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Smoothness is not an obstruction to realizability

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Abstract. A sequence of non-negative integers $(\phi_n)_{n=1}^{\infty}$ is said to be *realizable* if there is a map T of a set X such that $\phi_n = \#\{x : T^n x = x\}$. We prove that any realizable sequence can be realized by a C^∞ diffeomorphism of $\mathbb{T}^2$.

1. *Introduction*

There is a natural class of sequences of non-negative integers, called the *realizable* sequences, that arise as the sequence of the number of periodic points of period n for some dynamical system. This class of sequences was first introduced in the thesis of Puri [**4**], part of which appears as [**5**]. An intriguing consequence of this class of sequences for the Fibonacci recurrence can be found in [**6**] while further number theoretic consequences appear in [**1**].

Definition. Let $\phi = (\phi_n)_{n=1}^{\infty}$ be a sequence of non-negative integers. We say that ϕ is realizable if there is a set X and a map $T : X \to X$ such that

$$\phi_n = \#\{x \in X : T^n x = x\}$$

for all $n \geq 1$. In this case, we say that T realizes $(\phi_n)_{n=1}^{\infty}$.

Puri and Ward [**4, 5**] observe that the sole obstruction to being realizable is the natural restriction that the number of periodic orbits of length n is a non-negative integer, i.e.

$$\phi_n^o := \frac{1}{n} \sum_{d|n} \mu(n/d)\phi_d$$

is a non-negative integer for all $n \geq 1$. Here μ is the Möbius μ-function.

Furthermore, they observed that the class of realizable sequences is unchanged if we require T to be a homeomorphism of a compact metric space X. Puri [**4**] poses the question of whether the class of realizable sequences changes if we require that T be a C^∞ diffeomorphism of a smooth manifold M.

This question would seem to be harder because of the presence of non-trivial obstructions coming from both the topological and smooth structures. Hunt and Kaloshin [**2, 3**] prove that for a prevalent diffeomorphism the growth of periodic points is at most stretched exponential.

Depending on the manifold there may be restrictions arising from the Lefschetz fixed point theorem. The choice of $(\phi_n)_{n=1}^{\infty}$ therefore restricts the choice of manifolds on which even a homeomorphism could realize $(\phi_n)_{n=1}^{\infty}$. For example, every homeomorphism of the sphere must have at least one point of period 2.

Given a realizable sequence $(\phi_n)_{n=1}^{\infty}$ we construct a diffeomorphism T of $\mathbb{T}^2$ which realizes this sequence. For each period we introduce an invariant circle on which the induced diffeomorphism has the appropriate rational rotation number. The only complication is that these invariant circles must accumulate. They accumulate on another invariant circle on which the induced diffeomorphism has an irrational rotation number. All periodic points have index 0. The resulting diffeomorphism is not expansive and has zero topological entropy.

MAIN RESULT. *If $(\phi_n)_{n=1}^{\infty}$ is a realizable sequence of non-negative integers then there exists $T \in \mathrm{Diff}^{\infty}(\mathbb{T}^2)$ which realizes the sequence.*

2. *Preliminaries*

Our construction involves controlled perturbations in the space of C^{∞} diffeomorphisms on $\mathbb{T}^2$, which we denote by $\mathrm{Diff}^{\infty}(\mathbb{T}^2)$. This can be made into a complete metric space in the following manner:

$$d_{\infty}(S, T) := \max\{d^{\infty}(S, T), d^{\infty}(S^{-1}, T^{-1})\}$$

where d^{∞} is the complete metric on the space of C^{∞} mappings on $\mathbb{T}^2$, denoted by $C^{\infty}(\mathbb{T}^2, \mathbb{T}^2)$. This metric can be constructed by embedding $C^{\infty}(\mathbb{T}^2, \mathbb{T}^2)$ in $\prod C^k(\mathbb{T}^2, \mathbb{T}^2)$. It can also be computed directly as

$$d^{\infty}(S, T) = \sup_{k_1, k_2 \geq 0} \frac{\min\{d^0((\partial^{k_1+k_2}/\partial x_1^{k_1} \partial x_2^{k_2})S, (\partial^{k_1+k_2}/\partial x_1^{k_1} \partial x_2^{k_2})T), 1\}}{2^{k_1+k_2}},$$

where d^0 is the appropriate C^0 metric.

3. *Construction*

The main result is a corollary of a slightly more general theorem about maps of an annulus. Let $I := [0, 1]$ and $\mathbb{T} := \mathbb{R}/\mathbb{Z}$.

THEOREM. *Let $\epsilon > 0$ be arbitrary and let (ϕ_n) be a realizable sequence of non-negative integers. Define $m = \min\{n : \phi_n > 0\}$. Suppose that $\varphi \in C^{\infty}(I, \mathbb{R}^+)$ satisfies:*

(1) $\varphi(0), \varphi(1) \in \mathbb{R}\backslash\mathbb{Q}$;

(2) $\operatorname{osc} \varphi := \max \varphi - \min \varphi > 1/m$.

Define $T \in \mathrm{Diff}^{\infty}(\mathbb{T} \times I)$ by

$$T(x, y) := (x + \varphi(y), y).$$

Then there is a $\widetilde{T} \in \mathrm{Diff}^{\infty}(\mathbb{T} \times I)$ such that:

(1) $d_{\infty}(T, \widetilde{T}) < \epsilon$;

(2) *the number of periodic points of period n is ϕ_n;*

(3) $T(x, 0) = \widetilde{T}(x, 0)$ *and* $T(x, 1) = \widetilde{T}(x, 1)$;

(4) *all periodic points are isolated and have index* 0.

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This follows rather straightforwardly from a lemma about perturbations in a neighborhood of an invariant circle. We show that a small perturbation can be made to make the induced rational rotation have any given number of periodic orbits with all other points asymptotic to a periodic orbit.

LEMMA. *Let $\epsilon > 0$ be arbitrary, $\phi^o \in \mathbb{Z}^+$, and $\varphi \in C^\infty([-\delta, \delta], \mathbb{R}^+)$ satisfy $\varphi(0) = a/b$ with $(a, b) = 1$. Define $T \in \mathrm{Diff}^\infty(\mathbb{T} \times [-\delta, \delta])$ by*

$$T(x, y) := (x + \varphi(y), y)$$

Then there is a transformation $\widehat{T} \in \mathrm{Diff}^\infty(\mathbb{T} \times [-\delta, \delta])$ such that:
(1) $d_\infty(\widehat{T}, T) < \epsilon$;
(2) $\widehat{T} = T$ *in a neighborhood of the boundary;*
(3) $\widehat{T}(x, 0)$ *has precisely ϕ^o orbits of prime period b and no other periodic points.*

Proof. Indeed the transformation $\widehat{T}$ can be chosen with the form

$$\widehat{T}(x, y) = (x + \varphi(y) + f(x, y), y) \quad \text{with } f(x, y) = \xi(x)\zeta(y).$$

The function $\zeta \in C^\infty([-\delta, \delta], \mathbb{R})$ is a scaled bump function which is zero in a neighborhood of the boundary and positive at 0. We choose $\xi \in C^\infty(\mathbb{T}, \mathbb{R})$ to have the following properties:
(1) $\xi(x) \geq 0$;
(2) $\xi(x) = 0$ if and only if $x = i/(\phi^o b)$ for some $i \in \mathbb{Z}$.
The properties we have required depend only on the zeros and the sign of the function and not on the magnitude. Hence, we may multiply the function f by any positive constant and still get a function with all the required properties.

The map $\widehat{T}$ is invertible if we ensure that $f_x(x, y) > -1$. Then by the inverse function theorem we have that $\widehat{T}$ is a C^∞-diffeomorphism. By controlling the C^n norm of f we may ensure that $d_\infty(T, \widehat{T}) < \epsilon$, as required.

Thus $\widehat{T}$ is a circle diffeomorphism. Let $\widehat{x} = i/(\phi^o b)$ for some $i \in \mathbb{Z}$. Since $f(\widehat{x}, 0) = 0$ we have

$$\widehat{T}^j(\widehat{x}, 0) = \left(\widehat{x} + j\frac{a}{b}, 0\right) \mod 1 \tag{1}$$

and $\widehat{x}$ is periodic of prime period b. Therefore, $\widehat{T}$ has rotation number a/b and any periodic point must have period b. Now

$$\widehat{T}^b(x, 0) = (x + S_b f(x, 0), 0) \mod 1$$

where

$$S_b(x, 0) = \sum_{i=0}^{b-1} f(T^i(x, 0)).$$

If $x \neq i/(\phi^o b)$ then

$$0 < S_b f(x, 0) < \frac{1}{\phi^o} \leq 1$$

and thus x cannot be a periodic point. □

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Now we apply the lemma to a countable collection of invariant circles and then apply a perturbation to ensure that only these periodic points persist.

Proof of the Theorem. Let ϕ_n^o be the corresponding sequence of the number of orbits of prime period n. If $\phi_n^o > 0$ then we choose a circle $C_n := \{(x, y) : y = y_n\}$ such that $\varphi(y_n) = a/n$ with $(a, n) = 1$. This is always possible because of the hypothesis on the oscillation of φ. We can choose y_n such that y_n increases to 1 as $n \to \infty$.

For each such circle C_n we can find an annular neighborhood N_n of C_n such that the collection of neighborhoods $\{N_n\}$ is pairwise disjoint and disjoint from the boundary. Applying the lemma to each neighborhood with ϵ_n decreasing to 0, we construct a new diffeomorphism $\widehat{T}$ such that the required sequence of periodic points is realized by the restriction of $\widehat{T}$ to the family of circles $\{C_n\}$ and $d_\infty(T, \widehat{T}) < \epsilon/2$.

In order to ensure there are no further periodic points we introduce a second perturbative term. Define $\widetilde{T}$ by

$$\widetilde{T}(x, y) = (x + \varphi(y) + f(x, y), y + g(y)).$$

The function $g \in C^\infty(I, \mathbb{R})$ is chosen with the following properties:

(1) $g(y) \geq 0$;

(2) $g(y) = 0$ if and only if $y = y_n$, where y_n defines one of our circles C_n, $y = 0$, or $y = 1$.

For any n the map g can be chosen to have arbitrarily small C^n norm since we can multiply by any positive constant. In addition, although it is not used here, we notice that we could take g to be C^∞ flat at $y = 0$ and $y = 1$.

This map is invertible provided $f_x(x, y) > -1$ and $g_y(y) > -1$. The estimate on f is automatic as f comes from $\widehat{T}$ which is a diffeomorphism and we can choose g accordingly. If we choose g with a sufficiently small C^n norm then we may ensure that $d_\infty(\widetilde{T}, \widehat{T}) < \epsilon/2$.

On each circle C_n, and on the two boundary circles, we have $\widetilde{T} = \widehat{T}$. On the two boundary circles we have $\widetilde{T} = T$. Since the rotation induced by T on the boundary is irrational these circles contain no periodic points. Since $g(y) \geq 0$ the y-coordinate is non-decreasing under the action of $\widetilde{T}$. If (x, y) is not on the boundary or one of the circles C_n then $g(y) > 0$ and thus (x, y) cannot be periodic. Thus, the only periodic points occur on the specified circles C_n and these periodic points realize the sequence ϕ_n. □

Finally we state a corollary of the theorem and give a proof.

COROLLARY. *If $(\phi_n)_{n=1}^\infty$ is a realizable sequence of non-negative integers then there exists $T \in \mathrm{Diff}^\infty(\mathbb{T}^2)$ which realizes this sequence.*

Proof. Choose $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Define $m = \min\{n : \phi_n > 0\}$. Choose $\varphi \in C^\infty(I, \mathbb{R}^+)$ such that $\varphi(0) = \varphi(1) = \alpha$, φ is C^∞ flat at $y = 0$ and $y = 1$, and osc $\varphi > 1/m$. Then apply the main theorem to get a diffeomorphism $\widetilde{T}$, observing that the perturbation by $g(y)$ respects the C^∞ flatness on the boundary. Thus we may identify the boundaries to get a diffeomorphism of $\mathbb{T}^2$. □

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REFERENCES

[1] G. Everest, A. J. van der Poorten, Y. Puri and T. Ward. Integer sequences and periodic points. *J. Integer Seq.* **5**(2) (2002), Article 02.2.3, 10 pp (Electronic).

[2] V. Yu. Kaloshin and B. R. Hunt. A stretched exponential bound on the rate of growth of the number of periodic points for prevalent diffeomorphisms. I. *Electron. Res. Announc. Amer. Math. Soc.* **7** (2001), 17–27 (Electronic).

[3] V. Yu. Kaloshin and B. R. Hunt. A stretched exponential bound on the rate of growth of the number of periodic points for prevalent diffeomorphisms. II. *Electron. Res. Announc. Amer. Math. Soc.* **7** (2001), 28–36 (Electronic).

[4] Y. Puri. Arithmetic properties of periodic orbits. *PhD Thesis*, University of East Anglia, 2000.

[5] Y. Puri and T. Ward. Arithmetic and growth of periodic orbits. *J. Integer Seq.* **4**(2) (2001), Article 01.2.1, 18 pp (Electronic).

[6] Y. Puri and T. Ward. A dynamical property unique to the Lucas sequence. *Fibonacci Quart.* **39**(5) (2001), 398–402.