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Properties of co-operations: diagrammatic proofs

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We propose an alternative approach, based on diagram rewriting, for computations in bialgebras. We illustrate this graphical syntax by proving some properties of co-operations, including *coassociativity* and *cocommutativity*. This amounts to checking the confluence of some rewriting systems.

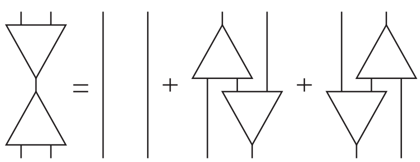
1. Introduction

Traditionally, terms taken together with *Sweedler notation* have been used to express computations in (generalised) bialgebras. Here, an *algebra* is a vector space equipped with an *operation* $\mu : \mathbb{A} \otimes \mathbb{A} \to \mathbb{A}$, and a *bialgebra* is an algebra $\mathbb{A}$ equipped with a *co-operation* $\delta : \mathbb{A} \to \mathbb{A} \otimes \mathbb{A}$. The operation μ must be (bi)linear and satisfy some properties, for instance, associativity and/or commutativity. Similarly, the co-operation δ must be linear and satisfy some co-properties, for instance, co-associativity and/or co-commutativity. Furthermore, a *compatibility relation* between μ and δ, such as the *Hopf identity*, must be satisfied. For more details, see Loday (2008).

We shall consider an example from Loday (2008): the definition of a *Lie*^{c}*-Lie-bialgebra* starting from an *Ass*^{c}*-Ass-bialgebra* (Section 4.4). To do this, we consider a vector space V, and we write $\overline{T}(V)$ for the (non-unital associative) algebra of non-commutative polynomials on V, and $Lie(V) \subseteq \overline{T}(V)$ for the algebra of *Lie polynomials* on V. We shall use the following equalities, expressed in Sweedler notation, where $X_1 \otimes X_2$ stands for *comultiplication* δ applied to X, as well as in diagrammatic notation, where co-operations, as well as operations, are represented by *gates*:

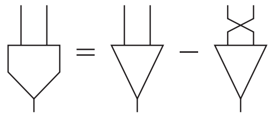
(1) *Non-unitary infinitesimal* compatibility relation, or the definition of *deconcatenation* (see Loday (2008, Section 4.2.1)):
For $X, Y \in \overline{T}(V)$,

$$\delta(XY) = X \otimes Y + X_1 \otimes X_2Y + XY_1 \otimes Y_2$$



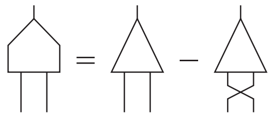
(2) Definition of *Lie bracket* $[\_,\_]: \overline{T}(V) \otimes \overline{T}(V) \to \overline{T}(V)$:
For $X, Y \in \overline{T}(V)$,

$$[X, Y] := XY - YX$$



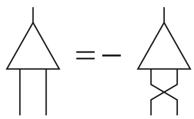
(3) Definition of *Lie cobracket* $\delta_{[,]}: \overline{T}(V) \to \overline{T}(V) \otimes \overline{T}(V)$:
For $X \in \overline{T}(V)$,

$$\delta_{[,]}(X) := X_1 \otimes X_2 - X_2 \otimes X_1$$



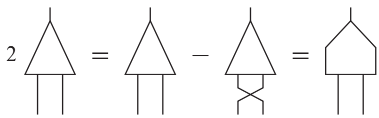
(4) *Anti-cocommutativity* of deconcatenation for *Lie polynomials*:
For $X \in Lie(V)$,

$$X_1 \otimes X_2 = -X_2 \otimes X_1$$



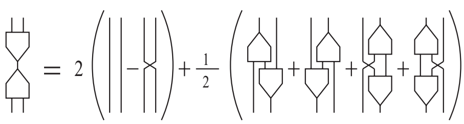
(5) Corollary of the previous two equalities:
For $X \in Lie(V)$,

$$2X_1 \otimes X_2 = X_1 \otimes X_2 - X_2 \otimes X_1 = \delta_{[,]}(X)$$



Our example is a proof of the *Lily compatibility relation*:
For $X, Y \in Lie(V)$,

$$\delta_{[,]}[X, Y] = 2(X \otimes Y - Y \otimes X) + \tfrac{1}{2}(X_{[1]} \otimes [X_{[2]}, Y] + [X, Y_{[1]}] \otimes Y_{[2]} + Y_{[1]} \otimes [X, Y_{[2]}] + [X_{[1]}, Y] \otimes X_{[2]})$$



Here, $X_{[1]} \otimes X_{[2]}$ stands for $\delta_{[,]}$ applied to X. This is Loday (2008, Proposition 4.4.4). In Figure 1, we translate the original proof of Proposition 4.4.4 into diagrammatic notation.

It should be noted that diagrams do appear in Loday (2008) (see, for instance, pages 105–106), but they are only used as convenient pictures. Here we consider diagrams as true mathematical objects to compute with.

In fact, a diagram represents a morphism in a *PRO*. Recall that a *PRO* is a strict monoidal category, whose objects are natural numbers and where the monoidal product of two objects p, q

$$
\begin{aligned}
\delta_{[,]}[X,Y] &= (\delta - \tau\delta)(XY - YX) \\
&= \delta(XY) - \delta(YX) - \tau\delta(XY) + \tau\delta(YX) \\
&= X\otimes Y + X_1\otimes X_2Y + XY_1\otimes Y_2 - Y\otimes X - Y_1\otimes Y_2X - YX_1\otimes X_2 \\
&\quad -Y\otimes X - X_2Y\otimes X_1 - Y_2\otimes XY_1 + X\otimes Y + Y_2X\otimes Y_1 + X_2\otimes YX_1 \\
&= X\otimes Y + X_1\otimes X_2Y + XY_1\otimes Y_2 - Y\otimes X - Y_1\otimes Y_2X - YX_1\otimes X_2 \\
&\quad -Y\otimes X + X_1Y\otimes X_2 + Y_1\otimes XY_2 + X\otimes Y - Y_1X\otimes Y_2 - X_1\otimes YX_2 \\
&= 2(X\otimes Y - Y\otimes X) + X_1\otimes X_2Y - X_1\otimes YX_2 + XY_1\otimes Y_2 - Y_1X\otimes Y_2 \\
&\quad +Y_1\otimes XY_2 - Y_1\otimes Y_2X + X_1Y\otimes X_2 - YX_1\otimes X_2 \\
&= 2(X\otimes Y - Y\otimes X) \\
&\quad +X_1\otimes[X_2,Y] + [X,Y_1]\otimes Y_2 + Y_1\otimes[X,Y_2] + [X_1,Y]\otimes X_2 \\
&= 2(X\otimes Y - Y\otimes X) \\
&\quad +\frac{1}{2}(X_{[1]}\otimes[X_{[2]},Y] + [X,Y_{[1]}]\otimes Y_{[2]} + Y_{[1]}\otimes[X,Y_{[2]}] + [X_{[1]},Y]\otimes X_{[2]})
\end{aligned}
$$

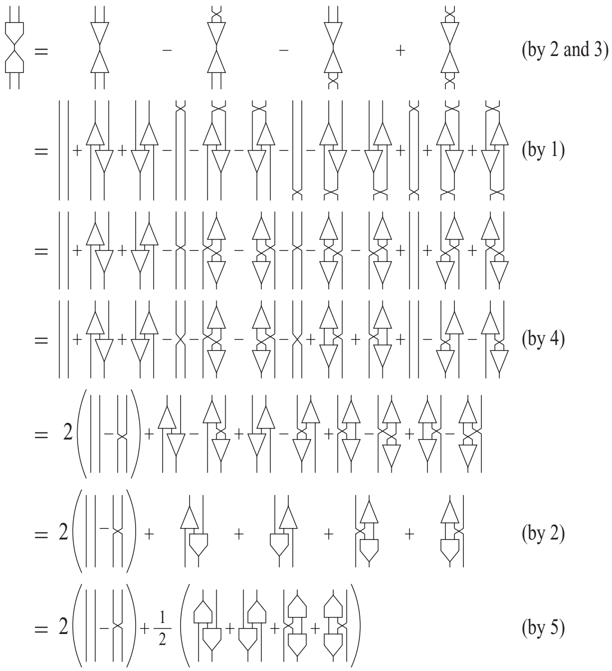


FIG. 1. Proof of the Lily compatibility relation

is $p+q$. Such a *PRO* defines an *operad*, which is a monadic Schur functor $\mathcal{P}$: $Vect \to Vect$, where $Vect$ is the category of vector spaces over some field: see Loday (2008). Sum and coefficients play a crucial role in $Vect$, so both should appear explicitly in diagrams.

Therefore, we use Σ-diagrams, which are formal sums of diagrams. These Σ-diagrams may not be familiar to mathematicians or computer scientists working in proof theory or in category theory, but they appear, for instance, in Ehrhard and Regnier (2006), where sum stands for non-determinism.

Organisation of the paper

In Section 2, we present basic algebraic notions, and define *deconcatenation*. In Section 3, we give a precise definition of *diagrams* and Σ-*diagrams*. In Section 4, we prove a well-known result, using the diagrammatic notation: coassociativity of deconcatenation for semi-groups. In Section 5, we prove the same result for monoids, using the previous result, and in Section 6, we study *shuffle* for monoids, and prove its coassociativity and its cocommutativity, using a similar method. Finally, we present our conclusions in Section 7.

2. Deconcatenation for semi-groups

Recall that a *semi-group* is a set with an associative operation, and a *monoid* is a semi-group with a *unit*. Let $\mathcal{A}$ be an alphabet. The elements of $\mathcal{A}$ are called *letters*.

Definition 1. We write $\mathcal{A}^+$ for the *free semi-group* generated by $\mathcal{A}$. Its elements are non-empty lists of letters. They are called *non-empty words*.

For instance, if our alphabet is $\mathcal{A} = \{a, b\}$, then $aabba$ is a non-empty word in $\mathcal{A}^+$. *Concatenation* is the operation $\cdot$ that, to each pair $(u, v) \in (\mathcal{A}^+)^2$, associates the word $uv \in \mathcal{A}^+$. For instance,

$$abba \cdot bba = abbabba.$$

Remark 1. Concatenation is associative. For instance,

$$(ab \cdot b) \cdot a = abb \cdot a = abba = ab \cdot ba = ab \cdot (b \cdot a).$$

Definition 2. The *free* $\mathbb{Q}$-*vector space* generated by a set $\mathcal{X}$ is the vector space $\mathbb{Q}\mathcal{X}$ whose elements are formal sums of elements of $\mathcal{X}$, with coefficients in $\mathbb{Q}$.

For instance, if $\mathcal{X} = \{x, y\}$, we have

$$x + y - x + y + y = y + y + y = 3y \in \mathbb{Q}\mathcal{X}.$$

Remark 2. If $\mathcal{X}$ is a finite set, $\mathbb{Q}\mathcal{X}$ is isomorphic to $\mathbb{Q}^n$, where n is the cardinality of X. For instance, $\mathbb{Q}\mathcal{X}$ is isomorphic to $\mathbb{Q}^2$ in the above example.

Definition 3. The *non-unital algebra* generated by a semi-group S is the free $\mathbb{Q}$-vector space $\mathbb{Q}S$ generated by the set S, equipped with the multiplication extending that of the semi-group S, and distributive over sum.

For instance, if $S = \mathcal{A}^+$ with $\mathcal{A} = \{a, b\}$, we get

$$(2abb - 3ba) \cdot aa = 2abbaa - 3baaa \in \mathbb{Q}S.$$

Definition 4. If U and V are $\mathbb{Q}$-vector spaces, the *tensor product* $U \otimes V$ is the free $\mathbb{Q}$-vector space generated by elements of the form $u \otimes v$ with $u \in U$ and $v \in V$, quotiented by the following

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equalities:

$$(u + u') \otimes v = u \otimes v + u' \otimes v$$
$$u \otimes (v + v') = u \otimes v + u \otimes v'$$
$$(\lambda u) \otimes v = \lambda(u \otimes v) = u \otimes (\lambda v) \quad \text{for all } \lambda \in \mathbb{Q}.$$

We write $U^{\otimes n}$ for the $\mathbb{Q}$-vector space $U \otimes \cdots \otimes U$ (n times).

Remark 3. By the universal property of tensor (a bilinear map from $U \times V$ to Z induces a unique linear map from $U \otimes V$ to Z), we have $(\mathbb{Q}X)^{\otimes n} = \mathbb{Q}X^n$. Hence, we get $u_1 \otimes \cdots \otimes u_n \in \mathbb{Q}X^n$ for any $u_1, \cdots, u_n \in \mathbb{Q}X$

Definition 5. The right and left actions of $\mathbb{Q}S$ on $\mathbb{Q}S^2$ are given as follows:

$$(u \otimes v) \cdot w = u \otimes (v \cdot w)$$
$$u \cdot (v \otimes w) = (u \cdot v) \otimes w$$

for any $u, v, w \in S$.

For instance,

$$(ab \otimes a) \cdot a = (ab \otimes aa).$$

Definition 6. Let $\mathcal{A}$ be an alphabet and let $S = \mathcal{A}^+$. *Deconcatenation* is the co-operation

$$\delta \colon \mathbb{Q}S \to \mathbb{Q}S \otimes \mathbb{Q}S = \mathbb{Q}S^2$$

defined recursively by:

$$\delta(a) := 0 \quad \text{for any } a \in \mathcal{A}$$
$$\delta(u \cdot v) := u \cdot \delta(v) + \delta(u) \cdot v + u \otimes v \quad \text{for any } u, v \in S.$$

Remark 4. In fact, $\delta(u) \cdot v$ consists of all terms of $\delta(u \cdot v)$ whose first component is a prefix of u and, similarly, $u \cdot \delta(v)$ consists of all terms of $\delta(u \cdot v)$ whose second component is a suffix of v.

Remark 5. The co-operation δ is described by the equality

$$\delta(w) := \sum_{w = u \cdot v} u \otimes v \text{ for any } w \in S.$$

For instance,

$$\delta(abaa) = a \otimes baa + ab \otimes aa + aba \otimes a.$$

Theorem 1. Deconcatenation is *coassociative*:

$$(\mathrm{id}_{\mathbb{Q}S} \otimes \delta) \circ \delta = (\delta \otimes \mathrm{id}_{\mathbb{Q}S}) \circ \delta,$$

or, in other words,

$$\text{for all } w \in S, \text{ if } \delta(w) = \sum u_i \otimes v_i, \text{ then } \sum u_i \otimes \delta(v_i) = \sum \delta(u_i) \otimes v_i.$$

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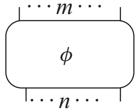
A diagrammatic proof of this classical result is given in section 4.

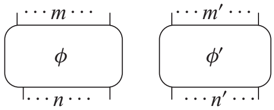
3. Diagrams and Σ-diagrams

For any $m, n \in \mathbb{N}$, a *diagram* $\phi\colon m \to n$, with m inputs and n outputs is drawn as follows:

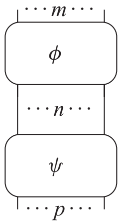


It is interpreted as a map $f\colon \mathcal{X}^m \to \mathcal{X}^n$, where $\mathcal{X}$ is some fixed set.
There are two operations on diagrams:

– The *parallel composition* of $\phi\colon m \to n$ and $\phi'\colon m' \to n'$ is $\phi * \phi'\colon m + m' \to n + n'$:



– The *sequential composition* of $\phi\colon m \to n$ and $\psi\colon n \to p$ is $\psi \circ \phi\colon m \to p$:

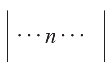


These operations are interpretated as follows:

– If $f\colon \mathcal{X}^m \to \mathcal{X}^n$ is the interpretation of $\phi\colon m \to n$ and $f'\colon \mathcal{X}^{m'} \to \mathcal{X}^{n'}$ is the interpretation of $\phi'\colon m' \to n'$, then $f \times f'\colon \mathcal{X}^{m+m'} \to \mathcal{X}^{n+n'}$ is the interpretation of the parallel composition $\phi * \phi'\colon m + m' \to n + n'$.

– If $f\colon \mathcal{X}^m \to \mathcal{X}^n$ is the interpretation of $\phi\colon m \to n$ and $g\colon \mathcal{X}^n \to \mathcal{X}^p$ is the interpretation of $\psi\colon n \to p$, then $g \circ f\colon \mathcal{X}^m \to \mathcal{X}^p$ is the interpretation of the sequential composition $\psi \circ \phi\colon m \to p$.

The *identity diagram* $Id_n\colon n \to n$ is drawn as follows:

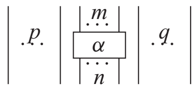


Atomic diagrams are called *gates* (or *generators*).

Definition 7. An *elementary diagram* is a formal composition

$$Id_p \otimes \alpha \otimes Id_q\colon p + m + q \to p + n + q$$

where $\alpha: m \to n$ is a gate:

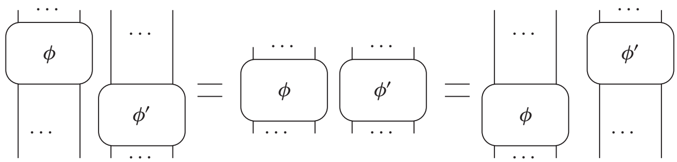


Definition 8. A diagram is a sequential composition

$$\phi_1 \circ \cdots \circ \phi_n$$

of elementary diagrams $\phi_1, \cdots, \phi_n$.

In fact, diagrams are defined modulo *interchange*:



For more information about diagrams, see Lafont (2003 ; 2010).

Remark 6. Diagrams are the morphisms of a free *PRO*. Moreover, any *PRO* is the quotient of a free *PRO* by some relations. Hence, diagrams are the syntax of *PROs*.

Definition 9. A Σ-*diagram* $\Phi: m \to n$ is a (finite) formal sum $\Sigma\lambda_i\phi_i$ where the coefficients λ_i are in $\mathbb{Q}$ and the $\phi_i: m \to n$ are diagrams with the same number m of inputs and the same number n of outputs.

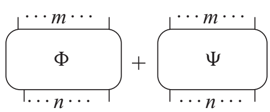
Parallel and sequential composition are extended to Σ-diagrams by distributivity over sum:

$$(\Sigma\lambda_i\phi_i) * \Psi = \Sigma\lambda_i(\phi_i * \Psi)$$
$$\Phi * (\Sigma\lambda_i\psi_i) = \Sigma\lambda_i(\Phi * \psi_i)$$
$$(\Sigma\lambda_i\phi_i) \circ \Psi = \Sigma\lambda_i(\phi_i \circ \Psi)$$
$$\Phi \circ (\Sigma\lambda_i\psi_i) = \Sigma\lambda_i(\Phi \circ \psi_i).$$

Remark 7. The field $\mathbb{Q}$ can be replaced by any field of characteristic 0, such as $\mathbb{R}$ or $\mathbb{C}$.

Note that we use uppercase greek letters Φ, Ψ for Σ-diagrams.

There is a binary sum on Σ-diagrams, which is drawn as follows:



Note that the Σ-diagrams Φ and Ψ have the same numbers of inputs m and the same number of outputs n. Similarly, we define the Σ-diagram $\lambda\Phi: m \to n$ for any scalar λ, and the *null* Σ-*diagram* $0: m \to n$.

A Σ-diagram $\Phi: m \to n$ is interpreted as a $\mathbb{Q}$-linear map $f: \mathbb{Q}\mathcal{X}^m \to \mathbb{Q}\mathcal{X}^n$. The interpretation of the operations is similar to the case of diagrams, except for parallel composition, which is interpreted by $\otimes$ instead of $\times$. The interpretation of + is straightforward.

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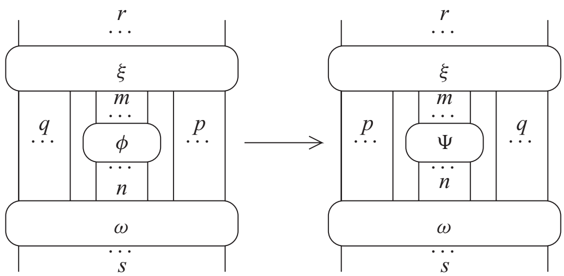
Definition 10. A *rewrite rule* is of the form $\phi \to \Psi$ where ϕ: $m \to n$ is a diagram and Ψ: $m \to n$ is a Σ-diagram.

Note that the left-hand member ϕ must be a diagram, not a Σ-diagram.

Elementary reduction, written $\to$, is defined as usual by applying a rule $\phi \to \Psi$ in a context given by two diagrams ξ: $r \to p + m + q$ and ω: $p + n + q \to s$:



Reduction is the *linear reflexive transitive closure* of elementary reduction, that is, the smallest relation $\to^*$ containing $\to$ such that:

$$\Phi \to^* \Phi \qquad \text{for any } \Sigma\text{-diagram } \Phi$$

$$\Phi \to^* \Phi'' \qquad \text{whenever } \Phi \to^* \Phi' \text{ and } \Phi' \to^* \Phi''$$

$$\Sigma\lambda_i\Phi_i \to^* \Sigma\lambda_i\Psi_i \qquad \text{whenever } \Phi_i \to^* \Psi_i \text{ for all } i.$$

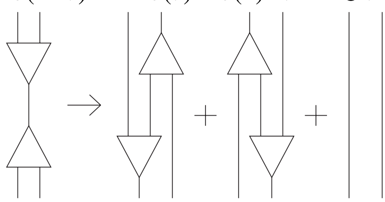
4. Diagrammatic proof of Theorem 1

We assume that X is the free semi-group $\mathcal{A}^+$. The gates are

concatenation $\cdot$ deconcatenation δ

Hence, we consider Σ-diagrams built on those gates. In other words, these gates are the generators of our free *PRO*. From Definition 6 (of deconcatenation), we deduce the following *interaction* rule:

$$\delta(u \cdot v) = u \cdot \delta(v) + \delta(u) \cdot v + u \otimes v$$



Recall that $u \cdot \delta(v)$ and $\delta(u) \cdot v$ are given by Definition 5. Similar kinds of rules are introduced in Lafont (1997) and Ehrhard and Regnier (2006).

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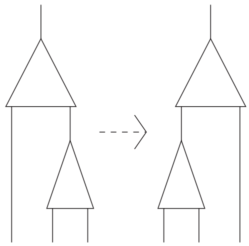
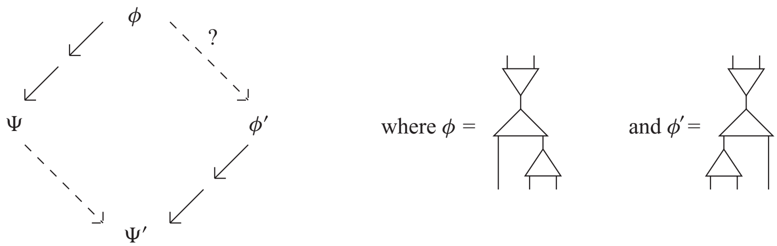
We will prove the *coassociativity* of deconcatenation, which corresponds to the following rewrite rule:



The key argument of our proof is described by a confluence diagram:



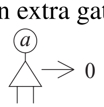
Note that there are two kinds of arrow:

– broken arrows for coassociativity;

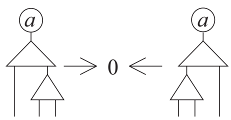
– solid arrows for interaction.

Proof. Formally, Theorem 1 is proved by induction on the length of the input word:

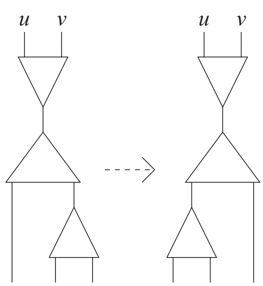
–Coassociativity obviously holds for letters since $\delta(a) = 0$ for any $a \in A$. In fact, this equality is expressed by the following rule (using an extra gate for each letter a):



Using this rule, we get:



–Let u and v be words in A^+ for which deconcatenation is coassociative. We will prove that deconcatenation is coassociative for $w = u \cdot v$. In other words, the following reduction holds:



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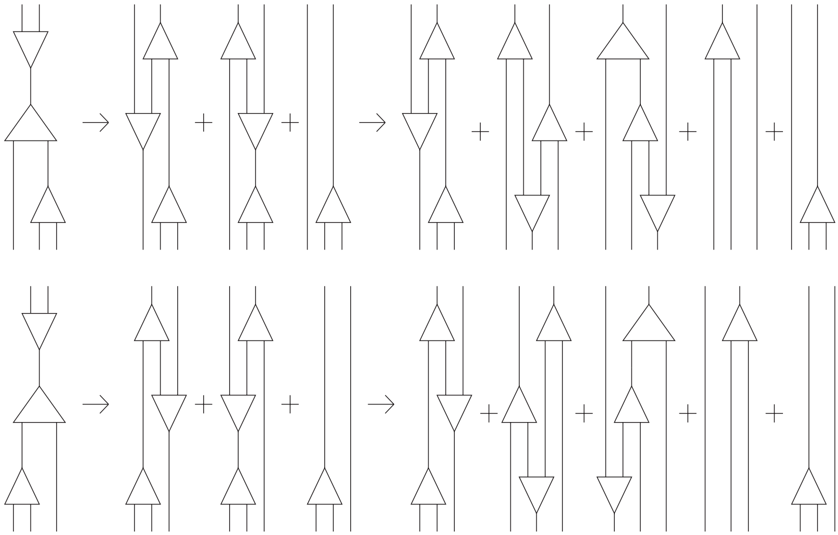
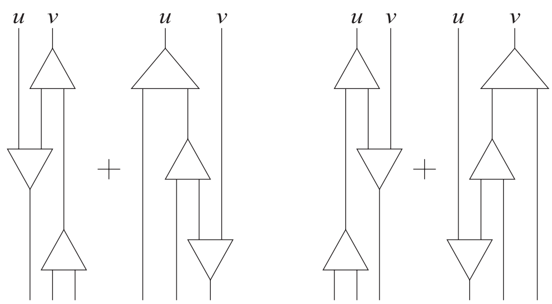
Applying interaction to both sides of the reduction, we get

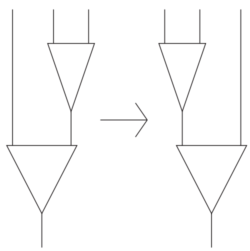


The two results differ in only two terms:



But, by the induction hypothesis, we can apply coassociativity to the Σ-diagram on the left, and get the one on the right. □

Remark 8. This proof expresses the confluence of the conflict between coassociativity and interaction. In fact, to get a complete rewrite system for (non-commutative and non-cocommutative) bialgebras, we need an associativity rule for concatenation:



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The resulting system has two *critical peaks*:
– the conflict between coassociativity and interaction;
– the conflict between associativity and interaction.
We have checked the confluence of the first one. For the second one, we simply reverse the diagrams. Hence, we get a confluent rewrite system. Termination is straightforward. Note that the latter argument is diagrammatic, and we do not need to consider the inputs of the diagrams.

5. Concatenation and deconcatenation for monoids

Let $\mathcal{A}$ be an alphabet.

Definition 11. We write $\mathcal{A}^*$ for the *free monoid generated* by $\mathcal{A}$. Its elements are those of $\mathcal{A}^+$ and the *empty word* ε.

Remark 9. The unit for concatenation is ε .

We write M for $\mathcal{A}^*$, and S for $\mathcal{A}^+$.

Definition 12. The *(unital)* $\mathbb{Q}$*-algebra* $\mathbb{Q}M$, is the free $\mathbb{Q}$-vector space generated by the set M, which is equipped with a multiplication $\cdot$ extending that of the monoid M and distributive over sum.

Definition 13. *Full deconcatenation* $\Delta\colon \mathbb{Q}M \to \mathbb{Q}M^2$ is defined as follows:

$$\Delta(w) := \sum_{w=u\cdot v} u \otimes v.$$

Remark 10. In particular, we get $\Delta(\varepsilon) = \varepsilon \otimes \varepsilon$.

Definition 14. *Primitive deconcatenation* $\delta\colon \mathbb{Q}M \to \mathbb{Q}M^2$ extending $\delta\colon \mathbb{Q}S \to \mathbb{Q}S^2$ is defined as follows:

$$\delta(w) = \sum_{\substack{w=u\cdot v\\ u,v\neq\varepsilon}} u \otimes v, \ \text{for } w \neq \varepsilon$$
$$\delta(\varepsilon) = -\varepsilon \otimes \varepsilon$$

Remark 11. The relation between the two deconcatenations is:

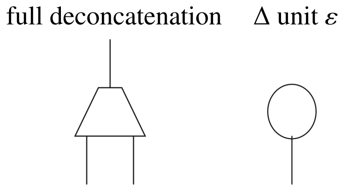
$$\Delta(u) = \delta(u) + u \otimes \varepsilon + \varepsilon \otimes u.$$

This explains why we defined $\delta(\varepsilon) := -\varepsilon \otimes \varepsilon$:

$$\Delta(\varepsilon) = \delta(\varepsilon) + \varepsilon \otimes \varepsilon + \varepsilon \otimes \varepsilon = -\varepsilon \otimes \varepsilon + 2\varepsilon \otimes \varepsilon = \varepsilon \otimes \varepsilon.$$

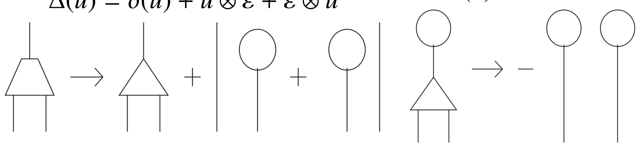
Theorem 2. Full deconcatenation is coassociative.

Proof. We introduce two new gates:

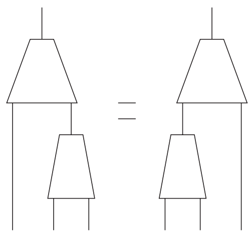


We also introduce two new rules:

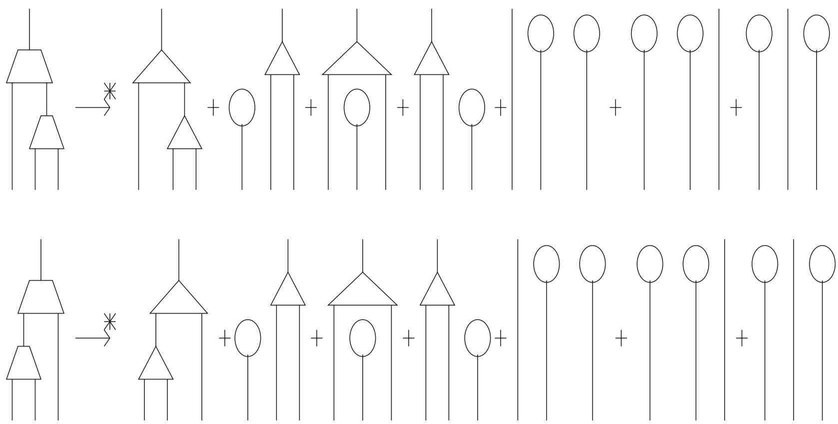
$$\Delta(u) = \delta(u) + u \otimes \varepsilon + \varepsilon \otimes u \qquad \delta(\varepsilon) = -\varepsilon \otimes \varepsilon$$



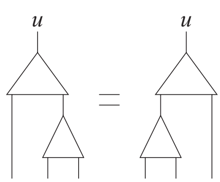
Coassociativity of full deconcatenation is drawn as follows:



Reducing these diagrams using the new rules gives



Hence, it just remains to show the following equality for $u \in \mathcal{A}^*$:



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There are two cases:
–If $u = \varepsilon$, we get $\varepsilon \otimes \varepsilon \otimes \varepsilon$ in both cases.
–If $u \in \mathcal{A}^+$, we apply Theorem 1. □

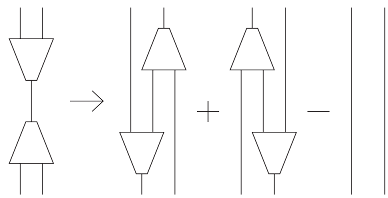
Remark 12. From

$$\Delta(u) = \delta(u) + u \otimes \varepsilon + \varepsilon \otimes u$$
$$\delta(u \cdot v) = u \cdot \delta(v) + \delta(u) \cdot v + u \otimes v$$

we can deduce

$$\Delta(u \cdot v) = u \cdot \Delta(v) + \Delta(u) \cdot v - u \otimes v.$$

This equality corresponds to the following interaction rule:



Using this rule, we can get an alternative proof of Theorem 2, which is very similar to the proof of Theorem 1.

6. Concatenation and shuffle for monoids

In this section, we also consider the monoid $M = \mathcal{A}^*$. Here, the syntax is also interpreted in $\mathbb{Q}$-vector spaces, but we only need diagrams (not Σ-diagrams).

Definition 15. *Shuffle* σ: $\mathbb{Q}M \to \mathbb{Q}M^2$ is defined as follows on a word $w = a_1 \cdots a_n$ of length n:

$$\sigma(w) := \sum_{(u,v)\in I_w} u \otimes v,$$

where I_w is the set of pairs (u, v) of words of the form:
– $u = a_{i_1} \cdots a_{i_p}$ with $1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant n$,
– $v = a_{j_1} \cdots a_{j_q}$ with $1 \leqslant j_1 < j_2 < \cdots < j_q \leqslant n$,
where $\{i_1, \cdots, i_p\} \cup \{j_1, \cdots, j_q\} = \{1, \cdots, n\}$ and $\{i_1, \cdots, i_p\} \cap \{j_1, \cdots, j_q\} = \varnothing$.

For instance,

$$\begin{aligned}\sigma(abaa) &= ab \otimes aa + 2aba \otimes a + abaa \otimes \varepsilon + 2aa \otimes ba + aaa \otimes b \\ &\quad + aa \otimes ab + 2a \otimes aba + \varepsilon \otimes abaa + 2ba \otimes aa + aaa \otimes b\end{aligned}$$

Shuffle is drawn as follows:

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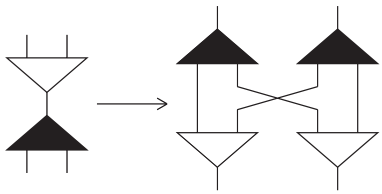
Remark 13. We have

$$\sigma(\varepsilon) = \varepsilon \otimes \varepsilon$$
$$\sigma(a) = \varepsilon \otimes a + a \otimes \varepsilon$$

for all $a \in \mathcal{A}$. Furthermore, we have

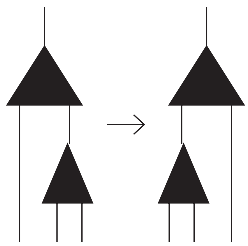
$$\sigma(w \cdot w') = \sum_{\substack{(u,v)\in I_w \\ (u',v')\in I_{w'}}} u \cdot u' \otimes v \cdot v' \text{ for all } w, w' \in \mathcal{A}.$$

The latter equality is expressed by the following *Hopf interaction* rule:



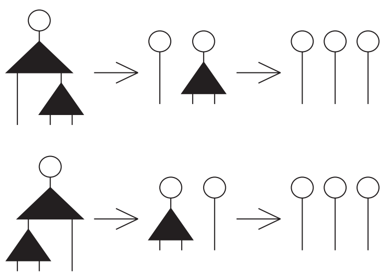
Theorem 3. Shuffle is *coassociative*.

Proof. Coassociativity of shuffle corresponds to the following rule:

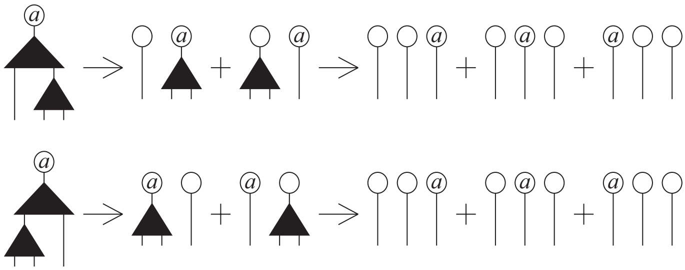


We prove this by induction on the length of the input word:

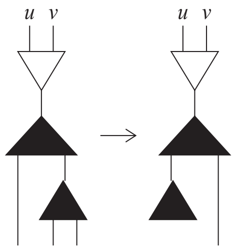
–For the empty word, we have



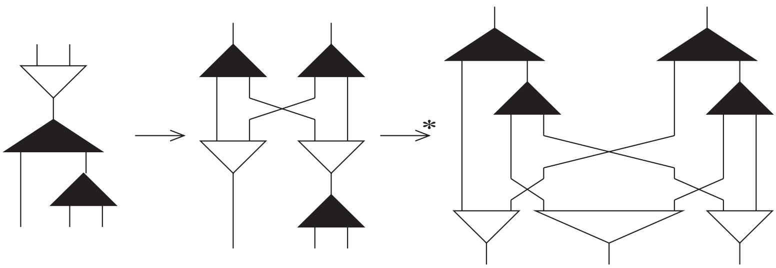
–For any $a \in \mathcal{A}$, we introduce a new gate and get

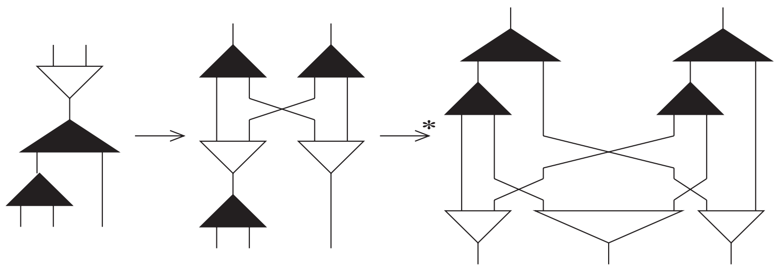


–Let u and v be two words in M for which shuffle is coassociative. We want to show that shuffle is also coassociative for $w = u \cdot v$. In other words, the following reduction holds:



We apply interaction to each side of the reduction:





By the induction hypothesis, we can apply coassociativity to the upper diagram, and get the lower one. □

Theorem 4. Shuffle is *cocommutative*:

$$\sigma(w) = \sum_{(u,v)\in I_w} v \otimes u \text{ for all } w \in M.$$

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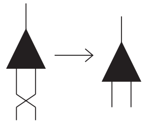
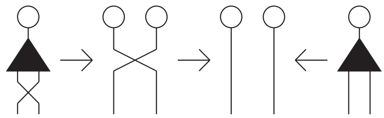
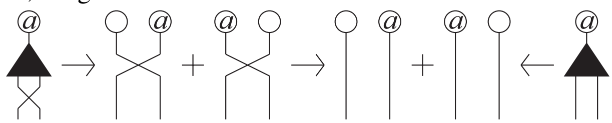
Proof. Cocommutativity of shuffle corresponds to the following rule:



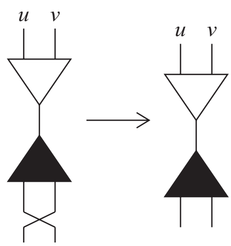
–For the empty word, we have



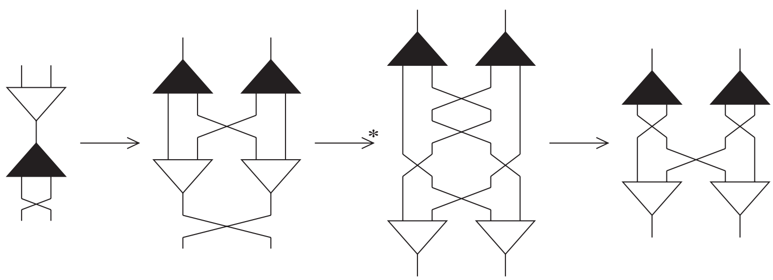
–For any letters $a \in \mathcal{A}$, we get

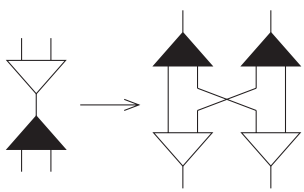


–Let u and v be two words in M for which shuffle is cocommutative. We want to show that shuffle is also cocommutative for $w = u \cdot v$. In other words, the following reduction holds:



We apply interaction to each side of the reduction:

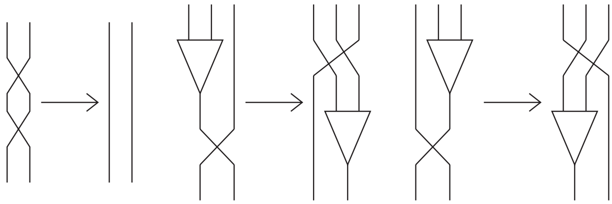


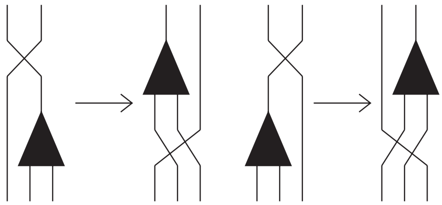


By the induction hypothesis, we can apply cocommutativity to the upper diagram, and get the lower one. □

Note that we have, in fact, used a new gate in this section:

And we should introduce the following new rules:





Crossing satisfies an extra equation (*Yang–Baxter*), but it is not needed here. For more details on this kind of rewriting, see Lafont (2003).

7. Conclusion

Σ-diagrams are used by mathematicians working on bialgebras to give a precise description of the relations between operations and co-operations. But the diagrammatic syntax is not usually formally defined, and is not used in computations or proofs.

In this paper, we have given a precise definition of these Σ-diagrams and some examples of computations using them. Note that:

- Computation with Σ-diagrams is well handled by rewriting: see Lafont (2003) for the case of diagrams.
- Σ-diagrams are very similar to *differential interaction nets*: see Ehrhard and Regnier (2006).

In future work, we shall develop a general theory of rewriting for Σ-diagrams as well as programs implementing these rewriting techniques.

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