z)\}$ is a normal family on $D^+(\beta_R)$; hence it contains a subsequence, which we denote by $\{U_n(z)\}$, that converges uniformly on compact subsets of $D^+(\beta_R)$ to a function $U(z)$ that is harmonic and positive on $D^+(\beta_R)$ (by Theorem A). We need to show that

$$\lim_{z\to\zeta,\, z\in D^+(\beta_R)} U(z)=0,$$

where ζ is any *finite* boundary point of $D^+(\beta_R)$. It is then clear that $U(z)$ is unbounded in $D^+(\beta_R)$.

Consider any finite boundary point ζ of $D^+(\beta_R)$. Since the boundary of $D^+(\beta_R)$ consists of a portion of a level curve of a harmonic function and a portion of $\{z: |z|=R\}$, there exists a positive number r such that, if we set $\mathscr{U}(\zeta)=\{z: |z-\zeta|<r\}$, then $\mathscr{U}(\zeta)\cap\partial D^+(\beta_R)$ consists of exactly one component. Map $\mathscr{U}(\zeta)\cap D^+(\beta_R)$ conformally by $z=g(w)$ onto the upper half-disk $H=\{w: |w|<1, \operatorname{Im} w>0\}$ so that $\mathscr{U}(\zeta)\cap\partial D^+(\beta_R)$ corresponds to $\{w: -1<\operatorname{Re} w<1\}$.

If we set $V_n(w)=U_n(g(w))$, we have that $V_n(w)$ is positive and harmonic in H and vanishes continuously on $\{w: -1<\operatorname{Re} w<1\}$. We also set $V(w)=U(g(w))$. It suffices to show that $V(w)$ vanishes continuously on $\{w: -2/3<\operatorname{Re} w<2/3\}$. Since $V_n(w)$ converges to $V(w)$ in H, the vanishing of $V(w)$ on $\{w: -2/3<\operatorname{Re} w<2/3\}$ is implied by the following standard result; this then completes the proof.

Lemma B. *We define $w=u+iv$, $H=\{w: |w|<1, v>0\}$, $H'=\{w: |w|<2/3, v>0\}$, and $\mathscr{H}$ the family of positive harmonic functions on H which are continuous on $\bar{H}$ and vanish on that part of ∂H on the u-axis. For any $h\in\mathscr{H}$ we have $h(w)\leqq 2000h(\tfrac{1}{2}i)v$, for $w\in H'$.*

Lemma B follows in a straightforward fashion from the Poisson integral representation for a function harmonic in the unit disc and continuous on its closure; see, for example, [**13**; Lemma 6.4, page 118]. The proof will be sketched since the authors do not know an easily accessible reference to it.

Since $h(w)$ is positive and harmonic in H, continuous on the closure of H, and vanishes at that part of ∂H on the u-axis, we have, using the Poisson integral representation for functions harmonic in the unit disc and the reflection principle, that

$$h(w)=\frac{2v}{\pi}\int_0^{\pi}\frac{(1-|w|^2)\sin\phi \quad h(e^{i\phi})\,d\phi}{|e^{i\phi}-w|^2\,|e^{-i\phi}-w|^2}$$

$$=\frac{2v}{\pi}\int_0^{\pi} K(\phi,w)\sin\phi \quad h(e^{i\phi})\,d\phi,$$

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for $w \in H$. Here the kernel

$$K(\phi, w) = \frac{(1-|w|^2)}{|e^{i\phi}-w|^2\,|e^{-i\phi}-w|^2}$$

satisfies the inequality $9/125 < K(\phi, w) < 81$ for $|w| < 2/3$ and the conclusion follows.

Theorem 2.2. *Let $u(z)$ be a non-constant entire function and let T be one component of the set $\{z: u(z) \neq c\}$, for some real c. Then $\mathbb{C} - T$ is connected at ∞.*

Proof. It suffices to prove the theorem when T is a component of $\{z: u(z) > 0\}$. Note that $\mathbb{C} - T$ consists of the union of $\{z: u(z) = 0\}$ and various components of $\{z: u(z) > 0\}$ and of $\{z: u(z) < 0\}$. The components of $\{z: u(z) > 0\}$ and $\{z: u(z) < 0\}$ are connected at ∞, by Theorem 2.1. We need only consider points of $\{z: u(z) = 0\}$. Since any neighbourhood of a point z_0 where $u(z_0) = 0$ contains points where $u(z) < 0$, Theorem 2.2 follows in a straightforward fashion.

In view of the Remark after Definition 2.1, Theorems 2.1 and 2.2 can be rephrased as follows:

Theorem 2.3. *For any asymptotic tract T, the sets $T \cup \{\infty\}$ and $(\mathbb{C} - T) \cup \{\infty\}$ are locally-connected (in $\hat{\mathbb{C}}$).*

In the next section we shall prove that asymptotic tracts are $\mathscr{T}$-tracts and that $\mathscr{T}$-tracts are quasi-tracts; however, note that it is obvious that asymptotic tracts are quasi-tracts.

We end this section with an example (related to Example 2.1) to show that a set may be connected at ∞ without its complement having the same property.

Example 2.2. We define the following sets:

$$D_1 = \{z: z = x + iy, x > 0, y < 0\},$$

$$D_2 = \{z: z = x + iy, x > 0, 0 < y < 1/(x+1)\} - \{z: z = x + iy, x \geqq 1, 1/(n+\tfrac{1}{2}) \leqq y \leqq 1/n, n \in \mathbb{N}, n \geqq 3\},$$

and

$$D = D_1 \cup D_2 \cup (0, 1).$$

Then the set $T = \mathbb{C} - \bar{D}$ is connected at ∞, but $\mathbb{C} - T$ and $\operatorname{Int}(\mathbb{C} - T)$ are not connected at ∞.

3. Asymptotic tracts, quasi-tracts and $\mathscr{T}$-tracts

In this section we discuss the relationships between asymptotic tracts, quasi-tracts and $\mathscr{T}$-tracts.

Trivially a tract is a quasi-tract, and Theorems 2.1, 2.2 and 3.1 will imply that an asymptotic tract is necessarily a $\mathscr{T}$-tract. The following example shows that a set may be a quasi-tract and a $\mathscr{T}$-tract without being an asymptotic tract.

Example 3.1. Let $T=\{z: z=x+iy, y>\sin(1/x) \text{ if } x\neq 0,\ y>1 \text{ for } x=0\}$. Map T onto $\{w: |w|<1\}$ by the function $w=f(z)$ so that ∞ corresponds to $w=1$. Let $u(z)=\operatorname{Re}((1+f(z))/(1-f(z)))$.

Then T is both a $\mathscr{T}$-tract and a quasi-tract, but consideration of $u(z)$ near $z=i$ shows that $u(z)$ cannot be the restriction to T of any function harmonic in all of $\mathbb{C}$ (so that T is *not* a tract); note also that $\mathbb{C}-T$ is *not* locally-connected at O. Thus this Example also shows that if T is a $\mathscr{T}$-tract or a quasi-tract, then $\mathbb{C}-T$ need not be locally-connected.

Example 2.1 showed that a quasi-tract is not necessarily a $\mathscr{T}$-tract; however, Theorem 3.2 below will show that a $\mathscr{T}$-tract is a quasi-tract. First we prove the following:

Theorem 3.1. *Let an unbounded simply-connected domain T in $\mathbb{C}$, where $T\neq\mathbb{C}$, be such that both T and $\mathbb{C}-T$ are connected at ∞ (in the sense of Arakelian). Then every prime end of T that contains ∞ in its impression is of the first kind (so that T is a $\mathscr{T}$-tract).*

Corollary. *Asymptotic tracts are $\mathscr{T}$-tracts.*

Proof of the Theorem. For prime ends we shall use the notation and terminology of [5; Chapter 9, pages 167–189]. Let P be a prime end of T that contains ∞ in its impression. Since ∞ is an accessible boundary point of T (as T is connected at ∞), ∞ is the only principal point of P [5; Theorem 9.7, page 177]. Let $\{q_n\}$ be a chain belonging to P, and let D_n be the subdomain of T defined by q_n that contains q_{n+1} (see [5; pages 169–179]). Since ∞ is the only principal point of P, it follows from [5; pages 171–172] that the q_n's may be chosen to be a sequence of circular arcs centred at the origin and whose radii strictly increase to ∞. Let the radius of q_n be $1/\delta_n$. Since T and $\mathbb{C}-T$ are connected at ∞, there is a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$, where $\varepsilon_n>0$ and $\varepsilon_n\downarrow 0$, such that any point of T or $\mathbb{C}-T$ lying in $\{z: |z|>1/\varepsilon_n\}$ can be connected to ∞ by a Jordan curve in $T\cap\{z: |z|>1/\delta_n\}$ or in $(\mathbb{C}-T)\cap\{z: |z|>1/\delta_n\}$, respectively. Now pick subsequences of the q_n and the ε_n, which for simplicity of notation we shall continue to denote by $\{q_n\}$ and $\{\varepsilon_n\}$, that satisfy the inequalities

$$1/\delta_n<1/\varepsilon_n<1/\delta_{n+1}<1/\varepsilon_{n+1}<\cdots.$$

Note that the redefined sequence $\{q_n\}$ is an equivalent chain to the original sequence $\{q_n\}$ [5; page 169].

There is a boundary point of T (a point of $\mathbb{C}-T$) at each end of each q_n; denote these two points in the order of increasing argument by a_n and b_n. From the argument in the previous paragraph, we see that a_n can be connected to ∞ by a Jordan curve α_n lying in $(\mathbb{C}-T)\cap\{z: |z|>1/\delta_{n-1}\}$ and that b_n can be connected to ∞ by a Jordan curve

β_n lying in $(\mathbb{C}-T)\cap\{z:|z|>1/\delta_{n-1}\}$. If we can show that D_n (the subdomain of T defined by q_n and containing q_{n+1}) is contained in $\{z:|z|>1/\delta_{n-1}\}$, then it will be clear that the impression of P only contains the point ∞ since the impression is a continuum [5; page 170].

To see that D_n is contained in $\{z:|z|>1/\delta_{n-1}\}$, let z_n be a point of $q_n\cap T$. Since T is connected at ∞, there exists a Jordan curve Ψ_n that connects z_n to ∞ in $T\cap\{z:|z|>1/\delta_{n-1}\}$. Now, since q_n is a cross-cut of T, by redefining the initial point of Ψ_n to be its last point of intersection $\tilde{z}_n$ with q_n we may assume that Ψ_n lies in $\{z:|z|>1/\delta_n\}$ (except for its initial point). It is clear that Ψ_n divides D_n into two components; denote the component with a_n in its boundary by A_n, and the other component by B_n. The Jordan curve

$$\alpha_n\cup\Psi_n\cup\{z:|z|=1/\delta_n, \arg a_n\leqq\arg z\leqq\arg\tilde{z}_n\}$$

divides the plane into two components; one of these components, $\tilde{D}_n$ say, is completely contained in $\{z:|z|>1/\delta_{n-1}\}$. Similarly, B_n is contained in $\{z:|z|>1/\delta_{n-1}\}$, and the proof is complete.

Theorem 3.2. *A $\mathcal{T}$-tract is a quasi-tract.*

Proof. We map T onto the unit disc in the w-plane in a one–one conformal fashion by a function f. By Carathéodory's Theorem [4], f can be extended to be a homeomorphism between the prime end compactification T^* of T and $\{w:|w|<1\}$ (see [5; pages 172–175]). We may normalise the mapping function f so that one of the prime ends with ∞ in its impression corresponds to the point $w=1$. It then follows readily that

$$u(z)=\operatorname{Re}(1+f(z))/(1-f(z))$$

is positive and unbounded in T and has limit zero at any finite boundary point of T. Hence T is a quasi-tract.

A natural way in which to proceed is to try to characterise topologically tracts and quasi-tracts. This seems very difficult. We begin with the question for tracts:

Question 3.1. If we add "the finite boundary of T consists of one or more locally analytic curves except for a set of isolated points where ∂T has tangents from the left and from the right and the angle between the tangents is a rational multiple of 2π" to the definition of a $\mathcal{T}$-tract, is T a $\mathcal{T}$-tract if and only if T is a component of the set $\{z:u(z)>0\}$ for some non-constant entire harmonic function $u(z)$?

Let us consider the simplest case—where ∂T consists of just one locally analytic Jordan curve. Surprisingly, in a beautiful paper [6] Flatto, Newman and Shapiro have shown that, if $u(z)$ is an entire harmonic function and vanishes on the curve $y=x^n$ for $n\geqq 3$, then $u(z)\equiv 0$. This shows that even in the simplest case the answer to the above question is negative, and that it will be very difficult to characterise topologically a tract of an entire harmonic function. However, given a $\mathscr{T}$-tract T that is bounded by one locally analytic curve, one can always approximate T as closely as one likes, using the standard approximation theorems, by a domain $\tilde{T}$ such that $\tilde{T}$ is a component of $\{z: u(z)>0\}$ for some entire harmonic function $u(z)$.

As Example 2.1 shows, an unbounded simply-connected domain can be a quasi-tract and need not be connected at ∞. Thus it appears that the topological characterisation of quasi-tracts will also be difficult.

It would also be interesting to relate whether an unbounded simply-connected domain D is an asymptotic tract or a quasi-tract to its prime end structure at ∞. If D has at least one prime end of the first type with ∞ in its impression, then the argument used in Theorem 3.2 may be used to show that D is a quasi-tract. However, Theorems 2.1, 2.2 and 3.1 show that D need not be an asymptotic tract. If all prime ends of D that contain ∞ in their impression are either of the third or fourth kinds, then ∞ is not an accessible boundary point and D cannot be a quasi-tract. If all the prime ends of D that contain ∞ in their impression are of the second kind and have ∞ as their principal point, then the situation is unclear. Let P be any such prime end; the argument used in Theorem 3.2 may be used to produce a curve tending to ∞ in D on which $u(z)\to+\infty$, and $u(z)\to 0$ as z approaches any other prime end of D. However, it is not clear to what value $u(z)$ tends, if any, as z approaches a subsidiary (finite) point of P.

4. Branching of tracts and quasi-tracts

In this Section we investigate how large the order of branching of tracts and quasi-tracts may be. If $u(z)=e^x \cos y+1$, we see that $\{z: u(z)>0\}$ is countably-infinitely branched. The following theorem shows that the order of branching of a tract may have the power of the continuum.

Theorem 4.1. *There exists an entire harmonic function $u(z)$ and a component T of $\{z: u(z)>0\}$ such that there is an uncountable number of nonhomotopic (in T) paths Γ_α such that the following are satisfied:*

(*i*) *Each path Γ_α connects a fixed point $b\in T$ to ∞ in T;*

(*ii*) *$u(z)\to+\infty$ as $|z|\to+\infty$ along each path Γ_α.*

Proof. The authors originally had a rather complicated example using Arakelian's Theorem, and wish to thank the referee for making a suggestion that made them realise that the following simple function has the desired properties.

Let

$$f(z)=\sum_{n=1}^{\infty}\frac{z^{4^n}}{(4^n)!}.$$

Since f is entire, the function

$$u(z) = \operatorname{Im} f(z) = \sum_{n=1}^{\infty} \frac{r^{4^n} \sin(4^n\theta)}{(4^n)!} \quad (\text{where } z = re^{i\theta})$$

is harmonic in the entire plane.

We now define $E_1 = \{\theta : \theta = \pi\alpha, 0 < \alpha < 1; \alpha$ has finitely many 0's and 2's but infinitely many 1's and infinitely many 3's when expanded in base 4$\}$, and $E_2 = \{\theta : \theta = \pi\alpha, 0 < \alpha < 1; \alpha$ has finitely many 1's and 3's but infinitely many 0's and infinitely many 2's when expanded in base 4$\}$. Then, if $\theta \in E_1$, we have that $\sin(4^n\theta) < 0$ for all sufficiently large n (depending on θ) so that

$$u(re^{i\theta}) \to -\infty, \text{ as } r \to +\infty;$$

similarly, if $\theta \in E_2$, we have that $\sin(4^n\theta) > 0$ for all sufficiently large n (depending on θ) so that

$$u(re^{i\theta}) \to +\infty, \text{ as } r \to +\infty.$$

Clearly E_1 and E_2 are both dense and uncountable in every subinterval of $(0, \pi)$. Since there are only a countable number of components of $\{z : u(z) > 0\}$ there must be at least one of these components, call it T, that contains an uncountable number of half-lines $L_\theta = \{re^{i\theta} : \theta \in E_2, r > r(\theta)\}$ on which $u \to +\infty$ as $r \to +\infty$. Since E_1 is dense in every subinterval of $(0, \pi)$, the existence of the desired family $\{\Gamma_\alpha\}$ of curves follows easily.

5. The mapping properties of a harmonic function in a tract with finite branching

The results of this Section shows that in a tract with branching of finite order, associated with a given harmonic function u, any analytic completion $\hat{f}$ of u has a particularly simple form.

We deal first with the case for branching of order 1.

Theorem 5.1. *Suppose that T is an unbranched quasi-tract which is also a $\mathcal{T}$-tract (in particular, an unbranched tract), and let $u(z)$ be harmonic and positive in T and approach zero as z approaches any finite boundary point of T. Let $v(z)$ be any harmonic conjugate of $u(z)$ in T. Then $\hat{f}(z) \equiv u(z) + iv(z)$ maps T one-to-one onto $H_w = \{w : w = u + iv, u > 0\}$ with ∞ in the z-plane corresponding to ∞ in the w-plane.*

Remark. This result is known at least in the simpler case where T is a Jordan domain (see, for example [**8**; Theorem 10.2, page 754]).

Proof. Recall that $u(z)$ must be unbounded in T and that there must exist at least one path Γ in T such that

$$\lim_{z \to \infty, z \in \Gamma} u(z) = \infty.$$

Now let $v(z)$ be any harmonic conjugate of $u(z)$ in T, and let $z=g(\zeta)$ map $H_\zeta=\{\zeta:\zeta=\xi+i\eta,\xi>0\}$ one-to-one conformally onto T with the prime end P to which Γ converges corresponding to ∞. Set

$$U(\zeta)=\mathrm{Re}\,\hat{f}(g(\zeta))=u(g(\zeta)).$$

Note that $U(\zeta)$ is defined and positive in H_ζ; we need to show that $U(\zeta)\to 0$ as $\zeta\to i\eta$ from within H_ζ.

Since $g(\zeta)$ maps H_ζ onto T, $g(\zeta)$ must approach a prime end P_η of T as $\zeta\to i\eta$ from within H_ζ. If the impression of P_η does not contain ∞, then $u(g(\zeta))\to 0$ as $g(\zeta)\to P_\eta$, by hypothesis. Thus we may now suppose that, on the other hand, the impression of P_η contains ∞; then, by the Corollary to Theorem 3.1, P_η coincides with ∞, since T is a $\mathscr{T}$-tract.

Hence $g(\zeta)\to\infty$ as $\zeta\to i\eta$ from within H_ζ. By letting ζ tend to $i\eta$ along the line segment $s=\{1-t+i\eta:0\leqq t<1\}$, we obtain a path $\tau=\{z:z=g(1-t+i\eta,0\leqq t<1\}$ in T along which $z\to\infty$. Since T is unbranched, z approaches the (unique) prime end at ∞ in T. Hence $\zeta=g^{-1}(z)$ approaches the prime end in the ζ-plane corresponding to $z=\infty$, namely ∞; this is contrary to our hypothesis. Thus $u(g(\zeta))\to 0$ as $\zeta\to i\eta$ from within H_η.

Now a standard uniqueness theorem of Bouligand (see, for example, [**8**; Theorem 10.1, page 752]) shows that $U(\zeta)=c\,\mathrm{Re}\,\zeta$ for some real constant c. Taking harmonic conjugates, we see that $f(g(\zeta))=c\zeta+id$ for some real constant d. Thus $g(\zeta)=f^{-1}(c\zeta+id)$, so that f^{-1} maps H_ζ one-to-one onto T and f maps T one-to-one onto H_ζ.

This completes the proof of Theorem 5.1.

For the case of finite branching of order higher than one we need an extension of the uniqueness theorem used above.

Theorem L. *Let $u(z)$ be harmonic and positive in $H_z=\{z:\mathrm{Re}\,z>0\}$, and let E be a finite subset of the imaginary axis I.*

If $\lim_{z\to\zeta,z\in H_z}u(z)=0$ for each point $\zeta\in I-E$, then

$$u(z)\equiv\mathrm{Re}\left(cz+\sum_{j=1}^{n}\frac{c_j}{z-it_j}\right),$$

where $E=\bigcup_{j=1}^{n}it_j$, $c\geqq 0$, and $c_j\geqq 0$ for $j=1,2,\ldots,n$.

Theorem L is a consequence of a more general theorem of Lohwater [**10**]; alternatively, Theorem L could be proved directly from the Poisson integral representation for u.

Theorem 5.2. *Suppose that T is a quasi-tract which is also a $\mathscr{T}$-tract and is branched of order $n>1$. Let $u(z)$ be harmonic and positive in T and approach zero as z approaches any finite boundary point of T. Suppose that v is any harmonic conjugate of u, and that $\hat{f}=u+iv$. Then:*

(*i*) *$\hat{f}$ maps T p-to-one ($1 \leqq p \leqq n$) onto $H_w = \{w\colon \operatorname{Re} w > 0\}$ with ∞ in the z-plane corresponding to p points on the imaginary axis in the w-plane, and*

(*ii*) *The number of zeros of $\hat{f}'$ in T is $p-1$.*

Remark. Recall that we proved in Section 2 that the tracts of entire harmonic functions are $\mathscr{T}$-tracts, so Theorem 5.2 applies to them.

Proof. As in the proof of Theorem 5.1, there is a path Γ in T such that $\lim_{z \to \infty, z \in \Gamma} u(z) = \infty$. By hypothesis there are n nonhomotopic curves $\{\Gamma_k\}_{k=1}^n$ which end at ∞ such that any other path in T which ends at ∞ is homotopic to one of the paths in the set $\{\Gamma_k\}$. Choose the notation so that Γ is homotopic to Γ_n. Let v be any harmonic conjugate of u in T, and let $z = g(\zeta)$ map $H_\zeta = \{\zeta\colon \zeta = \xi + i\eta, \xi > 0\}$ one-to-one and conformally onto T with the prime end P_η to which Γ_n converges corresponding to ∞. Set

$$U(\zeta) = \operatorname{Re} \hat{f}(g(\zeta)) = u(g(\zeta)).$$

Note that $U(\zeta)$ is defined and positive in H_ζ. Denote by it_k the point that corresponds to the prime end P_k to which Γ_k converges, for $k = 1, 2, \ldots, n-1$. In order to apply Theorem L we need to show that $U(\zeta) \to 0$ as $\zeta \to i\eta$ from within H_ζ, except possibly for $\eta = t_k$, $k = 1, 2, \ldots, n-1$. To see this, we note that the points $i\eta$ for $\eta \neq t_k$ correspond under the mapping $z = g(\zeta)$ to prime ends P_η whose impression does not contain ∞. Thus, just as in the proof of Theorem 5.1, we can show that $u(z) \to 0$ as $z \to P_\eta$, so that $U(\zeta) = u(g(\zeta)) \to 0$ as $\zeta \to i\eta$. Hence Theorem L applies, so that

$$U(\zeta) \equiv \operatorname{Re}\left(c\zeta + \sum_{j=1}^{n-1} \frac{c_j}{\zeta - it_j}\right)$$

where $c \geqq 0$, and $c_j \geqq 0$ for $j = 1, 2, \ldots, n-1$.

Let the degree of the rational function

$$R(\zeta) \equiv c\zeta + \sum_{j=1}^{n-1} \frac{c_j}{\zeta - it_j}$$

be p, where $1 \leqq p \leqq n$. Setting $w = u + iv$, we have that $R(\zeta)$ is a p-to-one mapping of the ζ-plane onto the w-plane. An elementary computation shows that $\operatorname{Re} R(\zeta) > 0$ if and only if $\operatorname{Re} \zeta > 0$; hence $R(\zeta)$ is a p-to-one mapping of H_ζ onto H_w. Since $g(\zeta)$ is a one-to-one mapping of H_ζ onto T, it follows that $\hat{f}(z)$ is a p-to-one mapping of T onto H_w as desired. Clearly ∞ in the z-plane corresponds to p points (including ∞) on the imaginary axis in the w-plane.

We now consider the number of zeros of $\hat{f}'$. Elementary calculations show that $R'(\zeta) > 0$ for $\zeta = i\eta$ except for $\eta = t_j$, where $j = 1, 2, \ldots, n-1$, and that $R'(\zeta_0) = 0$ if and only if $R'(-\bar{\zeta}_0)=0$. Since R' has $2p$-2 zeros, it follows that R' has p-1 zeros in H_ζ and that $\hat{f}'$ has p-1 zeros in T (because $g'(\zeta)\neq 0$ in H_ζ). The reason that p may be less than n is that some of the numbers c_j may be zero—that is, u may remain bounded in the "branch" of the tract "determined" by Γ_j.

We now study the existence of positive harmonic functions approaching ∞ along the various "branches" of T-tracts.

Theorem 5.3. *Suppose that T is a $\mathcal{T}$-tract which is branched of order n. Let $\{\Gamma_k\}_{k=1}^n$ be n nonhomotopic curves in T connecting a point $b\in T$ to ∞. Then there exist n functions $u_j(z), j=1,2,\ldots,n$, that are positive and harmonic in T such that, for each fixed j:*

(*i*) *$u_j(z)\to 0$ as z approaches any finite boundary point of T;*

(*ii*) *$\lim_{z\to\infty, z\in\Gamma_j} u(z)=+\infty$; and*

(*iii*) *$\lim_{z\to\infty, z\in\Delta} u(z)=0$, where Δ is any curve connecting b and ∞ in T which is not homotopic to Γ_j.*

Remark. It follows from this Theorem that T supports any function of the form

$$u(z)=\sum_{j=1}^{n} c_j u_j(z)$$

where $c_j\geqq 0$, and that we can make u approach ∞ along any desired subset of $\{\Gamma_k\}_{k=1}^n$ by an appropriate choice of the c_j's.

Proof. As in the proof of Theorem 3.1 we map T onto $\{\zeta:|\zeta|<1\}$ by a mapping $\zeta=g(z)$ with the prime end to which Γ_1 converges corresponding to the point $\zeta=1$. Suppose that the prime end to which Γ_j converges corresponds to the point $e^{i\theta_j}$. Then the functions

$$u_j(z)=\text{Re}\,\frac{e^{i\theta_j}+g(z)}{e^{i\theta_j}-g(z)},\quad j=1,2,\ldots,n,$$

have the desired properties, since if two curves in T ending at ∞ are nonhomotopic their images under g must end at different points of the unit circle.

6. The growth of harmonic functions along an asymptotic path in a tract which is branched of finite order

Suppose that $u(z)$ is harmonic and non-constant in the plane. It was shown in [2] that there exists a path Γ such that

$$\frac{u(z)}{|z|^\alpha}\to\infty \text{ as } z\to\infty \text{ along } \Gamma \tag{6.1}$$

for $\alpha<\frac{1}{2}$ but not necessarily for $\alpha\geqq\frac{1}{2}$ (see [**2**; Theorem 1, page 363] for a precise statement of the result). In [**1**; Theorem 3, page 216] it was shown that this result can be improved if one makes an assumption about the angular width of the tract that contains Γ. We now show that (6.1) can be improved if one assumes that the tract that contains Γ is branched of finite order (no assumption on the angular width of T is required).

Theorem 6.1. *Let T be a quasi-tract which is also a $\mathcal{T}$-tract (in particular, the tract of an entire harmonic function), and suppose that $O\in T$. Let $u(z)$ be supported by T. If T is finitely branched and has inner mapping radius m at O, then there exists a curve Γ in T joining O to ∞ along which*

$$\lim_{z\to\infty, z\in\Gamma}\inf\frac{u(z)}{|z|^{1/2}}\geqq\frac{2c}{m^{1/2}}\geqq 0, \tag{6.2}$$

where c is positive and depends on u. If T is unbranched, then $c=u(O)$.

Remark. To illustrate Theorem 6.1, let T be the plane slit along the negative real axis from -1 to $-\infty$, and let

$$u(z)=\operatorname{Re}(z+1)^{1/2}\quad(z\in T).$$

Then the mapping radius of T at 0 is 4 (cf. the Koebe function), and there exists a path Γ (viz. the positive real axis) in T that joins 0 to ∞ along which

$$\lim_{z\to\infty, z\in\Gamma}\frac{u(z)}{|z|^{1/2}}=1.$$

Thus Theorem 6.1 is best possible in the case that T is an unbranched quasi-tract. Of course, this particular domain T is not the tract of any entire harmonic function; however, in [**1**; pages 219–225] an entire harmonic function $\xi(t)$ was constructed with a tract T such that

$$M(r)\sim\frac{1}{L}r^{1/2},$$

where $M(r)=\max\{\xi(t): t\in T, |t|=r\}$ and $0<L<1$; it follows for this function that on any path Γ in T we must have

$$\overline{\lim_{t\to\infty, t\in\Gamma}}\frac{\xi(t)}{|t|^{1/2}}\leqq\frac{1}{L}.$$

Hence Theorem 6.1 is best possible in the unbranched case. An example showing that

Theorem 6.1 is best possible in the case that T is branched of order n can be constructed by adapting the methods in [**1**; pages 219–225], but we omit the details as they are rather lengthy.

Proof of Theorem 6.1. We know that there is at least one curve Δ such that $u(z)\to\infty$ as $z\to\infty$ along Δ. Let g map $\{\zeta:|\zeta|<1\}$ one-to-one and conformally onto T with $g(0)=0$ and the prime end P to which Δ converges corresponding to the point $\zeta=1$. (Recall that since T is a $\mathscr{T}$-tract, the impression of P must consist exactly of the point ∞.) Set $h(z)=g^{-1}(z)$. Transplanting the results of Theorem L to the unit disk and setting $U(\zeta)=u(g(\zeta))$, we deduce that

$$U(\zeta)=c\,\mathrm{Re}\frac{1+\zeta}{1-\zeta}+\sum_{j=1}^{n-1}c_j\,\mathrm{Re}\frac{e^{i\theta_j}+\zeta}{e^{i\theta_j}-\zeta}, \tag{6.3}$$

where $\theta_j\neq 0$, $c\geqq 0$ and $c_j\geqq 0$ for $j=1,2,\ldots,n-1$. Also by the Koebe Distortion Theorem,

$$|g(\zeta)|\leqq\frac{m|\zeta|}{(1-|\zeta|)^2}\quad(|\zeta|<1). \tag{6.4}$$

We denote by τ the segment $[0,1)$. For $\alpha\in\tau$, we write $\beta=g(\alpha)$, so that $\beta\in T$. As $\alpha\to 1$ along τ, $\beta\to\infty$ along some curve $\Gamma=g(\tau)$ lying in T because the impression of the prime end P that corresponds to the point $\zeta=1$ consists only of the point ∞. We will prove that

$$\lim_{\beta\to\infty,\,\beta\in\Gamma}\inf\frac{u(\beta)}{|\beta|^{1/2}}\geqq\frac{2u(0)}{m^{1/2}}, \tag{6.5}$$

which gives (6.2).

Let the left hand side of (6.5) be denoted by ρ; obviously $0\leqq\rho\leqq+\infty$. Let $\{\beta_k\}_{k=1}^{\infty}$ be a sequence of points on Γ with $\beta_k\to\infty$ and

$$\frac{u(\beta_k)}{|\beta_k|^{1/2}}\to\rho\quad(\text{as } k\to\infty). \tag{6.6}$$

Put $\alpha_k=g^{-1}(\beta_k)$; then $\alpha_k\in[0,1)$, and $\alpha_k\to 1$ as $k\to\infty$. Now choose any positive number ε. Then there exists some number K such that

$$\frac{u(\beta_k)}{|\beta_k|^{1/2}}\leqq\rho+\varepsilon\quad(\text{for } k\geqq K). \tag{6.7}$$

Set

$$w_k=u(\beta_k)=U(\alpha_k), \tag{6.8}$$

and note that (6.3) implies that

$$U(\alpha_k)=c\frac{1+\alpha_k}{1-\alpha_k}+\sum_{j=1}^{n-1}c_j\,\mathrm{Re}\frac{e^{i\theta_j}+\alpha_k}{e^{i\theta_j}-\alpha_k}. \tag{6.9}$$

Substituting for $u(\beta_k)$ in (6.7) and using (6.8) and (6.9), we obtain

$$c\frac{1+\alpha_k}{1-\alpha_k}+\sum_{j=1}^{n-1}c_j\,\mathrm{Re}\frac{e^{i\theta_j}+\alpha_k}{e^{i\theta_j}-\alpha_k}\leqq|\beta_k|^{1/2}(\rho+\varepsilon). \tag{6.10}$$

Putting $\alpha_k=\zeta$ and $\beta_k=g(\zeta)$ in (6.4), we see that

$$\begin{aligned}|\beta_k|&\leqq\frac{m|\alpha_k|}{(1-\alpha_k)^2}\\&<\frac{m}{(1-\alpha_k)^2}\quad(0\leqq\alpha_k<1).\end{aligned} \tag{6.11}$$

Substituting for $|\beta_k|$ from (6.11) into (6.10), we obtain that

$$c\frac{1+\alpha_k}{1-\alpha_k}+\sum_{j=1}^{n-1}c_j\,\mathrm{Re}\frac{e^{i\theta_j}+\alpha_k}{e^{i\theta_j}-\alpha_k}\leqq\frac{m^{1/2}(\rho+\varepsilon)}{(1-\alpha_k)}\quad(k\geqq K),$$

which we then rewrite in the form

$$c(1+\alpha_k)+(1-\alpha_k)\sum_{j=1}^{n-1}c_j\,\mathrm{Re}\frac{e^{i\theta_j}+\alpha_k}{e^{i\theta_j}-\alpha_k}\leqq m^{1/2}(\rho+\varepsilon)\quad(k\geqq K).$$

Letting $k\to\infty$, we obtain

$$2c\leqq m^{1/2}(\rho+\varepsilon),$$

since $\theta_j\neq 0$ for $j=1,2,\ldots,n-1$. Because ε is arbitrary, it follows that $\rho\geqq 2c/m^{1/2}$, as required.

REFERENCES

1. K. F. Barth, and D. A. Brannan, Asymptotic tracts of harmonic functions I, *Ann. Acad. Sci. Fenn. Series A.I. Math.* **11** (1986), 215–232.

2. K. F. Barth, D. A. Brannan and W. K. Hayman, The growth of plane harmonic functions along an asymptotic path, *Proc. London Math. Soc.* (3) **37** (1978), 363–384.

3. D. A. Brannan, W. H. J. Fuchs, W. K. Hayman and Ü. Kuran, A characterisation of harmonic polynomials in the plane, *Proc. London Math. Soc.* (3) **32** (1976), 213–229.

4. C. Carathéodory, Über die Begrenzung einfach zusammenhängender Gebiete, *Math. Ann.* **73** (1913), 323–370.

5. E. F. Collingwood and A. J. Lohwater, *The Theory of Cluster Sets* (Cambridge University Press, Cambridge, 1966).

6. L. Flatto, D. J. Newman and H. S. Shapiro, The level curves of harmonic functions, *Trans. Amer. Math. Soc.* **123** (1966), 425–436.

7. W. H. J. Fuchs, *Théorie de l'approximation des fonctions d'une variable complexe* (Séminaire de Mathématiques Supérieures, No. 26 (Été, 1967), Les Presses de l'Université de Montréal, Montréal, Canada, 1968).

8. W. K. Hayman, *Subharmonic Functions*, Volume 2 (Academic Press, London, 1989).

9. M. H. Heins, *Selected Topics in the Classical Theory of Functions of a Complex Variable* (Holt, Rinehart and Winston, New York, 1962).

10. A. J. Lohwater, A uniqueness theorem for a class of harmonic functions, *Proc. Amer. Math. Soc.* **3** (1952), 278–279.

11. G. R. MacLane, Asymptotic values of holomorphic functions (Rice University Studies, Vol. 49, No. 1, Winter 1963).

12. M. H. A. Newman, *Elements of the Topology of Plane Sets of Points* (Cambridge University Press, Cambridge, 1964).

13. P. J. Rippon, *The boundary behaviour of subharmonic functions*, Ph.D. Thesis, Imperial College, London (1977).

14. M. Tsuji, *Potential Theory in Modern Function Theory* (Chelsea Pub. Co., New York, 1975).

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