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ON ALMOST CONTINUOUS MAPPINGS AND BAIRE SPACES

BY

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ABSTRACT. It is proved, in particular, that a topological space X is a Baire space if and only if every real valued function $f: X \to R$ is almost continuous on a dense subset of X. In fact, in the above characterization of a Baire space, the range space R of real numbers may be generalized to any second countable, Hausdorff space that contains infinitely many points.

1. **Introduction.** In 1966 T. Husain [2] (see, also [3]) introduced the concept of almost continuous mappings and investigated some of their properties. Subsequently, many papers including Lin [5], Long and Carnahan [7], Long and McGehee [8], Singal and Singal [10] and Noiri [9], to name a few, have appeared. Following Husain [2], a mapping $f: X \to Y$, from a topological space to another, is said to be *almost continuous* at $x \in X$ if and only if for each neighborhood V of $f(x)$, $\operatorname{Int} \operatorname{Cl} f^{-1}(V)$ is a neighborhood of x; the function f is *almost continuous* on $A \subset X$, if it is almost continuous at every point $x \in A$. A *Baire space* is a topological space in which the intersection of each countable family of open dense subsets is dense [1], [4], [6].

In a previous paper [5], the first author has proved the following theorem which motivates the present article.

THEOREM 1. *If $f: X \to Y$ is a mapping from a Baire space X to a topological space Y that satisfies the second axiom of countability, then the mapping f is almost continuous on a dense subset of X.*

Proof. See [5].

Working on a converse of Theorem 1, we have come up with the following result.

THEOREM 2. *Let Y be an arbitrary infinite Hausdorff space. If X is a topological space such that every mapping $f: X \to Y$ is almost continuous on a dense subset $D(f)$ of X, then X is a Baire space.*

The proof of Theorem 2 is given in the next section. We observe that by taking the common ground of the range space Y in both Theorems 1 and 2,

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and combining these two theorems, results in the following new characterization of a Baire space.

THEOREM 3. *Let Y be an arbitrary second countable, infinite Hausdorff space. Then a topological space X is a Baire space if and only if every mapping $f: X \to Y$ is almost continuous on a dense subset of X.*

A particularly interesting special case of Theorem 3 is obtained by using the usual space R of real numbers in place of the space Y in Theorem 3. Thus,

THEOREM 4. *A topological space X is a Baire space if and only if every real valued function on X is almost continuous on a dense subset of X.*

2. **Proof of the main theorem.** Before proving Theorem 2, we shall need the following lemma which is taken from Problem 14, page 147 of Long [6].

LEMMA. *Every infinite Hausdorff space contains a countably infinite discrete subspace.*

Proof of Theorem 2. We shall prove, equivalently, that if X is not a Baire space, then there exists a mapping $f: X \to Y$ such that the set D of almost continuity of f is not dense in X. For this purpose, suppose now, on the contrary, that X is a topological space that does not satisfy the condition of Baire. Then, there exists a sequence of dense open sets

$$D_i, D_2, D_3, \ldots$$

such that the intersection $\bigcap_{i=1}^{\infty} D_1$ is not dense in X. Consequently, there exists a nonempty open subset, say U, of X such that

$$U \subset X \sim \bigcap_{i=1}^{\infty} D_1 = \bigcup_{i=1}^{\infty} (X \sim D_i),$$

where $\sim$ denotes the complementation of sets. Notice that each $X \sim D_i$ is nowhere dense in X: For,

$$\text{Int Cl}(X \sim D_i) = \text{Int}(X \sim D_i) = \square \quad \text{(the empty set)},$$

for all i.

Let $U_i = U \sim D_i$. Then $U = \bigcup_{i=1}^{\infty} U_i$. Without losing generality, we may assume that these U_i are pairwise disjoint (and not empty); for, otherwise, we may instead choose

$$U'_1 = U_1, \; U'_n = U_n \sim \bigcup_{i=1}^{n-1} U_i, \quad \text{for all } n$$

and drop the empty ones.

Since the space Y is Hausdorff and containing infinitely many points, by the lemma stated earlier, there exists a countably infinite discrete subspace S of Y which we exhibit as

$$S=\{y_1, y_2, y_3, \ldots, y_n, \ldots\}.$$

We then consider the mapping $f:X\to Y$ defined by:

$$f(x)=\begin{cases} y_{n+1}, & \text{if} \quad x\in U_n \quad \text{for some } n,\\ y_1, & \text{otherwise.}\end{cases}$$

It is readily seen that f is a well-defined mapping. Therefore, by the hypothsis of the theorem, this mapping $f:X\to Y$ is almost continuous on a dense subset $D(f)$ of X. Since the set U is not empty and open, we must have

$$U\cap D(f)\neq\square.$$

Choose an arbitrary fixed point $x_0\in U\cap D(f)$. Then, since $x_0\in U_m$ for some U_m, we have

$$f(x_0)=y_{m+1}.$$

Since S is a discrete subspace of Y, there exists an open neighborhood V_{m+1} of y_{m+1} such that $V_{m+1}\cap S=\{y_{m+1}\}$. Then, since $U_m\subset X\sim D_m$ and U_m is nowhere dense, for any neighborhood V of $f(x_0)$ such that $V\subset V_{m+1}$, $\operatorname{Int}\operatorname{Cl} f^{-1}(V)$ is an empty set, which cannot be a neighborhood of x_0. This shows that f is not almost continuous at $x_0\in D(f)$, a contradiction. Therefore, X is a Baire space.

3. **Open problems.** 1. Let $f:X\to Y$ be a mapping from a Baire space X to a second countable space Y. If f is almost continuous and has a closed graph; that is, the set $\{(x, f(x)) \mid x\in X\}$ is closed in the product space $X\times Y$. Is f necessarily continuous?

2. Do Theorems 1 and 3 remain true without assuming second countability on the range space Y?

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