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WKB EXPANSIONS FOR HYPERBOLIC BOUNDARY VALUE PROBLEMS IN A STRIP: SELFINTERACTION MEETS STRONG WELL-POSEDNESS

ANTOINE BENOIT

Université du Littoral Côte d'Opale, LMPA, 50 rue Ferdinand Buisson, CS 80699, 62228 Calais, France (antoine.benoit@univ-littoral.fr)

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Abstract In this article we are interested in the rigorous construction of WKB expansions for hyperbolic boundary value problems in the strip $\mathbb{R}^{d-1} \times [0, 1]$. In this geometry, a new inversibility condition has to be imposed to construct the WKB expansion. This new condition is due to selfinteraction phenomenon which naturally appear when several boundary conditions are imposed. More precisely, by selfinteraction we mean that some rays can regenerated themselves after some rebounds against the sides of the strip. This phenomenon is not new and has already been studied in Benoit (Geometric optics expansions for hyperbolic corner problems, I: self-interaction phenomenon, *Anal. PDE* **9**(6) (2016), 1359–1418), Sarason and Smoller (Geometrical optics and the corner problem, *Arch. Rat. Mech. Anal.* **56** (1974/75), 34–69) in the corner geometry. In this framework the existence of such selfinteracting rays is linked to specific geometries of the characteristic variety and may seem to be somewhat anecdotal. However for the strip geometry such rays become generic. The new inversibility condition, used to construct the WKB expansion, is a microlocalized version of the one characterizing the uniform in time strong well-posedness (Benoit, Lower exponential strong well-posedness of hyperbolic boundary value problems in a strip (preprint)). It is interesting to point here that such a situation already occurs in the half space geometry (Kreiss, Initial boundary value problems for hyperbolic systems, *Comm. Pure Appl. Math.* **23** (1970), 277–298).

Keywords: hyperbolic boundary value problems; geometric optics expansions; strip geometry; selfinteraction phenomenon

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Contents

1. Introduction

This article deals with the geometric optics expansion of the following highly oscillating hyperbolic problem in the strip $\mathbb{R}^{d-1} \times [0, 1]$

$$\begin{cases} L(\partial)u^\varepsilon := \partial_t u^\varepsilon + \sum_{j=1}^{d} A_j \partial_j u^\varepsilon = 0 & \text{for } (t, x', x_d) \in \mathbb{R} \times \mathbb{R}^{d-1} \times]0, 1[, \\ B_0 u^\varepsilon_{|x_d=0} = g^\varepsilon & \text{for } (t, x') \in \mathbb{R} \times \mathbb{R}^{d-1}, \\ B_1 u^\varepsilon_{|x_d=1} = 0 & \text{for } (t, x') \in \mathbb{R} \times \mathbb{R}^{d-1}, \\ u^\varepsilon_{|t\leqslant 0} = 0 & \text{for } (x', x_d) \in \mathbb{R}^{d-1} \times [0, 1], \end{cases} \tag{1}$$

where the coefficients in the interior, namely the A_j's, are in $\mathbf{M}_{N\times N}(\mathbb{R})$, the ones on the boundaries, namely B_0 and B_1 are respectively in $\mathbf{M}_{p\times N}(\mathbb{R})$ and $\mathbf{M}_{N-p\times N}(\mathbb{R})$ (the value of p will be made precise in Assumption 2.2). Consequently the solution u^ε of (1) lies in $\mathbb{R}^N$. In (1) the only non-zero source term[1] is on the boundary $\mathbb{R}^{d-1} \times \{0\}$ and is highly oscillating with respect to the parameter $0 < \varepsilon \ll 1$ (we refer to Section 5 for more details about the precise expression of g^ε).

[1]We could also consider problems (1) with a non-zero source term in the interior (and by linearity also on the boundary $\mathbb{R}^{d-1} \times \{1\}$). However, we are here mainly interested in the influence of the boundaries on the behavior of the solution of (1). That is why we decided, in order to simplify the computations, to set homogeneous source term on the boundary $\mathbb{R}^{d-1} \times \{1\}$ or/and in the interior.

The aim of geometric optics expansions is to construct an approximate solution of (1) in the high frequency asymptotic. Then we expect that some qualitative phenomenon can be easily observed on this approximate solution whereas they are not easily readable on the solution of (1).

Before to give some more comments about the strong well-posedness of (1) let us recall some elements about the analogous (well-known) situation in the half space. We consider the following boundary value problem in the half space geometry:

$$\begin{cases} L(\partial)u^{\varepsilon} = f^{\varepsilon} & \text{for } (t, x', x_d) \in \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}_+^*, \\ Bu^{\varepsilon}_{|x_d=0} = g^{\varepsilon} & \text{for } (t, x') \in \mathbb{R} \times \mathbb{R}^{d-1}, \\ u^{\varepsilon}_{|t\leqslant 0} = 0 & \text{for } (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}_+. \end{cases} \tag{2}$$

From the seminal work of [7] it is known that the strong well-posedness (here by strong well-posedness we mean existence, uniqueness and an energy estimate in some weighted (in time) L^2-norm) of (2) is equivalent to the fulfillment of the so-called uniform Kreiss–Lopatinskii condition. Roughly speaking this condition ensures that in the normal mode analysis no stable mode is solution of the homogeneous boundary condition on $\mathbb{R}^{d-1} \times \{0\}$. With more details, the uniform Kreiss–Lopatinskii condition states that for all (time–space) frequency parameter ζ in the normal mode analysis we have the decomposition

$$E^s(\zeta) \oplus \ker B = \mathbb{C}^N, \tag{3}$$

or equivalently that the restriction of B to the stable subspace $E^s(\zeta)$ is an isomorphism.

Then when one wants to construct the geometric optics expansion for (2) (see for example [13]) then he has to impose a microlocalized version of the uniform Kreiss–Lopatinskii condition. To explain this, in a formal setting, let us consider a situation in which a compactly supported interior source term f^{ε} induces some waves traveling (with fixed frequency $\underline{\zeta}$) to the boundary $\mathbb{R}^{d-1} \times \{0\}$. Then by finite speed of propagation arguments these waves will hit the boundary (this kind of traveling waves will be referred as outgoing waves) after a finite travel time and will be reflected back. To determine the reflections that go from the boundary to the interior (they will be referred as incoming waves) one needs to express the new incoming waves in terms of the outgoing ones (at frequency $\underline{\zeta}$) and it is exactly a microlocalized version of the uniform Kreiss–Lopatinskii condition at the frequency $\underline{\zeta}$. Indeed in such a situation incoming waves are elements of $E^s(\underline{\zeta})$ so that (3) microlocalized in $\zeta = \underline{\zeta}$ permits to invert B in the boundary condition.

The first goal of the article is roughly speaking to determine if such a situation also occurs in the strip geometry. That is does some condition (or a microlocalized version of a condition), coming from the characterization of the strong well-posedness of (1) can be observed in the construction of the WKB expansion of (1). This question has already been addressed for hyperbolic boundary value problems in a corner (see [3, 14]).

The second one is to give some properties about the growth in time of the solution of the strip problem. Indeed, for the half space geometry it is known from Kreiss [7] that the solution is lower exponentially increasing with time. Whereas for the strip problem the standard strong well-posedness proof (see bellow) indicates that there is a possible

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exponential growth in time of the solution. The question addressed here is: 'Does the geometric optics expansion show that this growth exists?'. 'If the answer is positive can we give some quantitative results about the rate of growth and can we explain its origin thanks to the geometric optics expansions?'

These aims are motivated by the following points:

- First, as already noticed in [1] the problem in a strip is really close to the one in the quarter space. Indeed the methods used in [1] to deal with the problem in a strip look *really closed* to the ones used in [12] to deal with the problem in the corner geometry. There is however a main difference between these two problems.
  While the characterization of strongly well-posed (uniformly in time) strip problems given in [1] involves matrices, the one used in [12] to characterize strongly well-posed corner problems involves Fourier Integral Operators and thus is much more technical. At present time, in the author's knowledge, the full characterization of strongly well-posed corner problems has not been achieved yet (we refer to [2] for some recent advances). So the better we understand the (simpler) problem in the strip, the more information and/or intuition we can obtain on the corner problem.
- Second the full characterization of lower exponentially strongly well-posed problems described in [1] seems really difficult to verify for a fixed hyperbolic operator. This is due to the fact that this condition requires uniform inversibility (compared with the frequency parameter lying in a half open sphere) of the trace operator. We believe that the simplest way (it was already the case for the uniform Kreiss–Lopatinskii condition [7]) to show these uniform bounds is to have a look at the boundary of the half sphere and to conclude by compactness/continuity arguments. So understand what happens for frequencies of the boundary (which are exactly the ones of the geometric optics regime) shall give some precious intuition to obtain an equivalent characterization of the one described in [1] but which involves the boundary frequency.
- Finally the last motivation for studying the strip problem (1) is linked to numerical simulation of Cauchy problems of waves propagation. Indeed due to the impossibility to model the full space $\mathbb{R}^d$ on a computer when one deals with the numerical simulation of some Cauchy problem, artificial boundary conditions have to be imposed. Consequently the implemented numerical problem is in fact defined in a (possibly) big box. Understand the (discrete) boundary conditions leading to a stable numerical scheme is thus a natural question.
  Historically, for the problem in the half space geometry, the full characterization of the boundary conditions leading to the strong well-posedness of (2) has been obtained before the one of the associated finite difference schemes (see (for example) [4, 5, 7]).
  The study of the stability of finite difference approximations in a strip is a first step in the study of the stability of finite difference approximations in a box. However compared to the continuous problem (2) a finite difference scheme necessitates that all the components of the solution are prescribed by the boundary conditions so that once again compared to the continuous problem some extra boundary conditions have to be imposed.
  The question of the effect(s) of the adding of such extra (and purely arbitrary) boundary conditions on the computed solution is thus natural. Indeed one may

perfectly imagine that the boundary value problem in the strip (1) is strongly well-posed and that its solution does not have any exponential growth compared to time (some that it is uniformly well-posed in time) but that by a bad choice of the extra discrete boundary conditions the computed approximation exhibits such a non-natural growth.

In the author's knowledge this stability question has not be widely considered in the literature (we however refer to [15]). But we believe that it is a question that deserves to be studied in future works. Once again the understanding of the geometric optics regime (which is the one permitting this exponential growth of the solution) should be of precious help for these future works.

About the strong well-posedness of (1) we first observe that from the result of [7], localization and stability by zero order terms arguments it is easy to show that if the strip problem (1) satisfies the uniform Kreiss–Lopatinskii condition on each side then there exists a unique solution u of (1) with bounded exponential growth in time (we refer to Definition 3.1 or to [1] for more details).

For problems with lower exponential growth in time we refer to [1] in which the author gives a new characterization of uniformly strongly well-posed hyperbolic boundary value problems in a strip in terms of the inversibility of some trace operator reading under the form $(I - T(\zeta))$, where $T(\zeta) := T_{1\to 0}(\zeta)T_{0\to 1}(\zeta)$. The operator $T_{0\to 1}(\zeta)$ (respectively $T_{1\to 0}(\zeta)$) takes the value of the trace of the solution on $\mathbb{R}^{d-1} \times \{0\}$ (respectively $\mathbb{R}^{d-1} \times \{1\}$) and gives in output the value of the trace of the solution on $\mathbb{R}^{d-1} \times \{1\}$ (respectively $\mathbb{R}^{d-1} \times \{0\}$).

Let us explain why such a condition is so natural. Consider two decoupled transport equations the first one traveling to the 'right' and the second to the 'left'. Choose boundary conditions in (1) coupling these two transport phenomena together. Intuitively the non-trivial source term g^{ε} induces a wave traveling to the side $\mathbb{R}^{d-1} \times \{1\}$. This wave will be reflected against this side and travel back to $\mathbb{R}^{d-1} \times \{0\}$ and after some time more travel time the same process is repeated periodically in time (this kind of phases will be referred as selfinteracting phases). If we denote by $\mathcal{R}$ the coefficient of amplification during the two reflections needed to regenerate back the first considered transport phenomenon, then intuitively the growth of the source term g^{ε} with respect to time should behave like $\mathcal{R}^t$ and we expect to have exponential growth in time when $|\mathcal{R}| > 1$. The conditions in [1], even if they seem to be somewhat technical, seem to be linked with this simple energy observation (we refer to [1] Paragraph 3.3.3 (first part by 'hand') for more details).

In this article for the construction of the geometric optics expansion for (1) we exhibit the fact that a microlocalized version of one of the conditions of [1] is necessary to initialize the resolution of the cascade of equations. With more details we ask the inversibility of an operator reading $(I - \mathcal{T}(\underline{\zeta}))$ (where $\underline{\zeta}$ is a (micro)-localization of the frequency) on some spaces $H^{\infty}_{\natural,\underline{\gamma}}$ (where $\underline{\gamma}$ stands for the maximal exponential growth in time of the solution). In particular the geometric optic expansions if lower exponentially growing in time if and only if $\underline{\gamma} = 0$ and in this framework we can explicit some results of [1].

An another point of interest is that while one of the inversibility conditions in [1] is asked to hold on the full subspace $E^s(\zeta)$, the one in this article only has to hold on the hyperbolic component of $E^s(\zeta)$). This observation will be explained through this article and is linked to the fact that non-hyperbolic modes are linked to boundary layers so that they do not propagate information from one side to the other. We postpone to § 10 for more details.

The article is organized as follows, § 2 contains some notations and the main assumptions, § 3 describes the main results. In § 4 we give a formal analysis of the phase generation process and in particular we explain in a formal setting why selfinteraction becomes generic in the strip geometry.

The construction of the geometric optics expansion is performed in Sections 5–7 and justified in § 8. This permits to show the main results. As already noticed this construction is made under a new inversibility condition which is studied in more details in § 9.

Finally § 10 contains some examples of application and gives some comments about the obtained results.

2. Notations and definitions

For simplicity we introduce the following notations for the strip and the time/space strip:

$$\Gamma := \mathbb{R}^{d-1}_{x'} \times]0,1[,\ \partial\Gamma_0 := \Gamma \cap \{x_d = 0\},\ \partial\Gamma_1 := \Gamma \cap \{x_d = 1\}$$
$$\Omega := \mathbb{R}_t \times \Gamma,\ \partial\Omega_0 := \mathbb{R}_t \times \partial\Gamma_0 \quad \text{and} \quad \partial\Omega_1 := \mathbb{R}_t \times \partial\Gamma_1.$$

The frequency space and its boundary are defined by:

$$\Xi := \{\zeta := (\sigma = \gamma + i\tau, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1} \setminus \gamma \geqslant 0\} \quad \text{and} \quad \Xi_0 := \Xi \cap \{\gamma = 0\}.$$

In order to state the energy estimates used in this article we define the following weighted Sobolev spaces. Let $s \in \mathbb{N}$, $X \subset \mathbb{R}_t \times \mathbb{R}^d_x$ and $\chi > 0$, the H^s_χ-weighted (in time) Sobolev space is defined by:

$$H^s_\chi(X) := \{u \in \mathscr{D}'(X) \setminus ue^{-\chi t} \in H^s(X)\},$$

equipped with the norm $\|\cdot\|_{H^s_\chi(X)} := \|\cdot e^{-\chi t}\|_{H^s(X)}$. We also denote $H^\infty_\chi(X) := \bigcap_{s\in\mathbb{N}} H^s_\chi(X)$ and finally for $s \in \mathbb{N} \cup \{\infty\}$ we define $H^s_{\flat,\chi}(X)$ as the set of functions of $H^s_\chi(X)$ vanishing for negative times.

In all this article we make the following assumptions on the strip problem (1). The first assumption ensures that the operator $L(\partial)$ is hyperbolic in the following sense:

Assumption 2.1 (Constantly hyperbolic operator). *The system (1) is constantly hyperbolic that is there exist $q \geqslant 1$, real valued analytic functions $\lambda_1, \ldots, \lambda_q$ on $\mathbb{R}^d \setminus \{0\}$ and positive integers $\mu_1, \ldots, \mu_q$ such that:*

$$\forall \xi \in \mathbb{S}^{d-1},\ \det\left(\tau + \sum_{j=1}^{d} \xi_j A_j\right) = \prod_{j=1}^{q} (\tau + \lambda_j(\xi))^{\mu_j},$$

with $\lambda_1(\xi) < \cdots < \lambda_q(\xi)$ and the eigenvalues $\lambda_j(\xi)$ of $\sum_{j=1}^{d} \xi_j A_j$ are semi-simple.

The second one imposes that the boundaries are not characteristics for $L(\partial)$ and that the number of boundary conditions imposed on each side of the boundary gives rise to a well-determined problem.

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Assumption 2.2 (Non-characteristic boundary conditions). *The matrix A_d is invertible. Let p be the number of positive eigenvalues (counted with multiplicity) of A_d then $B_0 \in \mathbf{M}_{p\times N}(\mathbb{R})$ and $B_1 \in \mathbf{M}_{N-p\times N}(\mathbb{R})$.*

With Assumption 2.2 in hand we can perform a Laplace transform in time ($t \leadsto \sigma$) and a Fourier transform in the tangential space variable ($x' \leadsto \eta$) so that (1) reads in the resolvent form:

$$\begin{cases} \dfrac{d}{dx_d}\widehat{u}(\zeta, x_d) = \mathscr{A}(\zeta)\widehat{u}(\zeta, x_d) \quad \text{for } x_d \in]0, 1[, \\ B_0\widehat{u}(\zeta, 0) = \widehat{g}(\zeta), \\ B_1\widehat{u}(\zeta, 1) = 0, \end{cases} \tag{4}$$

in which $\zeta \in \Xi$ acts like a parameter and where the so-called resolvent matrix $\mathscr{A}(\zeta)$ is defined by:

$$\mathscr{A}(\zeta) = A_d^{-1}\left(\sigma I + i\sum_{j=1}^{d-1}\eta_j A_j\right). \tag{5}$$

The following classical result due to Hersh [6] ensures that as soon as the Laplace parameter σ has non-vanishing real part then the elements in the spectrum of $\mathscr{A}(\sigma, \eta)$ are well-separated.

Lemma 2.1. *[6] Under Assumptions 2.1 and 2.2, for all frequency parameter $\zeta \in \Xi \setminus \Xi_0$, the resolvent matrix $\mathscr{A}(\zeta)$ only admits eigenvalues with non-zero real part, and thus does not have purely imaginary eigenvalues. We denote by $E^s(\zeta)$ (respectively $E^u(\zeta)$), the stable (respectively unstable) space of $\mathscr{A}(\zeta)$ that is the eigenspace associated with the negative (respectively positive) real part eigenvalues. Then independently of $\zeta \in \Xi \setminus \Xi_0$, $\dim E^s(\zeta) = p$ and $\dim E^u(\zeta) = N - p$ and we have the following decomposition:*

$$\mathbb{C}^N = E^s(\zeta) \oplus E^u(\zeta). \tag{6}$$

However for $\zeta \in \Xi_0$ then generically Lemma 2.1 is not satisfied anymore because of the possible degeneracy of some real parts of the eigenvalues. In this setting the result allowing to describe the situation is the so-called block structure theorem first shown by [7] for strictly hyperbolic systems and then extended by [10] for constantly hyperbolic systems (see also [11] for systems with non-constant multiplicities).

Theorem 2.1 (Block structure). *Under Assumptions 2.1 and 2.2, for all $\underline{\zeta} \in \Xi$, there exist a neighborhood $\mathscr{V}$ of $\underline{\zeta}$ in Ξ, an integer $L \geqslant 1$, a partition $N = \mu_1 + \cdots + \mu_L$, with $\mu_1, \ldots, \mu_L \geqslant 1$ and an invertible matrix T, regular on $\mathscr{V}$ such that:*

$$\forall \zeta \in \mathscr{V},\ T^{-1}(\zeta)\mathscr{A}(\zeta)T(\zeta) = \operatorname{diag}(\mathscr{A}_1(\zeta), \ldots, \mathscr{A}_L(\zeta))$$

where the blocks $\mathscr{A}_j(\zeta) \in \mathbf{M}_{\mu_j\times\mu_j}(\mathbb{C})$ satisfy one of the following alternatives:

(i) *All the elements in the spectrum of $\mathscr{A}_j(\underline{\zeta})$ have positive real part.*

(ii) *All the elements in the spectrum of $\mathscr{A}_j(\underline{\zeta})$ have negative real part.*

(iii) $\mu_j = 1$, $\mathscr{A}_j(\underline{\zeta}) \in i\mathbb{R}$, $\partial_\gamma \mathscr{A}_j(\underline{\zeta}) \in \mathbb{R} \setminus \{0\}$ *and* $\mathscr{A}_j(\zeta) \in i\mathbb{R}$ *for all* $\zeta \in \Xi_0 \cap \mathscr{V}$.

(iv) $\mu_j > 1$ *and there exists* $k_j \in i\mathbb{R}$ *such that*

$$\mathscr{A}_j(\underline{\zeta}) = \begin{bmatrix} k_j & i & 0 \\ & \ddots & i \\ 0 & & k_j \end{bmatrix},$$

the coefficient in the lower left corner of $\partial_\gamma \mathscr{A}_j(\underline{\zeta}) \in \mathbb{R} \setminus \{0\}$ *and for all* $\zeta \in \Xi_0 \cap \mathscr{V}$, $\mathscr{A}_j(\zeta) \in i\mathbf{M}_{\mu_j \times \mu_j}(\mathbb{R})$.

Consequently Theorem 2.1 permits to give the following decomposition of the boundary of the frequency space.

Definition 2.1. For $\underline{\zeta} \in \Xi_0$ we define:

- $\mathbb{E}$ the elliptic area which is the set of $\underline{\zeta}$ such that Theorem 2.1 is satisfied with blocks of type (i) and (ii) only.
- $\mathbb{EH}$ the mixed area which is the set of $\underline{\zeta}$ such that Theorem 2.1 is satisfied with blocks of type (i), (ii) and at least one block of type (iii).
- $\mathbb{H}$ the hyperbolic area which is the set of $\underline{\zeta}$ such that Theorem 2.1 is satisfied with blocks of type (iii) only.
- $\mathbb{G}$ the glancing area which is the set of $\underline{\zeta}$ such that Theorem 2.1 is satisfied with at least one block of type (iv).

We thus have the following decomposition of Ξ_0:

$$\Xi_0 = \mathbb{E} \cup \mathbb{EH} \cup \mathbb{H} \cup \mathbb{G}. \tag{7}$$

Moreover for all $\underline{\zeta} \in \Xi_0 \setminus \mathbb{G}$ the decomposition (6) still holds and we write:

$$\mathbb{C}^N = E^s(\underline{\zeta}) \oplus E^u(\underline{\zeta}), \tag{8}$$

where $E^s(\underline{\zeta})$ (respectively $E^u(\underline{\zeta})$) is the extension by continuity of $E^s(\zeta)$ (respectively $E^u(\zeta)$) up to the boundary Ξ_0.

Moreover we can decompose these spaces in the following way:

$$E^s(\underline{\zeta}) = E^s_e(\underline{\zeta}) \oplus E^s_h(\underline{\zeta}) \quad \text{and} \quad E^u(\underline{\zeta}) = E^u_e(\underline{\zeta}) \oplus E^u_h(\underline{\zeta}), \tag{9}$$

where $E^s_e(\underline{\zeta})$ (respectively $E^u_e(\underline{\zeta})$) is the generalized eigenspace associated to eigenvalues of $\mathscr{A}(\underline{\zeta})$ with negative (respectively positive) real part and where the $E^s_h(\underline{\zeta})$, $E^u_h(\underline{\zeta})$ are sums of eigenspaces associated to some purely imaginary eigenvalues of $\mathscr{A}(\underline{\zeta})$.

However for $\underline{\zeta} \in \mathbb{G}$ the decomposition (8) does not hold anymore because at glancing frequencies we have $E^s(\underline{\zeta}) \cap E^u(\underline{\zeta}) \neq \{0\}$. In this setting we define the following decompositions of the stable and unstable spaces $E^s(\underline{\zeta})$ and $E^u(\underline{\zeta})$.

$$E^s(\underline{\zeta}) = E^s_e(\underline{\zeta}) \oplus E^s_h(\underline{\zeta}) \oplus E^s_g(\underline{\zeta}) \quad \text{and} \quad E^u(\underline{\zeta}) = E^u_e(\underline{\zeta}) \oplus E^u_h(\underline{\zeta}) \oplus E^u_g(\underline{\zeta}), \tag{10}$$

where $E^s_g(\underline{\zeta})$ and $E^u_g(\underline{\zeta})$ are sum of eigenspaces associated to the Jordan's block(s) of type (iv) of $\mathscr{A}(\underline{\zeta})$ in Theorem 2.1 and consequently satisfying $E^s_g(\underline{\zeta}) \cap E^u_g(\underline{\zeta}) \neq \{0\}$.

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Geometric optics expansions involving glancing frequencies (that is to say frequencies such that $E_g^s(\underline{\zeta}) \cap E_g^u(\underline{\zeta}) \neq \{0\}$) have been studied in the half space geometry by [16, 17]. In these papers, in order to define a bounded projector on $E_g^s(\underline{\zeta})$ associated to the decomposition (10) (which is needed in order to define the boundary layer induced by glancing modes), the author assumes that $E_g^s(\underline{\zeta}) = \bigoplus_{j=1}^{M} G_j^s(\underline{\zeta})$, where for all $j = 1, \ldots, M$, $\dim G_j^s(\underline{\zeta}) = 1$.

Following [7–10], this assumption is equivalent to the fact that Theorem 2.1 is satisfied with block of type (iv) of size at most three. Indeed in this case the contribution in $E_g^s(\underline{\zeta})$ (respectively $E_g^u(\underline{\zeta})$) of one block of type (iv) is one dimensional (respectively one dimensional if the associated block is of size two, two dimensional if the associated block is of size three). Consequently the projector upon $E_g^s(\underline{\zeta})$ remains bounded.

In the following we shall define the projectors on both $E_g^s(\underline{\zeta})$ and $E_g^u(\underline{\zeta})$ (because glancing boundary layers are expected on both sides of the boundary) so that we make the following assumption:

Assumption 2.3. *Let $\underline{\zeta} \in \mathbb{G}$; then Theorem 2.1 is satisfied with blocks of type (iv) of size two only. In this setting we have that there exists $M \in \mathbb{N}$, $M \leqslant \frac{N}{2}$ and $(e_j)_{j=1,\ldots,M} \in \mathbb{C}^N$ such that:*

$$E_g^s(\underline{\zeta}) = E_g^u(\underline{\zeta}) = \bigoplus_{j=1}^{M} G_j(\underline{\zeta}) \quad \textit{where } G_j(\underline{\zeta}) := \mathrm{vect}\{e_j\}.$$

We now give some precisions about the spaces $E_h^s(\underline{\zeta})$, $E_h^u(\underline{\zeta})$, $E_g^s(\underline{\zeta})$ and $E_g^u(\underline{\zeta})$.

Let $i\underline{\xi}_m \in i\mathbb{R}$ be a purely imaginary eigenvalue of $\mathscr{A}(\underline{\zeta})$ (possibly with multiplicity more than two except for glancing modes thanks to Assumption 2.3); then

$$\det\left(\underline{\tau} I + \sum_{j=1}^{d-1} \eta_j A_j + \underline{\xi}_m A_d\right) = 0.$$

From Assumption 2.1 there exists an index k_m such that

$$\underline{\tau} + \lambda_{k_m}(\underline{\eta}, \underline{\xi}_m) = 0,$$

where λ_{k_m} is smooth in both variables. This motivates the following definition:

Definition 2.2. The set of incoming (respectively outgoing) phases for the side $\partial\Gamma_0$ denoted by $\mathscr{I}_0$ (respectively $\mathscr{O}_0$) is the set of indices m such that the group velocity $v_m := \nabla\lambda_{k_m}(\underline{\eta}, \underline{\xi}_m)$ satisfies $\partial_\xi \lambda_{k_m}(\underline{\eta}, \underline{\xi}_m) > 0$ (respectively $\partial_\xi \lambda_{k_m}(\underline{\eta}, \underline{\xi}_m) < 0$).

The set of incoming (respectively outgoing) phases for the side $\partial\Gamma_1$ denoted by $\mathscr{I}_1$ (respectively $\mathscr{O}_1$) is the set of indices m such that the group velocity $v_m := \nabla\lambda_{k_m}(\underline{\eta}, \underline{\xi}_m)$ satisfies $\partial_\xi \lambda_{k_m}(\underline{\eta}, \underline{\xi}_m) < 0$ (respectively $\partial_\xi \lambda_{k_m}(\underline{\eta}, \underline{\xi}_m) > 0$).

The set of glancing phases for the side $\partial\Gamma_0$ (or equivalently for the side $\partial\Gamma_1$) denoted by $\mathscr{G}$ is the set of indices m such that the group velocity $v_m := \nabla\lambda_{k_m}(\underline{\eta}, \underline{\xi}_m)$ satisfies $\partial_\xi \lambda_{k_m}(\underline{\eta}, \underline{\xi}_m) = 0$.

Clearly we have $\mathscr{I}_0 = \mathscr{O}_1$ and $\mathscr{O}_0 = \mathscr{I}_1$. So that in the following we will use the convention that an incoming (respectively outgoing) phase is incoming if it is incoming (respectively outgoing) for the side $\partial\Gamma_0$. Thus, with this convention in mind we set:

$$\mathscr{I} := \mathscr{I}_0 = \mathscr{O}_1 \quad \text{and} \quad \mathscr{O} := \mathscr{O}_0 = \mathscr{I}_1,$$

and for simplicity we also define $\mathscr{H} := \mathscr{I} \cup \mathscr{O}$, the set of indices associated to hyperbolic modes.

With this definition in hand we can give the following description of the spaces $E_h^s(\underline{\zeta})$, $E_h^u(\underline{\zeta})$, $E_g^s(\underline{\zeta})$ and $E_g^u(\underline{\zeta})$.

Lemma 2.2. *For all $\underline{\zeta} \in \Xi_0$ we have:*

$$E_h^s(\underline{\zeta}) = \bigoplus_{k\in\mathscr{I}} \ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \underline{\xi}_k), \quad E_h^u(\underline{\zeta}) = \bigoplus_{k\in\mathscr{O}} \ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \underline{\xi}_k)$$

$$\text{and} \quad E_g^s(\underline{\zeta}) = E_g^u(\underline{\zeta}) = \bigoplus_{k\in\mathscr{G}} \ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \underline{\xi}_k), \tag{11}$$

where $\mathscr{L}$ stands for the symbol of $L(\partial)$ defined for all $\omega = (\omega_0, \ldots, \omega_d) \in \mathbb{R}^{d+1}$ by $\mathscr{L}(\omega) := \omega_0 I + \sum_{j=1}^d \omega_j A_j$.

Consequently for $\underline{\zeta} \in \mathbb{G}$, (10) reads:

$$E^s(\underline{\zeta}) = \bigoplus_{k\in\mathscr{I}} \ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \underline{\xi}_k) \bigoplus_{k\in\mathscr{G}} \ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \underline{\xi}_k) \oplus E_e^s(\underline{\zeta}), \tag{12}$$

$$E^u(\underline{\zeta}) = \bigoplus_{k\in\mathscr{O}} \ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \underline{\xi}_k) \bigoplus_{k\in\mathscr{G}} \ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \underline{\xi}_k) \oplus E_e^u(\underline{\zeta}). \tag{13}$$

We now turn to the definition of the uniform Kreiss–Lopatinskii condition which is the condition ensuring the strong well-posedness of the boundary value problem in the half space (see [7]). It is not difficult to show (and to be convinced) that the strong well-posedness of (1) requires that each boundary condition on $\partial\Gamma_0$ and $\partial\Gamma_1$ satisfies the uniform Kreiss–Lopatinskii condition. So that in the WKB expansion construction we should assume that these conditions hold. More precisely we assume the following

Assumption 2.4 (Uniform Kreiss–Lopatinskii condition). *Under Assumptions 2.1 and 2.2 let $\zeta \in \Xi$ and as previously we still denote by $E^s(\zeta)$ (respectively $E^u(\zeta)$) the extension by continuity of $E^s(\zeta)$ up to $\zeta \in \Xi_0$ of the well-defined stable (respectively unstable) subspace of $\mathscr{A}(\zeta)$. Then each of the boundary $\partial\Gamma_0$ and $\partial\Gamma_1$ satisfies the uniform Kreiss–Lopatinskii condition that is to say that:*

$$\forall \zeta \in \Xi,\ \ker B_0 \cap E^s(\zeta) = \ker B_1 \cap E^u(\zeta) = \{0\}.$$

In other words, the restriction of B_0 (respectively B_1) to $E^s(\zeta)$ (respectively $E^u(\zeta)$) is invertible and we denote its inverse by $\phi_0(\zeta) := B_{0|E^s(\zeta)}^{-1}$ (respectively $\phi_1(\zeta) := B_{1|E^u(\zeta)}^{-1}$).

We conclude this section by defining some projectors that will be useful in the construction the WKB expansion.

Definition 2.3 (Interior projectors). Under Assumptions 2.1 and 2.2 for $\underline{\zeta} = i\underline{\tau} + \underline{\eta} \in \Xi_0$ we define:

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- $\Pi_e^s := \Pi_e^s(\underline{\zeta})$ (respectively $\Pi_e^u := \Pi_e^u(\underline{\zeta})$) the spectral projector on $E_e^s(\underline{\zeta})$ (respectively $E_e^u(\underline{\zeta})$).
- For $k \in \mathscr{H} \cup \mathscr{G}$, $\Pi^k := \Pi^k(\underline{\zeta})$ the orthogonal projector on $\ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \underline{\xi_k})$.
- For $k \in \mathscr{H} \cup \mathscr{G}$, we define $\Upsilon^k := \Upsilon^k(\underline{\zeta})$ the partial inverse of $\mathscr{L}(\underline{\tau}, \underline{\eta}, \underline{\xi_k})$ characterized by the relations:

$$\begin{cases} \Upsilon^k \mathscr{L}(\underline{\tau}, \underline{\eta}, \underline{\xi_k}) = I - \Pi^k, \\ \Upsilon^k \Pi^k = \Pi^k \Upsilon^k = 0. \end{cases} \tag{14}$$

Definition 2.4 (Boundary projectors). Under Assumptions 2.1, 2.2 and 2.3 for $\underline{\zeta} = i\underline{\tau} + \underline{\eta} \in \Xi_0$ we define:

- $P_e^s := P_e^s(\underline{\zeta})$ (respectively $P_e^u := P_e^u(\underline{\zeta})$) the projector on $E_e^s(\underline{\zeta})$ with respect to (12) (respectively (13)).
- For $k \in \mathscr{I}$ (respectively $k \in \mathscr{O}$), $P_h^k := P_h^k(\underline{\zeta})$ the projector on $\ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \underline{\xi}_k)$ with respect with the sums (12) (respectively(13)).
- For $k \in \mathscr{G}$ we define $P_{g,s}^k := P_{g,s}^k(\underline{\zeta})$ (respectively $P_{g,u}^k := P_{g,u}^k(\underline{\zeta})$) the projector on $\ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \underline{\xi}_k)$ with respect with the sum (12) (respectively (13)).

3. Main results

In this Section we state the main results of this paper but in order to do so we need to give some more details about the strong well-posedness of the strip problem (1).

Definition 3.1. Let $f \in L^2(\Omega)$, $g_0 \in L^2(\partial\Omega_0)$ and $g_1 \in L^2(\partial\Omega_1)$ be given source terms. The hyperbolic boundary value problem in the strip Γ

$$\begin{cases} L(\partial)u = f & \text{in } \Omega, \\ B_0 u_{|x_d=0} = g_0 & \text{on } \partial\Omega_0, \\ B_1 u_{|x_d=1} = g_1 & \text{on } \partial\Omega_1, \\ u_{|t\leqslant 0} = 0 & \text{on } \Gamma, \end{cases}$$

is said to be strongly well-posed if it admits a unique solution $u \in L^2(\Omega)$ with traces $u_{|x_d=0} \in L^2(\partial\Omega_0)$ and $u_{|x_d=1} \in L^2(\partial\Omega_1)$ satisfying the energy estimate that there exist $C > 0$ and $\gamma_0 \geqslant 0$ such that for all $\gamma > \gamma_0$:

$$\begin{aligned} &\gamma \|u\|^2_{L^2_\gamma(\Omega)} + \|u_{|x_d=0}\|^2_{L^2_\gamma(\partial\Omega_0)} + \|u_{|x_d=1}\|^2_{L^2_\gamma(\partial\Omega_1)} \\ &\quad \leqslant C\left(\frac{1}{\gamma}\|f\|^2_{L^2_\gamma(\Omega)} + \|g_0\|^2_{L^2_\gamma(\partial\Omega_0)} + \|g_1\|^2_{L^2_\gamma(\partial\Omega_1)}\right). \end{aligned} \tag{15}$$

In the particular setting where $\gamma_0 = 0$ the strip problem is said to be lower exponentially strongly well-posed.

As mentioned in the introduction, there exists $\gamma_0 > 0$ such that the strip problem is strongly well-posed in the sense of Definition 3.1 only requires the

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uniform Kreiss–Lopatinskii condition on each side of the boundary so that under Assumptions 2.1, 2.2 and 2.4 the strip problem (1) is automatically strongly well-posed in the sense of Definition 3.1.

The question of the lower exponential strong well-posedness of (1) is studied in [1]. In this article the author describes a particular framework, namely the one of strictly dissipative boundary conditions in which the lower exponential strong well-posedness of (1) as well as a full characterization of lower exponentially strongly well-posed problems. More precisely this characterization asks the inversibility of some trace operators reading under the form $(I - \mathscr{T}(\zeta))$ and $(I - \widetilde{\mathscr{T}}(\zeta))$: on the stable subspace $E^s(\zeta)$ for $(I - \mathscr{T}(\zeta))$ and on $\ker B_0$ for $(I - \widetilde{\mathscr{T}}(\zeta))$. This inversibility is asked to be uniform with respect to the frequency parameter $\zeta \in \Xi$. That is we have that there exists $C, \widetilde{C} > 0$ such that for all $\zeta \in \Xi$

$$\forall u \in E^s(\zeta), \quad |u| \leqslant C|(I - \mathscr{T}(\zeta))u|, \tag{16}$$

$$\forall v \in \ker B_0, \quad |v| \leqslant \widetilde{C}|(I - \widetilde{\mathscr{T}}(\zeta))v|, \tag{17}$$

where we stress that C and $\widetilde{C}$ do not depend on ζ. The precise expressions of $\mathscr{T}(\zeta)$ and $\widetilde{\mathscr{T}}(\zeta)$ can be found in [1] but we will also give it in Paragraph 9 in order to compare $\mathscr{T}(\zeta)$ with $\mathcal{T}^\varepsilon(\underline{\zeta})$. However these expressions are of little interest for the justification of the WKB expansion which only requires the (lower exponential) strong well-posedness of (1).

We sum up the known results about the strong well-posedness of (1) in the following Theorem.

Theorem 3.1 (Strong well-posedness of (1)). • *Under Assumptions 2.1, 2.2 and 2.4 there exists $\gamma_0 \geqslant 0$ such that the strip problem (1) is strongly well-posed in the sense of Definition 3.1.*

- *[1] Under Assumptions 2.1, 2.2 also assume that the matrices A_j are symmetric for all $j = 1, \dots, d$ and that the boundary conditions on $\partial\Gamma_0$ and $\partial\Gamma_1$ are strictly dissipative that is there exist $C_0, C_1, \varepsilon_0, \varepsilon_1 > 0$ such that*

$$\forall u \in \mathbb{R}^N \ \varepsilon_0|u|^2 + \langle A_d u, u\rangle - C_0|B_0 u|^2 < 0 \quad \text{and} \quad \varepsilon_1|u|^2 + \langle A_d u, u\rangle - C_1|B_1 u|^2 > 0, \tag{18}$$

then the strip problem (1) is lower exponentially strongly well-posed in the sense of Definition 3.1.

- *[1] Under Assumptions 2.1, 2.2, 2.4 also assume that the matrices A_j are symmetric for all $j = 1, \dots, d$ and that $\ker B_0 \cap \ker B_1 = \{0\}$. Then the strip problem (1) is lower exponentially strongly well-posed in the sense of Definition 3.1 if and only if the inversibility conditions (16) and (17) hold.*

The main results of this article are the following.

The first ones show the existence and the convergence of the geometric optics expansions in the case where the considered frequency is or is not a glancing frequency. These results hold under a new assumption (namely Assumption 6.1). Without going into details this assumption imposes a maximal growth in time for some operator.

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Theorem 3.2. *Under Assumptions 2.1, 2.2 and 2.4 also assume that (1) is strongly well-posed in the sense of Definition 3.1 for some* $\underline{\gamma}_0 \geqslant 0$ *with* $0 \leqslant \underline{\gamma}_0 \leqslant \gamma_0$*, that* $\mathscr{G} = \emptyset$ *and finally that Assumption 6.1 holds for* γ_0*. Then for all* $N_0 \in \mathbb{N}$ *there exists an approximate solution* $u^\varepsilon_{app,N_0} \in L^2_\gamma(\Omega)$ *(see (75) for a precise definition) of* u^ε *the solution of (1) in the sense that: there exists* $C > 0$ *such that for all* $\gamma > \gamma_0$

$$\|u^\varepsilon - u^\varepsilon_{app,N_0}\|_{L^2_\gamma(\Omega)} \leqslant C\varepsilon^{N_0+1}. \tag{19}$$

Theorem 3.3. *Under Assumptions 2.1, 2.2, 2.3 and 2.4 also assume that (1) is strongly well-posed in the sense of Definition 3.1 for some* $\gamma_0 \geqslant 0$ *and that Assumption 6.1 holds for* $\underline{\gamma}_0$ *with* $0 \leqslant \underline{\gamma}_0 \leqslant \gamma_0$*. Then there exists an approximate solution* $u^\varepsilon_{app,glan} \in L^2_\gamma(\Omega)$ *(see (76) for a precise definition) of* u^ε *the solution of (1) in the sense that: there exists* $C > 0$ *such that for all* $\gamma > \gamma_0$

$$\|u^\varepsilon - u^\varepsilon_{app,glan}\|_{L^2_\gamma(\Omega)} \leqslant C\varepsilon^{1/4}.$$

Before to give a formal justification of Theorems 3.2 and 3.3 let us give some more details about these results.

First remark that Theorem 3.2 gives a better approximation than Theorem 3.3 because the difference $u^\varepsilon - u^\varepsilon_{app,N_0}$ is $O(\varepsilon^{N_0+1})$ where N_0 is arbitrary while the maximal rate of convergence in Theorem 3.3 is $O(\varepsilon^{1/4})$. This is due to the fact that when glancing modes exist we are only able to construct one corrector while if they are not present we can define an arbitrary number of correctors (see [16, 17]).

Second about Assumption 6.1 appearing in Theorems 3.2 and 3.3. It asks the inversibility of some operator, reading under the form $(I - \mathcal{T}^\varepsilon(\underline{\zeta}))$ (we refer to (47) for a precise definition), to initialize the resolution of the WKB expansion. Crudely speaking the operator $\mathcal{T}^\varepsilon(\underline{\zeta})$ encodes the amplifications of the hyperbolic components of the WKB expansion by successive reflections against the sides $\partial\Gamma_0$ and $\partial\Gamma_1$.

It is interesting to remark that the operator $\mathcal{T}^\varepsilon(\underline{\zeta})$ is a microlocalized version of the operator $\mathscr{T}(\zeta)$ at the frequency $\zeta = \underline{\zeta}$. Note that it was already the case for the uniform Kreiss–Lopatinskii condition in the half space geometry.

Some more details about Assumption 6.1 are given in § 9 in which we show in particular that this assumption is always satisfied for a large enough threshold $\underline{\gamma}_0$. More precisely in § 9 the given sufficient condition for Assumption 6.1 to hold is that the sum of the coefficients of amplification during a complete circuit of reflection is bounded by $e^{\alpha\underline{\gamma}_0}$ for some strictly positive α encoding the time needed to perform a complete circuit. This result agrees with the intuition that if during a complete circuit of reflection the energy increases then the geometric optics expansion (and thus so do u^ε) has an exponential growth in time.

Finally, Assumption 2.3 used in Theorem 3.3 was concerned. This assumption states that all glancing modes are of size two and is used in a crucial way to define the projectors on $E^s_g(\underline{\zeta})$ and $E^u_g(\underline{\zeta})$ as explained in § 2. However, let us stress that Assumption 2.3 is not necessary to show the strong well-posedness (and even the lower exponential strong well-posedness) of (1) (which essentially require the uniform Kreiss–Lopatinskii on each side and possibly the inversibility conditions described in [1]).

Thus, in the author's opinion, Theorem 3.3 should also hold without Assumption 2.3[2]. However, the literature does not provide any information about the construction of geometric optics expansions without this size restriction on the glancing modes. Indeed, the only papers constructing these expansions are, in the author's knowledge, due to Williams [16, 17].

That is why we made this technical assumption to deal with glancing modes (and to define the associated boundary layers). Indeed concerning glancing modes we were not really interested in obtaining the most general possible theorem but rather in showing that Assumption 6.1 used to initialize the resolution of the cascade of equations does not involve glancing modes. More comments about this observation are made in Paragraph 10.2. The study of geometric optics expansion with glancing modes of size greater than two is postponed to future studies.

The main point in the proof of Theorems 3.2 and 3.3 is the construction of a geometric optics expansion. The existence of such an expansion is given by the following Theorems which are demonstrated in Sections 5–7:

Theorem 3.4. *Under Assumptions 2.1, 2.2 and 2.4 also assume that $\mathscr{G} = \emptyset$ and that Assumption 6.1 holds for some $\underline{\gamma}_0 \geqslant 0$, then for all $n \in \mathbb{N}$, for all $k \in \mathscr{H}$ there exist $u^{\varepsilon}_{h,n,k} \in H^{\infty}_{\natural,\gamma}(\Omega)$ for all $\gamma > \underline{\gamma}_0$ and $U^{\varepsilon}_{ev,n} \in \mathcal{P}_{ev}$, $U^{\varepsilon}_{ex,n} \in \mathcal{P}_{ex}$ satisfying the cascades of equations (28), (31), (32) and (33).*

Theorem 3.5. *Under Assumptions 2.1–2.4 also assume that Assumption 6.1 holds for some $\underline{\gamma}_0 \geqslant 0$, then for all $n = 0, 1$, for all $k \in \mathscr{H}$ there exist $u^{\varepsilon}_{h,n,k} \in H^{\infty}_{\natural,\gamma}(\Omega)$ for all $\gamma > \underline{\gamma}_0$, $U^{\varepsilon}_{ev,n} \in \mathcal{P}_{ev}$, $U^{\varepsilon}_{ex,n} \in \mathcal{P}_{ex}$ and for all $k \in \mathscr{G}$, $u^{\varepsilon}_{g,n,k} \in H^{\infty}_{\natural,\gamma}(\Omega)$ for all $\gamma > \underline{\gamma}_0$ the cascades of equations (28), (31), (32) and (33) written for $n = 0, 1$.*

4. Formal analysis

In this paragraph, we give a formal analysis describing the phases appearing in the WKB expansion as well as the selfinteraction phenomenon between the oscillating ones.

As the reader will notice, in comparison with the expansions for the quarter space geometry (see for example [3]), on the one hand the phase generation process in the strip geometry will not be richer than the one in the half space geometry. Indeed the number of generated phases will be the same as the one for the problem in the half space. This was not the case for the corner problem for which the number of considered phases was generically greater than the one in the half space (this number can even be infinite).

However on the other hand, the selfinteraction phenomenon (meaning that a phase can regenerate itself after a suitable number of rebounds against the sides of the domain of resolution), which can be seen as something somewhat anecdotal in the quarter space geometry (because it requires strong constraints on the geometry of the characteristic variety) becomes generic in the strip geometry. Indeed an incoming phase coming from

[2]Possibly with a rate of convergence which depends on the size of the glancing mode and which decreases as the glancing mode's size increases.

the side $\partial\Gamma_0$ will always be reflected back against the side $\partial\Gamma_1$ and will always regenerate itself after two reflections.

4.1. Source term induced phases

The first point of our discussion is to determine the source term induced phases. Note that the system (1) is hyperbolic, so that it satisfies the finite speed of propagation property, and that the only non-trivial information in (1) lies on the side $\partial\Gamma_0$. Consequently, this information cannot hit the side $\partial\Gamma_1$ immediately and we can (in a formal setting and at least during a short time) neglect the boundary condition on the side $\partial\Gamma_1$. By doing this we shall consider the following system of equations:

$$\begin{cases} L(\partial)u^\varepsilon = 0 & \text{for } (t, x', x_d) \in]-\infty, T] \times \mathbb{R}^{d-1} \times \mathbb{R}_+^*, \\ B_0 u^\varepsilon_{|x_d=0} = g^\varepsilon & \text{for } (t, x') \in]-\infty, T] \times \mathbb{R}^{d-1}, \\ u^\varepsilon_{|t\leqslant 0} = 0 & \text{for } (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}_+, \end{cases} \tag{20}$$

for $T > 0$ (possibly small). It is thus natural to choose for ansatz the one for the problem in the half space (20). More precisely if g^ε reads

$$g^\varepsilon(t, x') := e^{\frac{i}{\varepsilon}(t,x')\cdot(\underline{\tau},\underline{\eta})} g(t, x'),$$

where the amplitude $g \in H^\infty_\natural(\partial\Omega_0)$ is given and where the frequency parameters $\underline{\tau} \in \mathbb{R}$, $\underline{\eta} \in \mathbb{R}^{d-1}$ are fixed then the ansatz associated to (20) reads:

$$u^\varepsilon_{app} \approx \sum_{k=1}^{K} e^{\frac{i}{\varepsilon}((t,x')\cdot(\underline{\tau},\underline{\eta})+x_d\xi_k)} u^\varepsilon_k(t, x), \tag{21}$$

where $u^\varepsilon_k(t, x) := \sum_{n\geqslant 0} \varepsilon^n u_{n,k}(t, x)$, the $u_{n,k}$ are unknown amplitudes lying in some profile space. Moreover in (21) the terms ξ_k are roots in the ξ variable of the so-called dispersion relation $\mathbf{det}\,\mathscr{L}(\underline{\tau}, \underline{\eta}, \xi) = 0$ where we recall that $\mathscr{L}$ stands for the symbol of $L(\partial)$.

The behavior of the u^ε_k in (21) is thus given by the kind of phase that we are considering. That is to say that it depends on ξ_k and we have to discuss several cases:

⋄ $\xi_k \in \mathbb{C}$, $\mathrm{Im}\,\xi_k \neq 0$. In this case the factor $e^{\frac{i}{\varepsilon}((t,x')\cdot(\underline{\tau},\underline{\eta})+x_d\xi_k)}$ has a (real) exponential behavior with respect to the sign of $\mathrm{Im}\,\xi_k$. More precisely:

• $\mathrm{Im}\,\xi_k > 0$ (*evanescent for the side* $\partial\Gamma_0$). In this subcase the factor $e^{\frac{i}{\varepsilon}((t,x')\cdot(\underline{\tau},\underline{\eta})+x_d\xi_k)}$ induces an exponential decrease with respect to the normal variable x_d. The associated amplitude has exponential decrease so that when it hits the side $\partial\Gamma_1$ its contribution is $O(\varepsilon^\infty)$ with respect to ε and it will not contribute to the boundary condition on $\partial\Gamma_1$. Consequently it will not be reflected back.

• $\mathrm{Im}\,\xi_k < 0$ (*explosive for the side* $\partial\Gamma_0$). In this subcase the factor $e^{\frac{i}{\varepsilon}((t,x')\cdot(\underline{\tau},\underline{\eta})+x_d\xi_k)}$ induces an exponential growth with respect to the normal variable x_d. As in the half space geometry we decide, to simplify the discussion, to **initially** neglect these amplitudes in the ansatz (21) (recall that we are interested in solutions lying in $L^2_\gamma(\Omega)$ for some $\gamma > \gamma_0 \geqslant 0$).

⋄ $\xi_k \in \mathbb{R}$. In this case the factor $e^{\frac{i}{\varepsilon}((t,x')\cdot(\underline{\tau},\underline{\eta})+x_d\xi_k)}$ induces an oscillating behavior. Moreover Lax's lemma [8] should apply and the leading order term in the terms u^ε_k,

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namely the main amplitudes $u_{0,k}$ are expected to solve the transport equations:

$$\partial_t u_{0,k} + v_k \cdot \nabla_x u_{0,k} = 0, \tag{22}$$

where the velocity v_k is the so-called group velocity for ξ_k (we refer to § 2, Definition 2.2 for a precise definition). Depending on the sign of $v_{k,d}$ the transport equation (22) has to be completed by some boundary conditions. This leads us to the following study of subcases:

• $v_{k,d} < 0$ (*outgoing for the side* $\partial\Gamma_0$). In this subcase the transport in the equation (22) is made from the 'right to the left'. So that the transported informations can be the ones in the interior or the ones on the side $\partial\Gamma_1$. But in (1) these source terms are chosen to be zero. Consequently, $u_{0,k}$ is zero and this amplitude is **initially** neglected in (21).

• $v_{k,d} = 0$ (*glancing for the side* $\partial\Gamma_0$). In this subcase the transport equation (22) reads

$$\partial_t u_{0,k} + v'_k \cdot \nabla_{x'} u_{0,k} = 0, \tag{23}$$

equation which does not require any boundary condition on $\partial\Gamma_0$ or on $\partial\Gamma_1$. The only transportable information is the one in the interior, it is zero, so that the associated amplitude $u_{0,k}$ **shall** be zero and **shall** be neglected in (21).

However, to solve the boundary conditions for the WKB expansion of (1), with a suitable error, it will be necessary to consider a boundary layer (around $\partial\Gamma_0$) for $u_{0,k}$. We refer to [16] and [17] for more details. Consequently, the $u_{0,k}$ **are not neglected** in (21). However, due to the special form of the transport equation (23), this boundary layer cannot be propagated to the side $\partial\Gamma_1$, it will not be reflected against this side and will not contribute to the boundary condition on $\partial\Gamma_1$.

• $v_{k,d} > 0$ (*incoming for the side* $\partial\Gamma_0$). Finally in this subcase the transport is made from the 'left to the right'. Consequently, the non-trivial information on the side $\partial\Gamma_0$, is transported. The associated amplitude $u_{0,k}$ is not zero, it is not neglected in the ansatz (21). Moreover, this non-trivial information will hit the side $\partial\Gamma_1$ after some travel time. It will be reflected and we have to determine it(s) reflection(s). It is the aim of the following paragraph.

In conclusion, the source term induced phases are the glancing ones, the incoming ones and the evanescent ones. Only the incoming ones spread some non-trivial information from the side $\partial\Gamma_0$ to the side $\partial\Gamma_1$ and only their reflections have to be considered. The situation is summarized in Figure 1.

4.2. The first reflection

We assume that there exists at least one incoming phase[3], that is that $\det \mathscr{L}(\underline{\tau}, \underline{\eta}, \xi) = 0$ admits at least one root $\xi_{\underline{k}}$ such that the associated group velocity $v_{\underline{k}}$ satisfies $v_{\underline{k},d} > 0$.

We have justified in the previous paragraph that the amplitude $u_{0,\underline{k}}$, after some travel time, induces a non-trivial information on the side $\partial\Gamma_1$. Once again by finite speed of propagation arguments, this information cannot go back to the side $\partial\Gamma_0$ immediately, so

[3]This assumption is clearly not necessary at all. But we can easily show that if it is not satisfied, then the WKB expansion for (1) is the same as the one for the problem in the half space $\{x_d \geqslant 0\}$ and this case is of little interest.

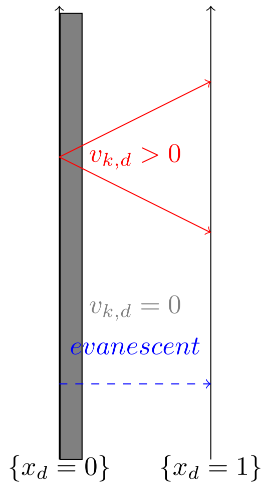

Figure 1. Appearance of the source term induced phases.

that, in a formal setting and at least during a short time, we can consider the problem $L(\partial)u^\varepsilon = 0$ defined on the half space $\{x_d \leqslant 1\}$ with a boundary condition on $\partial\Gamma_1$ involving the amplitude $u_{0,\underline{k}_{|x_d=1}}$ and with homogeneous initial condition. We shall describe the amplitudes induced by the source term on $\partial\Gamma_1$.

Note that because we are still working in a half space indexed by x_d, the possible induced amplitudes satisfy the same dispersion relation as the one for the source term induced phases[4]. That is the ξ_k are roots in the ξ-variable of the dispersion relation $\det \mathscr{L}(\underline{\tau}, \underline{\eta}, \xi) = 0$. So that the discussion of the previous paragraph can also be performed to determine the reflections during the first rebound.

However, due to the change of orientation in the x_d variable, the sign in the discussion has to be reverse. More precisely, let ξ_k be such that $\det \mathscr{L}(\underline{\tau}, \underline{\eta}, \xi_k) = 0$ we distinguish:

◇ $\xi_k \in \mathbb{C}$, $\operatorname{Im} \xi_k \neq 0$. Then the amplitude $u_{0,k}$ is associated to a non-trivial real exponential factor. And depending on the sign of $\operatorname{Im} \xi_k$ we have:

• $\operatorname{Im} \xi_k < 0$ (*evanescent for the side* $\partial\Gamma_1$). These amplitudes have been initially neglected in the amplitudes induced by the source term. But at this step of the analysis they are evanescent for the side $\partial\Gamma_1$ (or equivalently explosive for the side $\partial\Gamma_0$) so that we reintroduce these amplitudes in the ansatz (21). They are associated to boundary layer around the side $\partial\Gamma_1$ which propagate to $\partial\Gamma_0$ and hit this side as $O(\varepsilon^\infty)$ so that they are not reflected against $\partial\Gamma_0$ and do not contribute in the boundary condition on $\partial\Gamma_0$.

• $\operatorname{Im} \xi_k > 0$ (*explosive for the side* $\partial\Gamma_1$). These amplitudes are evanescent for the side $\partial\Gamma_0$. So that they are still present in the ansatz (21) and there is no need to add them.

[4]It is not the case in the corner geometry, see [3], for which the dispersion relation changes at each rebound. This explains why in the strip geometry, the phase generation process is not as rich as in the quarter space.

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◇ $\xi_k \in \mathbb{R}$. Then the associated amplitude is oscillating, Lax's lemma [8] applies so we expect to solve the transport equation (22) and we have to reiterate the discussion of the previous paragraph depending on the sign of the dth component of the group velocity v_k:

• $v_{k,d} > 0$ (*outgoing for the side* $\partial\Gamma_1$). These amplitudes are already present in (21) because they are incoming for the side $\partial\Gamma_0$.

• $v_{k,d} = 0$ (*glancing for the side* $\partial\Gamma_1$ *(or equivalently for* $\partial\Gamma_0$*)*). In this case, once again the transport equation (22) degenerates in (23) and we have justified already that even if this equation is homogeneous we chose to keep $u_{0,k}$ as a boundary layer in the neighborhood of $\partial\Gamma_0$ (in order to solve the boundary conditions up to an acceptable error term). In order to solve the boundary condition on $\partial\Gamma_1$ (which at this step of the analysis is not homogeneous anymore because it depends on $u_{0,\underline{k}_{|x_d=1}}$) we will introduce in $u_{0,k}$ a boundary layer in the neighborhood of $\partial\Gamma_1$. However this new layer cannot be propagated to $\partial\Gamma_0$ (because of the degeneracy of the transport equation) so that it will not contribute to the boundary condition on $\partial\Gamma_0$ and will not be reflected against this side.

• $v_{k,d} < 0$ (*incoming for the side* $\partial\Gamma_1$). We recall that these amplitudes have **initially** been neglected in (21) and that they are associated to the transport equation for the 'right to the left'. But at this step of the discussion, the information lying on the side $\partial\Gamma_1$ is not trivial anymore, so these amplitudes propagate this information from $\partial\Gamma_1$ to $\partial\Gamma_0$ and are not zero anymore. Consequently they have to be considered in (21). These phases will hit the side $\partial\Gamma_0$ after some positive travel time and we have to determine their rebounds. This is done in the next paragraph.

To sum up, the first rebound makes us consider the explosive and outgoing (for the side $\partial\Gamma_0$) phases which has been initially discarded. So that all the possible phases are now taken into account in (21). Moreover, we also add a boundary layer in the neighborhood of $\partial\Gamma_1$ to deal with glancing modes. However the only phases carrying some non-trivial information from the side $\partial\Gamma_1$ to the side $\partial\Gamma_0$ are the outgoing ones. The generated phases during the first reflection are described on Figure 2.

4.3. Selfinteraction phenomenon

Once again we assume that there exists an outgoing phase associated to some $\xi_{\underline{\ell}}$ satisfying $\det \mathscr{L}(\underline{\tau}, \underline{\eta}, \xi_{\underline{\ell}}) = 0$ and $v_{\underline{\ell},d} < 0$. Then the information carried by the amplitude $u_{0,\underline{\ell}}$ hits the side $\partial\Gamma_0$ after some travel time and we have to determine it(s) reflection(s) against this side.

However reiterating exactly the same arguments as in Paragraph 4.1 (that is finite speed of propagation property to restrict the problem to the study of the problem (20)), we obtain that the reflections are associated to the ξ_k satisfying $\det \mathscr{L}(\underline{\tau}, \underline{\eta}, \xi_k) = 0$ and one of the following alternatives:

(i) $\operatorname{Im} \xi_k > 0$,

(ii) $\xi_k \in \mathbb{R}$, $v_{k,d} = 0$,

(iii) $\xi_k \in \mathbb{R}$, $v_{k,d} > 0$.

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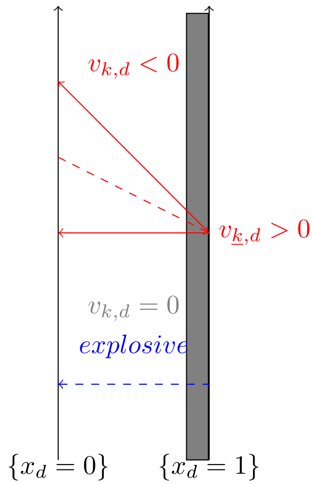

Figure 2. The first rebound.

Recall that all of these phases are already considered in the ansatz (21). Consequently we do not have to add any phase in (21).

But let us remark that the amplitude $u_{0,\underline{k}}$ considered at the beginning of Paragraph 4.2 satisfies (iii) so that this phase has regenerated itself after two rebounds. It is what we mean by selfinteraction.

This phenomenon will be crucial in the construction of the geometric optics expansion for (1) and will lead to an inversibility condition imposed to initialize the resolution of the cascade of equations (see Assumption 6.1).

Let us make some other remarks. In this discussion we followed the path of phases $\underline{k} \hookrightarrow \underline{\ell}$ but if one changes $\underline{\ell}$ and considers a path of phases $\underline{k} \hookrightarrow \underline{\ell}'$ then during the second rebound the phase $\underline{k}$ is still generated. So that each path of the form $\underline{k} \hookrightarrow \ell$, where ℓ is associated to an outgoing phase (for the side $\partial\Gamma_0$) gives a contribution to the regeneration of the phase associated to $\underline{k}$.

Moreover a path of phases of the form $k \hookrightarrow \ell$ where $k \neq \underline{k}$ is associated to an incoming phase (for the side $\partial\Gamma_0$) and ℓ to an outgoing phase (for the side $\partial\Gamma_0$) will also generate the phase associated to $\underline{k}$. [5]

We conclude this section by Figure 3 illustrating the several amplitudes in the WKB expansion and the selfinteraction phenomenon.

[5]Consequently, compared to the corner geometry see [3], the selfinteraction phenomenon is here a bit more complicated because there is *a priori* more than one path of phases that regenerate a fixed phase. Moreover, once again compared to the corner geometry, here the selfinteraction phenomenon is generic because to hold it only requires the existence of an incoming phase and an outgoing phase. Whereas in the corner geometry, some really restrictive assumptions have to be made on the geometry of the characteristic variety (we refer to [3] for more details).

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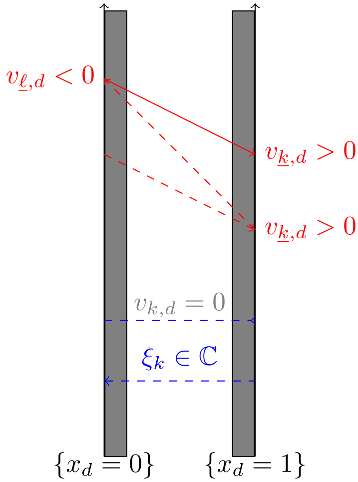

Figure 3. Phases in the WKB expansion and selfinteraction.

5. The cascades of equations

We consider the following system of equations

$$\begin{cases} L(\partial)u^\varepsilon = \partial_t u^\varepsilon + \sum_{j=1}^d A_j \partial_j u^\varepsilon = 0 & \text{in } \Omega, \\ B_0 u^\varepsilon_{|x_d=0} = g^\varepsilon & \text{on } \partial\Omega_0, \\ B_1 u^\varepsilon_{|x_d=1} = 0 & \text{on } \partial\Omega_1, \\ u^\varepsilon_{|t\leqslant 0} = 0 & \text{on } \Gamma. \end{cases} \tag{24}$$

Let $\underline{\zeta} := (i\underline{\tau}, \underline{\eta}) \in \Xi_0$ be a fixed frequency parameter. We define the phase functions

$$\psi(t, x') := \underline{\tau} t + \underline{\eta} \cdot x' \text{ and for } k \in \mathscr{H} \cup \mathscr{G},\ \varphi_k(t, x) := \psi(t, x') + \underline{\xi}_k x_d, \tag{25}$$

where the $\underline{\xi}_k$ stands for the real roots of $\det \mathscr{L}(\underline{\tau}, \underline{\eta}, \xi)$ in the ξ variable.

In (24) the source term on the boundary $\partial\Omega_0$ reads:

$$g^\varepsilon := g^\varepsilon(t, x') := e^{\frac{i}{\varepsilon}\psi(t,x')} g(t, x'), \tag{26}$$

where the amplitude $g \in H^\infty_\natural(\partial\Omega_0)$.

We define the ansatz[6]

$$u^\varepsilon(t, x) \sim \sum_{k\in\mathscr{H}} e^{\frac{i}{\varepsilon}\varphi_k(t,x)} \sum_{n\geqslant 0} \varepsilon^n u^\varepsilon_{h,n,k}(t, x) + \sum_{k\in\mathscr{G}} e^{\frac{i}{\varepsilon}\varphi_k(t,x)} \sum_{n=0}^{1} \varepsilon^n u^\varepsilon_{g,n,k}(t, x)$$
$$+ \sum_{n\geqslant 0} e^{\frac{i}{\varepsilon}\psi(t,x')} \varepsilon^n U^\varepsilon_{ev,n}\left(t, x, \frac{x_d}{\varepsilon}\right) + \sum_{n\geqslant 0} e^{\frac{i}{\varepsilon}\psi(t,x')} \varepsilon^n U^\varepsilon_{ex,n}\left(t, x, \frac{x_d - 1}{\varepsilon}\right), \tag{27}$$

[6]Remark that in (27) we take an arbitrary number of correctors for the non-glancing modes while we take only one corrector for the glancing ones. This choice is motivated by [16, 17] and will be explained in Paragraphs 6.2 and 7.3.

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where for all $0 < \varepsilon \ll 1$, for all $n \in \mathbb{N}$ and for all $k \in \mathscr{H}$ (respectively $k \in \mathscr{G}$) the profiles $u_{h,n,k}$ (respectively $u_{g,n,k}$) are in $H^{\infty}_{\natural,\gamma}(\Omega)$ for all $\gamma > \underline{\gamma}_0$ for some $\underline{\gamma}_0 \geqslant 0$ to be determined and where the evanescent (respectively explosive) profiles $U^{\varepsilon}_{ev,n}$ (respectively $U^{\varepsilon}_{ex,n}$) are in the following profile spaces:

Definition 5.1. The space $\mathcal{P}_{ev}$ (respectively $\mathcal{P}_{ex}$) of evanescent (respectively explosive) profiles is the set of functions $U(t, x, X_d) \in H^{\infty}_{\natural}(\Omega \times \mathbb{R}_+)$ (respectively $H^{\infty}_{\natural}(\Omega \times \mathbb{R}_-)$) satisfying that there exists $\delta > 0$ such that $e^{\delta X_d} U(t, x, X_d) \in H^{\infty}_{\natural}(\Omega \times \mathbb{R}_+)$ (respectively $H^{\infty}_{\natural}(\Omega \times \mathbb{R}_-)$).

In the ansatz (27) let us stress that depending on the kind of the frequency $\underline{\zeta}$ some (but not all) sums can be zero. Indeed for example if $\underline{\zeta} \in \mathbb{E}$ then the sums on $\mathscr{H}$ and on $\mathscr{G}$ are zeros. We also insist on the fact that the sum on $\mathscr{H}$ can always be zero when $\underline{\zeta} \notin \mathbb{H}$.

Plugging the ansatz (27) in the evolution equation of (24) leads, by identification on the ε^n, to the following cascade of equation

$$\begin{cases} \mathscr{L}(d\varphi_k) u^{\varepsilon}_{h,0,k} = 0 & \forall k \in \mathscr{H}, \\ i\mathscr{L}(d\varphi_k) u^{\varepsilon}_{h,n+1,k} + L(\partial) u^{\varepsilon}_{h,n,k} = 0 & \forall k \in \mathscr{H}, \forall n \in \mathbb{N}, \\ \mathscr{L}(d\varphi_k) u^{\varepsilon}_{g,0,k} = 0 & \forall k \in \mathscr{G}, \\ i\mathscr{L}(d\varphi_k) u^{\varepsilon}_{g,1,k} + L(\partial) u^{\varepsilon}_{g,0,k} = 0 & \forall k \in \mathscr{G}, \\ L(\partial) u^{\varepsilon}_{g,1,k} = 0 & \forall k \in \mathscr{G} \\ L(\partial_{X_d}) U^{\varepsilon}_{ev,0} = L(\partial_{X_d}) U^{\varepsilon}_{ex,0} = 0, & \\ \left(L(\partial_{X_d}) U^{\varepsilon}_{ev,n+1} + L(\partial) U^{\varepsilon}_{ev,n} \right)(t, x, X_d) = 0 & \forall n \in \mathbb{N},\ X_d > 0, \\ \left(L(\partial_{X_d}) U^{\varepsilon}_{ex,n+1} + L(\partial) U^{\varepsilon}_{ex,n} \right)(t, x, \widetilde{X_d}) = 0 & \forall n \in \mathbb{N},\ \widetilde{X_d} < 0, \end{cases} \tag{28}$$

where the operator of differentiation with respect to the fast variable is defined by

$$L(\partial_{X_d}) = A_d(\partial_{X_d} - \mathscr{A}(\underline{\zeta})).$$

Then plugging the ansatz (27) in the boundary conditions of (24) gives

$$B_0 \left[\sum_{k \in \mathscr{H}} u^{\varepsilon}_{h,n,k}(t, x', 0) + \sum_{k \in \mathscr{G}} u^{\varepsilon}_{g,n,k}(t, x', 0) + U^{\varepsilon}_{ev,n}(t, x', 0, 0) + U^{\varepsilon}_{ex,n}\left(t, x', 0, -\frac{1}{\varepsilon}\right) \right] = \delta_{n,0} g, \tag{29}$$

and

$$B_1 \left[\sum_{k \in \mathscr{H}} e^{\frac{i}{\varepsilon}\xi_k} u^{\varepsilon}_{h,n,k}(t, x', 1) + \sum_{k \in \mathscr{G}} e^{\frac{i}{\varepsilon}\xi_k} u^{\varepsilon}_{g,n,k}(t, x', 1) + U^{\varepsilon}_{ev,n}\left(t, x', 1, \frac{1}{\varepsilon}\right) + U^{\varepsilon}_{ex,n}(t, x', 1, 0) \right] = 0, \tag{30}$$

where $\delta_{n,p}$ stands for the Kronecker's symbol.

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However by definition of $\mathcal{P}_{ev}$ and $\mathcal{P}_{ex}$, the terms $U^{\varepsilon}_{ex,n}(t,x',0,-\frac{1}{\varepsilon})$ and $U^{\varepsilon}_{ev,n}(t,x',1,\frac{1}{\varepsilon})$ appearing in (29) and (30) respectively are $O(\varepsilon^{\infty})$ so that the boundary conditions (29) and (30) can be simplified into

$$B_0\left[\sum_{k\in\mathscr{H}} u^{\varepsilon}_{h,n,k}(t,x',0)+\sum_{k\in\mathscr{G}} u^{\varepsilon}_{g,n,k}(t,x',0)+U^{\varepsilon}_{ev,n}(t,x',0,0)\right]=\delta_{n,0}g, \tag{31}$$

and

$$B_1\left[\sum_{k\in\mathscr{H}} e^{\frac{i}{\varepsilon}\xi_k} u^{\varepsilon}_{h,n,k}(t,x',1)+\sum_{k\in\mathscr{G}} e^{\frac{i}{\varepsilon}\xi_k} u^{\varepsilon}_{g,n,k}(t,x',1)+U^{\varepsilon}_{ex,n}(t,x',1,0)\right]=0. \tag{32}$$

Finally plugging the ansatz (27) in the initial condition of (24) leads to

$$\begin{cases} u^{\varepsilon}_{h,n,k_{|t\leqslant 0}}=0 & \forall k\in\mathscr{H}, \forall n\in\mathbb{N},\\ u^{\varepsilon}_{g,n,k_{|t\leqslant 0}}=0 & \forall k\in\mathscr{G}, \forall n\in\mathbb{N},\\ U^{\varepsilon}_{ev,n_{|t\leqslant 0}}=0 & \forall n\in\mathbb{N},\\ U^{\varepsilon}_{ex,n_{|t\leqslant 0}}=0 & \forall n\in\mathbb{N}. \end{cases} \tag{33}$$

So that to construct an approximate solution of (24) one shall solve the cascades of equations (28)–(31)–(32) and (33) up to some order. The construction of the leading order term, that is the one associated to ε^0, in the expansion is performed in the following section.

6. Construction of the leading order term

To initialize the construction of the leading order term of the geometric optics expansion we study the behavior of the oscillating and glancing amplitudes in (27), that is $u_{h,0,k}$ and $u_{g,0,k}$. The first (respectively the third) equation of (28) implies that for all $k\in\mathscr{H}$ (respectively $k\in\mathscr{G}$), $u_{h,0,k}\in\ker\mathscr{L}(d\varphi_k)$ (respectively $u_{g,0,k}\in\ker\mathscr{L}(d\varphi_k)$). Consequently we have the well-known polarization condition

$$\forall k\in\mathscr{H}\cup\mathscr{G},\ \Pi^k u_{h,0,k}=u_{h,0,k} \tag{34}$$

where we recall that the projectors Π^k are introduced in Definition 2.3.

Using the polarization condition (34) and composing the second (respectively the fourth) equation of (28) by Π^k gives

$$\Pi^k L(\partial)\Pi^k u_{h,0,k}=0 \text{ (respectively } \Pi^k L(\partial)\Pi^k u_{g,0,k}=0),$$

so that we are in a position to apply Lax's lemma [8]:

Lemma 6.1 [8]. *Under Assumption 2.1 we have the equalities*

$$\forall k\in\mathscr{H}\cup\mathscr{G}, \Pi^k L(\partial)\Pi^k=(\partial_t+v_k\cdot\nabla_x)\Pi^k, \tag{35}$$

where we recall that v_k is the group velocity associated to k introduced in Definition 2.2.

As a consequence the leading order oscillating and glancing amplitudes are expected to satisfy transport equations and we have to consider several cases depending on k:

• $k \in \mathscr{I}$. By definition of $\mathscr{I}$ the group velocity v_k satisfies $v_{k,d} > 0$ so that the transport equation (35) only requires a boundary condition on the side $\partial\Gamma_0$ and an initial condition to be solved (the value of the trace on $\partial\Gamma_1$ is deduced by integration along the characteristics).

• $k \in \mathscr{O}$. By definition of $\mathscr{O}$ the group velocity v_k satisfies $v_{k,d} < 0$ so that the transport equation (35) only requires a boundary condition on the side $\partial\Gamma_1$ and an initial condition to be solved (the value of the trace on $\partial\Gamma_0$ is deduced by integration along the characteristics).

• $k \in \mathscr{G}$. By definition of $\mathscr{G}$ the group velocity v_k satisfies $v_{k,d} = 0$ so that the transport operator in (35) reads $\partial_t + v_k' \cdot \nabla_{x'}$ and no boundary conditions are required so that we just only require the initial condition.

These remarks lead us to study the boundary conditions (31) and (32) written for $n = 0$, they read:

$$B_0 \left[\sum_{k\in\mathscr{I}} u^{\varepsilon}_{h,0,k_{|x_d=0}} + \sum_{k\in\mathscr{G}} u^{\varepsilon}_{g,0,k_{|x_d=0}} + U^{\varepsilon}_{ev,0_{|x_d=X_d=0}} \right] = g - B_0 \sum_{\ell\in\mathscr{O}} u^{\varepsilon}_{h,0,\ell_{|x_d=0}}, \tag{36}$$

and

$$B_1 \left[\sum_{\ell\in\mathscr{O}} e^{\frac{i}{\varepsilon}\xi_\ell} u^{\varepsilon}_{h,0,\ell_{|x_d=1}} + \sum_{\ell\in\mathscr{G}} e^{\frac{i}{\varepsilon}\xi_\ell} u^{\varepsilon}_{g,0,\ell_{|x_d=1}} + U^{\varepsilon}_{ex,0_{|x_d=X'_d=1}} \right] = -B_1 \sum_{k\in\mathscr{I}} e^{\frac{i}{\varepsilon}\xi_k} u^{\varepsilon}_{h,0,k_{|x_d=1}}. \tag{37}$$

We remark, from (12) (respectively (13)), that the term in the left hand side of (36) (respectively (37)) lies in $B_0 E^s(\underline{\zeta})$ (respectively $B_1 E^u(\underline{\zeta})$), so that, by the uniform Kreiss–Lopatinskii condition (see Assumption 2.4), we can multiply (36) (respectively (37)) by $\phi_0(\underline{\zeta})$ (respectively $\phi_1(\underline{\zeta})$) and then by P_h^k, $P_{g,s}^k$, P_{ev} respectively (respectively P_h^k, $P_{g,u}^k$, P_{ex}) (recall that these projectors are those of Definition 2.4) to obtain

$$\begin{cases} u^{\varepsilon}_{h,0,k_{|x_d=0}} = P_h^k \phi_0(\underline{\zeta}) \left(g - B_0 \sum_{\ell\in\mathscr{O}} u^{\varepsilon}_{h,0,\ell_{|x_d=0}} \right) & \forall k \in \mathscr{I}, \\ u^{\varepsilon}_{g,0,k_{|x_d=0}} = P_{g,s}^k \phi_0(\underline{\zeta}) \left(g - B_0 \sum_{\ell\in\mathscr{O}} u^{\varepsilon}_{h,0,\ell_{|x_d=0}} \right) & \forall k \in \mathscr{G}, \\ U^{\varepsilon}_{ev,0_{|x_d=X_d=0}} = P_{ev} \phi_0(\underline{\zeta}) \left(g - B_0 \sum_{\ell\in\mathscr{O}} u^{\varepsilon}_{h,0,\ell_{|x_d=0}} \right), \end{cases} \tag{38}$$

and

$$\begin{cases} u^{\varepsilon}_{h,0,\ell_{|x_d=1}} = -e^{-\frac{i}{\varepsilon}\xi_\ell} P_h^\ell \phi_1(\underline{\zeta}) B_1 \sum_{k\in\mathscr{I}} e^{\frac{i}{\varepsilon}\xi_k} u^{\varepsilon}_{h,0,k_{|x_d=1}} & \forall \ell \in \mathscr{O}, \\ u^{\varepsilon}_{g,0,\ell_{|x_d=1}} = -e^{-\frac{i}{\varepsilon}\xi_\ell} P_{g,u}^\ell \phi_1(\underline{\zeta}) B_1 \sum_{k\in\mathscr{I}} e^{\frac{i}{\varepsilon}\xi_k} u^{\varepsilon}_{h,0,k_{|x_d=1}} & \forall \ell \in \mathscr{G}, \\ U^{\varepsilon}_{ex,0_{|x_d=X'_d=1}} = -P_{ev} \phi_1(\underline{\zeta}) B_1 \sum_{k\in\mathscr{I}} e^{\frac{i}{\varepsilon}\xi_k} u^{\varepsilon}_{h,0,k_{|x_d=1}}, \end{cases} \tag{39}$$

where we used the fact that $P_h^k \Pi^k = \Pi^k$, $P_{g,s}^k \Pi^k = P_{g,u}^k \Pi^k = \Pi^k$ combined with the polarization condition (34).

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The main observation is that in (38) and (39), to determine the traces of the glancing amplitudes, the evanescent amplitude or the explosive amplitude it is sufficient to first determine the traces of the amplitudes associated to indices in $\mathscr{H}$. Consequently we shall determine the amplitudes associated to indices in $\mathscr{H}$ before the other ones to initialize the resolution of the cascade of equations.

However in (38) to determine the traces associated to the indices in $\mathscr{I}$ we have to determine the traces of the amplitudes for the indices in $\mathscr{O}$, which depend on the traces of the amplitudes for the indices in $\mathscr{I}$ by (39). So that (38) and (39) show that the traces of the amplitudes for the indices in $\mathscr{I}$ (or $\mathscr{O}$) depend on themselves which agree with the selfinteraction phenomenon described formally in § 4. The rigorous determination of these amplitudes is made in the next paragraph.

6.1. Construction of the leading order selfinteracting amplitudes

In this paragraph we show that the determination of the amplitudes associated to the indices in $\mathscr{I}$ necessitates a new inversibility condition. We consider $\ell \in \mathscr{O}$ so that the group velocity v_ℓ is outgoing and the resolution of the transport equation (35) only requires a boundary condition on $\partial\Gamma_1$. More precisely from (39), the equation to solve is:

$$\begin{cases} (\partial_t + v_\ell \cdot \nabla_x) u^\varepsilon_{h,0,\ell} = 0, \\ u^\varepsilon_{h,0,\ell_{|x_d=1}} = -e^{-\frac{i}{\varepsilon}\xi_\ell} P^\ell_h \phi_1(\underline{\zeta}) B_1 \sum_{k\in\mathscr{I}} e^{\frac{i}{\varepsilon}\xi_k} u^\varepsilon_{h,0,k_{|x_d=1}}, \\ u^\varepsilon_{h,0,\ell_{|t\leqslant 0}} = 0. \end{cases} \tag{40}$$

Let us assume that in (40) the right hand side of the boundary condition, namely $\sum_{k\in\mathscr{I}} e^{\frac{i}{\varepsilon}\xi_k} u^\varepsilon_{h,0,k_{|x_d=1}}$, is a known function. Then is it easy to solve (40) by integration along the characteristics to determine $u^\varepsilon_{h,0,\ell}$. More precisely, we have:

$$u^\varepsilon_{h,0,\ell}(t,x) = -e^{-\frac{i}{\varepsilon}\xi_\ell} \left(P^\ell_h \phi_1(\underline{\zeta}) B_1 \sum_{k\in\mathscr{I}} e^{\frac{i}{\varepsilon}\xi_k} u^\varepsilon_{h,0,k_{|x_d=1}} \right) \left(t + \frac{1}{v_{\ell,d}}(1-x_d), x' + \frac{v'_\ell}{v_{\ell,d}}(1-x_d) \right), \tag{41}$$

where we used the notation $v_\ell = (v'_\ell, v_{\ell,d}) \in \mathbb{R}^{d-1} \times \mathbb{R}^*_-$. We easily determine the value of $u^\varepsilon_{h,0,\ell_{|x_d=0}}$ for $\ell \in \mathscr{O}$

$$u^\varepsilon_{h,0,\ell_{|x_d=0}}(t,x') = -e^{-\frac{i}{\varepsilon}\xi_\ell} \left(P^\ell_h \phi_1(\underline{\zeta}) B_1 \sum_{k\in\mathscr{I}} e^{\frac{i}{\varepsilon}\xi_k} u^\varepsilon_{h,0,k_{|x_d=1}} \right) \left(t + \frac{1}{v_{\ell,d}}, x' + \frac{v'_\ell}{v_{\ell,d}} \right). \tag{42}$$

Using (42) we can compute the right hand side of the first equation of (38). For $k \in \mathscr{I}$ we have:

$$\begin{aligned} u^\varepsilon_{h,0,k_{|x_d=0}}(t,x') = P^k_h \phi_0(\underline{\zeta}) g(t,x') + P^k_h \phi_0(\underline{\zeta}) B_0 \sum_{\ell\in\mathscr{O}} e^{-\frac{i}{\varepsilon}\xi_\ell} P^\ell_h \phi_1(\underline{\zeta}) B_1 \\ \times \sum_{k'\in\mathscr{I}} e^{\frac{i}{\varepsilon}\xi_{k'}} u^\varepsilon_{h,0,k'_{|x_d=1}} \left(t + \frac{1}{v_{\ell,d}}, x' + \frac{v'_\ell}{v_{\ell,d}} \right). \end{aligned} \tag{43}$$

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Because $k \in \mathscr{I}$, $u^{\varepsilon}_{h,0,k}$ solves the incoming transport equation (35). We deduce by integration along the characteristics that:

$$\begin{aligned} u^{\varepsilon}_{h,0,k}(t,x) = {} & P^k_h \phi_0(\underline{\zeta}) g\left(t - \frac{1}{v_{k,d}} x_d, x' - \frac{v'_k}{v_{k,d}} x_d\right) \\ & + P^k_h \phi_0(\underline{\zeta}) B_0 \sum_{\ell \in \mathscr{O}} e^{-\frac{i}{\varepsilon}\xi_\ell} P^\ell_h \phi_1(\underline{\zeta}) B_1 \sum_{k' \in \mathscr{I}} e^{\frac{i}{\varepsilon}\xi_{k'}} u^{\varepsilon}_{h,0,k'_{|x_d=1}} \\ & \times \left(t + \frac{1}{v_{\ell,d}} - \frac{1}{v_{k,d}} x_d, x' + \frac{v'_\ell}{v_{\ell,d}} - \frac{v'_k}{v_{k,d}} x_d\right), \end{aligned} \tag{44}$$

from which we immediately obtain the value of the trace of $u^{\varepsilon}_{h,0,k}$ on $\partial\Gamma_1$:

$$\begin{aligned} u^{\varepsilon}_{h,0,k_{|x_d=1}}(t,x') = {} & P^k_h \phi_0(\underline{\zeta}) g\left(t - \frac{1}{v_{k,d}}, x' - \frac{v'_k}{v_{k,d}}\right) \\ & + P^k_h \phi_0(\underline{\zeta}) B_0 \sum_{\ell \in \mathscr{O}} e^{-\frac{i}{\varepsilon}\xi_\ell} P^\ell_h \phi_1(\underline{\zeta}) B_1 \sum_{k' \in \mathscr{I}} e^{\frac{i}{\varepsilon}\xi_{k'}} u^{\varepsilon}_{h,0,k'_{|x_d=1}} \\ & \times \left(t + \frac{1}{v_{\ell,d}} - \frac{1}{v_{k,d}}, x' + \frac{v'_\ell}{v_{\ell,d}} - \frac{v'_k}{v_{k,d}}\right). \end{aligned} \tag{45}$$

Equation (45) holds for all $k \in \mathscr{I}$ so that we can multiply by $e^{\frac{i}{\varepsilon}\xi_k}$ and sum over $k \in \mathscr{I}$ to derive the following condition on $\mathcal{U}^{\varepsilon}_{\mathscr{I}} := \sum_{k \in \mathscr{I}} e^{\frac{i}{\varepsilon}\xi_k} u^{\varepsilon}_{h,0,k_{|x_d=1}}$:

$$(I - \mathcal{T}^{\varepsilon}(\underline{\zeta}))\mathcal{U}^{\varepsilon}_{\mathscr{I}} = \mathcal{G}^{\varepsilon}(\underline{\zeta}) g, \tag{46}$$

where we set for f a function defined on $\mathbb{R}_t \times \mathbb{R}^{d-1}_{x'}$

$$\begin{aligned} (\mathcal{T}^{\varepsilon}(\underline{\zeta}) f)(t,x') := {} & \sum_{k \in \mathscr{I}} e^{\frac{i}{\varepsilon}\xi_k} P^k_h \phi_0(\underline{\zeta}) B_0 \sum_{\ell \in \mathscr{O}} e^{-\frac{i}{\varepsilon}\xi_\ell} P^\ell_h \phi_1(\underline{\zeta}) \\ & \times B_1 f\left(t + \frac{1}{v_{\ell,d}} - \frac{1}{v_{k,d}}, x' + \frac{v'_\ell}{v_{\ell,d}} - \frac{v'_k}{v_{k,d}}\right), \end{aligned} \tag{47}$$

and

$$(\mathcal{G}^{\varepsilon}(\underline{\zeta}) f)(t,x') := \sum_{k \in \mathscr{I}} e^{\frac{i}{\varepsilon}\xi_k} P^k_h \phi_0(\underline{\zeta}) f\left(t - \frac{1}{v_{k,d}}, x' - \frac{v'_k}{v_{k,d}}\right). \tag{48}$$

Note that in the definitions of $\mathcal{T}^{\varepsilon}(\underline{\zeta})$ and $\mathcal{G}^{\varepsilon}(\underline{\zeta})$ the evaluations in the time variable are of the form $t - \alpha_{k,\ell}$ where $\alpha_{k,\ell} > 0$ because by definition for $k \in \mathscr{I}$, $v_{k,d} > 0$ and for $\ell \in \mathscr{O}$, $v_{\ell,d} < 0$. Consequently, the form of the operator $\mathcal{T}^{\varepsilon}(\underline{\zeta})$ agrees with the intuition given in § 4 that the selfinteraction phenomenon needs some time (more precisely at least the minimum of the times needed to make two reflections) to appear. We will give more comments about the operator $\mathcal{T}^{\varepsilon}(\underline{\zeta})$ in § 9.

Moreover from (47) and (48) it is clear that $H^{\infty}_{\natural,\gamma}(\mathbb{R}_t \times \mathbb{R}^{d-1}_{x'})$ is an invariant set for $\mathcal{T}^{\varepsilon}(\underline{\zeta})$ and $\mathcal{G}^{\varepsilon}(\underline{\zeta})$.

Equation (46) combined with the fact that $\mathcal{U}^{\varepsilon}_{\mathscr{I}} \in E^s_h(\underline{\zeta})$ and $\mathcal{G}^{\varepsilon}(\underline{\zeta}) g \in E^s_h(\underline{\zeta})$ lead us to the following assumption:

Assumption 6.1. *Let $\underline{\gamma}_0 \geqslant 0$ we assume that for all $0 < \varepsilon \ll 1$ the operator $(I - \mathcal{T}^\varepsilon(\underline{\zeta}))$, where $\mathcal{T}^\varepsilon(\underline{\zeta})$ is defined in (47) is invertible from $H^\infty_{\natural,\gamma}(\partial\Omega_1, E^s_h(\underline{\zeta}))$ to $H^\infty_{\natural,\gamma}(\partial\Omega_1, E^s_h(\underline{\zeta}))$ for all $\gamma > \underline{\gamma}_0$.*

Remark. Let $A := \inf_{k\in\mathscr{I},\ell\in\mathscr{O}} \frac{1}{v_{k,d}} - \frac{1}{v_{\ell,d}}$ and note that from the special form of (46) it is trivially invertible on $H^\infty_{\natural,\gamma}([0, A[\times\mathbb{R}^{d-1})$, because for $f \in H^\infty_{\natural,\gamma}(]-\infty, A[\times\mathbb{R}^{d-1})$, the term $\mathcal{T}^\varepsilon(\underline{\zeta})$ vanishes so that $(I - \mathcal{T}^\varepsilon) = I$. So in fact a more precise version of Assumption 6.1 is to assume that $(I - \mathcal{T}^\varepsilon(\underline{\zeta}))$ is invertible from $H^\infty_{\natural,\gamma}(\mathbb{R}\times\mathbb{R}^{d-1}, E^s_h(\underline{\zeta}))$ to $H^\infty_{\natural,\gamma}(]A, \infty]\times\mathbb{R}^{d-1}, E^s_h(\underline{\zeta}))$ for all $\gamma > \underline{\gamma}_0$.

This is due to the fact that before the time A the wave packets have not performed a complete reflection so that there is no selfinteraction and only the identity component of $(I - \mathcal{T}^\varepsilon(\underline{\zeta})$ prescribes the behavior of the wave.

However Assumption 6.1 has the advantage of simplicity compared to the other weaker version described above. This is why even if it is not the sharpest one we used to use it. We refer to Paragraph 9.2 for more details about the differences between these two assumptions.

We refer to § 9 for a partial study of Assumption 6.1 and to § 10 for explicit examples.

With Assumption 6.1 in hand it is now easy to determine the amplitudes associated to indices in $\mathscr{H}$. From Assumption 6.1 we obtain:

$$\mathcal{U}^\varepsilon_{\mathscr{I}} = (I - \mathcal{T}^\varepsilon(\underline{\zeta}))^{-1}_{|E^s_h(\underline{\zeta})}\mathcal{G}^\varepsilon(\underline{\zeta})g,$$

and we can use this expression in (41) and (44) to obtain that for all $\ell \in \mathscr{O}$:

$$\begin{aligned} u^\varepsilon_{h,0,\ell}(t,x) &= -e^{-\frac{i}{\varepsilon}\xi_\ell} P^\ell_h \phi_1(\underline{\zeta}) B_1 \left((I - \mathcal{T}^\varepsilon(\underline{\zeta}))^{-1}_{|E^s_h(\underline{\zeta})}\mathcal{G}^\varepsilon(\underline{\zeta})g \right) \\ &\quad \times \left(t + \frac{1}{v_{\ell,d}}(1-x_d), x' + \frac{v'_\ell}{v_{\ell,d}}(1-x_d) \right), \end{aligned} \tag{49}$$

and for all $k \in \mathscr{I}$:

$$\begin{aligned} u^\varepsilon_{h,0,k} &= P^k_h \phi_0(\underline{\zeta}) g \left(t - \frac{1}{v_{k,d}} x_d, x' - \frac{v'_k}{v_{k,d}} x_d \right) \\ &\quad + P^k_h \phi_0(\underline{\zeta}) B_0 \sum_{\ell\in\mathscr{O}} e^{-\frac{i}{\varepsilon}\xi_\ell} P^\ell_h \phi_1(\underline{\zeta}) B_1 \left((I - \mathcal{T}^\varepsilon(\underline{\zeta}))^{-1}_{|E^s_h(\underline{\zeta})}\mathcal{G}^\varepsilon(\underline{\zeta})g \right) \\ &\quad \times \left(t + \frac{1}{v_{\ell,d}} - \frac{1}{v_{k,d}} x_d, x' + \frac{v'_\ell}{v_{\ell,d}} - \frac{v'_k}{v_{k,d}} x_d \right), \end{aligned} \tag{50}$$

equations which uniquely determine $u^\varepsilon_{h,0,k}$ for $k \in \mathscr{H}$ in terms of the known source term g. Also note that due to the fact that $g \equiv 0$ for negative times, the initial condition (33) written for $n = 0$ is satisfied for $k \in \mathscr{H}$.

This concludes the construction of the leading order amplitudes for selfinteracting phases. It remains to consider the other kinds of phase. The construction is made in the following paragraphs.

To sum up we give the following proposition:

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Proposition 6.1. *Under Assumptions 2.1, 2.2, 2.4 and 6.1. For all $k \in \mathscr{H}$ and for all $0 < \varepsilon \ll 1$ there exists $u^{\varepsilon}_{h,0,k} \in H^{\infty}_{\natural,\gamma}(\Omega)$ for all $\gamma > \underline{\gamma}_0$ (the one fixed in Assumption 6.1) satisfying the cascade of equations (28),(31),(32) and (33) written for $n = 0$.*

6.2. Construction of the leading order glancing amplitudes

To simplify the following we introduce the notations:

$$\mathcal{U}^{\varepsilon}_{n,\mathscr{I}} := \sum_{k\in\mathscr{I}} e^{\frac{i}{\varepsilon}\xi_k} \Pi^k u^{\varepsilon}_{h,n,k_{|x_d=1}} \quad \text{and} \quad \mathcal{U}^{\varepsilon}_{n,\mathscr{O}} := \sum_{\ell\in\mathscr{O}} \Pi^{\ell} u^{\varepsilon}_{h,n,\ell_{|x_d=0}}. \tag{51}$$

Let $k \in \mathscr{G}$; then the associated amplitude $u^{\varepsilon}_{g,0,k}$ shall solve the transport equation (see (35),(38),(39) and (33))

$$\begin{cases} (\partial_t + v'_k \cdot \nabla_{x'}) u^{\varepsilon}_{g,0,k} = 0 & \text{for } (t,x) \in \Omega, \\ u^{\varepsilon}_{g,0,k_{|x_d=0}} = P^k_{g,s} \phi_0(\underline{\zeta})(g - B_0 \mathcal{U}^{\varepsilon}_{0,\mathscr{O}}) & \text{on } \partial\Omega_0, \\ u^{\varepsilon}_{g,0,k_{|x_d=1}} = -e^{-\frac{i}{\varepsilon}\xi_k} P^k_{g,u} \phi_1(\underline{\zeta}) B_1 \mathcal{U}^{\varepsilon}_{0,\mathscr{I}} & \text{on } \partial\Omega_1, \\ u^{\varepsilon}_{g,0,k_{|t\leqslant 0}} = 0 & \text{on } \Gamma, \end{cases} \tag{52}$$

where from Paragraph 6.1 the right hand sides in the boundary conditions of (52) are known functions in $H^{\infty}_{\natural,\gamma}(\mathbb{R}_t \times \mathbb{R}^{d-1}_{x'})$, for all $\gamma > \underline{\gamma}_0$ depending on g (their precise expression in terms of g can be made precise from (49) and (50) but is of little interest in the following).

As noticed in [16, 17], the main issue in the resolution of (52) is due to the fact that the group velocity v_k is tangent to the boundary, the couple of equations

$$\begin{cases} (\partial_t + v'_k \cdot \nabla_{x'}) u^{\varepsilon}_{g,0,k} = 0, \\ u^{\varepsilon}_{g,0,k_{|t\leqslant 0}} = 0, \end{cases} \tag{53}$$

already determines the solution $u^{\varepsilon}_{g,0,k}$. Moreover with homogeneous initial condition and interior forcing term it shall be zero. Consequently with the boundary conditions the system (52) is overdetermined (and the boundary conditions cannot be satisfied because $u^{\varepsilon}_{g,0,k} \equiv 0$).

However we stress that we need to solve theses boundary conditions to obtain a suitable error on the boundary in the energy estimate.

To overcome this difficulty induced by glancing modes, we follow the method of [16] that is we decompose $u^{\varepsilon}_{g,0,k} = u^{\varepsilon,\sharp}_{g,0,k} + u^{\varepsilon,\flat}_{g,0,k}$ where $u^{\varepsilon,\sharp}_{g,0,k}$ solves the transport equation (53) (in our study we can choose $u^{\varepsilon,\sharp}_{g,0,k} \equiv 0$) and where $u^{\varepsilon,\flat}_{g,0,k}$ is a boundary layer satisfying the boundary conditions of (52). Indeed, note that if $u^{\varepsilon,\flat}_{g,0,k}$ does not satisfy the boundary conditions (38) and (39) then because boundary conditions (31) and (32) are decoupled compared to n the error on the boundaries for glancing modes will be $O(1)$ with respect to ε which is not a suitable error rate for the justification of the WKB expansion (see § 8).

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Following [16] let $\chi \in \mathscr{C}^\infty(]0,1[)$ satisfying $\chi(x)=1$ for $x \leqslant \frac{1}{4}$ and $\chi(x)=0$ for $x \geqslant \frac{3}{4}$, we define[7]:

$$u^{\varepsilon}_{g,0,k}(t,x) = u^{\varepsilon,\flat}_{g,0,k}(t,x) := \chi\left(\frac{x_d}{\sqrt{\varepsilon}}\right) P^k_{g,s}\phi_0(\underline{\zeta})\left(g - B_0\mathcal{U}^{\varepsilon}_{0,\mathscr{O}}\right)(t,x') \\ - \left(1-\chi\left(\frac{x_d}{\sqrt{\varepsilon}}\right)\right) e^{-\frac{i}{\varepsilon}\xi_k} P^k_{g,u}\phi_1(\underline{\zeta}) B_1 \mathcal{U}^{\varepsilon}_{0,\mathscr{I}}(t,x'). \tag{54}$$

It is clear that such a $u^{\varepsilon}_{g,0,k}$ satisfies the boundary conditions (38) and (39). Moreover, by construction $u^{\varepsilon}_{g,0,k} \in \ker \mathscr{L}(d\varphi_k)$, so that the third equation of (28) is satisfied and by definition of g, the initial condition (33) written for $n=0$ is satisfied for $k \in \mathscr{G}$.

The construction of a corrector term is postponed to Paragraph 7.3. In the last paragraph of this section we conclude the construction by the one of evanescent/explosive amplitudes.

6.3. Construction of evanescent and explosive leading order amplitudes

The only remaining leading order amplitudes to be constructed are the evanescent and the explosive ones. We recall that the evanescent amplitude of leading order satisfies the equations (see (28),(38) and (33))

$$\begin{cases} L(\partial_{X_d}) U^{\varepsilon}_{ev,0}(X_d) = 0 & \text{for } X_d \geqslant 0, \\ U^{\varepsilon}_{ev,0_{|x_d=X_d=0}} = P_{ev}\phi_0(\underline{\zeta})(g - B_0\mathcal{U}^{\varepsilon}_{0,\mathscr{O}}), \\ U^{\varepsilon}_{ev,0_{|t\leqslant 0}} = 0, \end{cases} \tag{55}$$

and that the explosive amplitude of leading order satisfies the equations (see (28),(39) and (33))

$$\begin{cases} L(\partial_{\widetilde{X}_d}) U^{\varepsilon}_{ex,0}(\widetilde{X}_d) = 0 & \text{for } \widetilde{X}_d \leqslant 0, \\ U^{\varepsilon}_{ex,0_{|\widetilde{x}_d=\widetilde{X}_d=0}} = -P_{ex}\phi_1(\underline{\zeta}) B_1 \mathcal{U}^{\varepsilon}_{0,\mathscr{I}}, \\ U^{\varepsilon}_{ex,0_{|t\leqslant 0}} = 0, \end{cases} \tag{56}$$

where we set $\widetilde{X}_d = X_d - 1$, $\widetilde{x}_d = x_d - 1$. From Paragraph 6.1 (see Proposition 6.1) for all $0 < \varepsilon \ll 1$ the right hand side of the boundary condition in (55) (respectively (56)) is a known function in $H^{\infty}_{\natural,\gamma}(\partial\Omega_0)$ (respectively $H^{\infty}_{\natural,\gamma}(\partial\Omega_1)$ for all $\gamma > \underline{\gamma}_0$ depending only on g.

To solve these systems of equation we follow the method introduced by [9] that is we first determine the value of the double traces $x_d = X_d = 0$ and $\widetilde{x}_d = \widetilde{X}_d = 0$ and then we extend these traces for $x_d \neq 0$ and $\widetilde{x}_d \neq 0$ as boundary layers in the normal variable.

The following Lemma is a trivial generalization of the one dealing only with evanescent modes in [9] to explosive modes:

[7] The scaling $\varepsilon^{-1/2}$ for the size of the boundary layers comes from [16] and is explained in Paragraph 7.3. It permits to construct a corrector for glancing modes such that the error in the interior in $O(\varepsilon^{1/4})$ is $L^2_{\gamma}(\Omega)$. Note that this is the sharpest possible error rate.

Lemma 6.2 [9]. *We define for $X_d \geqslant 0$*

$$\mathbf{P}_{ev} U(X_d) := e^{X_d \mathscr{A}(\underline{\zeta})} \Pi_e^s U(0), \tag{57}$$

$$\mathbf{Q}_{ev} F(X_d) := \int_0^{X_d} e^{(X_d - y)\mathscr{A}(\underline{\zeta})} \Pi_e^s A_d^{-1} F(y)\, dy - \int_{X_d}^{\infty} e^{(X_d - y)\mathscr{A}(\underline{\zeta})} \Pi_e^u A_d^{-1} F(y)\, dy, \tag{58}$$

and for $\widetilde{X}_d \leqslant 0$

$$\mathbf{P}_{ex} U(\widetilde{X}_d) := e^{\widetilde{X}_d \mathscr{A}(\underline{\zeta})} \Pi_e^u U(0), \tag{59}$$

$$\mathbf{Q}_{ex} F(\widetilde{X}_d) := \int_{-\infty}^{\widetilde{X}_d} e^{(\widetilde{X}_d - y)\mathscr{A}(\underline{\zeta})} \Pi_e^s A_d^{-1} F(y)\, dy - \int_{\widetilde{X}_d}^{0} e^{(\widetilde{X}_d - y)\mathscr{A}(\underline{\zeta})} \Pi_e^u A_d^{-1} F(y)\, dy, \tag{60}$$

then for all $F \in \mathcal{P}_{ev}$ (respectively $F \in \mathcal{P}_{ex}$) the equation

$$L(\partial_{X_d})U = F \text{ for } X_d \geqslant 0, \ (\text{respectively } L(\partial_{\widetilde{X}_d})U = F \text{ for } \widetilde{X}_d \leqslant 0),$$

admits a solution reading $U = \mathbf{P}_{ev} U + \mathbf{Q}_{ev} F$ (respectively $U = \mathbf{P}_{ex} U + \mathbf{Q}_{ex} F$).

Lemma 6.2 combined with equations (55) and (56) implies that we have the conditions $U^{\varepsilon}_{ev,0} = \mathbf{P}_{ev} U^{\varepsilon}_{ev,0}$ and $U^{\varepsilon}_{ex,0} = \mathbf{P}_{ex} U^{\varepsilon}_{ex,0}$ which are comparable to the polarization condition (34) for oscillating amplitudes. We describe in the following the way to construct the evanescent amplitude $U^{\varepsilon}_{ev,0}$. The arguments are exactly the same for the explosive amplitude $U^{\varepsilon}_{ex,0}$.

From the definition of $\mathbf{P}_{ev}$ and the 'polarization' condition $U^{\varepsilon}_{ev,0} = \mathbf{P}_{ev} U^{\varepsilon}_{ex,0}$ to determine $U^{\varepsilon}_{ev,0}$ it is sufficient to determine its trace on $\{X_d = 0\}$. However by (55) we do not know this trace but only the double trace on $\{x_d = X_d = 0\}$, so that we follow the method of [9] consisting in extending this double trace for $x_d > 0$ as a boundary layer. Consequently

$$U^{\varepsilon}_{ev,0}(t, x, X_d) = \chi(x_d) e^{X_d \mathscr{A}(\underline{\zeta})} P_{ev} U^{\varepsilon}_{ev,0}(t, x', 0, 0) = \chi(x_d) e^{X_d \mathscr{A}(\underline{\zeta})} P_{ev} \phi_0(\underline{\zeta})(g - B_0 \mathcal{U}^{\varepsilon}_{0,\mathscr{O}}),$$

where $\chi \in \mathscr{C}^{\infty}(]0, 1[)$ satisfies[8] $\chi(x) = 1$ for $x \leqslant \frac{1}{4}$ and $\chi(x) = 0$ for $x \geqslant \frac{3}{4}$. So that $U^{\varepsilon}_{ev,0}$ is a solution of (55). The same kind of formula also holds for explosive amplitude.

Moreover clearly by definition of $\mathbf{P}_{ev}$ (respectively $\mathbf{P}_{ex}$) it is clear that $U^{\varepsilon}_{ev,0}(t, x, X_d) \in \mathcal{P}_{ev}$ (respectively $U^{\varepsilon}_{ex,0}(t, x, X_d) \in \mathcal{P}_{ex}$).

We sum up the results of this section in the following proposition:

Proposition 6.2. *Under Assumptions 2.1, 2.2, 2.4, assume that $\underline{\zeta} \notin \mathbb{G}$ and that Assumption 6.1 holds for $\underline{\gamma}_0$. For all $k \in \mathscr{H}$ and for all $0 < \varepsilon \ll 1$ there exist $u^{\varepsilon}_{h,0,k} \in H^{\infty}_{\natural,\gamma}(\Omega)$ for all $\gamma > \underline{\gamma}_0$ satisfying the cascade of equations (28),(31),(32),(33) written for $n = 0$ and there exist $U^{\varepsilon}_{ev,0} \in \mathcal{P}_{ev}$ and $U^{\varepsilon}_{ex,0} \in \mathcal{P}_{ex}$ satisfying the cascade of equations (28), (31), (32), (33) written for $n = 0$.*

So at this step we have determined all the amplitudes of the leading order in the ansatz (27). The following section aims to show that this construction can be repeated to higher orders to obtain an approximate solution of (24) (we postpone the justification to § 8).

[8]Note that compared to the boundary layer for glancing modes the size of the boundary layer for elliptic modes can be made independent on ε.

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7. Construction of higher order terms

In this paragraph we first sketch the construction of the amplitudes of order one in the WKB expansion. As the reader will notice, the construction for selfinteracting amplitudes is rather classical, that is we first determine the unpolarized part of the amplitudes (which only depends on the leading order amplitude) and then reiterate the construction described in Paragraph 6.1 to determine the polarized part. The determination of the evanescent or explosive amplitudes follows more or less the same ideas. More precisely we decompose the evanescent/explosive amplitude in some ‘unpolarized part’ depending only on the leading order evanescent/explosive amplitude and some ‘polarized’ part which is determined as described in Paragraph 6.3.

Finally we show in Paragraph 7.4 that these constructions can be performed at any order for selfinteracting and evanescent/explosive amplitudes if $\underline{\zeta} \notin \mathbb{G}$.

However, the situation is not so ideal when glancing modes exist. Indeed as mentioned in Paragraph 6.2, the glancing amplitudes cannot solve simultaneously the boundary conditions and the equation in the interior. This fact implies that we are able to define only one corrector ensuring a suitable rate of convergence. We refer to Paragraph 7.3 for more details.

7.1. Selfinteracting amplitudes of order one

First, in a classical setting (see for example [13]), we determine the unpolarized part of the hyperbolic amplitudes of order one, namely the $u^{\varepsilon}_{h,1,k}$ for $k \in \mathscr{H}$. In order to do so, we apply the pseudo-inverse Υ^k (see Definition 2.3) to the second equation of (28) written for $n = 0$. By definition of Υ^k we obtain that

$$\forall k \in \mathscr{H},\ (I - \Pi^k)u^{\varepsilon}_{h,1,k} = i\Upsilon^k L(\partial)u^{\varepsilon}_{h,0,k}. \tag{61}$$

The right hand side of (61) has been determined in Paragraph 6.1, so that (61) uniquely determines the unpolarized part of the selfinteracting amplitudes (moreover they are in $H^{\infty}_{\natural,\gamma}(\Omega)$ for all $\gamma > \underline{\gamma}_0$). So to conclude the construction it only remains to determine the polarized parts, namely the $\Pi^k u^{\varepsilon}_{h,1,k}$ for $k \in \mathscr{H}$.

Consider the second equation of (28) written for $n = 1$, compose by Π^k and use the trivial decomposition $I = I - \Pi^k + \Pi^k$ leads to:

$$\Pi^k L(\partial)\Pi^k u^{\varepsilon}_{h,1,k} = -\Pi^k L(\partial)(I - \Pi^k)u^{\varepsilon}_{h,1,k} \Longleftrightarrow \Pi^k L(\partial)\Pi^k u^{\varepsilon}_{h,1,k} = -i\Pi^k L(\partial)\Upsilon^k_h L(\partial)u^{\varepsilon}_{h,0,k}.$$

We can apply Lax’s lemma [8] to rewrite this equation as:

$$(\partial_t + v_k \cdot \nabla_x)\Pi^k u^{\varepsilon}_{h,1,k} = -i\Pi^k L(\partial)\Upsilon^k_h L(\partial)u^{\varepsilon}_{h,0,k}. \tag{62}$$

We again have to solve a transport equation so we reiterate the discussion depending on the type of the phase.

$\diamond$ $k \in \mathscr{I}$. In that case the transport phenomenon is incoming so that to be solved (62) only requires a boundary condition on $\partial\Gamma_0$. To determine this boundary condition we consider (31) written for $n=1$ that we write under the form:

$$B_0\left[\sum_{k\in\mathscr{I}}\Pi^k u^\varepsilon_{h,1,k}+\sum_{k\in\mathscr{G}}\Pi^k u^\varepsilon_{h,1,k}+U^\varepsilon_{ev,1|x_d=0}\right]_{|x_d=0}$$

$$=-B_0\left[\sum_{k\in\mathscr{I}}(I-\Pi^k)u^\varepsilon_{h,1,k}+\sum_{\ell\in\mathscr{O}}(I-\Pi^\ell)u^\varepsilon_{h,1,\ell}\right.$$

$$\left.+\sum_{k\in\mathscr{G}}(I-\Pi^k)u^\varepsilon_{g,1,k}+\sum_{\ell\in\mathscr{O}}\Pi^\ell u^\varepsilon_{h,1,\ell}\right]_{|x_d=0}, \tag{63}$$

and we remark that all the terms in the right hand side of (63), except the last one, are known functions in $H^\infty_{\natural,\gamma}(\partial\Omega_0)$ for all $\gamma>\underline{\gamma}_0$. Consequently applying the uniform Kreiss–Lopatinskii condition (see Assumption 2.4) and the projector P^k_h (see Definition 2.4) to (63) shows that the polarized part $\Pi^k u^\varepsilon_{h,1,k}$ for $k\in\mathscr{I}$ satisfies the transport equation (note that $P^k_h\Pi^k=\Pi^k$):

$$\begin{cases}(\partial_t+v_k\cdot\nabla_x)\Pi^k u^\varepsilon_{h,1,k}=\mathcal{F}^\varepsilon_{1,k,\mathscr{I}},\\ \Pi^k u^\varepsilon_{h,1,k_{|x_d=0}}=-P^k_h\phi_0(\underline{\zeta})B_0\left(\mathcal{U}^\varepsilon_{1,\mathscr{O}}+\mathcal{G}^\varepsilon_{1,\mathscr{I}}\right),\\ \Pi^k u^\varepsilon_{h,1,k_{|t\leqslant 0}}=-(I-\Pi^k)u^\varepsilon_{h,1,k_{|t\leqslant 0}}=0,\end{cases} \tag{64}$$

where we recall that $\mathcal{U}^\varepsilon_{1,\mathscr{O}}$ is defined in (51) and where the source terms are given by:

$$\mathcal{F}^\varepsilon_{1,k,\mathscr{I}}:=-i\Pi^k L(\partial)\Upsilon^k L(\partial)u^\varepsilon_{h,0,k},$$

$$\mathcal{G}^\varepsilon_{1,\mathscr{I}}:=\left(\sum_{k\in\mathscr{I}}(I-\Pi^k)u^\varepsilon_{h,1,k}+\sum_{\ell\in\mathscr{O}}(I-\Pi^\ell)u^\varepsilon_{h,1,\ell}+\sum_{k\in\mathscr{G}}(I-\Pi^k)u^\varepsilon_{g,1,k}\right)_{|x_d=0}.$$

$\diamond$ $\ell\in\mathscr{O}$. In that case the transport phenomenon is outgoing so that to be solved (62) only requires a boundary condition on $\partial\Gamma_1$. Reiterating essentially the same computations as the ones for the case $k\in\mathscr{I}$ we easily obtain that the polarized part of an outgoing amplitude satisfies the transport equation:

$$\begin{cases}(\partial_t+v_\ell\cdot\nabla_x)\Pi^\ell u^\varepsilon_{h,1,\ell}=\mathcal{F}^\varepsilon_{1,\ell,\mathscr{O}},\\ \Pi^\ell u^\varepsilon_{h,1,k_{|x_d=1}}=-e^{-\frac{i}{\varepsilon}\xi_\ell}P^\ell_h\phi_1(\underline{\zeta})B_1\left(\mathcal{U}^\varepsilon_{1,\mathscr{I}}+\mathcal{G}^\varepsilon_{1,\mathscr{O}}\right),\\ \Pi^\ell u^\varepsilon_{h,1,\ell_{|t\leqslant 0}}=0,\end{cases} \tag{65}$$

where the source terms are given by:

$$\mathcal{F}^\varepsilon_{1,\ell,\mathscr{O}}:=-i\Pi^\ell L(\partial)\Upsilon^\ell L(\partial)u^\varepsilon_{h,0,\ell},$$

$$\mathcal{G}^\varepsilon_{1,\mathscr{O}}:=\left(\sum_{k\in\mathscr{I}}e^{\frac{i}{\varepsilon}\xi_k}(I-\Pi^k)u^\varepsilon_{h,1,k}+\sum_{\ell\in\mathscr{O}}e^{\frac{i}{\varepsilon}\xi_\ell}(I-\Pi^\ell)u^\varepsilon_{h,1,\ell}+\sum_{k\in\mathscr{G}}e^{\frac{i}{\varepsilon}\xi_k}(I-\Pi^k)u^\varepsilon_{g,1,k}\right)_{|x_d=1}.$$

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We can repeat the same arguments as the ones described in Paragraph 6.1 to obtain a compatibility condition on $\mathcal{U}^{\varepsilon}_{1,\mathscr{I}} = \sum_{k\in\mathscr{I}} e^{\frac{i}{\varepsilon}\xi_k}\Pi^k u^{\varepsilon}_{h,1,\ell_{|x_d=1}}$. Integrating (65) along the characteristics gives:

$$\begin{aligned}\Pi^\ell u^{\varepsilon}_{h,1,\ell}(t,x) = &- e^{-\frac{i}{\varepsilon}\xi_\ell} P^\ell_h \phi_1(\underline{\zeta}) B_1 \mathcal{U}^{\varepsilon}_{1,\mathscr{I}}\left(t+\frac{1}{v_{\ell,d}}(1-x_d), x' + \frac{v'_\ell}{v_{\ell,d}}(1-x_d)\right)\\ &- e^{-\frac{i}{\varepsilon}\xi_\ell} P^\ell_h \phi_1(\underline{\zeta}) B_1 \mathcal{G}^{\varepsilon}_{1,\mathscr{O}}\left(t+\frac{1}{v_{\ell,d}}(1-x_d), x' + \frac{v'_\ell}{v_{\ell,d}}(1-x_d)\right)\\ &- \int_0^{1-x_d} \mathcal{F}^{\varepsilon}_{1,\ell,\mathscr{O}}\left(t+\frac{1}{v_{\ell,d}}(1-x_d-s), x' + \frac{v'_\ell}{v_{\ell,d}}(1-x_d-s), 1-s\right) ds,\end{aligned} \tag{66}$$

and consequently the right hand side of the boundary condition of (64) depends on $\mathcal{U}^{\varepsilon}_{1,\mathscr{I}}$, $\mathcal{G}^{\varepsilon}_{1,\mathscr{O}}$ and $\mathcal{F}^{\varepsilon}_{1,\ell,\mathscr{O}}$. Integrating again along the characteristics the transport equation gives (by linearity):

$$\begin{aligned}\Pi^k u^{\varepsilon}_{h,1,k}(t,x) = & P^k_h \phi_0(\underline{\zeta}) B_0 \sum_{\ell\in\mathscr{O}} e^{-\frac{i}{\varepsilon}\xi_\ell} P^\ell_h \phi_1(\underline{\zeta}) B_1 \mathcal{U}^{\varepsilon}_{1,\mathscr{I}}\\ &\times \left(t+\frac{1}{v_{\ell,d}} - \frac{1}{v_{k,d}}x_d, x' + \frac{v'_\ell}{v_{\ell,d}} - \frac{v'_k}{v_{k,d}}x_d\right)\\ &+ P^k_h \phi_0(\underline{\zeta}) B_0 \sum_{\ell\in\mathscr{O}} e^{-\frac{i}{\varepsilon}\xi_\ell} P^\ell_h \phi_1(\underline{\zeta}) B_1 \mathcal{G}^{\varepsilon}_{1,\mathscr{O}}\\ &\times \left(t+\frac{1}{v_{\ell,d}} - \frac{1}{v_{k,d}}x_d, x' + \frac{v'_\ell}{v_{\ell,d}} - \frac{v'_k}{v_{k,d}}x_d\right)\\ &- P^k_h \phi_0(\underline{\zeta}) B_0 \mathcal{G}^{\varepsilon}_{1,\mathscr{I}}\left(t-\frac{1}{v_{k,d}}x_d, x' - \frac{v'_k}{v_{k,d}}x_d\right)\\ &+ P^k_h \phi_0(\underline{\zeta}) B_0 \sum_{\ell\in\mathscr{O}} \int_0^1 \mathcal{F}^{\varepsilon}_{1,\ell,\mathscr{O}}(\mathbf{t}_{k,\ell}(s,x_d), \mathbf{x}'_{k,\ell}(s,x_d), 1-s)\, ds\\ &+ \int_0^{x_d} \mathcal{F}^{\varepsilon}_{1,k,\mathscr{I}}\left(t-\frac{1}{v_{k,d}}(x_d-s), x' + \frac{v'_k}{v_{k,d}}(x_d-s), s\right) ds,\end{aligned} \tag{67}$$

where we defined:

$$\mathbf{t}_{k,\ell}(s,x_d) := t + \frac{1}{v_{\ell,d}}(1-s) + \frac{1}{v_{k,d}}x_d \quad \text{and} \quad \mathbf{x}'_{k,\ell}(s,x_d) := x' + \frac{v'_\ell}{v_{\ell,d}}(1-s) + \frac{v'_k}{v_{k,d}}x_d.$$

Multiplying (67) by $e^{\frac{i}{\varepsilon}\xi_k}$ and summing over $k \in \mathscr{I}$ we obtain the compatibility condition:

$$\begin{aligned}(I - \mathcal{T}^{\varepsilon}(\underline{\zeta}))\mathcal{U}^{\varepsilon}_{\mathscr{I},1} = & \mathcal{T}^{\varepsilon}(\underline{\zeta})\mathcal{G}^{\varepsilon}_{1,\mathscr{O}} - \sum_{k\in\mathscr{I}} e^{\frac{i}{\varepsilon}\xi_k} P^k_h \phi_0(\underline{\zeta}) B_0 \mathcal{G}^{\varepsilon}_{1,\mathscr{I}}\left(t-\frac{1}{v_{k,d}}, x' - \frac{v'_k}{v_{k,d}}\right)\\ &+ \sum_{k\in\mathscr{I}} e^{\frac{i}{\varepsilon}\xi_k} P^k_h \phi_0(\underline{\zeta}) B_0 \sum_{\ell\in\mathscr{O}} \int_0^1 \mathcal{F}^{\varepsilon}_{1,\ell,\mathscr{O}}(\mathbf{t}_{k,\ell}(s,x_d), \mathbf{x}'_{k,\ell}(s,x_d), 1-s)\, ds\\ &+ \sum_{k\in\mathscr{I}} e^{\frac{i}{\varepsilon}\xi_k} \int_0^1 \mathcal{F}^{\varepsilon}_{1,k,\mathscr{I}}\left(t-\frac{1}{v_{k,d}}(1-s), x' + \frac{v'_k}{v_{k,d}}(1-s), s\right) ds,\end{aligned} \tag{68}$$

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where $\mathcal{T}^{\varepsilon}(\underline{\zeta})$ is defined in (47). Remark that all the terms in the right hand side of (68) are in $E_h^s(\underline{\zeta})$ (recall that by definition $\mathcal{F}^{\varepsilon}_{1,k,\mathscr{I}}$ reads $\mathcal{F}^{\varepsilon}_{1,k,\mathscr{I}} = \Pi^k \widetilde{F^{\varepsilon}_{1,k,\mathscr{I}}}$ for $k \in \mathscr{I}$) so we can use Assumption 6.1 in (68) to determine the value of $\mathcal{U}^{\varepsilon}_{\mathscr{I},1}$ (in terms of some known functions in $H^{\infty}_{\natural,\gamma}(\partial\Omega_1, E_h^s(\underline{\zeta}))$, namely $\mathcal{G}^{\varepsilon}_{1,\mathscr{O}}$, $\mathcal{G}^{\varepsilon}_{1,\mathscr{I}}$, $\mathcal{F}^{\varepsilon}_{1,k,\mathscr{I}}$ and $\mathcal{F}^{\varepsilon}_{1,\ell,\mathscr{O}}$). Plugging this value in (66) and (67) gives the value of the polarized part of the amplitude for selfinteracting phases namely the $\Pi^k u_{n,1,k}$ for $k \in \mathscr{H}$. This concludes the construction of the selfinteracting amplitudes of order one.

7.2. Evanescent and explosive amplitudes of order one

We now turn to the determination of evanescent and explosive amplitudes of order one. Considering (28) written for $n = 0$ we obtain that

$$L(\partial_{X_d})U^{\varepsilon}_{ev,1} = -L(\partial)U^{\varepsilon}_{ev,0} \text{ for } X_d \geqslant 0 \quad \text{and} \quad L(\partial_{\widetilde{X}_d})U^{\varepsilon}_{ex,1} = -L(\partial)U^{\varepsilon}_{ex,0} \quad \text{for } \widetilde{X}_d \leqslant 0. \tag{69}$$

From Proposition 6.2 the known function $U^{\varepsilon}_{ev,0} \in \mathcal{P}_{ev}$ (respectively $U^{\varepsilon}_{ex,0} \in \mathcal{P}_{ex}$) and it is clear that $\mathcal{P}_{ev}$ (respectively $\mathcal{P}_{ex}$) is stable by $L(\partial)$. Consequently the right hand sides in (70) are in $\mathcal{P}_{ev}$ and $\mathcal{P}_{ex}$ respectively, and we can apply Lemma 6.2 to obtain the decompositions:

$$U^{\varepsilon}_{ev,1} = \mathbf{P}_{ev}U^{\varepsilon}_{ev,1} - \mathbf{Q}_{ev}L(\partial)U^{\varepsilon}_{ev,0} \quad \text{and} \quad U^{\varepsilon}_{ex,1} = \mathbf{P}_{ex}U^{\varepsilon}_{ex,1} - \mathbf{Q}_{ex}L(\partial)U^{\varepsilon}_{ex,0}. \tag{70}$$

To determine $U^{\varepsilon}_{ev,1}$ (respectively $U^{\varepsilon}_{ex,1}$) it is sufficient to determine $\mathbf{P}_{ev}U^{\varepsilon}_{ev,1}$ (respectively $\mathbf{P}_{ex}U^{\varepsilon}_{ex,1}$). This is done mainly in the same way that in Paragraph 6.3. We briefly sketch the construction of $\mathbf{P}_{ev}U^{\varepsilon}_{ev,1}$ for completeness. Recall that by definition of $\mathbf{P}_{ev}$ (see (57)) $\mathbf{P}_{ev}U^{\varepsilon}_{ev,1}$ is known if and only if we know the value of its trace on $\{X_d = 0\}$ to determine this trace we consider the cascade of equation (31) written for $n = 1$ and then we extend the double trace on $\{X_d = x_d = 0\}$ as a trace on $\{X_d = 0\}$ only. The second equation of (31) written for $n = 1$ reads (after decomposition on the stable/unstable subspaces and by the uniform Kreiss–Lopatinskii condition on $\partial\Gamma_0$ and the composition by P_e^s):

$$U^{\varepsilon}_{ev,1_{|x_d=X_d=0}} = -P_e^s\phi_0(\underline{\zeta})B_0 \sum_{k\in\mathscr{O}} u^{\varepsilon}_{h,1,k_{|x_d=0}},$$

where from Paragraph 7.1 the right hand side is a known function in $H^{\infty}_{\natural,\gamma}(\Omega)$ for all $\gamma > \underline{\gamma}_0$.

Consequently by definition of $\mathbf{P}_{ev}$ we obtain (by using a function χ as before) that:

$$\mathbf{P}_{ev}U^{\varepsilon}_{ev,1}(t, x, X_d) = -\chi(x_d)e^{X_d\mathscr{A}(\underline{\zeta})}P_e^s\phi_0(\underline{\zeta})B_0 \sum_{k\in\mathscr{O}} u^{\varepsilon}_{h,1,k_{|x_d=0}},$$

which concludes the construction of evanescent amplitude of order one. Once again by definition of $\mathbf{P}_{ev}$ and $\mathbf{Q}_{ev}$ it is clear that $U^{\varepsilon}_{ev,1} \in \mathcal{P}_{ev}$. The same permits to show that $U^{\varepsilon}_{ex,1} \in \mathcal{P}_{ex}$.

7.3. A corrector for glancing amplitudes

In this paragraph we follow the method of [16] to construct a corrector for glancing modes such that the geometric optics expansion is a good approximation of the exact solution up to an admissible rate of convergence (that is $O(\varepsilon^{1/4})$).

To this aim we recall the equations governing glancing amplitudes of order one in (28) (namely the fourth and the fifth equation of (28) in which we reintroduced the power of ε for convenience) that is:

$$\begin{cases} \varepsilon^0 \left(i\mathscr{L}(d\varphi_k) u^\varepsilon_{g,1,k} + L(\partial) u^\varepsilon_{g,0,k} \right) = 0 & \forall k \in \mathscr{G}, \\ \varepsilon L(\partial) u^\varepsilon_{g,1,k} = 0 & \forall k \in \mathscr{G}. \end{cases} \tag{71}$$

We decompose the first equation of (71) as:

$$\varepsilon^0 (i\mathscr{L}(d\varphi_k) u_{g,1,k} + L(\partial)\Pi^k u^\varepsilon_{g,0,k}) = \varepsilon^0 (i\mathscr{L}(d\varphi_k) u^\varepsilon_{g,1,k} + \Pi^k L(\partial)\Pi^k u^\varepsilon_{g,0,k} + (I-\Pi^k) L(\partial) u^\varepsilon_{g,0,k}) = 0.$$

This equation has exactly the same form as the one for hyperbolic amplitudes except that we chose the leading order glancing mode in such a way that it satisfies the boundary condition (to ensure an error at least of size $O(\varepsilon)$ on the boundary) but not the interior equation so that compared to hyperbolic modes $\Pi^k L(\partial)\Pi^k u_{g,0,k}$ is not zero and gives rise to an extra error in the interior.

However as for oscillating modes we compose the first equation of (71) by Υ^k the partial inverse of $\mathscr{L}(d\varphi_k)$ and we define:

$$u^\varepsilon_{g,1,k} = (I-\Pi^k) u^\varepsilon_{g,1,k} := i\Upsilon^k L(\partial) u^\varepsilon_{g,0,k}. \tag{72}$$

By doing this we obtain that for all $k \in \mathscr{G}$

$$L(\partial)(e^{i\frac{\varphi_k}{\varepsilon}} (u^\varepsilon_{g,0,k} + \varepsilon u^\varepsilon_{g,1,k})) = \Pi^k L(\partial)\Pi^k u^\varepsilon_{g,0,k} + \varepsilon L(\partial) u^\varepsilon_{g,1,k}. \tag{73}$$

The term of order ε^0 in the right hand side of (73) may seem to be alarming to obtain a good error estimate for glancing modes but thanks to the choice of the boundary layer in (54) it is not. Indeed, from (54) and using the fact that for all $k \in \mathscr{G}$, $\Pi^k L(\partial)\Pi^k = \partial_t + v'_k \cdot \nabla_{x'}$ (so that $\Pi^k L(\partial)\Pi^k$ does not act on the x_d variable) we have that:

$$\Pi^k L(\partial)\Pi^k u^\varepsilon_{g,0,k} = \chi_\varepsilon(x_d)\mathscr{B}_0(t,x') + (1-\chi_\varepsilon(x_d))\mathscr{B}_1(t,x'),$$

where $\chi_\varepsilon(x_d) := \chi(\varepsilon^{-1/2} x_d)$ and where from Proposition 6.1 $\mathscr{B}_0(t,x')$, $\mathscr{B}_1(t,x')$ are in $H^\infty_{\natural,\gamma}(\mathbb{R}_t \times \mathbb{R}^{d-1}_{x'})$ for all $\gamma > \underline{\gamma}_0$. So a simple change of variables shows that $\Pi^k L(\partial)\Pi^k u^\varepsilon_{g,0,k}$ is $O(\varepsilon^{1/4})$ in $L^2_\gamma(\Omega)$ for all $\gamma > \underline{\gamma}_0$.

We now turn to the term $\varepsilon L(\partial) u^\varepsilon_{g,1,k}$ in the right hand side of (73). From (54) and (72), $u^\varepsilon_{g,1,k}$ reads under the form:

$$u^\varepsilon_{g,1,k} = \varepsilon^{-1/2} \left(\chi'_\varepsilon(x_d)\widetilde{\mathscr{B}}_0(t,x') + (1-\chi'_\varepsilon(x_d))\widetilde{\mathscr{B}}_1(t,x') \right) + \text{h.o.t.},$$

where $\widetilde{\mathscr{B}}_0, \widetilde{\mathscr{B}}_1 \in H^\infty_{\natural,\gamma}(\mathbb{R}_t \times \mathbb{R}^{d-1}_{x'})$ for all $\gamma > \underline{\gamma}_0$. From which we immediately deduce that $\varepsilon L(\partial) u^\varepsilon_{g,1,k}$ is $O(\varepsilon^{1/4})$ in $L^2_\gamma(\Omega)$ for all $\gamma \geqslant \underline{\gamma}_0$.

By construction of $u^\varepsilon_{g,0,k_{|x_d=0}} = g^\varepsilon$ it follows:

Proposition 7.1. *Assume that the hyperbolic strip problem (24)*[9] *satisfies Assumptions 2.1, 2.2, 2.4, 2.3 and 6.1 for some* $\underline{\gamma}_0 \geqslant 0$. *Then with* $u^{\varepsilon}_{g,0,k}$ *defined in (54) and* $u^{\varepsilon}_{g,1,k}$ *defined in (72) we have*

$$\begin{cases} L(\partial)\left(\sum_{k\in\mathscr{G}} e^{i\frac{\varphi_k}{\varepsilon}}(u^{\varepsilon}_{g,0,k}+\varepsilon u^{\varepsilon}_{g,1,k})\right) = O_{\Omega}(\varepsilon^{1/4}) & \text{in } \Omega, \\ B_0\left(\sum_{k\in\mathscr{G}} e^{i\frac{\varphi_k}{\varepsilon}}(u^{\varepsilon}_{g,0,k}+\varepsilon u^{\varepsilon}_{g,1,k})\right)_{|x_d=0} = O_{\partial\Omega_0}(\varepsilon) & \text{on } \partial\Omega_0, \\ B_1\left(\sum_{k\in\mathscr{G}} e^{i\frac{\varphi_k}{\varepsilon}}(u^{\varepsilon}_{g,0,k}+\varepsilon u^{\varepsilon}_{g,1,k})\right)_{|x_d=1} = O_{\partial\Omega_1}(\varepsilon) & \text{on } \partial\Omega_1, \\ \left(\sum_{k\in\mathscr{G}} e^{i\frac{\varphi_k}{\varepsilon}}(u^{\varepsilon}_{g,0,k}+\varepsilon u^{\varepsilon}_{g,1,k})\right)_{|t\leqslant 0} = 0 & \text{on } \Gamma, \end{cases} \tag{74}$$

where $O_X(\cdot)$ *is understand in* $L^2_{\gamma}(X)$ *for all* $\gamma > \underline{\gamma}_0$.

7.4. Higher order non-glancing amplitudes

As mentioned in Paragraph 7.3 when the frequency $\underline{\zeta}$ admits glancing modes then we can construct a first order corrector such that the error (in the interior) is $O(\varepsilon^{1/4})$ in $L^2_{\gamma}(\Omega)$. However it seems difficult to reiterate this method to construct a second order corrector giving rise to an admissible error (the reason remains that glancing modes cannot solve the interior and the boundary equations simultaneously).

However when $\mathscr{G} = \emptyset$ we can repeat the construction made in Paragraphs 7.1 and 7.2 to define an arbitrary number of correctors. In this paragraph we briefly describe the way to proceed.

Assume that all the terms $u^{\varepsilon}_{h,n,k}$, $k \in \mathscr{H}$ and $U^{\varepsilon}_{ev,n}$, $U^{\varepsilon}_{ex,n}$ appearing in (27) have been constructed up to some order $n_0 \geqslant 1$. We sketch the construction of $u^{\varepsilon}_{h,n_0+1,k}$ for $k \in \mathscr{H}$, $U^{\varepsilon}_{ev,n_0+1}$ and $U^{\varepsilon}_{ex,n_0+1}$.

- First the second equation of (28) written for $n = n_0$ gives the unpolarized part of the hyperbolic amplitude $u^{\varepsilon}_{h,n_0+1,k}$ (so that it is sufficient to determine the polarized part) and the 7th (respectively 8th) equation of (28) combined with Lemma 6.2 implies that to determine $U^{\varepsilon}_{ev,n_0+1}$ (respectively $U^{\varepsilon}_{ex,n_0+1}$) it is sufficient to determine $\mathbf{P}_{ev} U^{\varepsilon}_{ev,n_0+1}$ (respectively $\mathbf{P}_{ex} U^{\varepsilon}_{ex,n_0+1}$) (see (57) and (59)).
- From Lax's lemma [8] and Lemma 6.2 each of the terms mentioned above require only a boundary condition (on $\partial\Gamma_0$ for the $u^{\varepsilon}_{h,n_0+1,k}$, $k \in \mathscr{I}$ and $U^{\varepsilon}_{ev,n_0+1}$ and on $\partial\Gamma_1$ for the $u^{\varepsilon}_{h,n_0+1,\ell}$, $\ell \in \mathscr{O}$ and $U^{\varepsilon}_{ex,n_0+1}$). Identify in (31) and (32) (written for $n = n_0$) the stable and the unstable parts of the traces show that the 'double trace' of evanescent and explosive amplitudes only depends on the trace of the hyperbolic amplitudes. Consequently we shall determine the traces of the hyperbolic amplitudes first.
- To determine the trace of the oscillating amplitudes we remark that by the uniform Kreiss–Lopatinskii condition on each side the boundary conditions (31) and (32)

[9] In fact in this setting as we only construct a first order corrector in (27) it is in fact sufficient to take $g \in H^2_{\sharp}(\partial\Omega_0)$ to ensure that the $u^{\varepsilon}_{g,1,k} \in H^1_{\gamma}(\Omega)$ for all $\gamma > \underline{\gamma}_0$ in such a way that the previous discussion makes sense.

(written for $n = n_0$) can be written under the form:

$$\begin{cases} \Pi^k u^\varepsilon_{h,n_0+1,k_{|x_d=0}} = -P^k_n \phi_0(\underline{\zeta}) B_0 \mathcal{U}^\varepsilon_{n_0+1,\mathscr{O}} + F_0 & k \in \mathscr{I} \\ \Pi^\ell u^\varepsilon_{h,n_0+1,\ell_{|x_d=1}} = -P^\ell_n \phi_1(\underline{\zeta}) B_1 \mathcal{U}^\varepsilon_{n_0+1,\mathscr{I}} + F_1 & \ell \in \mathscr{O}, \end{cases}$$

where F_0 and (respectively F_1) is a given source term that depends on the $u^\varepsilon_{h,n,k}$ for $k \in \mathscr{H}$ and $n \leqslant n_0$, on $(I - \Pi^k) u^\varepsilon_{h,n_0+1,k_{|x_d=0}}$ (respectively $(I - \Pi^\ell) u^\varepsilon_{h,n_0+1,\ell_{|x_d=1}}$, $\ell \in \mathscr{O}$) but not on $\Pi^k u^\varepsilon_{h,n_0+1,k_{|x_d=0}}$ (respectively $\Pi^\ell u^\varepsilon_{h,n_0+1,\ell_{|x_d=1}}$).
Reiterate exactly the same kind of computations as the ones described in Paragraphs 6.1 and 7.1 leads to the compatibility condition:

$$(I - \mathcal{T}(\underline{\zeta})) \mathcal{U}^\varepsilon_{\mathscr{I},n_0+1} = F_{n_0+1},$$

where F_{n_0+1} is a known function in $H^\infty_{\natural,\gamma}(\partial\Omega_1, E^s_h(\underline{\zeta}))$ for all $\gamma > \underline{\gamma}_0$. From Assumption 6.1 we determine $\mathcal{U}^\varepsilon_{\mathscr{I},n_0+1}$ and then each oscillating amplitude $u^\varepsilon_{h,n_0+1,k}$ for $k \in \mathscr{H}$ by resolution of transport equations.

- The final step is to construct the 'polarized' parts of the evanescent/explosive amplitude (that is $\mathbf{P}_{ev} U^\varepsilon_{ev,n_0+1}$ and $\mathbf{P}_{ex} U^\varepsilon_{ex,n_0+1}$). Is it done exactly as it has been done in Paragraphs 6.3 and 7.2. More precisely the knowledge of the traces of the oscillating amplitudes gives the knowledge of the 'double' traces of U^ε_{ev,n_0+1} and U^ε_{ex,n_0+1} then we are free to extend these double traces in simple ones thanks to the cut-off function χ.

This concludes the construction of the amplitudes at any order in the particular framework where $\mathscr{G} = \emptyset$ to sum up we give the following proposition:

Proposition 7.2. *Under Assumptions 2.1, 2.2 and 2.4 also assume that $\mathscr{G} = \emptyset$ and that Assumption 6.1 holds for some $\underline{\gamma}_0 \geqslant 0$. Then for all $n \in \mathbb{N}$, for all $k \in \mathscr{H}$ there exist $u^\varepsilon_{h,n,k} \in H^\infty_{\natural,\gamma}(\Omega)$ for all $\gamma > \underline{\gamma}_0$ and $U^\varepsilon_{ev,n} \in \mathcal{P}_{ev}$, $U^\varepsilon_{ex,n} \in \mathcal{P}_{ex}$ satisfying the cascades of equations (28), (31), (32) and (33).*

8. Proofs of the main results

In this paragraph we give two justifications of the geometric optics expansion depending on the kind of the frequency $\underline{\zeta}$.

As explained in Paragraph 7.3, when the frequency $\underline{\zeta}$ involves glancing modes (that is to say $\mathscr{G} \neq \emptyset$) then the error between the approximate solution given by the WKB expansion and the exact solution of (24) is $O(\varepsilon^{1/4})$ because of the glancing amplitudes of order one, namely the $u^\varepsilon_{g,1,k}$.

Whereas when the frequency $\underline{\zeta}$ does not involve glancing modes, the arguments described in Paragraphs 7.1, 7.2 and 7.4 show that we can construct the amplitudes at any order so that the error between this expansion and the exact solution of (24) is of order $O(\varepsilon^{N_0+1})$, where N_0 stands for the number of terms in the geometric optics expansion.

We first consider the case where $\mathscr{G} = \emptyset$. In this framework we define an approximate solution of u^ε by: for $N_0 \in \mathbb{N}$

$$u^\varepsilon_{app,N_0} := \sum_{n=0}^{N_0} \sum_{k\in\mathscr{H}} e^{\frac{i}{\varepsilon}\varphi_k(t,x)} \varepsilon^n u^\varepsilon_{h,n,k}(t,x) + \sum_{n=0}^{N_0} e^{\frac{i}{\varepsilon}\psi(t,x')} \varepsilon^n \left(U^\varepsilon_{ev,n}\left(t,x,\frac{x_d}{\varepsilon}\right) + U^\varepsilon_{ex,n}\left(t,x,\frac{x_d-1}{\varepsilon}\right)\right), \tag{75}$$

where the terms appearing in the right hand side of (75) are defined in Proposition 7.2. We are now in a position to show Theorem 3.3.

By construction of u^ε_{app,N_0}, $u^\varepsilon_{app,N_0+1} - u^\varepsilon$ satisfies the hyperbolic boundary value problem

$$\begin{cases} L(\partial)(u^\varepsilon_{app,N_0+1} - u^\varepsilon) = \varepsilon^{N_0+1} f^\varepsilon_{N_0+1} & \text{in } \Omega, \\ B_0(u^\varepsilon_{app,N_0+1} - u^\varepsilon)_{|x_d=0} = 0 & \text{on } \partial\Omega_0, \\ B_1(u^\varepsilon_{app,N_0+1} - u^\varepsilon)_{|x_d=1} = 0 & \text{on } \partial\Omega_1, \\ (u^\varepsilon_{app,N_0+1} - u^\varepsilon)_{|t\leqslant 0} = 0 & \text{on } \Gamma, \end{cases}$$

where we defined

$$f^\varepsilon_{N_0+1} := \sum_{k\in\mathscr{H}} e^{\frac{i}{\varepsilon}\varphi_k} L(\partial) u^\varepsilon_{h,N_0+1,k} + e^{\frac{i}{\varepsilon}\psi}\left(L(\partial) U^\varepsilon_{ev,N_0+1}\left(t,x,\frac{x_d}{\varepsilon}\right) + L(\partial) U^\varepsilon_{ex,N_0+1}\left(t,x,\frac{x_d-1}{\varepsilon}\right)\right).$$

By construction and from Assumption 6.1 the terms composing $f^\varepsilon_{N_0+1}$ are $H^\infty_{\natural,\gamma}(\Omega)$ for all $\gamma > \underline{\gamma}_0$ (because the $u^\varepsilon_{h,N_0+1,k} \in H^\infty_{\natural,\gamma}(\Omega)$ for all $\gamma > \underline{\gamma}_0$ independently on ε). Consequently $f^\varepsilon_{N_0+1}$ is in $H^\infty_{\natural,\gamma}(\Omega)$ for all $\gamma > \underline{\gamma}_0$ so that from the energy estimate (15) we obtain:

$$\|u^\varepsilon - u^\varepsilon_{app,N_0+1}\|^2_{L^2_\gamma(\Omega)} \leqslant C\varepsilon^{N_0+1},$$

for all $\gamma > \gamma_0$. We then conclude to (19) by the triangle inequality.

We now turn to the case where $\mathscr{G} \neq \emptyset$. We include in (75) the contribution of glancing modes and restrict the expansion to the order one to define:

$$u^\varepsilon_{app,glan} := \sum_{n=0}^{1} \sum_{k\in\mathscr{H}} e^{\frac{i}{\varepsilon}\varphi_k(t,x)} \varepsilon^n u^\varepsilon_{h,n,k}(t,x) + \sum_{n=0}^{1} \sum_{k\in\mathscr{G}} e^{\frac{i}{\varepsilon}\varphi_k(t,x)} \varepsilon^n u^\varepsilon_{g,n,k}(t,x) + \sum_{n=0}^{1} e^{\frac{i}{\varepsilon}\psi(t,x')} \varepsilon^n \left(U^\varepsilon_{ev,n}\left(t,x,\frac{x_d}{\varepsilon}\right) + U^\varepsilon_{ex,n}\left(t,x,\frac{x_d-1}{\varepsilon}\right)\right), \tag{76}$$

where the extra terms $u^\varepsilon_{g,0,k}$ and $u^\varepsilon_{g,1,k}$ are defined in (54) and (72), respectively.

Then theorem 3.3 is an immediate corollary of Theorem 3.2 and Proposition 7.1.

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9. Study of Assumption 6.1

9.1. Sufficient conditions

In this paragraph we study Assumption 6.1, that is to say that there exists some $\overline{\gamma}_0 \geqslant 0$ such that the operator $(I - \mathcal{T}^{\varepsilon}(\underline{\zeta}))$ is invertible on $H^{\infty}_{\natural,\gamma}(\mathbb{R}_t \times \mathbb{R}^{d-1}_{x'}, E^s_h(\underline{\zeta}))$ with values in $H^{\infty}_{\natural,\gamma}(\mathbb{R}_t \times \mathbb{R}^{d-1}_{x'}, E^s_h(\underline{\zeta}))$ for all $\gamma > \overline{\gamma}_0$. For convenience we recall that $\mathcal{T}^{\varepsilon}(\underline{\zeta})$ is defined by:

$$(\mathcal{T}^{\varepsilon}(\underline{\zeta})f)(t,x') := \sum_{k\in\mathscr{I},\ell\in\mathscr{O}} e^{\frac{i}{\varepsilon}(\xi_k-\xi_\ell)}\mathcal{R}_{k,\ell}(\underline{\zeta})f\left(t-\alpha_{k,\ell}, x'+\beta_{k,\ell}\right) \tag{77}$$

where we set

$$\mathcal{R}_{k,\ell}(\underline{\zeta}) := P^k_h\phi_0(\underline{\zeta})B_0P^{\ell}_h\phi_1(\underline{\zeta})B_1,\ \alpha_{k,\ell} := \frac{1}{v_{\ell,d}} - \frac{1}{v_{k,d}} \quad \text{and} \quad \beta_{k,\ell} := \frac{v'_\ell}{v_{\ell,d}} - \frac{v'_k}{v_{k,d}}.$$

Clearly when $f \in H^{\infty}_{\natural,\gamma}(\mathbb{R}_t \times \mathbb{R}^{d-1}_{x'}, E^s_h(\underline{\zeta}))$ then so do $(I - \mathcal{T}^{\varepsilon}(\underline{\zeta}))f$ (because the derivatives only apply on f and because by definition $v_{\ell,d} < 0$ for $\ell \in \mathscr{O}$ and $v_{k,d} > 0$ for $k \in \mathscr{I}$ so that $t - \alpha_{k,\ell} < t$).

First let us note that when there is no selfinteracting phase then the geometric optics expansion does not have any exponential growth in time. Consequently, this seems to indicate that the exponential growth in time of the exact solution is linked to the selfinteracting modes.

Moreover it is clear from (77) that $\mathcal{T}^{\varepsilon}(\underline{\zeta})$ is compact so that Fredholm alternative applies and we have the following proposition.

Proposition 9.1. *Assumption 6.1 holds if and only if there exists $\overline{\gamma}_0 > 0$ such that for all $0 < \varepsilon \ll 1$, $(I - \mathcal{T}^{\varepsilon}(\underline{\zeta}))$ is one to one on $H^{\infty}_{\natural,\gamma}(\mathbb{R}_t \times \mathbb{R}^{d-1}_{x'}, E^s_h(\underline{\zeta}))$ for all $\gamma \geqslant \overline{\gamma}_0$.*

The simplest way to show that $(I - \mathcal{T}^{\varepsilon}(\underline{\zeta}))$ is invertible over $H^{\infty}_{\natural,\gamma}(\mathbb{R}_t \times \mathbb{R}^{d-1}_{x'}, E^s_h(\underline{\zeta}))$ with values in $H^{\infty}_{\natural,\gamma}(\mathbb{R}_t \times \mathbb{R}^{d-1}_{x'}, E^s_h(\underline{\zeta}))$ is of course to show that $\mathcal{T}(\underline{\zeta})$ is a contraction on $H^{\infty}_{\natural,\gamma}(\mathbb{R}_t \times \mathbb{R}^{d-1}_{x'}, E^s_h(\underline{\zeta}))$. In Paragraphs 10.1.1 and 10.1.2 we give some examples of such a situation.

From the particular expression of $\mathcal{T}^{\varepsilon}(\underline{\zeta})$ it is sufficient to consider the $L^2_\gamma(\mathbb{R}_t \times \mathbb{R}^{d-1}_{x'})$-norm.

Proposition 9.2. *Let $\overline{\gamma}_0$ be such that*[10]

$$\sqrt{\sum_{k\in\mathscr{I},\ell\in\mathscr{O}} \|\mathcal{R}_{k,\ell}(\underline{\zeta})\|^2} < e^{\overline{\gamma}_0 \min_{k\in\mathscr{I},\ell\in\mathscr{O}} \alpha_{k,l}}, \tag{78}$$

then $\mathcal{T}^{\varepsilon}(\underline{\zeta})$ is a contraction on $H^{\infty}_{\natural,\gamma}(\mathbb{R}_t \times \mathbb{R}^{d-1}_{x'}, E^s_h(\underline{\zeta}))$ for all $\gamma > \overline{\gamma}_0$ and consequently Assumption 6.1 is satisfied.

[10] Such $\overline{\gamma}_0$ always exists because $\min_{k\in\mathscr{I},\ell\in\mathscr{O}} \alpha_{k,l} > 0$.

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Proof. As already mentioned it is sufficient to consider the $L^2_\gamma(\mathbb{R}_t \times \mathbb{R}^{d-1}_{x'})$-norm of $\mathcal{T}^\varepsilon(\underline{\zeta})$. We have for $f \in L^2_\gamma(\mathbb{R}_t \times \mathbb{R}^{d-1}_{x'})$

$$\begin{aligned}\|\mathcal{T}^\varepsilon(\underline{\zeta}) f\|^2_{L^2_\gamma(\mathbb{R}_t\times\mathbb{R}^{d-1}_{x'})} &\leqslant \sum_{k\in\mathscr{I},\ell\in\mathscr{O}} \int_{\mathbb{R}_+\times\mathbb{R}^{d-1}} e^{-2\gamma t}|\mathcal{R}_{k,\ell}(\underline{\zeta}) f(t-\alpha_{k,\ell}, x'-\beta_{k,\ell})|^2\,dt\,dx',\\ &\leqslant \sum_{k\in\mathscr{I},\ell\in\mathscr{O}} e^{-2\alpha_{k,\ell}\gamma}\int_{\mathbb{R}_+\times\mathbb{R}^{d-1}} e^{-2\gamma t}|\mathcal{R}_{k,\ell}(\underline{\zeta}) f(t,x')|^2\,dt\,dx',\\ &\leqslant \|f\|^2_{L^2_\gamma(\mathbb{R}_t\times\mathbb{R}^{d-1}_{x'})}\sum_{k\in\mathscr{I},\ell\in\mathscr{O}} e^{-2\alpha_{k,\ell}\gamma}\|\mathcal{R}_{k,\ell}(\underline{\zeta})\|^2.\end{aligned}$$

So that if we choose $\overline{\gamma}_0 \geqslant 0$ large enough such that $\sqrt{\sum_{k\in\mathscr{I},\ell\in\mathscr{O}}\|\mathcal{R}_{k,\ell}(\underline{\zeta})\|^2} < e^{\overline{\gamma}_0 \min_{k\in\mathscr{I},\ell\in\mathscr{O}}\alpha_{k,l}}$ then $\mathcal{T}^\varepsilon(\underline{\zeta})$ is a contraction on $H^\infty_{\natural,\gamma}(\mathbb{R}_t\times\mathbb{R}^{d-1}_{x'}, E^s_h(\underline{\zeta}))$ for all $\gamma > \overline{\gamma}_0$. □

Remark. • We note that if $\overline{\gamma}_0 = 0$ in (78) then Assumption 6.1 holds with $\overline{\gamma}_0 = 0$ and consequently the approximate solution given by the geometric optics expansion (76) or (75) admits a lower exponential growth in time (so that it can be a good approximation of a solution which is lower exponentially strongly well-posed).

- In (78) the term $\min_{k\in\mathscr{I},\ell\in\mathscr{O}}\alpha_{k,\ell}$ is the minimal time to perform a full regenerating reflection.
- In the particular setting where $\#\mathscr{I} = \#\mathscr{O} = 1$ (meaning that there is only one selfinteraction path of phases) then (78) becomes $\|\mathcal{R}(\underline{\zeta})\| < e^{\alpha\overline{\gamma}_0}$, where α is the time needed to perform a full regenerating reflection. In particular when $\overline{\gamma}_0 = 0$ this condition is nothing but asking that the coefficient of reflection for a complete circuit is less than one so that the energy decreases after a complete circuit.
  This condition agrees with the intuition that if the energy increases after one complete circuit then the associated solution should have an exponential growth in time depending on the time needed to perform a complete circuit.
- In [1] one of the conditions characterizing the lower exponentially strongly well-posed problems, namely the uniform inversibility of $(I - \mathscr{T}(\zeta))$ on $E^s(\zeta)$, can be explicit as:

$$\mathscr{T}(\zeta) = \phi_0(\zeta)B_0 e^{-\mathscr{A}(\zeta)}\phi_1(\zeta)B_1 e^{\mathscr{A}(\zeta)}.$$

  So that from this expression it immediately follows that the condition used to construct the WKB expansion (that is Assumption 6.1) is a microlocalized version of the condition (16) on hyperbolic modes (and only on hyperbolic modes).

Proposition 9.2 has the interesting counterpart to show that the solution of the strip problem (1) admits a WKB expansion. Indeed consider a problem which is strongly well-posed (in the sense of Definition 3.1) for some $\gamma_0 > 0$ then it is also strongly well-posed for all $\widetilde{\gamma}_0 \geqslant \gamma_0$. In particular it holds for the $\widetilde{\gamma}_0$ satisfying $\widetilde{\gamma}_0 \geqslant \overline{\gamma}_0$, where $\overline{\gamma}_0 \geqslant 0$ satisfies Proposition 9.2. Consequently Theorem 3.2 or 3.3 applies.

However let us stress that in this argument the maximal exponential growth in time of the solution may not be sharp because we are assuming that $\widetilde{\gamma}_0$ is large enough. Consequently the lowest $\overline{\gamma}_0$ is, the best (in terms of the L^2_γ spaces) the approximation given by the geometric optics expansion is. The following Paragraph is devoted to this study in the simplest possible case of selfinteraction.

To conclude this Paragraph let us notice that if one considers geometric optics expansions for hyperbolic strip problems in finite time that is to say for the following equation

$$\begin{cases} L(\partial)u^\varepsilon := \partial_t u^\varepsilon + \sum_{j=1}^d A_j \partial_j u^\varepsilon = 0 & \text{for } (t, x', x_d) \in]-\infty, T] \times \mathbb{R}^{d-1} \times]0, 1[, \\ B_0 u^\varepsilon_{|x_d=0} = g^\varepsilon & \text{for } (t, x') \in]-\infty, T] \times \mathbb{R}^{d-1}, \\ B_1 u^\varepsilon_{|x_d=1} = 0 & \text{for } (t, x') \in]-\infty, T] \times \mathbb{R}^{d-1}, \\ u^\varepsilon_{|t\leqslant 0} = 0 & \text{for } (x', x_d) \in \mathbb{R}^{d-1} \times [0, 1], \end{cases} \tag{79}$$

where $T > 0$ stands for a finite time of resolution. Then in this setting Assumption 6.1 is trivially satisfied and consequently to construct the WKB expansion only the uniform Kreiss–Lopatinskii condition on both sides is necessary. Indeed from the expression of $\mathcal{T}^\varepsilon(\underline{\zeta})$ it is clear that for $p = p(T)$ large enough $(\mathcal{T}^\varepsilon(\underline{\zeta}) f)^p \equiv 0$ for all $f \in H^\infty_{\natural,\gamma}(]-\infty, T] \times \mathbb{R}^{d-1}, E^s_h(\underline{\zeta}))$ so that the Neumann series expansion is finite and equals $(I - \mathcal{T}^\varepsilon(\underline{\zeta}))^{-1}$. This remark is coherent with the fact that in finite time imposing the uniform Kreiss–Lopatinskii condition on both sides of the strip is sufficient to ensure strong well-posedness.

9.2. The case $\#\mathscr{I} = \#\mathscr{O} = 1$

In the particular setting where $\#\mathscr{I} = \#\mathscr{O} = 1$ (which is automatically satisfied when $N = 2$ but also includes systems with $N > 2$) then we can give a full characterization of systems satisfying Assumption 6.1 in terms of the reflection coefficient. For simplicity we write

$$(\mathcal{T}^\varepsilon(\underline{\zeta}) f)(t, x') := e^{\frac{i}{\varepsilon}\xi} \mathcal{R}(\underline{\zeta}) f(t - \alpha, x' + \beta) \tag{80}$$

instead of (77). The result is the following.

Proposition 9.3. *Let $\gamma_0 \geqslant 0$ and assume that $\mathcal{R}(\underline{\zeta}) > e^{\alpha\gamma_0}$ then there exists a non-trivial $f \in H^\infty_{\natural,\gamma}(\mathbb{R}_t \times \mathbb{R}^{d-1}, E^s_h(\underline{\zeta}))$ for all $\gamma \geqslant \frac{\ln R}{\alpha}$ such that $(I - \mathcal{T}^\varepsilon(\underline{\zeta}))f = 0$ in $H^\infty_{\natural-\alpha,\gamma}(\mathbb{R}_t \times \mathbb{R}^{d-1}_{x'}, E^s_h(\underline{\zeta}))$.*

In particular the (weak version of) Assumption 6.1 is not satisfied for all $\gamma \geqslant \gamma_0$.

Proof. Performing a Fourier transform $x' \leftrightsquigarrow \eta$ in the equation $(I - \mathcal{T}^\varepsilon(\underline{\zeta}))f(t, x) = 0$, gives the functional equation

$$\widehat{f}(t, \eta) - e^{\frac{i}{\varepsilon}\xi} \mathcal{R}(\underline{\zeta}) e^{i\beta\eta} \widehat{f}(t - \alpha, \eta) = 0, \tag{81}$$

and we are looking from a solution of (81) under the form $\widehat{f}(t, \eta) = e^{\sigma t} \mathbf{1}_{[0,\infty[}(t) \widehat{g}(\eta) v$ where g lies in $H^\infty(\mathbb{R}^{d-1}_{x'})$, where $v \in E^s_h(\underline{\zeta})$ and where $\sigma := \lambda + i\tau \in \mathbb{C}$ has to be fixed. Injecting such an ansatz in (81) gives:

$$\left(\mathbf{1}_{[0,\infty[}(t) - e^{\frac{i}{\varepsilon}\xi} \mathcal{R}(\underline{\zeta}) e^{i\beta\eta} e^{-\alpha\sigma} \mathbf{1}_{[\alpha,\infty[}(t)\right) e^{\sigma t} \widehat{g}(\eta) v = 0. \tag{82}$$

Recall that from Remark 6.1 it is sufficient to solve section 9.2 on the time interval $[\alpha, \infty[$ one which section 9.2 simplifies into:

$$\left(1 - e^{\frac{i}{\varepsilon}\xi}\mathcal{R}(\underline{\zeta})e^{i\beta\eta}e^{-\alpha\sigma}\right)e^{\sigma t}\widehat{g}(\eta)v = 0,$$

so that we choose

$$\begin{cases} \tau := \tau(\varepsilon, \eta) = \frac{1}{\alpha}\left(\frac{\xi}{\varepsilon} + \beta\eta\right), \\ \lambda = \frac{\ln\mathcal{R}(\underline{\zeta})}{\alpha}, \end{cases} \tag{83}$$

in such a way that is satisfied. The constructed Fourier transform is in $L^2(\mathbb{R}^{d-1})$ independently on $t \in \mathbb{R}_+$. So that by inverse Fourier transform the function $f(t, x) = e^{\frac{\ln\mathcal{R}(\underline{\zeta})}{\alpha}t}\mathbf{1}_{[0,\infty[}(t)\mathcal{F}^{-1}_{\eta\to x}(e^{\frac{i}{\alpha}(\frac{\xi}{\varepsilon}+\beta\eta)t}\widehat{g}(\eta),)v$ is solution of $(I - \mathcal{T}^{\varepsilon}(\underline{\zeta}))f = 0$ lying in $H^{\infty}_{\natural-\alpha,\gamma}(\mathbb{R}_t \times \mathbb{R}^{d-1}_{x'}, E^s_h(\underline{\zeta})$ for all $\gamma \geqslant \frac{\ln\mathcal{R}(\underline{\zeta})}{\alpha} > 0$ (because $\mathcal{R}(\underline{\zeta}) > 1$). □

10. Examples and comments

10.1. Examples

10.1.1. The wave equation in two dimensions. In this first example we consider the wave equation in two dimensions

$$\begin{cases} \partial_t u^{\varepsilon} + A_1\partial_1 u^{\varepsilon} + A_2\partial_2 u^{\varepsilon} = 0 & \text{for } (t, x) \in \Omega, \\ B_0 u^{\varepsilon}_{|x_2=0} := \begin{bmatrix} 1 & -\alpha_0 \end{bmatrix} u^{\varepsilon}_{|x_2=0} = g^{\varepsilon} & \text{on } (t, x_1) \in \partial\Omega_0, \\ B_1 u^{\varepsilon}_{|x_2=1} := \begin{bmatrix} -\alpha_1 & 1 \end{bmatrix} u^{\varepsilon}_{|x_2=1} = 0 & \text{on } (t, x_1) \in \partial\Omega_1, \\ u^{\varepsilon}_{|t\leqslant 0} = 0 & \text{for } x \in \Gamma, \end{cases} \tag{84}$$

where $\alpha_0, \alpha_1 \in \mathbb{R}$ and where the coefficients A_1, A_2 are given by:

$$A_1 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad A_2 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In (84) the source term g^{ε} reads

$$g^{\varepsilon}(t, x_1) := e^{\frac{i}{\varepsilon}(\underline{\tau}t + \underline{\eta}x_1)}g(t, x_1), \tag{85}$$

where $g \in H^{\infty}_{\natural}(\partial\Omega_0)$ and where $\underline{\tau}, \underline{\eta} \in \mathbb{R}$ are fixed frequency parameters.

We can easily check that the boundary conditions in (84) are strictly dissipative (see (18)) if and only if the parameters α_0, α_1 satisfy $\alpha_0 < 0$, $\alpha_1 > 0$. So for such parameters Theorem 3.1 applies and (84) is exponentially strongly well-posed. We also recall that from [6], as in the case $N = 2$ the uniform Kreiss–Lopatinskii condition is equivalent to the strict dissipativity of the boundary condition. Consequently the restrictions $\alpha_0 < 0$, $\alpha_1 > 0$ are the only ones leading to an exponentially strongly well-posed problem.

We are now interested in the fulfillment of Assumption 6.1 in order to construct a geometric optics expansion by Theorem 3.2 or Theorem 3.3 (depending on the frequency $(\underline{\tau}, \underline{\eta})$).

The resolvent matrix associated to (84) for $\zeta = (\sigma, \eta)$ is

$$\mathscr{A}(\zeta) = \begin{bmatrix} 0 & -\sigma + i\eta \\ -(\sigma + i\eta) & 0 \end{bmatrix}.$$

So that for $\underline{\zeta} = (i\underline{\tau}, \underline{\eta})$ we deduce that if X is an eigenvalue of $\mathscr{A}(\underline{\zeta})$ then it satisfies the dispersion relation

$$X^2 = \underline{\eta}^2 - \underline{\tau}^2.$$

Consequently the partition of the boundary of the frequency space Ξ_0 in (7) reads:

$$\begin{aligned} \mathbb{E} &= \{(\tau, \eta) \in \mathbb{R}^2 \setminus |\eta| > |\tau|\}, \quad \mathbb{H} = \{(\tau, \eta) \in \mathbb{R}^2 \setminus |\tau| > |\eta|\}, \\ \mathbb{G} &= \{(\tau, \eta) \in \mathbb{R}^2 \setminus |\eta| = |\tau|\} \quad \text{and} \quad \mathbb{E}\mathbb{H} = \emptyset. \end{aligned}$$

Without loss of generality let us assume that $\underline{\tau} > 0$ and in order to study Assumption 6.1 we assume that $\underline{\zeta} \in \mathbb{H}$ (if $\underline{\zeta} \in \mathbb{E} \cup \mathbb{G}$ then clearly Theorem 3.2 or 3.3 applies independently on Assumption 6.1). In this setting the stable (respectively unstable) eigenvalue $X^s := X^s(\underline{\tau}, \underline{\eta})$ (respectively $X^u := X^u(\underline{\tau}, \underline{\eta})$) is given by:

$$X^s := i\underline{\xi} = -i\sqrt{\underline{\tau}^2 - \underline{\eta}^2} \text{ (respectively } X^u = -X^s), \tag{86}$$

from which we immediately deduce that the stable subspace $E^s(\underline{\zeta})$ and the unstable subspace $E^u(\underline{\zeta})$ are parametrized by:

$$E^s(\underline{\zeta}) = \mathrm{vect}\{(-\underline{\xi}, \underline{\tau} + \underline{\eta})^t\} \quad \text{and} \quad E^u(\underline{\zeta}) = \mathrm{vect}\{(\underline{\xi}, \underline{\tau} + \underline{\eta})^t\}.$$

We now study Assumption 6.1, in the setting of (84) the restriction of the operator $\mathcal{T}^\varepsilon(\underline{\zeta})$ to $E^s(\underline{\zeta}) = E^s_h(\underline{\zeta})$ is:

$$\mathcal{T}^\varepsilon(\underline{\zeta}) \begin{bmatrix} -\underline{\xi} \\ \underline{\tau} + \underline{\eta} \end{bmatrix} = e^{2\frac{i}{\varepsilon}\underline{\xi}} \frac{-\underline{\xi} + \alpha_0(\underline{\tau} + \underline{\eta})}{\underline{\xi} + \alpha_0(\underline{\tau} + \underline{\eta})} \cdot \frac{\alpha_1 \underline{\xi} + \underline{\tau} + \underline{\eta}}{-\alpha_1 \underline{\xi} + \underline{\tau} + \underline{\eta}} \begin{bmatrix} -\underline{\xi} \\ \underline{\tau} + \underline{\eta} \end{bmatrix}. \tag{87}$$

Consequently Assumption 6.1 is automatically satisfied for all boundary parameters α_0, α_1 leading to strictly dissipative boundary conditions (for all $\underline{\zeta} \in \mathbb{H}$) because in such a framework one can easily check that Proposition 9.2 applies with $\overline{\gamma}_0 = 0$ so that $\mathcal{T}^\varepsilon(\underline{\zeta})$ is a contraction.

However, it is also interesting to note that in fact $\mathcal{T}^\varepsilon(\underline{\zeta})$ is a contraction for more boundary parameters than the ones leading to strictly dissipative boundary conditions. Indeed, it is not difficult to check that we have the following equivalence:

$$\left| \mathcal{T}^\varepsilon(\underline{\zeta}) \begin{bmatrix} -\underline{\xi} \\ \underline{\tau} + \underline{\eta} \end{bmatrix} \right| < \left| \begin{bmatrix} -\underline{\xi} \\ \underline{\tau} + \underline{\eta} \end{bmatrix} \right| \Leftrightarrow \alpha_1 \alpha_0 < 1,$$

so that Assumption 6.1 is satisfied for more parameters than the strictly dissipative ones.

The aim of the next example is to give more details about this observation.

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10.1.2. A modification of the wave equation . In this second example we consider the following modification of the classical wave equation:

$$\begin{cases} \partial_t u^\varepsilon + A_1\partial_1 u^\varepsilon + A_2\partial_2 u^\varepsilon = 0 & \text{for } (t,x)\in\Omega, \\ B_0 u^\varepsilon_{|x_2=0} = g^\varepsilon & \text{on } (t,x_1)\in\partial\Omega_0, \\ B_1 u^\varepsilon_{|x_2=1} = 0 & \text{on } (t,x_1)\in\partial\Omega_1, \\ u^\varepsilon_{|t\leqslant 0} = 0 & \text{for } x\in\Gamma, \end{cases} \tag{88}$$

where the coefficients A_1, A_2 are given by:

$$A_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & a \end{bmatrix}, \quad A_2 := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -b \end{bmatrix},$$

for fixed parameters $a\in\mathbb{R}$, $b\in\mathbb{R}_+^*$. So the evolution equation of (88) is a wave equation (for the components u_1 and u_2) combined with an uncoupled transport phenomenon for the component u_3. The source term g^ε is of the form (85).

The boundary matrices in (88) are defined by (note that A_2 admits only one positive eigenvalue):

$$B_0 := \begin{bmatrix} 1 & -\alpha_0 & -\alpha_1 \end{bmatrix}, \quad B_1 := \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -\delta \end{bmatrix},$$

where $\alpha_0, \alpha_1, \delta\in\mathbb{R}$. Consequently in (88) the coupling between u_1, u_2 and u_3 is made in the boundary conditions.

As in Paragraph 10.1.1 in order to study Assumption 6.1 for (88) we are interested in the hyperbolic area of (88). The system is decoupled and the transport equation added on u_3 is hyperbolic whatever the frequency parameter is. So that we have the following decomposition of the boundary of the frequency space:

$$\begin{aligned} \mathbb{EH} &= \{(\tau,\eta)\in\mathbb{R}^2\setminus|\eta|>|\tau|\}, \quad \mathbb{H} = \{(\tau,\eta)\in\mathbb{R}^2\setminus|\tau|>|\eta|\}, \\ \mathbb{G} &= \{(\tau,\eta)\in\mathbb{R}^2\setminus|\eta|=|\tau|\} \quad \text{and} \quad \mathbb{E}=\emptyset, \end{aligned}$$

consequently in the following we will assume that $|\underline{\tau}|>|\underline{\eta}|$ to be in the hyperbolic area.[11]

Reiterating essentially the same computations as the ones performed in Paragraph 10.1.1, we can easily show that the stable subspace $E^s(\underline{\zeta})$ and the unstable subspace $E^u(\underline{\zeta})$ associated to (88) are given by:

$$\begin{aligned} & E^s(\underline{\zeta}) := \text{vect}\{e_s\} = \text{vect}\{(-\underline{\xi}, \underline{\tau}+\underline{\eta}, 0)^t\} \\ \text{and} \quad & E^u(\underline{\zeta}) := \text{vect}\{e_{u,1}, e_{u,2}\} = \text{vect}\{(\underline{\xi}, \underline{\tau}+\underline{\eta}, 0)^t, (0,0,1)^t\}, \end{aligned}$$

where $\underline{\xi}$ is defined in (86).

It is also easy to show that the boundary condition on $\partial\Gamma_0$ is strictly dissipative if and only if $\alpha_0<0$ and $\alpha_1^2+2\alpha_0 b<0$. This condition satisfies the uniform Kreiss–Lopatinskii

[11] In this example, the mixed area is of little interest because $\mathscr{A}(\underline{\zeta})$ has two elliptic roots and only one hyperbolic root so that the selfinteraction phenomenon cannot occur.

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if and only if $\alpha_0 < 0$, independently on α_1. With such a choice of α_0, the inverse given by the uniform Kreiss–Lopatinskii condition is given by: $\phi_0(\underline{\zeta}) : \mathbb{C} \to E^s(\underline{\zeta})$

$$\phi_0(\underline{\zeta})x := \frac{-x}{\underline{\xi} + \alpha_0(\underline{\tau} + \underline{\eta})} e_s.$$

The boundary condition on $\partial\Gamma_1$ satisfies the uniform Kreiss–Lopatinskii condition for all $\delta \neq 0$ and is strictly dissipative if and only if we have $\delta^2 > \frac{b}{2}$. For $\delta \neq 0$ the inverse given by the uniform Kreiss–Lopatinskii condition is: $\phi_1(\underline{\zeta}) : \mathbb{C}^2 \to E^u(\underline{\zeta})$

$$\phi_1(\underline{\zeta}) := \begin{bmatrix} \frac{\underline{\xi}}{-\underline{\xi}+\underline{\tau}+\underline{\eta}} & 0 \\ \frac{\underline{\tau}+\underline{\eta}}{-\underline{\xi}+\underline{\tau}+\underline{\eta}} & 0 \\ \frac{\underline{\tau}+\underline{\eta}}{\delta(-\underline{\xi}+\underline{\tau}+\underline{\eta})} & -\frac{1}{\delta} \end{bmatrix}.$$

With these expressions in hand it is easy to show that the operator $\mathcal{T}^\varepsilon(\underline{\zeta})$ applied to e_s reads:

$$\mathcal{T}^\varepsilon(\underline{\zeta})e_s = \left(e^{2\frac{i\underline{\xi}}{\varepsilon}} \frac{\underline{\xi}+\underline{\tau}+\underline{\eta}}{-\underline{\xi}+\underline{\tau}+\underline{\eta}} \cdot \frac{-\underline{\xi}+\alpha_0(\underline{\tau}+\underline{\eta})}{\underline{\xi}+\alpha_0(\underline{\tau}+\underline{\eta})} \right.$$
$$\left. + e^{\frac{i}{\varepsilon}(\underline{\xi}-\frac{1}{b}(\underline{\tau}+a\underline{\eta}))} \frac{2\underline{\xi}(\underline{\tau}+\underline{\eta})}{\delta(-\underline{\xi}+\underline{\tau}+\underline{\eta})} \cdot \frac{\alpha_1}{\underline{\xi}+\alpha_0(\underline{\tau}+\underline{\eta})} \right) e_s \tag{89}$$
$$:= (\varrho_1 + \varrho_2)e_s. \tag{90}$$

Let us first remark that if in B_0 one chooses $\alpha_0 < 0$ and $\alpha_1 = 0$ (so that the boundary on $\partial\Gamma_0$ is strictly dissipative) then from (90) and Paragraph 10.1.1, $\mathcal{T}^\varepsilon(\underline{\zeta})$ is a contraction on $E^s(\underline{\zeta})$ and consequently Assumption 6.1 holds for $\overline{\gamma}_0 = 0$ and Theorem 3.2 applies independently on δ. Choose $0 < \delta < \sqrt{\frac{b}{2}}$ shows that Theorem 3.2 applies for non-strictly dissipative boundary condition on $\partial\Gamma_1$.

Then it is easy to show that for all strictly dissipative boundary conditions on $\partial\Gamma_0$ and $\partial\Gamma_1$ we have $|\varrho_1| < 1$ and $|\varrho_2| < 1$ independently on $\underline{\zeta}$. Unfortunately this result is not sufficient to conclude that $\mathcal{T}^\varepsilon(\underline{\zeta})$ is a contraction and that Assumption 6.1 holds for $\overline{\gamma}_0 = 0$ for all possible $\underline{\zeta} \in \mathbb{H}$.

However numerical results seem to indicate that $|\varrho_1 + \varrho_2| < 1$ for all strictly dissipative boundary conditions independently on $\underline{\zeta} \in \mathbb{H}$. We refer to Figure 4 for an illustration when an explicit computation[12] with $\alpha_0 = -\frac{1}{2}, b = 1, \alpha_1 = \sqrt{-2\alpha_0 b} + 10^{-2}$ and $\delta = \sqrt{\frac{b}{2}} - 10^{-2}$.

10.2. Conclusion and comments

In this article we show that to construct the geometric optics expansion associated to a hyperbolic boundary value problem defined in a strip a new inversibility condition has

[12]Note that in Paragraph 10.1.2 we make a crude estimate in the sense that we do not take into account the oscillating factors and the dependency of $\underline{\xi} < 0$ with respect to $(\underline{\tau}, \underline{\eta})$.

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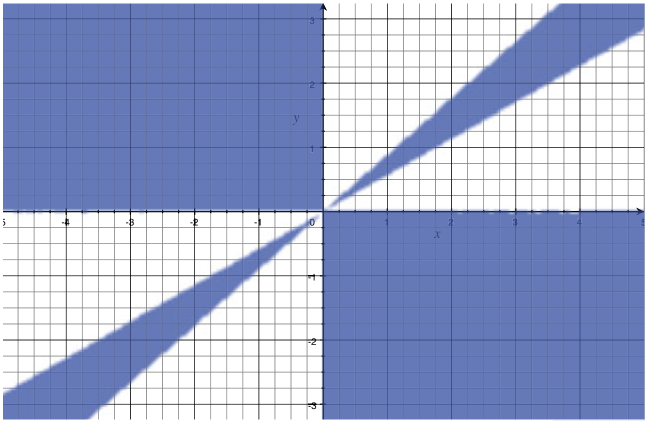

Figure 4. The set (in blue) of $(x, y) \in \mathbb{R}^2$ such that $|\frac{y+x}{-y+x}\frac{-y+\alpha_0 x}{y+\alpha_0 x} + \frac{2xy}{\delta(-y+x)}\frac{\alpha_1}{y+\alpha_0 x}| < 1$.

to be imposed (see Assumption 6.1). This condition involves the traces of the hyperbolic components of the geometric optics expansion.

This condition is shown to be automatically satisfied when the considered localization frequency does not involve selfinteracting phases, when the strip problem is finite in time and also if during a full circuit of reflection the coefficients of reflection ensure that the energy does not increase. Moreover in the particular setting where there are only two selfinteracting modes this non-increasing property is equivalent to the fulfillment of Assumption 6.1.

As a consequence, this condition meets the intuition that if after a full reflection the boundary conditions are such that the energy increases then as the full reflection is periodically repeated in time the associated ansatz should have an exponential growth in time (with some rate depending on the time needed to perform a full reflection).

This seems to indicate that the maximal exponential growth in time of the solution is linked to the time needed to perform a full reflection and to the maximum of the reflection coefficients for all selfinteracting frequencies (that is to say boundary frequencies involving at least an incoming and an outgoing phase).

Moreover the examples described in Paragraphs 10.1.1, 10.1.2 seem to indicate that Assumption 6.1 is trivially (in the sense that $\mathcal{T}(\underline{\zeta})$ is a contraction) satisfied for all strictly dissipative boundary conditions (for which the (lower exponential) strong well-posedness of (24) is known to hold).

A point of interest is that in the expansions described so far the inversibility condition used to construct the WKB expansions does not involve the elliptic or the glancing parts of the ansatz. This point meets the intuition that these parts of the ansatz are linked to boundary layers so that they cannot propagate the information from one side to the other and consequently they should behave like they do in the half space geometry.

However, in the author's opinion, this observation has an important counterpart. More precisely, in [1] the author obtains a full characterization of lower exponentially strongly well-posed problems (see Definition 3.1) in terms of new inversibility conditions involving the traces of the solution of each side of the strip. Nevertheless compared to the inversibility condition Assumption 6.1 one of the inversibility conditions used in [1] differs by the following:

- first, as it is not at the microlocalized level, it has to hold uniformly in terms of the frequency parameter $\zeta \in \Xi \setminus \Xi_0$.
- Second this condition has to be imposed on the full stable subspace $E^s(\zeta)$ and not only on the hyperbolic part of this space that is $E^s_h(\zeta)$ (note that by Hersh's lemma [6] this space is empty for $\zeta \in \Xi \setminus \Xi_0$).

The main issue with the characterization used in [1] is its uniformity in terms of $\zeta \in \Xi \setminus \Xi_0$ which seems really difficult to check in practice. To overcome this difficulty the natural strategy is to have a look to the boundary frequencies $\zeta \in \Xi_0$ to obtain the uniform bound by compactness arguments (it is the classical method of [7]).

First let us remark that the extension of the condition made in [1] to hyperbolic frequency $\zeta \in \Xi_0$ is nothing but Assumption 6.1 for hyperbolic frequencies. Consequently Assumption 6.1 is a microlocalized version of the condition ensuring the lower exponential strong well-posedness. This phenomenon already appeared for the geometric optics expansions of boundary value problems in the half space. So we believe that it is interesting to notice that such a situation also occurs in more complex geometries.

Second as pointed in [1], the condition ensuring the lower exponential strong well-posedness cannot hold for glancing modes. So the fact that Assumption 6.1 only holds on $E^s_h(\zeta)$ seems to indicate that in fact in the extension to Ξ_0 only the hyperbolic part of the solution should be considered. So probably the extension of the characterization in [1] up to Ξ_0 does not require any inversibility property on glancing modes. Meaning that it may be possible to extend the symmetrizer construction of [1] up to Ξ_0 (except at glancing modes) to recover the uniformity of the bound. We expect to have further results about this conjecture in some forthcoming publications.

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