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A multiplicative dual of nil-clean rings

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Abstract. The goal of this note is to completely determine the rings for which every nonunit is a product of a nilpotent and an idempotent (in either order).

1 Introduction

Throughout, R is an associative ring with unity, and $U(R)$, $\mathrm{idem}(R)$ and $\mathrm{nil}(R)$ denote, respectively, the group of units, the set of idempotents and the set of nilpotents in R. In the literature, an extensive knowledge has been developed for rings R satisfying, respectively, $R = \mathrm{idem}(R) + U(R)$, $R = \mathrm{idem}(R) + \mathrm{nil}(R)$ and $R\backslash(0) = U(R) + \mathrm{nil}(R)$. A ring R with $R = \mathrm{idem}(R) + U(R)$ is called a clean ring, a notion first appeared in 1977 in the prominent paper [10] by Nicholson. A ring R with $R = \mathrm{idem}(R) + \mathrm{nil}(R)$ is called a nil-clean ring, introduced by Diesl [5] in 2013. A ring R with $R\backslash(0) = U(R) + \mathrm{nil}(R)$ is called a fine ring, introduced by Călugăreanu and Lam more recently in [3]. These notions can be defined elementwise: an element $a \in R$ is called a clean element if $a \in \mathrm{idem}(R) + U(R)$, and one defines nil-clean elements and fine elements in a similar manner.

All these notions have natural multiplicative duals. An element in a ring is unit-regular if it is a product of a unit and an idempotent (in either order), and a ring is unit-regular if each of its elements is unit-regular. Thus, unit-regular elements and unit-regular rings are multiplicative duals of clean elements and clean rings. In other words, clean elements and clean rings are additive duals of unit-regular elements and unit-regular rings. An element in a ring is a UN-element if it is a product of a unit and a nilpotent, and a ring is a UN-ring if every nonunit is a product of a unit and a nilpotent. Thus, UN-elements and UN-rings are multiplicative duals of fine elements and fine rings. Unit-regular rings have been well studied in the literature, and UN-rings is a topic discussed recently by Călugăreanu in [2].

While nil-clean rings are widely investigated (for example, see [1, 5, 6, 7, 8, 9, 12]), there has been no discussion of their multiplicative dual. Our interest is to fill up what is missing. As a multiplicative dual of a nil-clean element, an element $a \in R$ is called dual nil-clean if $a = be$ where b is a nilpotent and e is an idempotent. Because a unit cannot be dual nil-clean, we define a ring R to be dual nil-clean if every nonunit of R is dual nil-clean. We will see that the order of the factors in the product does not

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matter for a dual nil-clean ring, but matters for a single dual nil-clean element. Here, we completely determine dual nil-clean rings, and our main result states that a ring is dual nil-clean if and only if it is either a local ring with nil Jacobson radical or a 2×2 matrix ring over a division ring.

For a ring R, the Jacobson radical of R is denoted by $J(R)$. We write $\mathbb{M}_n(R)$ for the ring of $n \times n$ matrices over R. For an element a in a ring R, $a^{\perp}$ (resp., ${}^{\perp}a$) denotes the right (resp., left) annihilator of a in R. A ring is called abelian if each of its idempotents is central.

2 The result

A ring R is called dual nil-clean if every nonunit a in R is dual nil-clean, i.e., $a = be$ where $b \in \text{nil}(R)$ and $e^2 = e \in R$.

Lemma 2.1 *Let R be a dual nil-clean ring. If $a^{\perp} = 0$ or ${}^{\perp}a = 0$, then $a \in U(R)$.*

Proof Assume $a^{\perp} = 0$ and $a \notin U(R)$, and write $a = be$ where $b \in \text{nil}(R)$ and $e^2 = e \in R$. Then $a(1-e) = be(1-e) = 0$, so $1 - e \in a^{\perp}$, and hence $e = 1$. So $a = b$ is nilpotent. Choose $n \geq 1$ such that $a^n \neq 0$ but $a^{n+1} = 0$. Then $0 \neq a^n \in a^{\perp}$, a contradiction.

Assume ${}^{\perp}a = 0$ and $a \notin U(R)$, and write $a = be$ where $b \in \text{nil}(R)$ and $e^2 = e \in R$. Then $b \neq 0$. Let us say $b^{n+1} = 0$ but $b^n \neq 0$. Then $b^n a = b^{n+1}e = 0$, so $0 \neq b^n \in {}^{\perp}a$, a contradiction. ■

Lemma 2.2 [11] *Let R be a ring and $n \geq 2$. Then R is isomorphic to some $n \times n$ matrix ring if and only if R contains elements $a_1, \ldots, a_n$ and f such that $1 = \sum_{i=1}^{n} f^{i-1} a_i f^{n-i}$ and $f^n = 0$.*

Dual nil-clean rings can be completely determined.

Theorem 2.3 *A ring R is dual nil-clean if and only if R is either a local ring with $J(R)$ nil or the 2×2 matrix ring over a division ring.*

Proof ($\Leftarrow$). If R is a local ring with $J(R)$ nil, then R is clearly dual nil-clean. Let D be a division ring and let $A \in \mathbb{M}_2(D)$ be a nonunit. By Gaussian elimination, there is a unit $U \in \mathbb{M}_2(D)$ such that $UA = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$. Thus, $UAU^{-1} = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$ for some $x, y \in D$. If $y \neq 0$, then $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ y^{-1}x & 1 \end{pmatrix}$ is a product of a sqaure-zero matrix and an idempotent. If $y = 0$, then $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ is a product of a square-zero matrix and an idempotent. Therefore, in $\mathbb{M}_2(D)$, UAU^{-1} is a product of a square-zero matrix and an idempotent, and so is A. Hence, $\mathbb{M}_2(D)$ is a dual nil-clean ring.

($\Rightarrow$). First assume that R is an abelian ring. Let $a \in R$ be a nonunit, and let $x \in aR$. Since R is abelian, x is a nonunit. Write $x = be$ where $b \in \text{nil}(R)$ and $e^2 = e \in R$. So

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$x^n = b^n e$ for all $n \geq 1$. As b is nilpotent, x is nilpotent. Thus, aR is nil and hence $a \in J(R)$. It follows that R is local with $J(R)$ nil.

Next assume that R is not abelian. Then R has a noncentral idempotent e. With $e' = 1 - e$, we show:

(1) There exist $x_0 \in eRe'$ and $y_0 \in e'Re$ such that $x_0 y_0 = e$.

(2) Whenever $xy = e$, $x \in eRe'$ and $y \in e'Re$, we have $yx = e'$.

Proof of (1). The Peirce decomposition of R with respect to e gives $R = \begin{pmatrix} eRe & eRe' \\ e'Re & e'Re' \end{pmatrix}$. Let $A := \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$ and write $A = BE$ where $B = (b_{ij})$ is a nilpotent and $E = (e_{ij})$ is an idempotent. Then $A = AE$ and it follows that $e_{11} = e$ and $e_{12} = 0$. From $A = BE$ it follows that $b_{11} = e - b_{12}e_{21}$, $b_{21} = -b_{22}e_{21}$. Thus, $B = \begin{pmatrix} e - b_{12}e_{21} & b_{12} \\ -b_{22}e_{21} & b_{22} \end{pmatrix}$, so $1 - B = \begin{pmatrix} b_{12}e_{21} & -b_{12} \\ b_{22}e_{21} & e' - b_{22} \end{pmatrix}$. Hence, $C := (1 - B)\begin{pmatrix} e & 0 \\ e_{21} & e' \end{pmatrix} = \begin{pmatrix} 0 & -b_{12} \\ e_{21} & e' - b_{22} \end{pmatrix}$, which is an invertible matrix with inverse, say $Y := (y_{ij})$. So, $1 = YC$. That is,

$$\begin{pmatrix} e & 0 \\ 0 & e' \end{pmatrix} = \begin{pmatrix} y_{12}e_{21} & -y_{11}b_{12} + y_{12}(e' - b_{22}) \\ y_{22}e_{21} & -y_{21}b_{12} + y_{22}(e' - b_{22}) \end{pmatrix}.$$

Thus, $x_0 y_0 = e$ where $x_0 = y_{12} \in eRe'$ and $y_0 = e_{21} \in e'Re$.

Proof of (2). Suppose that $xy = e$, $x \in eRe'$, and $y \in e'Re$. By (1), with e replaced by e' we have $y'x' = e'$ where $x' \in eRe'$ and $y' \in e'Re$. Let $U = \begin{pmatrix} 0 & x' \\ y & 0 \end{pmatrix}$. If $UX = 0$ where $X = (x_{ij}) \in R$, then $0 = \begin{pmatrix} 0 & x' \\ y & 0 \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} x'x_{21} & x'x_{22} \\ yx_{11} & yx_{12} \end{pmatrix}$, so

$$x'x_{21} = 0, \quad x'x_{22} = 0, \quad yx_{11} = 0, \quad yx_{12} = 0.$$

Thus, $x_{11} = ex_{11} = xyx_{11} = 0$, $x_{12} = ex_{12} = xyx_{12} = 0$, $x_{21} = e'x_{21} = y'x'x_{21} = 0$, and $x_{22} = e'x_{22} = y'x'x_{22} = 0$. So the right annihilator of U in R is zero. Hence, $U \in R$ is a unit by Lemma 2.1. Let $V = (v_{ij})$ be the inverse of U. Then

$$UV = \begin{pmatrix} 0 & x' \\ y & 0 \end{pmatrix}\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} = \begin{pmatrix} x'v_{21} & x'v_{22} \\ yv_{11} & yv_{12} \end{pmatrix},$$

so $e' = yv_{12}$. But we have $v_{12} = ev_{12} = xyv_{12} = xe' = x$, and hence $yx = e'$.

Next we show that R is a 2×2 matrix ring over a division ring. Consider the nonunit $A := \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \in R$ and write $A = BE$ where $B = (b_{ij}) \in R$ is nilpotent and $E = (e_{ij}) \in R$ is an idempotent. Then $A = AE$ and it follows that $E = \begin{pmatrix} e & 0 \\ e_{21} & e_{22} \end{pmatrix}$. From

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$A = BE$ it follows that $B = \begin{pmatrix} e - b_{12}e_{21} & b_{12} \\ -b_{22}e_{21} & b_{22} \end{pmatrix}$, so $1 - B = \begin{pmatrix} b_{12}e_{21} & -b_{12} \\ b_{22}e_{21} & e' - b_{22} \end{pmatrix}$. Hence,

$$C := (1 - B)\begin{pmatrix} e & 0 \\ e_{21} & e' \end{pmatrix} = \begin{pmatrix} 0 & -b_{12} \\ e_{21} & e' - b_{22} \end{pmatrix},$$

which is an invertible matrix with inverse, say $Y := (y_{ij})$. Thus, $1 = YC$. That is,

$$\begin{pmatrix} e & 0 \\ 0 & e' \end{pmatrix} = \begin{pmatrix} y_{12}e_{21} & -y_{11}b_{12} + y_{12}(e' - b_{22}) \\ y_{22}e_{21} & -y_{21}b_{12} + y_{22}(e' - b_{22}) \end{pmatrix}.$$

It follows that $y_{12}e_{21} = e$. By (2), $e_{21}y_{12} = e'$. Therefore, $1 = e + e' = y_{12}e_{21} + e_{21}y_{12}$ with $e_{21}^2 = 0$. So, by Lemma 2.2, R is a 2×2 matrix ring. Write $R = \mathbb{M}_2(S)$ for some ring S. We verify that S is a division ring. If $x \in S$ is not a unit, then $A := \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \in \mathbb{M}_2(S)$ is not a unit, so it is dual nil-clean in R. Write $A = BE$, where $B = (b_{ij}) \in \mathbb{M}_2(S)$ is nilpotent and $E = (e_{ij}) \in \mathbb{M}_2(S)$ is an idempotent. We have $A = AE = \begin{pmatrix} e_{11} & e_{12} \\ xe_{21} & xe_{22} \end{pmatrix}$, so

$$e_{11} = 1,\quad e_{12} = 0,\quad xe_{21} = 0 \ \text{and}\ x = xe_{22}. \tag{2.1}$$

Thus, $E = \begin{pmatrix} 1 & 0 \\ e_{21} & e_{22} \end{pmatrix}$. From $A = BE$, we have $\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} b_{11} + b_{12}e_{21} & b_{12}e_{22} \\ b_{21} + b_{22}e_{21} & b_{22}e_{22} \end{pmatrix}$, so $b_{11} = 1 - b_{12}e_{21}$ and $b_{21} = -b_{22}e_{21}$. Thus, $B = \begin{pmatrix} 1 - b_{12}e_{21} & b_{12} \\ -b_{22}e_{21} & b_{22} \end{pmatrix}$. As B is nilpotent, $I_2 - B$ is invertible. So $(I_2 - B)\begin{pmatrix} 1 & 0 \\ e_{21} & 1 \end{pmatrix} = \begin{pmatrix} 0 & -b_{12} \\ e_{21} & 1 - b_{22} \end{pmatrix}$ is invertible. It follows that $e_{21} \in U(R)$. So, by (2.1), $x = 0$. Therefore, S is a division ring. ■

Corollary 2.4 *Let $n \geq 2$ be a fixed integer. The following are equivalent for a ring R:*

(1) *For each nonunit $a \in R$, $a = be$ where $b^n = 0$ and $e^2 = e$.*
(2) *R is either a local ring with $j^n = 0$ for all $j \in J(R)$ or the 2×2 matrix ring over a division ring.*

Proof (1) ⇒ (2). Assume that R is not the 2×2 matrix ring over a division ring. Then, by Theorem 2.3, R is a local ring. For $j \in J(R)$, $j = be$ where $b^n = 0$ and $e^2 = e$. As R is local, $e = 0$ or $e = 1$. It follows that $j^n = 0$.

(2) ⇒ (1). We may assume that $R = \mathbb{M}_2(D)$ where D is a division ring. Let $A \in \mathbb{M}_2(R)$ be a nonunit. Then, by Gaussian elimination, there is a unit $U \in \mathbb{M}_2(R)$ such that $UA = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$. Thus, $UAU^{-1} = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$ for some $x, y \in R$. If $y \neq 0$, then $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ y^{-1}x & 1 \end{pmatrix}$ is a product of a sqaure-zero matrix and an idempotent. If $y = 0$, then $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ is a product of a square-zero matrix and

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an idempotent. Therefore, in $\mathbb{M}_2(R)$, UAU^{-1} is a product of a square-zero matrix and an idempotent, and so is A. ■

By Theorem 2.3, for a ring R, every element of R is a product of a nilpotent and an idempotent if and only if every element of R is a product of an idempotent and a nilpotent. We end with an example of an element a in a ring R such that $a = be$ where b is nilpotent and $e^2 = e$, but $a \neq fc$ for any nilpotent c and any idempotent f in R.

Example 2.5 Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ and $A = \begin{pmatrix} -4 & -2 \\ 0 & 0 \end{pmatrix}$. We see that $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -4 & -2 \\ 8 & 4 \end{pmatrix}$, a product of an idempotent and a nilpotent. Assume that $A = BE$ where $B \in R$ is nilpotent and $E^2 = E \in R$. It is clear that E can not be trivial, so $E = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$ where $bc = a - a^2$ (see [4, Lemma 1.5]). Thus, $A = AE = \begin{pmatrix} -4a-2c & -2+2a-4b \\ 0 & 0 \end{pmatrix}$, and it follows that $-4a - 2c = -4$ and $-2 + 2a - 4b = -2$. That is, $a = 2b$ and $c = 2 - 4b$. As $c \in 4\mathbb{Z}$, we deduce that $2 = 4b + c$ is divided by 4. This is a contradiction.

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