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A NOTE ON DETERMINING VIABLE ECONOMIC STATES IN A DYNAMIC MODEL OF TAXATION

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Viability theory is the study of dynamical systems that asks what set of initial conditions will generate evolutions that obey the laws of motion of a system and some state constraints, for the length of the evolution. We apply viability theory to Judd's dynamic tax model [The welfare cost of factor taxation in a perfect-foresight model, *Journal of Political Economy* 95(4), 675–709 (1987)] to identify which economic states today are sustainable under only slightly constrained tax-rate adjustments in the future, when the dynamic budget constraint and the consumers' transversality condition at infinity are satisfied. We call the set of such states the economic viability kernel. In broad terms, knowledge of the viability kernel can tell the planner what economic objectives are achievable and assist in the choice of suitable controls to realize them. We observe that high consumption levels can only be sustained when capital is abundant and, unsurprisingly, that a very high consumption economy lies outside such kernels, at least for annual tax-adjustment levels limited by 20 percentage points. Furthermore, we notice that by and large the sizes of the kernel slices do not diminish as the tax rate rises; hence high-taxation economies are not necessarily more prone to explode, or implode, than their low-taxation counterparts. In fact, higher tax rates are necessary to keep many consumption choices viable, especially when capital approaches the constraint-set boundaries.

Keywords: Taxation Policy, Macroeconomic Modeling, Dynamic Systems, Viability Theory, VIKAASA

1. INTRODUCTION

This paper uses viability theory [Aubin et al. (2011)] to examine basic problems in dynamic public finance.[1] For specificity, we use the model studied in Judd (1987).

Viability theory is the study of dynamical systems that asks what set of possible paths obey the system's laws of motion and remain in some state-constraint set. In one example in our paper, we compute the set of possible consumption

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levels today that remains invariant under loose restrictions on tax policy and given a fixed level of government expenditure in the future. Another way of putting this is that we perform a kind of robustness analysis to answer the question "*What are the sustainable consumption levels today if all we know is that tax policy will satisfy the dynamic budget constraint and that consumers' transversality conditions at infinity will be satisfied?*" The usual perfect foresight analysis specifies one future path for taxes. The viability theory approach relaxes this assumption and puts some (loose) restrictions on tax policy. This enables one to ask how much the perfect foresight result depends on having perfect foresight.

For example, suppose that we have some debt today and know the future path of tax rates and government expenditure. Then, under the classical approach, there would (likely) be only one consumption and capital combination that would be *viable*; i.e., only one equilibrium path could originate from this combination. In that case, viability reduces to equilibrium. On the other hand, a viability analysis can establish the set of *all* pairs of consumption and capital (c, k) that represent initial conditions such that there is some future tax-rate path that obeys the restrictions we put on the change in tax rate, and is consistent with equilibrium and with initial conditions (c, k). We assert that the collection of all such initial conditions, which we call the *viability kernel*, generalizes the notion of equilibrium, which is one theme of viability theory.

We find that if the only tax is a proportional income tax, then uncertainty about future tax policy does not affect consumption much. However, in other tax systems, such as one that taxes labor and capital differently, uncertainty about future tax policy may lead to much greater uncertainty about current consumption. In fact, our framework, which enables us to deal with a *slight* constraint on tax-rate adjustments in the future, can be complementary to Guo and Krause (2014)'s, where *loose* government commitment to income taxation is considered. Overall, provided that the model calibration is believable, we contend that studying implications of various taxation regimes in this setting is a useful exercise for politicians and economists.

This paper focuses on some specific questions in a simple dynamic model of expenditure and taxation. However, there is a much more ambitious agenda behind this paper, which is to present viability theory as an important tool for the solution of economic problems.[2] Its main machinery consists of the formulation and solution of differential inclusions. That is, in viability theory the system's dynamics is represented as a *set* of the directions of motion of the system that depend on the state at any moment. The concept of a solution is a path of sets instead of a path of points, where the "tube" formed by these sets is the union of all possible paths that stay in the tube but also satisfy the usual terminal constraints and some additional state restriction. Viability theory is therefore part of set-valued analysis.

Solving viability problems is computationally intensive. However, thanks to some specialized software, solving simple models, of 2–4 state variables and 1–2 controls, is possible. The software we use is VIKAASA [see Krawczyk and Pharo (2011, 2014)].

Here is how the paper is organized. We expound viability theory in Section 2. Following Judd (1987), we introduce a simple model of expenditure and taxation in Section 3. In Section 4, we make the assumption that the only tax charged in this model will be a proportional income tax and calibrate the model according to this assumption. Further, in Section 5, we compute viability kernels and comment on their topology. The paper ends with concluding remarks.

2. A BRIEF ON VIABILITY THEORY AND VIABLE SOLUTIONS

2.1. An Introduction to Viability Theory

Viability theory is a relatively new part of mathematics; see, e.g., Aubin et al. (2011). Viability problems concern systems that evolve over time, where the concern is to identify *viable evolutions*—trajectories that do not violate some set of viability constraints over a given (possibly infinite) time-frame. A *viability domain* is a set of initial states from which viable trajectories originate and the *viability kernel* is the *largest* viability domain. These are the basic tools for analyzing constrained evolutions, also known as viability problems.

The basic feature of the viability kernel is that it provides us with the information necessary to determine whether or not a given state-space position has a viable trajectory proceeding from it, i.e., whether starting at that position, the system can be maintained within its constraints, or not. In what follows, we give a more technical explanation of viability theory, including a formal definition of the *viability kernel*.

The core ingredients of a viability problem are [compare Krawczyk and Pharo (2011)]

(1) A continuum of time[3] values, $\Theta \equiv [0, T] \subseteq \mathbf{R}_+$, where T can be finite or infinite.
(2) A vector of n real-valued state variables, $x(t) \equiv [x_1(t), x_2(t), \ldots, x_n(t)]' \in \mathbf{R}^n$, $t \in \Theta$, that together represent the dynamic system in which we are interested.
(3) A *constraint set*, $K \subset \mathbf{R}^n$, which is a closed set representing some normative constraints to be imposed on these state variables. Violation of these constraints means that the system has become nonviable. Thus in seeking viable trajectories, we want to ensure that $\forall t\ (t \in \Theta)\ x(t) \in K$.
(4) A vector of real-valued controls, $u(t) \equiv [u_1(t), u_2(t), \ldots, u_m(t)]' \in \mathbf{R}^m$, $t \in \Theta$.
(5) Some normative constraints on the controls. In this paper, we assume that $u \in U$, where U is the set of control vectors available in each state. (In general, the set U can depend on x.)

(6) A set of real-valued first-order differential inclusions,

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} \in \left\{ \psi(x,u) = \begin{bmatrix} \psi_1(x,u) \\ \psi_2(x,u) \\ \vdots \\ \psi_n(x,u) \end{bmatrix} \right\}_{u \in U}. \qquad \textbf{(1)}$$

Each function $\psi_i : \mathbf{R}^n \times \mathbf{R}^m \mapsto \mathbf{R}$, $i = 1, 2, \ldots, n$, specifies the range of velocities of the corresponding variable x_i, at the state position $x(t) \in \mathbf{R}^n$, where $u \in U \subset \mathbf{R}^m$ is a control choice available at this position. Some, but not all, inclusions in (1) can be equalities.

Note that we have formulated these viability problems in terms of *differential inclusions* whereby the evolution of some or all of the system's variables is *set-valued*. That is, for a given $x(t)$, we have an array of possible controls U to choose from and hence a *set* of velocities $\psi(x(t), u)$, $u \in U$, associated with state $x(t)$. The symbol ψ denotes a point-to-set map, or correspondence, from states x to velocities $\psi(x, U)$. We will abbreviate the notation and write $\Psi(x)$ instead of $\psi(x, U)$.[4]

Given problem formulation (1), we can attempt to find one or more *viability domains*, $D \subseteq K$, where each viability domain is a set of initial conditions $x(0)$ for which there exist viable trajectories. That is, for every element $x \in D$, $D \subseteq K \subset \mathbf{R}^n$, there must exist a trajectory that originates at x and is a solution to (1) in D. The problem's *viability kernel*, $\mathcal{V} \subseteq K$, is then the *largest* possible viability domain (or the union of all viability domains), giving all initial conditions in K, for which a viable evolution exists.

We will characterize a viability domain using the Viability Theorem from Cardaliaguet et al. (1999) :

PROPOSITION 1. *Assume that D is a closed set in* $\mathbf{R}^N$. *Suppose that* $\psi : \mathbf{R}^N \times U \to \mathbf{R}^N$ *is a continuous function, Lipschitz in the first variable; furthermore, for every x, we define a set-valued map* $\psi(x, U) = \{\psi(x, u); u \in U\}$, *which is assumed to be Lipschitz continuous with convex, compact, nonempty values.*

Then the two following assertions are equivalent:[5]

(i)

$$\forall x \in D, \quad \forall p \in \mathcal{NP}_D(x), \quad \min_u \langle \psi(x,u), p \rangle \leq 0 \qquad \textbf{(2)}$$

(*respectively*, $\max_u \langle \psi(x,u), p \rangle \leq 0$);

(ii) *there exists a function* $u : \Theta \mapsto U$ *such that (respectively, for all such functions) the solution of*

$$\begin{cases} \dot{x}(s) = \psi(x(s), u(s)) \text{ for almost every } s \\ x(t) = x \end{cases} \qquad \textbf{(3)}$$

remains in D.

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To be precise, Proposition 1 merges two results first proved in Veliov (1997) (concerning $\exists\, u$) and in Krastanov (1995) (concerning $\forall\, u$).

Notice that the inequality $\min_u \langle \psi(x, u), p \rangle \leq 0$ in (2) means that there *exists* a control for which the system's velocity $\dot{x}$ "points inside" the set D. Respectively, $\max_u \langle \psi(x, u), p \rangle \leq 0$ means that the system's velocity $\dot{x}$ "points inside" the set D for *all* controls from U.

When (i) or (ii) holds, we say that D is a *viability domain* (or, respectively, an *invariance domain*) for the dynamics Ψ.

This introduces the classical notion of the viability (respectively, invariance) domain [Aubin et al. (2011)], as opposed to viability domains in problems with *targets*; see Quincampoix and Veliov (1998).

DEFINITION 2.1. *Let K be a closed set in $\mathbf{R}^N$. We define the* viability kernel *in K for the dynamics Ψ,*

$$\mathcal{V}_\Psi(K),$$

as the largest closed subset of K, which is a viability domain for Ψ.

It was proved [see, e.g., Quincampoix and Veliov (1998)] that $\mathcal{V}_\Psi(K)$ is the set of x such that there exists $x(\cdot)$, a solution of

$$\dot{x}(s) \in \Psi(x(s)) \tag{4}$$

starting from x, which is defined on $[0, \infty)$ and $x(s) \in K$ for all $s \geq 0$.

If Ψ is the collective vector of right-hand sides, as in (1), then the problem that we want to solve is

$$\text{establish the viability kernel} \quad \mathcal{V}_\Psi(K) \quad \text{for the dynamics } \Psi\,. \tag{5}$$

We will approximate $\mathcal{V}_\Psi(K)$ by looking for solutions to (4).

2.2. A Method for the Determination of Viability Kernels

In Gaitsgory and Quincampoix (2009), we can find a basis for how to approximate $\mathcal{V}_\Psi(K)$ using the solutions to (4). In broad terms, they say that if a constrained optimal control problem, subjected to the system's dynamics $\Psi(\cdot)$ and the constraint set K, can be solved for $x \in K$ and $x(t) \in K\ \forall t$, then x is viable.

VIKAASA[6] is a computational tool that computes viability kernel approximations (actually, domains) for the class of viability problems introduced in Section 2.1, using a user-selected algorithm. In this paper, we have selected one that solves a truncated optimal stabilization problem, rather than a general optimal control problem, for each $x^h \in K^h \subset K$, where K^h is a suitably discretized K.

For each $x^h \in K^h$, VIKAASA assesses whether a dynamic evolution originating at x^h can be controlled to a (nearly) steady state without leaving the constraint set in finite time. Those points that can be brought close enough to such a state are included in the kernel by the algorithm, whereas those that are not are excluded.[7]

In Section 5 we present some results from running the algorithm on the taxation problem, introduced in the next section.

3. THE TAX MODEL

Our goal in this paper is to use viability theory for an analysis of a tax model based on Judd (1987). In that model capital, labor, consumption, debt, marginal utility of consumption, and tax rates are all functions of time. However, to unburden the notation, we will drop the time argument on each of them.

The fundamental law of motion for capital k is determined by net output, i.e., $y - \delta k$, where y is output, $\delta > 0$ is the rate of depreciation, diminished by consumption $c > 0$, and government expenditure is $g \geq 0$. If so and assuming a Cobb–Douglas type production function for output, we get, in continuous time,

$$\frac{dk}{dt} = Ak^{\alpha}\ell^{1-\alpha} - \delta k - c - g\,. \tag{6}$$

As usual, $\ell > 0$ is labor, $A > 0$ is total factor productivity, and $\alpha \in (0,\ 1)$ is output elasticity of capital. In this model, expenditure g is assumed constant, but several values of g will be checked in the computations.

Let the utility of consumption and the disutility of labor of a representative agent be, respectively,

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \qquad v(\ell) = V\frac{\ell^{1+\eta}}{1+\eta}, \tag{7}$$

where V, γ, η are positive. If $\lambda > 0$ is the private marginal value of capital at time t, then it follows from maximization of the utility function $u(c) - v(\ell)$, on an infinite horizon with some discount rate $\rho > 0$ that[8]

$$\frac{d\lambda}{dt} = \lambda(\rho - \bar{r})\,. \tag{8}$$

Here, $\bar{r} = (1-\tau_{K})(\frac{\partial y}{\partial k} - \delta)$ is the after-tax marginal product of capital, where $\tau_{K} \in [0,\ 1]$ is the capital tax. Expanding $\bar{r}$ in (8) yields

$$\frac{d\lambda}{dt} = \lambda\left(\rho - (1-\tau_{K})\left(\alpha A\left(\frac{\ell}{k}\right)^{1-\alpha} - \delta\right)\right)\,. \tag{9}$$

To characterize the economy at hand, we will also use government debt B, which grows in g and diminishes with tax T as follows:

$$\frac{dB}{dt} = \bar{r}B - T + g, \tag{10}$$

where, as before, $\bar{r}$ is the net-of-tax interest rate. In this economy, tax rates on capital and labor are τ_{K} and τ_{L} ($\tau_{L},\ \tau_{K} \in [0,\ 1]$); if so, the expression for total tax T in (10) at time t becomes $T = \tau_K \alpha A k^{\alpha} \ell^{1-\alpha} + \tau_L (1-\alpha) A k^{\alpha} \ell^{1-\alpha} = (\alpha(\tau_K - \tau_L) + \tau_L) A k^{\alpha} \ell^{1-\alpha}$. Combining the last two expressions results in the debt dynamics

$$\frac{dB}{dt} = \bar{r} B - \big(\alpha(\tau_K - \tau_L) + \tau_L\big) A k^{\alpha} \ell^{1-\alpha} + g\,, \tag{11}$$

where $\bar{r}$ will be expanded later. In simple terms, we see that debt can diminish if output is large or if the tax rates are high (and when output is not too small).

Although the private marginal value of capital, λ, can adequately characterize the consumer's behavior, it lacks an easy economic interpretation. We will replace the equation for $\frac{d\lambda}{dt}$, (8), with a differential equation for consumption, easily interpretable.

The marginal utility of consumption [see (7)] is $\frac{du}{dc} = \frac{1}{c^{\gamma}}$; and, on the other hand, λ is the marginal utility of consumption, $\lambda = \frac{du}{dc}$. Hence,

$$c = \frac{1}{\lambda^{1/\gamma}}\,, \tag{12}$$

which, after differentiation in the time domain, yields

$$\frac{dc}{dt} = \frac{-1}{\gamma} \cdot \frac{1}{\lambda^{1+1/\gamma}} \cdot \frac{d\lambda}{dt} = \frac{-1}{\gamma}\, c^{1+\gamma}\, \frac{d\lambda}{dt}\,. \tag{13}$$

Using (9), after some simplifications, we get

$$\frac{dc}{dt} = -\,c \cdot \frac{\rho + \big(\delta - \alpha\, A\, k^{\alpha-1}\, \ell^{1-\alpha}\big)\ (1-\tau_K)}{\gamma}. \tag{14}$$

We can see that consumption has one trivial steady state and will grow if ρ (discount rate) and/or δ (depreciation) are "small."

The three equations of motion (6), (14), (11) jointly constitute the basic representation of the economy at hand, for which we want to establish the viability kernel, i.e., the loci of economic states from which moderate tax adjustments can guarantee a balanced evolution of the economy.

We recognize that this system is nonlinear with multiple steady states. We can see that, as one would expect, the consumption growth or decline can be moderated by adjusting the capital tax rate, whereas debt will (mainly) depend on the labor tax rate. If the rates were identical ($\tau_L = \tau_K$), then increasing them would slow down the consumption rate and diminish debt. With a high taxation rate, consumption and debt will naturally diminish and capital will grow (because labor increases; discussed later). We also notice that debt will grow very fast for large B and nonexcessive capital taxation.

We now want to express labor ℓ through capital and consumption and thus "close" the dynamic system (6), (14), (11).

Let w denote (time-dependent) wages; they equal the marginal product of labor:

$$w = \frac{dy}{d\ell} = \frac{(1-\alpha)k^{\alpha}\,A}{\ell^{\alpha}}. \tag{15}$$

In equilibrium, the marginal utility of consumption weighted by the after-tax wages must be equal to the marginal disutility from labor:

$$\frac{(1-\tau_{L})w}{c^{\gamma}} = \ell^{\eta}V\,. \tag{16}$$

Substituting wages and solving for labor yields

$$\ell = \left(\frac{(1-\tau_{L})(1-\alpha)Ak^{\alpha}}{c^{\gamma}V}\right)^{\frac{1}{\alpha+\eta}}\,, \tag{17}$$

from which we see that labor can be determined by capital and consumption.

We could now use (17) to substitute for labor in (6), (14), (11), but the resulting formulae would appear more complicated than the original equations, even if they contained one variable less. We will not show them here. We will, however, use them in the computations after we have calibrated the equations. Here, we can observe that if $\gamma > \alpha$ then labor decreases in consumption faster than it grows in capital. Allowing for this tells us that the sign of (14) will be negative for high discount and depreciation rates; hence high consumption levels will quickly diminish. High consumption will also contribute to a decline of capital and a rise of debt. However, this multiple downturn may be avoided by an "early" (preemptive) drop of taxes on capital. We will see from which states such a preventive drop can be efficient after we have computed the viability kernel for this economy in Section 5 .

To fully describe the tax model dynamics, the equations (6), (14), and (11) [with (17)] need be completed by two differential inclusions for the two tax rates τ_{L} and τ_{K}:

$$\frac{d\tau_{L}}{dt} = u_{L} \in [-d_{L},\, d_{L}] = U_{L} \quad \text{and} \quad \frac{d\tau_{K}}{dt} = u_{K} \in [-d_{K},\, d_{K}] = U_{K}, \tag{18}$$

where d_{L}, d_{K} are positive numbers. The inclusions represent bounds on the speed at which tax rates can change. This corresponds to the government policy of "smooth" tax rates adjustments determined by d_{L} and d_{K}.

In the current version of the model we will assume that the only tax is a proportional income tax, so the tax rates on labor and capital are equal, i.e., $\tau_{L} = \tau_{K} = \tau$. Therefore, the two inclusions in (18) collapse to

$$\frac{d\tau}{dt} = u \in [-d,\, d] = U, \quad d \geq 0\,. \tag{19}$$

4. MODEL CALIBRATION

We propose that neglecting depreciation will not greatly affect the economic dynamics and so we set δ to zero. Government expenditure g is assumed to be constant. We will construct a couple of different calibrations for the model, each with a different level of government expenditure.

First, we set g at 10% of no-tax steady-state output. We will assume $\rho = 0.04$, $\alpha = 0.3$, $\eta = 1$, and $\gamma = 0.5$, which, in broad terms, characterize a reasonably industrialized economy composed of rational agents interested in the near future [notably, $\exp(-0.04 \cdot 10) = 0.67$ and $\exp(-0.04 \cdot 50) = 0.13$], drawing a fair satisfaction from consumption and feeling, quite strongly, the burden of labor.

We will use a stylized steady state $\underline{k} = \underline{\ell} = 1$ with no taxes and no government expenditure to calibrate A and V. Setting the right-hand sides of (6) and (8) to zero yields

$$A = \underline{c} \quad \text{and} \quad A = \frac{\rho}{\alpha}, \quad \text{hence} \quad A = \underline{c} = 0.1333, \tag{20}$$

where $\underline{c}$ is the no-tax consumption steady state. Then we get from (17) that

$$V = (1-\alpha)\left(\frac{\rho}{\alpha}\right)^{1-\gamma}, \quad \text{hence} \quad V = 0.2556\,. \tag{21}$$

Finally, in our initial calibration, $g = 0.1A = 0.0133$.

As said in Section 2, we also need to set boundaries[9] that the economy should *not* cross. We propose that

(I) *capital* should be between 10% and 200% of no-tax steady state capital stock; i.e., $k \in [0.1,\ 2]$;
(II) *consumption* should range between 1/5 of and 5 times the no-tax steady state consumption $\underline{c}$; i.e., $c \in [0.0267,\ 0.6667]$;
(III) *debt* may be allowed to grow to 150–200% of the maximum steady-state capital stock and also may drop below zero, so, in this study, $B \in [-1,\ 3.5]$;
(IV) *tax rate* $\tau \in [0.\,0.8]$;
(V) *tax-rate adjustment speed*, i.e., the amount by which the regulator can change the current tax-rate level within a year, will be between -20 and 20 percentage points, $u \in [-0.2,\ 0.2]$.

The calibrated system's movements can be learned from Figure 1, which presents vector fields in the *capital–consumption* state space, for no debt, for two different tax levels.

The no-tax, no-government-expenditure steady state is shown as the big dot in the left panel. We observe in each panel that the closer we are to the center, the slower the system will be moving, so, for a large central area of consumption choices, the economy appears stabilizable. We also notice that consumption above 0.2 appears unsustainable in the long run because it causes capital to diminish or vanish quickly. With this observation, we will reduce the top consumption level to 0.225.

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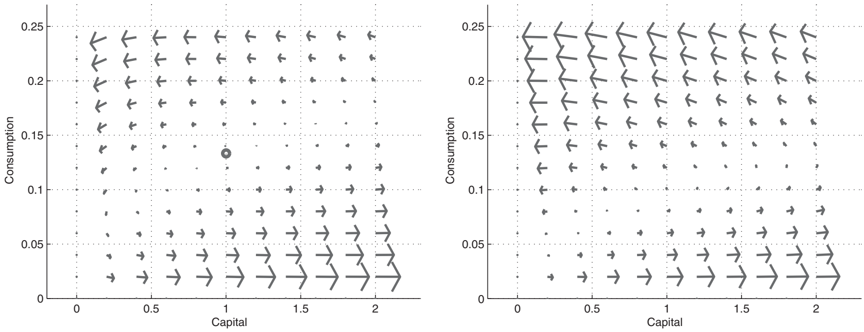

FIGURE 1. k, c-vector fields for $\tau = 0$, $g = 0$, left panel, and $\tau = 0.4$, right panel.

Finally, the constraint set K, for which we will seek the viability kernel, is

$$K = [0.1,\ 2] \times [0.0267,\ 0.225] \times [-1,\ 3.5] \times [0,\ 0.8]\,. \qquad \textbf{(22)}$$

The viability problem is then to determine the kernel $\mathcal{V} \in K \subset \mathbf{R}^4$ for the dynamics $\Psi(\cdot)$ defined through the vector differential inclusion (6), (14), (11), (19) [with (17)]. We will use VIKAASA to compute $\mathcal{V}$.

5. THE VIABILITY KERNEL

We will show several viability kernel *slices* for the following two situations:

- $\overline{B} = 3.5$ and $g = 0.0133$, as introduced in Section 4;
- government expenditure doubles to $g = 0.0266$.

5.1. How to Interpret 3D *Slices* of the 4D Kernel?

Given that $\mathcal{V} \subset K \subset \mathbf{R}^4$, where we cannot display sets, the analysis will be conducted using 3D (sometimes 2D) cross-sections, or "slices," of $\mathcal{V}$.

To analyze the tax policy, we will use 3D slices of the 4D space (k, c, B, τ) where evolutions of the economy "live." The first such slice is shown in Figure 2, left panel. The three dimensions for which the slice is cut are labeled along the respective axes (here: capital, consumption, and tax rate); the fourth dimension is kept constant (here: debt = 1.25). The rectangular box in each figure delimits a 3D projection of $K \subset \mathbf{R}^4$, where K is the constraint set within which the economy is assumed to remain. A 3D body ("boulder") is a snapshot of the viability kernel taken for a particular value of the fourth dimension, written down in the caption or as the figure's title. If there is a line (trajectory) shown in the figure as in the right panel, then each point of this line corresponds to a different value of the fourth dimension; i.e., the 3D line is parameterized in the fourth dimension.

We also note that by the kernel definition,

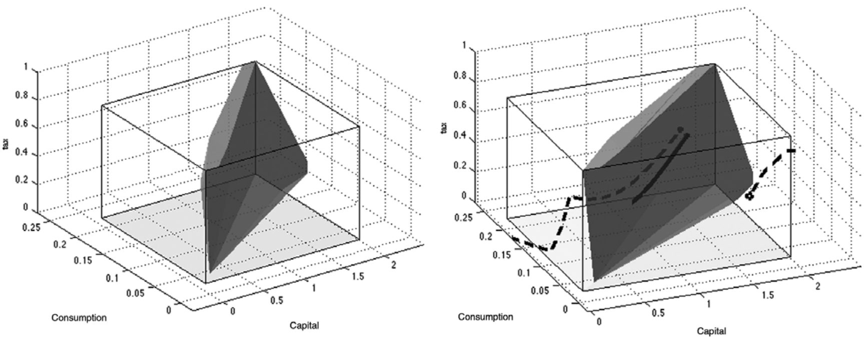

FIGURE 2. Kernel slices for $B = 1.25$.

- for each economic state represented as a point in the boulder, there exists a smooth tax-rate policy ($u \in [-0.2,\ 0.2]$) that maintains the economy in the constraint set K;
- the points outside the boulder are economic states that cannot be controlled by this policy to remain in K.

Obviously, a tax-rate policy that maintains the economy in K also keeps it in $\mathcal{V}$. Henceforth, given the restrictions we put on the change in tax rate, we can apprise where the economy will be in the future even if our knowledge about the economy today is only of debt and capital.

5.2. Maximum Allowable Debt $B = 3.5$

Figure 2 shows two kernel slices for a medium debt level, which differ only in the "camera elevation." We first observe that only the consumption levels that are broadly aligned with the main diagonal in the capital-consumption space are viable. Also, while many consumption choices right of this diagonal are supported by medium taxation rates, there are no taxation rates that would support consumption significantly left of the diagonal. Consumption left of the diagonal leads to decapitalization of the economy.

This is visible from the right panel. Three representative evolutions show what can happen to the economy depending on the "initial" state. If the state is $[1.6833,\ \mathbf{0.2085},\ 1.2500,\ 0.4000] \in \mathcal{V}$, then there are smooth[10] tax-rate strategies, for which the evolution remains contained in $\mathcal{V} \in K \subset \mathbf{R}^4$; see the solid line.

If the evolution starts at $[1.6833,\ \mathbf{0.2250},\ 1.2500,\ 0.4000] \notin \mathcal{V}$, which is left of the diagonal, then even the fastest tax-rate decrease (i.e., $u = -0.2$) cannot prevent the dramatic capital reduction to below its lower bound $k = 0.2$; see the dashed line on the left. However, if the evolution starts at $[1.6833,\ \mathbf{0.0433},\ 1.2500, 0.4000] \notin \mathcal{V}$, which is far to the right of the diagonal, then even the fastest tax-rate growth

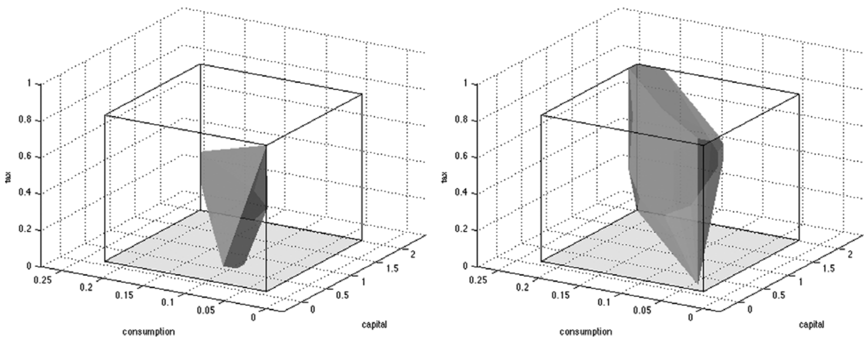

FIGURE 3. Kernel slices for $B = -0.55$, left panel, and $B = 2.6$, right panel.

(i.e., $u = 0.2$) cannot prevent overcapitalization and the economy violates the capital upper bound $k = 2$; see the dashed line on the right.[11]

Furthermore, the Figure 2 slice's projections onto the planes *tax-consumption* and *tax-capital*, not shown but easy to visualize, are almost rectangular. This implies that, for this moderate debt level (i.e., $B = 1.25$), the income-tax-rate "initial" conditions are nonessential for the consumption choices.

Figure 3 shows two kernel slices: for an economy with savings, $B = -0.55$, left panel, and for a high debt economy, $B = 2.6$, right panel. Overall, we notice that whereas the left slice is slanted toward higher consumption, with respect to the position of the medium-debt slice in Figure 2, the right panel slice (high debt) is slanted toward lower consumption (for high taxation rates).

Moreover, the kernel slice for an economy without debt (left panel) appears smallest among the so far analyzed slices. This implies that when the debt level is low, many consumption-capital combinations that could be supported by high taxation in an indebted economy, here would lead to over-savings. (For some policy makers, this probably is not a realistic constraint.) When debt is high (see the right panel), high taxation rates can be applied and more (low) consumption-capital choices are viable.

The slice projections onto the planes of *tax-consumption* and *tax-capital* (not shown) appear less rectangular than those for $B = 1.25$. This implies that, for these debt levels (i.e., $B = -0.55$ and $B = 2.6$), the income-tax-rate "initial" conditions need to be taken into account when the consumption choices are made. This is exemplified in Figure 4, where the slices' cuts are shown for capital $k = 1.525$. The big (light colored) shape is for a high debt economy; the small one (black) is for an economy with savings. We can see how viable consumption choices depend on debt. When the economy has savings there are fewer consumption choices than when the economy is with debt. Visibly, allowing for higher debt allows for higher consumption.

One might ask why it is not "viable" to have even lower consumption than $c = 0.0928$, which is on the left boundary of the high-debt economy slice.

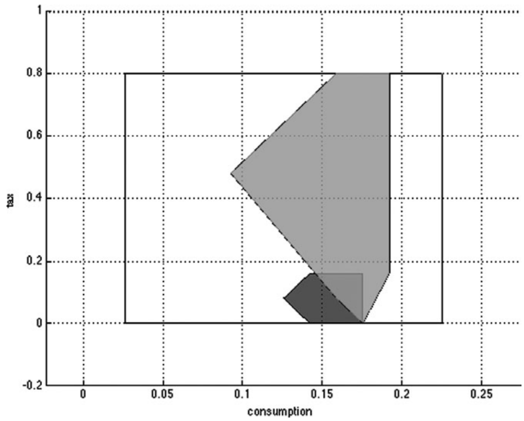

FIGURE 4. Kernel slices for $c = 1.525$ for $B = -0.55$ and 2.6.

In broad terms, the reason is that low consumption *now*, combined with the restrictions that must be satisfied along the *future* path, which include the rate at which future taxes can change, would put the capital accumulation process on an explosive path, which would violate the capital upper bound and TVC infinity[12] (i.e., the transversality condition when the optimization horizon tends to infinity).[13]

Now, we will have a closer look at the impact of tax-rate levels on viable consumption choices. Figure 5 shows two kernel slices for low ($\tau = 0$) and high ($\tau = 0.8$) tax rates; the vertical axis is debt. (We have chosen a different "elevation" for these slices to better illustrate what happens outside the slice.)

As in Figure 2, we see that the kernel slices are aligned with the main diagonal in the consumption-capital space. The slices show that very high debt is nonviable for low taxation and that low debt levels for high taxation are viable only for low consumption and capital.

We show two economic evolutions in this figure. In the left panel we start a viable evolution from [1.683, 0.192, −0.55, **0**]. It starts inside V and stabilizes at medium range values of capital and consumption. The evolution in the right panel begins at $[1.683,\ 0.192,\ -0.55,\ \mathbf{0.8}] \notin \mathcal{V}$ and crashes through the debt lower boundary for any allowable taxation policy. This is so because the tax rate could not drop sufficiently fast to start increasing debt.

Figure 6 shows that high debt levels are incompatible with low tax. (Here again, we see the slice through $\tau = 0$, but the graph "elevation" is different.) Notice two evolutions starting at $[1.05,\ 0.1259,\ \mathbf{0.35},\ 0] \in \mathcal{V}$ and $[1.05,\ 0.1259,\ \mathbf{3.06},\ 0] \notin \mathcal{V}$. Thus, the evolutions start, respectively, from low debt, inside slice, and high debt, outside slice. We see that the high-debt trajectory rises fast in debt and crashes through its upper boundary. This is because the smooth taxation policy cannot generate enough tax to curb the increasing debt. On the other hand, the initially low-debt economy remains almost stationary.

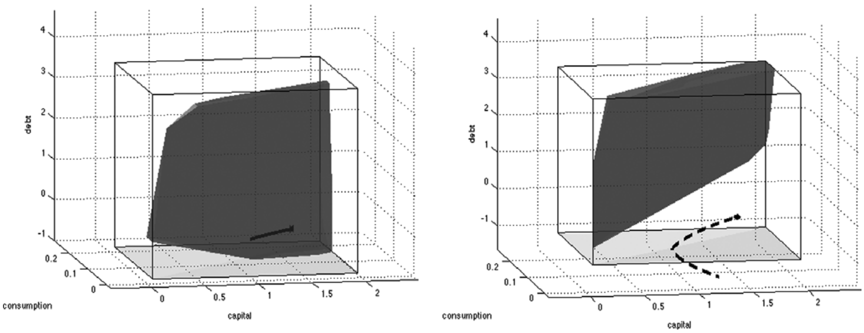

FIGURE 5. Kernel slices for $\tau = 0$ and 0.8.

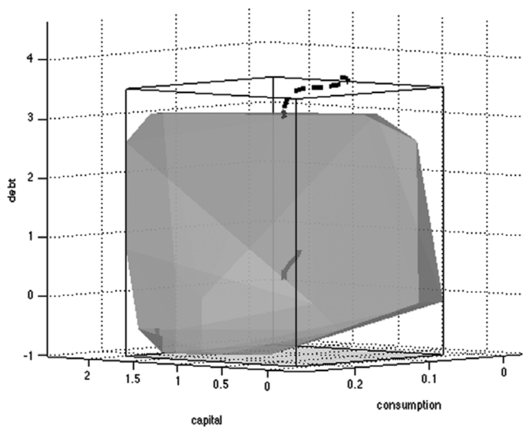

FIGURE 6. Kernel slice for $\tau = 0$.

5.3. Higher Government Expenditure

Here we have computed the kernel when the government expenditure is doubled, so that $g = 0.0266$. The other parameters are as in Section 5.2.

In Figure 7, we observe that even if the kernel slice for $2g$ (in the right panel) appears slightly bigger than in the left, more viable consumption positions above the consumption-capital diagonal are viable when the government expenditure is g. However, for $2g$, more lower consumption choices can be supported by high taxation rates. This means that if the economy is in credit, i.e., $B = -0.55$, increasing the government expenditure can reduce maximum achievable consumption. Indeed, the top consumption in the right panel reaches 0.2085 while it attains 0.25 in the left.

Figure 8 shows the kernel slices for a high-debt economy, $B = 2.6$. The right panel is for the doubled government expenditure, and the left is as in Section 5.2.

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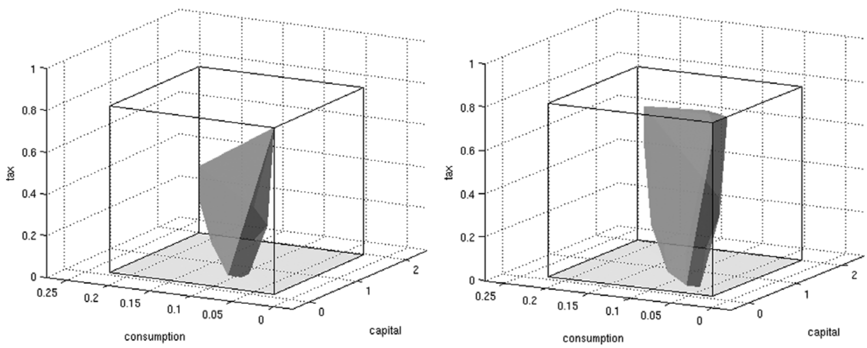

FIGURE 7. Kernel slices for $B = -0.55$. The left panel is as in Figure 3; the right-panel kernel slice is computed for the doubled g.

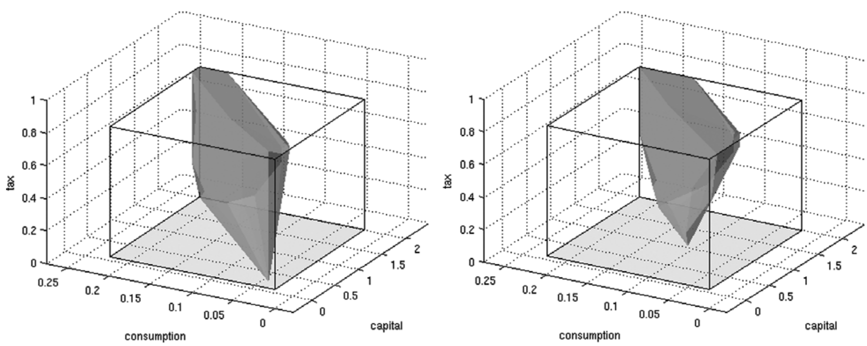

FIGURE 8. Kernel slices for $B = 2.6$. The left panel is as in Figure 3; the right-panel kernel slice is computed for the doubled g.

The right panel's empty space below the slice suggests that low taxation rates cannot be used in conjunction with high government expenditure. Also, fewer consumption choices are viable for this case.

Also, there is a feature of the kernel slice in the right panel, i.e., when the government expenditure is $2g$, which is absent from the left panel: the kernel's projections onto the spaces tax-consumption and tax-capital are less rectangular. In particular, low capital levels (approximately for $k < 0.5$) can be supported only by higher taxes (approximately for $\tau > 0.4$).

6. CONCLUDING REMARKS

We have presented a computational method based on viability theory for the discovery of consumption choices that are compatible with the state variables of the economy at hand. The compatibility means that viable consumption and capital choices will generate a nearly steady-state path for a smooth tax-rate adjustment policy.

Among other findings, we report that increasing government expenditure implies that higher tax rates will be needed to preserve the viability of many consumption choices when capital levels approach the constraint set boundaries.

NOTES

1. This paper draws from Krawczyk and Judd (2012); some extensions are provided in Krawczyk and Judd (2014).

2. So far, viability theory has been applied to a handful of economic and financial problems. For various applications see De Lara et al. (2007)—environmental economics; Pujal and Saint-Pierre (2006)—finance; Krawczyk et al. (2012)—managerial economics; Bonneuil and Boucekkine (2008), Krawczyk and Kim (2009), and Krawczyk and Sethi (2007), Clément-Pitiot and Doyen (1997)—macroeconomics; and Krawczyk and Serea (2013)—microeconomics. However, several of these publications are working papers of limited circulation.

3. A similar formulation could be made for a viability problem in discrete time.

4. In a numerical algorithm commented on in Section 2.2, we seek controls from U for which the trajectories are *viable*, i.e., $x(t) \in K$ for all $t \in \Theta$. For existence and characterization of feedback controls ensuring viability, see Veliov (1993).

5. Here $\mathcal{NP}_D(x)$ denotes the set of *proximal normals* to D at x, i.e., the set of $p \in \mathbf{R}^N$ such that the distance from $x + p$ to D is equal to $||p||$.

6. See Krawczyk and Pharo (2011) and Krawczyk and Pharo (2014); also Krawczyk et al. (2013).

7. This algorithm [called the *inclusion* algorithm; see Krawczyk et al. (2013)], employed by VIKAASA , will miss any viable points that cannot reach a steady state, e.g., because they form (large) "orbits." However, experimenting with the tax model (6), (14), (11), and (19), which consisted of using different discretization grids and trying various controls, did not lead to discovery of a point such as that.

8. Except where stated otherwise, all settings in our model are the same as in Judd (1987), which can also be traced down to Brock and Turnovsky (1981). In particular, the private marginal value of capital λ (or agent's marginal utility of consumption) is the adjoint state in the perfect-foresight household utility $u(c) - v(\ell)$ maximization problem. Part of its specification is a request for the satisfaction of the consumers' transversality condition at infinity. To obtain optimal consumption, it is sufficient to solve the underlying optimal control problem and use (12). Solving the viability problem will tell us which such optimal consumption decisions are compatible with current capital, labor, and a limited-variation (hence only "near-perfect" foresight) tax policy. When we say that the viability kernel is nonempty we imply that the consumers' transversality condition at infinity is fulfilled.

9. In a "real world" calibration, constraints would come from a combination of positive and normative sources, as well as from the requirement to close K. For instance, the lower bound on capital might be tied to a normative requirement concerning the nation's GDP, whereas the upper bound might be based simply on the observation that capital would never realistically fluctuate very far from its steady state. Bounds on consumption, debt, and tax would be similarly determined. In general, normative requirements might be determined through some auxiliary optimization procedure, or they might be externally given (e.g., politically).

10. I.e., $u \in [-0.2, 0.2]$.

11. In Krawczyk and Judd (2014), we have computed the *debt-to-GDP* ratio [see, e.g., Baker et al. (1999)] for each of these evolutions and conjectured that debt-to-GDP ratio cannot be used as a proxy for viability. On the other hand, a viable evolution can imply a diminishing debt-to-GDP ratio.

12. Unless *crisis control* was undertaken; see Cardaliaguet et al. (1999).

13. In Krawczyk and Judd (2014), we display several nonviable evolutions that originate from states that are "slightly" outside $\mathcal{V}$, but in K. We show that despite the use of the fastest drops or increases of the tax rate, they leave K in finite time.

REFERENCES

Aubin, J.P., A.M. Bayen, and P. Saint-Pierre (2011) *Viability Theory: New Directions*, 2nd ed. Berlin: Springer.

Baker, B., L.J. Kotlikoff, and W. Leibfritz (1999) Generational accounting in New Zealand. In A.J. Auerbach, L.J. Kotlikoff, and L. Willi (eds.), *Generational Accounting around the World*, pp. 347–368. Chicago: University of Chicago Press.

Bonneuil, N. and R. Boucekkine (2008) Sustainability, Optimality and Viability in the Ramsey Model. Center for Operations Research and Econometrics, Louvain La Neuve.

Brock, W.A. and S.J. Turnovsky (1981) The analysis of macroeconomic policies in perfect foresight equilibrium. *International Economic Review* 22, 179–209.

Cardaliaguet, P., M. Quincampoix, and P. Saint-Pierre (1999) Set valued numerical analysis for optimal control and differential games. *Stochastic and Differential Games: Theory and Numerical Methods, Ann. Internat. Soc. Dynam. Games* 4, 177–274.

Clément-Pitiot, H. and L. Doyen (1997) Exchange rate dynamics, target zone and viability. Communication, Association Francaise de Science Economique Conference, 18 Sept. Paris.

De Lara, M., L. Doyen, T. Guilbaud, and M.J. Rochet (2007) Is a management framework based on spawning stock biomass indicators sustainable? A viability approach. *ICES Journal of Marine Science* 64(4), 761–767.

Gaitsgory, V. and M. Quincampoix (2009) Linear programming approach to deterministic infinite horizon optimal control problems with discounting. *SIAM Journal on Control and Optimization* 48, 2480–2512.

Guo, J.T. and A. Krause (2014) Optimal dynamic nonlinear income taxation under loose commitment. *Macroeconomic Dynamics* 18, 1403–1427.

Judd, K.L. (1987). The welfare cost of factor taxation in a perfect-foresight model. *Journal of Political Economy* 95(4), 675–709.

Krastanov, M. (1995). Forward invariant sets, homogeneity and small-time local controllability. In *Nonlinear Control and Differential Inclusions*, pp. 298–300. Banach Center publication 32, Polish Academy of Science, Warsaw.

Krawczyk, J.B. and K.L. Judd (2012) Viable Economic States in a Dynamic Model of Taxation. Presented at the 18th International Conference on Computing in Economics and Finance, Prague, Czech Republic.

Krawczyk, J.B. and K.L. Judd (2014) Which Economic States Today Are Sustainable under a Slightly Constrained Tax-Rate Adjustment Policy. Munich Personal RePEc Archive. Available at http://mpra.ub.uni-muenche.de/59207/.

Krawczyk, J.B. and K. Kim (2009) "Satisficing" solutions to a monetary policy problem: A viability theory approach. *Macroeconomic Dynamics* 13, 46–80.

Krawczyk, J.B. and A. Pharo (2011) Viability Kernel Approximation, Analysis and Simulation Application—VIKAASA Manual. SEF working paper, Victoria University of Wellington. Available at http://hdl.handle.net/10063/1878.

Krawczyk, J.B. and A. Pharo (2014) Viability Kernel Approximation, Analysis and Simulation Application—VIKAASA Code. Available at http://code.google.com/p/vikaasa/.

Krawczyk, J.B., A. Pharo, O.S. Serea, and S. Sinclair (2013) Computation of viability kernels: A case study of by-catch fisheries. *Computational Management Science* 10, 365–396.

Krawczyk, J.B. and O.S. Serea (2013) When can it be not optimal to adopt a new technology? A viability theory solution to a two-stage optimal control problem of new technology adoption. *Optimal Control Applications and Methods* 34, 127–144.

Krawczyk, J.B. and R. Sethi (2007) Satisficing Solutions for New Zealand Monetary Policy. Technical report DP2007/03, Reserve Bank of New Zealand. Available at http://www.rbnz.govt.nz/research_and_publications/articles/details.aspx?id=3968.

Krawczyk, J.B., C. Sissons, and D. Vincent (2012) Optimal versus satisfactory decision making: A case study of sales with a target. *Computational Management Science* 9, 233–254.

Pujal, D. and P. Saint-Pierre (2006) Capture Basin Algorithm for Evaluating and Managing Complex Financial Instruments. Presented at the 12th International Conference on Computing in Economics and Finance, Cyprus.

Quincampoix, M. and V.M. Veliov (1998) Viability with a target: Theory and applications. In B. Cheshankov and M. Todorov (eds.), *Applications of Mathematical Engineering*, pp. 47–58. Sofia: Heron Press.

Veliov, V. (1997) Stability-like properties of differential inclusions. *Set-Valued Analysis* 5, 73–88.

Veliov, V.M. (1993) Sufficient conditions for viability under imperfect measurement. *Set-Valued Analysis* 1, 305–317.