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ITP, ISP, AND SCH

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Abstract. ISP cannot hold at the first or second successor of a singular strong limit of countable cofinality; on the other hand, we force a failure of "strong SCH" across a cardinal where ITP holds. We also show that ITP does not imply that there are stationary many internally unbounded models.

§1. Introduction and background.

The *tree property at κ* holds if every tree of height κ with levels of size less than κ has a cofinal branch. For an inaccessible cardinal, the tree property is equivalent to weak compactness. On the other hand, the tree property can consistently hold at successor cardinals. Mitchell [6] showed that starting from a weakly compact cardinal, there is a generic extension in which the tree property holds at $\aleph_2$. Silver showed that the large cardinal hypothesis is necessary. Thus, the tree property captures the combinatorial essence of weakly compact cardinals.

In his thesis (also see [16]), Weiß isolated strengthenings of the tree property, called TP and ITP, which in turn can be viewed as capturing the combinatorics of strongly compact and supercompact cardinals. ITP was originally (implicitly) defined by Magidor [7].

Definition 1.1. Let $\kappa \leq \lambda$ be cardinals. We say that $\langle d_a \mid a \in \mathcal{P}_\kappa(\lambda)\rangle$ is a $\mathcal{P}_\kappa(\lambda)$-*list* if each $d_a \subset a$. A $\mathcal{P}_\kappa(\lambda)$-list $\langle d_a \mid a \in \mathcal{P}_\kappa(\lambda)\rangle$ is *thin* if for club many $c \in \mathcal{P}_\kappa(\lambda)$, $|\{d_a \cap c \mid c \subset a\}| < \kappa$.

For example, note that if κ is inaccessible, every $\mathcal{P}_\kappa(\lambda)$-list is thin.

Definition 1.2. Suppose $\langle d_a \mid a \in \mathcal{P}_\kappa(\lambda)\rangle$ is a $\mathcal{P}_\kappa(\lambda)$-list and $b \subset \lambda$. Then,

- b is a *cofinal branch* if for all $a \in \mathcal{P}_\kappa(\lambda)$, there is $c \supset a$ in $\mathcal{P}_\kappa(\lambda)$, such that $d_c \cap a = b \cap a$,
- b is an *ineffable branch* if $\{a \mid d_a = b \cap a\}$ is stationary in $\mathcal{P}_\kappa(\lambda)$.

$\mathsf{TP}(\kappa, \lambda)$ holds if every thin $\mathcal{P}_\kappa(\lambda)$-list has a cofinal branch. Note that $\mathsf{TP}(\kappa, \kappa)$ is equivalent to the tree property at κ. We say that κ has the *strong tree property* if for all $\lambda > \kappa$, $\mathsf{TP}(\kappa, \lambda)$ holds.

The *super tree property* at κ, $\mathsf{ITP}(\kappa)$, holds if for all $\lambda > \kappa$, every thin $\mathcal{P}_\kappa(\lambda)$-list has an ineffable branch.

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The following is originally due to Jech and Magidor; but for an explicit proof with the above terminology, see Weiß's thesis.

Fact 1.3. *Suppose that κ is an inaccessible cardinal. Then κ is strongly compact if and only if the strong tree property at κ holds; and κ is supercompact if and only if* $\mathsf{ITP}(\kappa)$ *holds.*

Like in the case of the tree property, starting from a strongly compact (or supercompact) cardinal and forcing with the Mitchell poset, one can obtain the strong tree property at ω_2 (or $\mathsf{ITP}(\omega_2)$, respectively). Moreover, Spencer Unger [12] and Laura Fontanella [2] independently showed that in the Cummings-Foreman model [1], ITP holds at $\aleph_n$ for all $n > 1$.

An old project in set theory is to obtain the tree property at every regular cardinal greater than ω_1. The larger motivation is to obtain via forcing models of set theory with as much compactness as can consistently exist in the universe. The construction of such models would require large cardinals and many violations of the singular cardinals hypothesis (SCH). An even more ambitious question is whether we can obtain either the strong tree property or ITP at every (or at least at many consecutive) regular cardinals above ω_1. The results in this paper are motivated by the following question:

Question 1.4. *Does* $\mathsf{ITP}(\kappa)$ *imply SCH above* κ?

The motivation is two-fold. On one hand is Solovay's theorem that SCH holds above a strongly compact cardinal. On the other hand, by a theorem of Specker [11], obtaining ITP at the double successor of a singular strong limit cardinal requires violating SCH.

Viale and Weiß [15] have also asked whether a further strengthening of ITP called ISP implies SCH. This is related to Viale's theorem that PFA implies SCH.

Definition 1.5. A list $\langle d_a \mid a \in \mathcal{P}_\kappa(\lambda)\rangle$ is *slender* if for all sufficiently large θ, for club many $M \in \mathcal{P}_\kappa(H_\theta)$, for all $b \in M \cap \mathcal{P}_{\aleph_1}(\lambda)$, $d_{M\cap\lambda} \cap b \in M$. $\mathsf{ISP}(\kappa)$ holds if for every $\lambda \geq \kappa$, every slender $\mathcal{P}_\kappa(\lambda)$ list has an ineffable branch.

We have that ISP implies ITP. Viale and Weiß showed that PFA implies $\mathsf{ISP}(\aleph_2)$. A useful characterization from [15] of ISP uses guessing models:

Definition 1.6. Let $M \prec H_\theta$. M is an $\aleph_1$-*guessing model* if whenever $z \in M$ and $a \subseteq z$, if a is $\aleph_1$-*approximated by* M in the sense that

$$\{a \cap x \mid x \in \mathcal{P}_{\aleph_1}(z) \cap M\} \subseteq M,$$

then a is M-*guessed*, i.e., for some $b \in M$, $b \cap M = a \cap M$.

Theorem 1.7 ([15]). $\mathsf{ISP}(\kappa)$ *holds if and only if for all sufficiently large θ, there are stationary many $\aleph_1$-guessing models in $\mathcal{P}_\kappa(H_\theta)$.*

Viale also showed [14] that $\mathsf{ISP}(\aleph_2)$ together with stationary many *internally unbounded* models imply that SCH holds; here we say M is internally unbounded if the countable sets in M are $\subseteq$-cofinal in $\mathcal{P}_{\aleph_1}(M)$.

This leads to the following questions:

(1) At what other small cardinals can ISP hold?
(2) Does $\mathsf{ISP}(\kappa)$ imply SCH above κ?
(3) Is ISP or ITP consistent with the set of internally unbounded models being nonstationary?

In this paper we show that ISP cannot hold at the first or second successor of a singular strong limit of countable cofinality. On the other hand, we give two different constructions where ITP holds at the double successor of a singular strong limit cardinal. Using the first construction, we prove a failure of "strong SCH" across a cardinal where ITP holds. The second construction uses extender-based forcing, and can be brought down to $\aleph_\omega$. We also show that in both constructions there are club many models in $\mathcal{P}_{\kappa^{++}}(H_\theta)$ that are not internally unbounded, where κ is the singular cardinal. In particular, it follows that $\mathsf{ITP}(\lambda)$ does not imply that there are stationary many internally unbounded models in $\mathcal{P}_\lambda(H_\theta)$.

§2. Failure of ISP at first and second successor, and internally unbounded models.

Theorem 2.1. *Let $\kappa < \mu$ be cardinals with $2^{\aleph_0} < \kappa$, $\operatorname{cf}\kappa = \omega$, μ regular, and $\kappa^\omega \geq \mu$. Then $\mathsf{ISP}(\mu)$ fails.*

Corollary 2.2. *If κ is strong limit and $\operatorname{cf}(\kappa) = \omega$, then $\mathsf{ISP}(\kappa^+)$ and $\mathsf{ISP}(\kappa^{++})$ both fail.*

Proof. For the second claim, suppose $\mathsf{ISP}(\kappa^{++})$ holds; this implies the tree property at κ^{++}, and by a result of Specker, we must have $(\kappa^+)^{<\kappa^+} \geq \kappa^{++}$. Since $(\kappa^+)^{<\kappa^+} = 2^\kappa$, we have $2^\kappa \geq \kappa^{++}$, but this contradicts the theorem with $\mu = \kappa^{++}$. ⊣

Proof of Theorem 2.1. Letting κ, μ be as in the theorem, we show $\mathsf{ISP}(\mu)$ must fail. Suppose not. By Theorem 1.7, $\mathsf{ISP}(\mu)$ implies there is some $M \prec H_\theta$ with $|M| < \mu$, $\mu \in M$, $\kappa + 1 \subseteq M$, and $M \cap \mu$ an ordinal, such that M is $\aleph_1$-guessing.

So suppose $x \subset \kappa$ is countable. For each countable $y \in M$, we have $y \cap x \in M$, since $2^{\aleph_0} < \kappa \subseteq M$. So x is $\aleph_1$-approximated, and since M is $\aleph_1$-guessing, we have $x \in M$. Thus $\mathcal{P}_{\aleph_1}(\kappa) \subseteq M$. But $|\mathcal{P}_{\aleph_1}(\kappa)| = \kappa^\omega \geq \mu$, contradicting $|M| < \mu$. ⊣

Next we show a key abstract lemma on the existence of club many noninternally unbounded models.

Lemma 2.3. *Suppose that $2^\omega < \kappa$ and $\kappa^\omega \geq \kappa^{++}$. Then there are club many models of size $< \kappa^{++}$ that are not internally unbounded.*

Proof. Suppose that M is a model of size less that κ^{++} with $\kappa \subset M$. Then $|\mathcal{P}_{\aleph_1}(M)| \geq \kappa^{++}$. On the other hand for any countable $c \in M$ can cover at most 2^ω many countable sets x. So, there are at most $2^\omega \cdot |M| < \kappa^{++}$ many $x \in \mathcal{P}_{\aleph_1}(M)$, such that there is a countable $y \in M$ with $x \subset y$. So, M is not $\aleph_1$-internally unbounded. ⊣

Applying Specker's theorem, we immediately get the following corollary:

Corollary 2.4. *If κ is a singular strong limit cardinal with $\operatorname{cf}(\kappa) = \omega$, then $\mathsf{ITP}(\kappa^{++})$ implies that there are club many not internally unbounded models of size $< \kappa^{++}$.*

§3. A failure of strong SCH across a cardinal where ITP holds. The general strategy of obtaining tree properties at successor cardinals is by starting with some large cardinal embedding j in the ground model, and forcing in such a way that the embedding j can be extended to $V[G]$. The embedding is used to define a branch in $V[j(G)]$, and usually the hardest part of the argument is pulling this branch back to $V[G]$. This amounts to proving an approximation property of the quotient poset.

Definition 3.1. Let $\mathbb{P} \in V$ be a poset and G be $\mathbb{P}$-generic over V. We say a set of ordinals $a \in V[G]$ is *λ-approximated* if for all $x \in V$ with $|x|^V < \lambda$, $x \cap a \in V$.

$\mathbb{P}$ has the *λ-approximation property* if every λ-approximated set of ordinals $a \in V[\mathbb{P}]$ belongs to V.

$\mathbb{P}$ has the *thin λ-approximation property* if whenever $a \in V[\mathbb{P}]$ is λ-approximated, and furthermore for all $x \in V$ with $|x|^V < \lambda$, $|\{b \in V \mid p \Vdash b = x \cap \dot{a}$ for some $p \in \mathbb{P}\}|^V < \lambda$, we have $a \in V$.

We say *strong SCH holds* if for all singular cardinals κ, if $2^{\mathrm{cf}\,\kappa} < \kappa$, then $2^\kappa = \kappa^+$.

Theorem 3.2. *Let $\kappa < \lambda$ be supercompact cardinals. Then there is a poset $\mathbb{R}$ so that if G is generic for $\mathbb{R}$, then the following holds in $V[G]$:*

- $\mathsf{ITP}(\lambda)$.
- *κ is a singular strong limit cardinal with* $\mathrm{cf}(\kappa) = \omega$ *and* $\kappa^{++} = \lambda$.
- $2^\kappa = \lambda^{+\omega+2}$.

In particular, strong SCH fails at $\kappa^{+\omega}$.

Proof. By forcing with the Laver preparation if necessary, we may assume that in V, the supercompactness of κ is indestructible by κ-directed closed forcing.

We will have $\mathbb{R} = \mathbb{M} * \dot{\mathbb{P}}$, where $\mathbb{M}$ is Mitchell's poset to force $\kappa^{++} = \lambda$ and $\mathsf{TP}(\lambda)$, modified to first blow up 2^κ to $\lambda^{+\omega+2}$; in particular, conditions are (a, f), where:

- $a \in \mathbb{A} := \mathrm{Add}(\kappa, \lambda^{+\omega+2})$;
- $\mathrm{dom}(f) \subset \lambda \setminus \kappa^+$, $|\mathrm{dom}(f)| < \kappa^+$ and for all $\alpha \in \mathrm{dom}(f)$, $\Vdash_{\mathbb{A}\restriction\alpha} f(\alpha) \in \dot{\mathrm{Add}}(\kappa^+, 1)$.

$\mathbb{M}$ is ordered by letting $(a', f') \leq (a, f)$ iff $a' \leq a$ and for all $\alpha \in \mathrm{dom}(f) \subset \mathrm{dom}(f')$, $a' \restriction \alpha \Vdash f'(\alpha) \leq f(\alpha)$.

We have in the extension by $\mathbb{M}$ that $2^\kappa = \lambda^{+\omega+2}$ and $\kappa^{++} = \lambda$. Also $\mathbb{M}$ is the projection of a poset $\mathbb{A} \times \mathbb{Q}$, where as above $\mathbb{A} = \mathrm{Add}(\kappa, \lambda^{+\omega+2})$, and $\mathbb{Q}$ is κ^+-directed closed. Note that $\mathbb{M}$ is κ-directed closed. It follows that κ remains measurable (indeed, fully supercompact) in $V[\mathbb{M}]$. So let $\dot{\mathbb{P}}$ be an $\mathbb{M}$-name for Prikry forcing to singularize κ using any normal measure in the extension by $\mathbb{M}$.

Now by standard arguments, forcing with $\dot{\mathbb{P}}$ in $V[\mathbb{M}]$ preserves cardinals and singularizes κ to cofinality ω, and adds no bounded subsets of κ. So κ is strong limit and has cofinality ω, and so $\kappa^\omega = 2^\kappa = (\lambda^{+\omega})^\omega = \lambda^{+\omega+2}$.

It only remains to show that $\mathsf{ITP}(\lambda)$ holds. Let $\mu \geq \lambda^{+\omega+2}$ be a cardinal. Working in V, let $j : V \to M$ be an elementary embedding witnessing μ-supercompactness of λ. Fix an $\mathbb{M}$-generic filter G over V and a $\mathbb{P}$-generic H over $V[G]$. Since $\mathbb{M} * \dot{\mathbb{P}}$ has the λ-c.c., by standard arguments we can lift j to $j : V[G][H] \to M[G^*][H^*]$ in $V[G^*][H^*]$, where $G^* * H^*$ is generic for $j(\mathbb{M} * \dot{\mathbb{P}})/G * H$.

In $V[G]$, let U be the normal measure on κ used to define $\mathbb{P}$. Then $j(U)$ extends U and $j(\mathbb{P})$ is Prikry forcing singularizing κ with respect to the measure $j(U)$. Conditions in $j(\mathbb{P})$ have the same stems as conditions in $\mathbb{P}$, but there are more measure one sets.

Let $d \in V[G][H]$ be a thin $\mathcal{P}_\lambda(\mu)$-list. $j(d)$ is then a thin $\mathcal{P}_{j(\lambda)}(j(\mu))$-list. We define

$$B = j^{-1}[j(d)_{(j[\mu])}].$$

Claim 3.3. *B is an ineffable branch through d.*

Proof of Claim. We need to show that the set

$$\{x \in \mathcal{P}_\lambda(\mu) \mid d(x) = B \cap x\}$$

is stationary. So suppose $C \subseteq \mathcal{P}_\lambda(\mu)$ is a club in $V[G][H]$. Then $j[\mu] \in j(C)$. Also, by definition of B, $j(d)_{(j[\mu])} = j(B) \cap j[\mu]$. So, by elementarity, $\{x \in C \mid d_x = B \cap x\}$ is nonempty. ⊣

Claim 3.4. *B is λ-approximated by $V[G][H]$, that is, $x \cap B \in V[G][H]$ whenever $x \in (\mathcal{P}_\lambda(\mu))^{V[G][H]}$.*

Proof. Let $x \in (\mathcal{P}_\lambda(\mu))^{V[G][H]}$. Then $|\text{Lev}_x(d)| < \lambda$ by thinness of d, and since $\text{crit}(j) = \lambda$ we have $j(\text{Lev}_x(d)) = j[\text{Lev}_x(d)]$. Since $j(d)_{j[\mu]} \cap j(x) \in j(\text{Lev}_x(d)) = \text{Lev}_{j(x)}(j(d))$, there must be $z \subset x$ in $V[G][H]$ such that $j(z) = j(d)_{j[\mu]} \cap j(x)$. Then $z = B \cap x \in V[G][H]$, as needed. ⊣

Clearly $B \in V[G^*][H^*]$; we need $B \in V[G][H]$, that is, B is not added by forcing with the quotient $j(\mathbb{R})/\mathbb{R}$.

Sinapova and Unger [10] show that forcings of this type have the λ-approximation property. For completeness, below we outline the argument.

Lemma 3.5. *$\mathbb{N} := j(\mathbb{R})/G * H$ has the λ-approximation property.*

Proof. Suppose that $\tau : \mu \to 2$ in the extension by $\mathbb{N}$, such that for all $x \in V[G][H] \cap \mathcal{P}_\lambda(\mu)$, $\tau \restriction x \in V[G][H]$. Suppose for contradiction that τ is not in $V[G][H]$. We will denote conditions in $\mathbb{N}$ by $(p, f, \dot{r})$, where $p \in j(\mathbb{A})$, $f \in j(\mathbb{Q})$, r is forced to be in $j(\mathbb{P})/\mathbb{P}$. For a Prikry condition r (in $\mathbb{P}$ or $j(\mathbb{P})$), we use the notation $r = (s(r), A(r))$.

Note that $j(\mathbb{Q})$ is κ^+-closed in $V[G]$.

Claim 3.6. *There is a condition $(p, f, \dot{r}) \in \mathbb{N}$, such that for each $x \in \mathcal{P}_\lambda(\mu)$ and function $\sigma : x \to 2$ in $V[G][H]$, and for every $(p', f', \dot{r}') \leq_{\mathbb{N}} (p, f, \dot{r})$, if $f' \leq_{j(\mathbb{Q})} f$ and $(p', f', \dot{r}') \Vdash \dot{\tau} \restriction x = \sigma$, then there is some $f'' \leq_{j(\mathbb{Q})} f'$ such that $(p, f'', \dot{r}) \Vdash \dot{\tau} \restriction x = \sigma$.*

Proof. Otherwise, in $V[G]$, let $\bar{r} \in \mathbb{P}$ force the negation of the conclusion. Then whenever $\bar{r} \Vdash (p, f, \dot{r}) \in \dot{\mathbb{N}}$, densely often below $\bar{r}$, there are conditions $\bar{r}' \in \mathbb{P}$, such that there are $p_0, p_1 \in j(\mathbb{A})$, $f^* \leq_{j(\mathbb{Q})} f$, $j(\mathbb{M})/G$-names for elements in $j(\mathbb{P})$, $\dot{r}_0, \dot{r}_1, x \in V[G][H] \cap \mathcal{P}_\lambda(\mu)$, and $\mathbb{P}$-names σ_0, σ_1 such that

- for $i \in \{0, 1\}$, $\bar{r}' \Vdash (p_i, f^*, \dot{r}_i) \leq_{\mathbb{N}} (p, f, \dot{r})$,
- for $i \in \{0, 1\}$, $\bar{r}' \Vdash$ “$(p_i, f^*, \dot{r}_i) \Vdash_{\mathbb{N}} \dot{\tau} \restriction x = \sigma_i$”
- σ_0, σ_1 are forced to be distinct.

Moreover we can choose the above, so that if $g \leq_{j(\mathbb{Q})} f^*$, then a direct extension of $\bar{r}'$ forces that $(p_i, g, \dot{r}_i) \in \mathbb{N}$.

By induction construct $p^i_\alpha, \sigma^i_\alpha, \sigma_\alpha, f_\alpha, \dot{r}^i_\alpha, \bar{r}_\alpha$ and $\dot{x}_\alpha$ for $\alpha < \kappa^+, i \in 2$, such that $\langle f_\alpha \mid \alpha < \kappa^+\rangle$ is $\leq_{j(\mathbb{Q})}$-decreasing, and for each α, i, $\bar{r}_\alpha \in \mathbb{P}$ forces that:

- $\langle \dot{x}_\alpha \mid \beta < \alpha\rangle$ is a $\subset$- increasing sequence of elements in $V[G][\dot{H}] \cap \mathcal{P}_\lambda(\mu)$,
- $(p^i_\alpha, f_\alpha, \dot{r}^i_\alpha) \in \mathbb{N}$, $(p^i_\alpha, f_\alpha, \dot{r}^i_\alpha) \Vdash \dot{\tau} \restriction \bigcup_{\beta<\alpha} \dot{x}_\beta = \sigma_\alpha$,
- $(p^i_\alpha, f_\alpha, \dot{r}^i_\alpha) \Vdash \dot{\tau} \restriction x_\alpha = \sigma^i_\alpha$, where $\sigma^0_\alpha \neq \sigma^1_\alpha$, and
- (p^i_α, f_α) decides $s(\dot{r}^i_\alpha)$, and $s(\bar{r}_\alpha)$ extends it.

Since there are only κ many possible stems and $\mathbb{A} \times \mathbb{A}$ has the κ^+-c.c., there are $\beta < \beta' < \kappa^+$, such that $s(\bar{r}_\beta) = s(\bar{r}_{\beta'})$, and for $i \in 2$, $s(\dot{r}^i_\beta) = s(\dot{r}^i_{\beta'})$, and p^i_β is compatible with $p^i_{\beta'}$. Then for $i \in 2$, let p^i be the weakest lower bound for p^i_β and $p^i_{\beta'}$ and let $\dot{r}^i$ be a name for a common extension of $\dot{r}^i_\beta$ and $\dot{r}^i_{\beta'}$ with the same stem.

The following sufficient condition for forcing conditions into the quotient appears in [1].

Lemma 3.7. *Working in* $V[G]$, *let* $\bar{r} \in \mathbb{P}$, $m \in j(\mathbb{M})/G$ *and let* $\dot{r}$ *be a* $j(\mathbb{M})/G$-*name for a condition in* $j(\mathbb{P})$ *such that*

(1) *m decides the value of* $s(\dot{r})$,
(2) $s(\bar{r})$ *extends* $s(\dot{r})$ *and*
(3) *m forces that points in* $s(\bar{r})$ *above* $s(\dot{r})$ *are in* $A(\dot{r})$.

Then there is a direct extension of $\bar{r}$ *which forces* $(m, \dot{r}) \in j(\mathbb{R})/(G * H)$.

Now by Lemma 3.7, there is a direct extension r of $\bar{r}_\beta$ and $\bar{r}_{\beta'}$ which forces that each $(p^i, f_{\beta'}, \dot{r}^i)$ is in $\mathbb{N}$. Force with $\mathbb{P}$ below r to get a contradiction. ⊣

Work in $V[G]$. Let $r^* \in \mathbb{P}$ force that $(p, f, \dot{r})$ is as in Claim 3.6. Using the claim, inductively construct splitting sequences $\langle\langle f_s, \alpha^h_s\rangle \mid s \in 2^{<\kappa}, h \text{ is a stem}\rangle$, such that:

(1) if $s' \supset s$, then $f_{s'} \leq_{j(\mathbb{Q})} f_s$,
(2) for all $s \in 2^{<\kappa}$, stems h, and $i \in 2$, there is some Prikry condition with stem h forcing that $(p, f_{s^\frown i}, \dot{r}) \Vdash \dot{\tau}(\alpha^h_s) = i$.

Note in particular that if $s \perp t$, then f_s, f_t are forced to be incompatible: Otherwise we would have a Prikry condition, say with some stem h, forcing compatibility, and taking a strong enough direct extension contradicts (2).

Let $a^* = \{\alpha^h_s \mid h \text{ a stem}, s \in 2^{<\kappa}\}$; note $|a^*| < \lambda$.

Still working in $V[G]$, note that $j(\mathbb{M})/G$ is forced to add an Add$(\kappa, 1)$-generic set; let $\dot{g}$ be a name for this. Then in the extension by Add$(\kappa, 1)$, $\langle f_{\dot{g}\restriction\eta} \mid \eta < \kappa\rangle$ is forced to be $\leq_{j(\mathbb{Q})}$-decreasing.

We claim there is an element f^* of $j(\mathbb{Q})$ that is forced to be a lower bound of $\langle f_{\dot{g}\restriction\eta} \mid \eta < \kappa\rangle$. This is done by, for each γ that can be forced in Add$(\kappa, 1)$ to belong to some dom$(f_{\dot{g}\restriction\eta})$, defining a name for a lower bound of $\langle f_{\dot{g}\restriction\eta}(\gamma)\rangle_{\eta<\kappa}$. By the κ^+-c.c. of Add$(\kappa, 1)$, we may cover the union of possible domains

$$\{\gamma < \lambda \mid p' \Vdash \gamma \in \mathrm{dom}(f_{\dot{g}\restriction\eta}) \text{ for some } p' \in \mathrm{Add}(\kappa, 1) \text{ and } \eta < \kappa\}$$

by a set Y in $V[G]$ with $|Y| = \kappa$. We define f^* so that for all $\gamma \in Y$, $f^*(\gamma)$ is a $\mathbb{A} \restriction \gamma$-name such that if $p' \in$ Add$(\kappa, 1)$ forces $\gamma \in \mathrm{dom}(f_{\dot{g}\restriction\eta})$ for any η, then p' forces $f^*(\gamma)$ to be a lower bound for $\langle f_{\dot{g}\restriction\eta}(\gamma) \mid \eta < \kappa\rangle$. Finally, $(p, f^*, \dot{r})$ may be forced into the quotient $\mathbb{N}$.

Now working in $V[G][H]$, by the fact that τ is λ-approximated, we can (using Claim 3.6 to extend f^*, if necessary) assume $(p, f^*, \dot{r})$ decides $\dot{\tau} \restriction a^*$; say $\tau \restriction a^* = \sigma$. Let h be the stem of a Prikry condition below r^* forcing this to hold over $V[G]$.

Define $g : \kappa \to 2$ by inductively letting $g(\eta) = \sigma(\alpha^h_{g \restriction \eta})$. We have g is unique such that $f^* \leq f_{g \restriction \eta}$ for all η; by construction, g is $\mathrm{Add}(\kappa, 1)$-generic over $V[G]$. But $g \in V[G][H]$ was added by forcing with $\mathbb{P}$, a contradiction. ⊣

This completes the proof of Theorem 3.2. ⊣

Note by Theorem 2.1, $\mathsf{ISP}(\lambda)$ must fail in this model. This is related to the following remark on the extent of approximation in $j(\mathbb{R})/\mathbb{R}$.

Proposition 3.8. *$j(\mathbb{R})/\mathbb{R}$ does not have the $\aleph_1$-approximation property.*

Proof. Let x be any subset of κ in $V[j(\mathbb{R})]$. For any countable $a \subseteq \kappa$ in $V[\mathbb{R}]$, we trivially have $a \cap x \in V[\mathbb{R}]$, since no reals are added by the quotient $j(\mathbb{R})/\mathbb{R}$. So any subset of κ added by the quotient is a witness to the failure of $\aleph_1$-approximation. ⊣

Applying corollary 2.4, the above models yields:

Theorem 3.9. *From a supercompact, it is consistent to have $\mathsf{ITP}(\lambda)$, for λ the double successor of a singular strong limit cardinal, together with club many non $\aleph_1$-internally unbounded models of size less than λ.*

§4. Extender based forcing and ITP.

In this section we describe another model where ITP at the double successor of a singular strong limit. We use it to show that it is consistent to have ITP at the double successor of a singular together with the set of internally unbounded models being nonstationary. This is a partial result towards showing that ITP does not imply SCH.

Theorem 4.1. *Suppose that $\langle \kappa_n \mid n < \omega \rangle$ are strong cardinals, $\kappa = \sup_n \kappa_n$ and λ is a supercompact cardinal above κ. Then there is a forcing extension in which $|\prod_n \kappa_n| = \lambda = \kappa^{++}$ and ITP holds at λ.*

We take $\mathbb{P}$ to be the long extender forcing from Section 2 of [4]. This is almost the same poset from Section 2 of Gitik's Handbook chapter [3] with one modification: the Cohen parts of conditions are allowed to be Prikry names. For completeness, we briefly describe the poset. Let $E_n = \langle E_{n,\alpha} \mid \alpha < \lambda \rangle$ be an extender on κ_n of length λ. We have that Lemmas 2.1–2.4 from Section 2 of [3] hold.

As in [3], define $\mathbb{Q}_{n1}$ to be the poset of partial functions $f : \lambda \rightharpoonup \kappa_n$, with $|f| \leq \kappa$ (equivalently, $\mathrm{Add}(\kappa^+, \lambda)$). Also, for $\alpha < \lambda$, *the Prikry forcing at α* refers to the diagonal forcing with respect to the measures $\langle E_{n,\alpha} \mid n < \omega \rangle$ to add a sequence $\langle \rho_n \mid n < \omega \rangle$ in $\prod_n \kappa_n$.

The extender based forcing from [3] adds an unbounded $F \subset \lambda$ (in the notation below, $a^p := \bigcup_{n \geq \mathrm{lh}(p)} a^p_n$ and let $F = \bigcup_{p \in G} a^p$ and for every $\alpha \in F$, ω-sequences $t_\alpha \in \prod_n \kappa_n$ (in the notation below $t_\alpha(n) = f^p_n(\alpha)$ for some (equivalently all) $p \in G$, such that $\alpha \in \mathrm{dom}(f^p_n)$). Each such t_α is generic for the the Prikry forcing at α. In particular, below a condition forcing that $\alpha \in \dot{F}$, $\mathbb{P}$ projects to this forcing, and we denote the projection map by π_α.

Conditions are of the form $p = \langle p_n \mid n < \omega \rangle$, where for $n < \mathrm{lh}(p)$, $p_n = f_n \in \mathbb{Q}_{n1}$, and $n \geq \mathrm{lh}(p)$, $p_n = (a_n, A_n, f_n)$, such that:

- for $n \geq \mathrm{lh}(p)$, $a_n \in [\lambda]^{<\kappa_n}$, $A_n \in E_{n,\max(a_n)}$, $a_n \subset a_{n+1}$, f_n is a Prikry name for a condition in $\mathbb{Q}_{n1}$ with domain disjoint from a_n.

- for $n \geq \mathrm{lh}(p)$, for $\alpha \in a_n$, for $m < n$ $f_m \restriction \lambda \setminus (\alpha+1)$ is forced to be a condition in $\mathbb{Q}_{m1}$ by the Prikry forcing at α.

We also require that $\langle \mathrm{dom}(f_n) \mid n < \omega \rangle \in V$ and $\langle f_n(\alpha) \mid n < \omega \rangle \in V$ whenever $\alpha \in \mathrm{dom}(f_n)$ for all large n.

For p as above, we use the notation $p_n = f^p_n$ for $n < \mathrm{lh}(p)$ and $p_n = (a^p_n, A^p_n, f^p_n)$ for $n \geq \mathrm{lh}(p)$.

The order $q \leq p$ is as in [3] with the natural modification corresponding to the last item of the definition: if $\alpha \in a^q_n$, then $\pi_\alpha(q)$ forces that $f^q_n \restriction \lambda \setminus (\alpha + 1)$ is stronger than $f^p_n \restriction \lambda \setminus (\alpha + 1)$.

REMARK 4.2. The last item in the definition above is the difference between $\mathbb{P}$ and the usual long extender based forcing. The point of this modification is to collapse cardinals between κ^+ and λ. More formally, we can define the f^p_n's to be functions from finite sequences (i.e., Prikry stems) from $\prod_{\max(n,\mathrm{lh}(p)) \leq i < \mathrm{lh}(p)+k} A^p_i$, so that each $f_n(\vec{v}) \in \mathbb{Q}_{n1}$. A similar construction was first done in Assaf Sharon's thesis, Chapter IV, [8], and is also described in [9]. Then, given $\mathbb{P}$-generic filter G, for any $\alpha \in F$, $\{f^p_{n_0}(t_\alpha \restriction k) \mid p \in G, k > \omega\}$ will collapse α to κ^+.

We say that q is a direct extension of p, $q \leq^* p$, if $q \leq p$ and they have the same length. We say that q is an n-step extension of p if $q \leq p$ and $\mathrm{lh}(q) = \mathrm{lh}(p) + n$. Also, as usual, given p and $\vec{v} \in \prod_{\mathrm{lh}(p) \leq i < \mathrm{lh}(p)+n} A^p_i$, we write $p^\frown \vec{v}$ to denote the weakest n-step extension of p obtained from $\vec{v}$. I.e., if $r \leq p$ is with length at least $\mathrm{lh}(p) + n$ and for $\mathrm{lh}(p) \leq i < \mathrm{lh}(p) + n$, $f^r_i(\max(a^p_i)) = v_i$, then $r \leq p^\frown \vec{v}$.

$\mathbb{P}$ has the Prikry property, and more generally:

LEMMA 4.3 (Prikry lemma). *Suppose that D is a dense open set and p is a condition. Then there is $q \leq^* p$ and n, such that every n-step extension of q is in D.*

For the proof, see [3]. When the dense set above is of the form $\{r \mid r \parallel \phi\}$ for some formula ϕ, then $n = 0$. In particular there is a direct extension of p deciding ϕ. Then, since $\leq^*$ restricted to conditions of length n is κ_n-closed, the forcing does not add bounded subsets of κ and preserves κ^+. It also has the λ-chain condition. Forcing with this poset adds λ-many Prikry sequences in $\prod_n \kappa_n$, making $\kappa^\omega = \lambda = \kappa^{++}$ (see [3], [4]). In [4], it is also shown that in the generic extension by $\mathbb{P}$, λ has the tree property. Here we show that ITP_λ holds.

Let G be $\mathbb{P}$-generic. Suppose that for some $\theta \geq \lambda$, in $V[G]$, $\langle d_x \mid x \in \mathcal{P}_\lambda(\theta) \rangle$ is a thin $\mathcal{P}_\lambda(\theta)$-list. I.e., each $d_x \subset x$ and for club many $c \in \mathcal{P}_\lambda(\theta)$, $|\{d_x \cap c \mid c \subset x\}| < \lambda$.

Let $j : V \to M$ be a θ-supercompact embedding with critical point λ. By standard arguments, we have that $j(\mathbb{P})$ projects to $\mathbb{P}$. So, we can extend j to $j : V[G] \to M^*$. Then $d := j^{-1}[j(d)_{j[\theta]}]$ is an ineffable branch in the extension by $j(\mathbb{P})$ for the list.

We have to show that d cannot have been added by $j(\mathbb{P})/G$, i.e., that this poset has the thin λ-approximation property. Suppose for contradiction that $d \notin V[G]$.

Work in V. Let $\pi : j(\mathbb{P}) \to \mathbb{P}$ be the projection. Fix a $j(\mathbb{P})$-name for this branch $\dot{d}$, so that $1_{j(\mathbb{P})} \Vdash$ "$\forall x \in \mathcal{P}_\lambda(\theta) \dot{d} \cap x \in V[\dot{G}_{\mathbb{P}}]$". Note that for every $x \in \mathcal{P}_\lambda(\theta)$, in $V[G]$, there is $y \in \mathcal{P}_\lambda(\theta)$ in V, such that $x \subset y$. That is by the λ chain condition of $\mathbb{P}$. So we can restrict our attention to elements of $\mathcal{P}^V_\lambda(\theta)$.

Below we will say that a condition $p \in j(\mathbb{P})$ decides a value for $\dot{d} \cap x$ (or simply decides $\dot{d} \cap x$), if for some $\mathbb{P}$-name a, $p \Vdash_{j(\mathbb{P})} \dot{d} \cap x = a_{\dot{G}}$.

Lemma 4.4. *For any $x \in \mathcal{P}_\lambda(\theta)$ and $p \in j(\mathbb{P})$, there is $q \leq^* p$ and n, such that every n-step extension of q decides a value for $\dot{d} \cap x$. Moreover, for any $k \geq \mathrm{lh}(p)$, we can obtain q as above so that for all $\mathrm{lh}(p) \leq i \leq k$, $a_i^q = a_i^p$.*

Proof. We apply the Prikry lemma for $j(\mathbb{P})$ to the dense set $D = \{q \mid (\exists\, \mathbb{P}\text{-name } a) q \Vdash \dot{d} \cap x = a_{\dot{G}}\}$ to obtain q. The statement in the 'moreover' follows by the proof of the Prikry lemma. See for example Section 2 of [4]. ⊣

Lemma 4.5. *There is $\bar{n} < \omega$ and a condition $p' \in j(\mathbb{P})$, such that for all $p \leq^* p'$, there is $x \in \mathcal{P}_\lambda(\theta)$, such that for all $y \in \mathcal{P}_\lambda(\theta)$ with $x \subset y$, there is $q \leq^* p$, such that each $\bar{n}$-step extension of q decides a value for $\dot{d} \cap y$.*

Proof. Suppose otherwise. Then inductively build a $\leq^*$-decreasing sequence $\langle p_n \mid n < \omega\rangle$, an increasing $\langle k_n \mid n < \omega\rangle$ and a $\subset$-increasing sequence $\langle y_n \mid n < \omega\rangle$ in $\mathcal{P}_\lambda(\theta)$, such that for all n, there is no $q \leq^* p_n$, such that every k_n-step extension of q decides a value for $\dot{d} \cap y_n$. Now let $y = \cup_n y_n$ and $p \leq^* p_n$ for all n. Let $q \leq^* p$ and $k < \omega$ be such that every k-step extension of q decides a value for $\dot{d} \cap y$.

Pick n, such that $k \leq k_n$. But then any k_n-step extension of q decides a value for $\dot{d} \cap y_n$. Contradiction. ⊣

Remark 4.6. In the above lemma, for any $k < \omega$, we can get such a q, so that the $a_i^q = a_i^p$ for all $\mathrm{lh}(q) \leq i \leq k$

Fix $\bar{n}$ and p' as in the conclusion of the lemma. From now on work below p'. The following lemma is an adaptation of Lemma 2.7 of Gitik's paper [4].

Lemma 4.7. *Let $p \in j(\mathbb{P})$ and let δ be regular, such that $2^{\kappa_k} < \delta < \kappa_{k+1}$, where $k \geq \bar{n} + \mathrm{lh}(p)$. Then there is $x \in \mathcal{P}_\lambda(\theta)$, $\bar{q} \in \mathbb{P}$ and a sequence $\langle p_\xi \mid \xi < \delta\rangle$ of direct extensions of p, such that:*

1. *For all $\mathrm{lh}(p) \leq i \leq k$, for all ξ, $a_i^{p_\xi} = a_i^p$, $A_i^{p_\xi} = A_i^p$;*
2. *$\bar{q} = \pi(p_\xi)$ for all ξ;*
3. *every $\bar{n}$-step extension of p_ξ decides $\dot{d} \cap x$;*
4. *for $\xi \neq \xi'$, if r, r' are two $\bar{n}$-step extensions of p_ξ and $p_{\xi'}$, respectively, then $\bar{q}$ forces that the values decided by r and r' are different.*

Proof. This is a modification of Lemma 2.7 in [4], so we only focus on the main points.

Using the above lemma, construct $\langle q_\xi, M_\xi \mid \xi \leq \delta\rangle$, with each $|M_\xi| < \lambda$, such that:

1. $\langle q_\xi \mid \xi \leq \delta\rangle$ is $\leq^*$-decreasing sequence in $j(\mathbb{P})$, such that for all $\mathrm{lh}(p) \leq i \leq k$, for all ξ, $a_i^{q_\xi} = a_i^p$, $A_i^{q_\xi} = A_i^p$;
2. $\langle M_\xi \mid \xi \leq \delta\rangle$ is a $\in$-increasing continuous chain of elementary submodels, such that $M_0^{\kappa_k} \subset M_0$ and for each ξ, $M_{\xi+1}^{\kappa_k} \subset M_{\xi+1}$;
3. for all ξ, $q_\xi \in M_{\xi+1}$;
4. for all ξ, $y_\xi := M_\xi \cap \theta$ and each $\bar{n}$-step extension of q_ξ decides a value of $\dot{d} \cap y_\xi$;

To obtain $A_i^{q_\xi} = A_i^p$ for every $\xi < \delta$, we use that $2^{\kappa_k} < \delta$ and pass through an unbounded subset of δ if necessary.

Set $q := q_\delta$, $y := y_\delta = \bigcup_{\xi<\delta} y_\xi$.

Let X_ξ be the set of all values of $\dot{d} \cap y_\xi$ decided by an $\bar{n}$-step extension of q_ξ. I.e., X_ξ is a set of $\mathbb{P}$-names and is of size at most $\kappa_{\mathrm{lh}(p)+\bar{n}-1} < \delta$. Denote $X := X_\delta$. For $t \in X$, we identify $t \cap y_\xi$ with the $\mathbb{P}$-name a, such that $\Vdash_{\mathbb{P}} a = t \cap y_\xi$. For simplicity of notation, we identify elements in X_ξ as equal or distinct whenever $\pi(q_\xi)$ forces them to be so. Similarly if $\pi(q_\xi)$ forces $s = s'$ for some $s \in X_\xi$, we will identify s with s' and simply write $s' \in X_\xi$.

We have the following:

(1) (Coherence) If $t \in X$, then for all $\xi < \delta$, $t \cap y_\xi \in X_\xi$.
(2) (Splitting) If t, s are incompatible elements of X, then there is $\xi < \delta$, such that $t \cap y_\xi \neq s \cap y_\xi$.

Using that δ is greater than the possible instances of splitting, we fix some $\bar{\xi} < \delta$, after which there is no more splitting. I.e., for distinct t, s in X, there $\xi < \bar{\xi}$ with $t \cap y_\xi \neq s \cap y_\xi$.

Claim 4.8. *For all $\xi < \delta$, for all $\bar{r} \leq \pi(q_\xi)$, there is $z \in M_{\xi+1}$ with $y_\xi \subset z \subset y_{\xi+1}$ and $r \leq q_\xi$, such that $\pi(r) \leq \bar{r}$ and r decides a value for $\dot{d} \cap z$ incompatible with every value in $X_{\xi+1}$. More precisely, setting $r \Vdash \dot{d} \cap z = s$, we have that for all $t \in X_{\xi+1}$, $t \cap z \neq s$.*

Proof. By elementarity of the models and since we have assumed that the branch is new. Namely, if we suppose otherwise, we get that $M_{\xi+1} \models q_\xi$ forces that the branch is in $V[\dot{G}_{\mathbb{P}}]$. $\dashv$

Let $\xi < \delta$. Apply the above claim inductively to all $\bar{n}$-step extensions $q_\xi{}^\frown\vec{v}$ of q_ξ. Then we get $r_\xi^{\vec{v}} \leq^* q_\xi{}^\frown\vec{v}$ that decides $\dot{d} \cap y_{\xi+1}$ in a way that is incompatible with all the values in $X_{\xi+1}$.

As in the proof of the Prikry lemma, diagonalize $r_\xi^{\vec{v}}$ for each such $\vec{v}$ to obtain a condition $r_\xi \leq^* q_\xi$, with $a_i^{r_\xi} = a_i^{q_\xi} = a_i^p$ for $i \leq k$, such that every $\bar{n}$-step extension of r_ξ is stronger that some $r_\xi^{\vec{v}}$. By passing to an unbounded subset of δ, we may assume that for all ξ and $i \leq k$, $A_i^{r_\xi} = A_i^{q_\xi} = A_i^p$.

Also, doing this by induction on $\xi < \delta$, we can shrink the q_ξ's and then the r_ξ's, so that for each $\xi < \delta$, $\pi(q) = \pi(r_\xi)$. Finally, let $p_\xi \leq^* r_\xi$, so that $\pi(p_\xi) = \pi(q)$, for $i \leq k$, $a_i^{p_\xi} = a_i^{r_\xi}$, and p_ξ decides a value of $\dot{d} \cap y$. Then $\langle p_\xi \mid \xi < \delta \rangle$, $\pi(q)$ and y are as desired. $\dashv$

Let $\langle \delta_n \mid n < \omega \rangle$ be a cofinal sequence of measurable cardinals in κ, such that $2^{\kappa_n} < \delta_n < \kappa_{n+1}$ for each n. For each n, let U_n be a measure on δ_n.

Build a $\subset$-increasing sequence $\langle x_n \mid n < \omega \rangle$ in $\mathcal{P}_\lambda(\theta)$, a $\leq^*$-decreasing sequence of conditions $\langle q_n \mid n < \omega \rangle$ in $\mathbb{P}$, and $\langle p_\sigma \mid \sigma \in \prod_{n \leq k} Y_n, k < \omega \rangle$ in $j(\mathbb{P})$, where each $Y_n \in U_n$, such that:

(1) For all n, for all $\mathrm{lh}(p) \leq i \leq n$, $a_i^{q_n} = a_i^{q_{n+1}}$;
(2) For all σ, $\pi(p_\sigma) = q_{|\sigma|}$;
(3) If σ' extends σ, then $p_{\sigma'} \leq^* p_\sigma$;
(4) For all n, for all $\sigma \in \prod_{m<n} Y_m$ and $\xi \in Y_n$, for all $\mathrm{lh}(p) \leq i \leq n$, $a_i^{p_\sigma} = a_i^{p_\sigma{}^\frown\xi}$ and $A_i^{p_\sigma} = A_i^{p_\sigma{}^\frown\xi}$;
(5) All $\bar{n}$-stem extensions of each p_σ decide $\dot{d} \cap x_{|\sigma|}$;

(6) If σ_1 and σ_2 are incompatible, then any two $\bar{n}$-step extensions of p_{σ_1} and p_{σ_2} decide incompatible values for $\dot{d} \cap x_{|\sigma_1|}$ and $\dot{d} \cap x_{|\sigma_2|}$ (as forced by $q_{|\sigma_1 \cap \sigma_2|}$).

We do this by induction on $|\sigma|$. For simplicity, all conditions will have length 0.

First let $\langle p_\xi \mid \xi < \delta_0 \rangle$, x_0, q_0 be given by the above lemma applied to δ_0. Then for each $\xi < \delta_0$, inductively apply Lemma 4.7 to p_ξ, δ_1 to obtain sequences $\langle p'_{\xi,\eta} \mid \eta < \delta_1 \rangle$, q_ξ, and x_ξ. Let $x_1 = \bigcup_\xi x_\xi$. Then let $p''_{\xi,\eta} \leq^* p'_{\xi,\eta}$ be such that $a_i^{p''_{\xi,\eta}} = a_i^{p'_{\xi,\eta}}$ for $i = 0, 1$ and every $\bar{n}$-step extension of $p''_{\xi,\eta}$ decides $\dot{d} \cap x_1$.

Note that, by construction we have that for $i = 0, 1$, $a_i^{p''_{\xi,\eta}}$ are constant for all $\langle \xi, \eta \rangle$.

Next we use the measurability of δ_1 to fix the measure one sets in the first two coordinates, in order to be able to define q_1.

For each $\xi < \delta_0$, consider the map $\phi_\xi : \eta \mapsto \langle A_0^{p''_{\xi,\eta}}, A_1^{p''_{\xi,\eta}} \rangle$. Since $2^{\kappa_1} < \delta_1$, let $B_\xi \in U_1$ be such ϕ_ξ is constant on B_ξ, say with value $\langle A_0^\xi, A_1^\xi \rangle$. Let $B_1 = \bigcap_{\xi < \delta_0} B_\xi \in U_1$, and let $A_1 = \bigcap_{\xi < \delta_0} A_1^\xi$. For the latter we use that $\delta_0 < \kappa_1$. Now consider the map $\xi \mapsto A_0^\xi$. Since $2^{\kappa_0} < \delta_0$, fix $B_0 \in U_0$, on which this map is constant, say with value A_0.

Then, for $\xi \in B_0, \eta \in B_1$, let $p_{\langle \xi,\eta \rangle}$ be obtained from $p''_{\xi,\eta}$, so that $\pi(p_{\langle \xi,\eta \rangle})$ is constant. (in particular, each $A_1^{p_{\langle \xi,\eta \rangle}} = A_1$). Then we can define $q_1 = \pi(p_{\langle \xi,\eta \rangle})$ for some (equivalently all) $\langle \xi, \eta \rangle \in B_0 \times B_1$.

Continue in the same way for the rest of the construction. At the end each Y_n will be the intersection of countably many measure one sets in U_n.

Let q be a lower bound for the q_n's. Let G be $\mathbb{P}$ generic containing q and work in $V[G]$. For each $f \in \prod_n Y_n$, let $p_f \leq^* p_{f \restriction n}$ for all n. Let $c \supset \cup_n x_n$ in $\mathcal{P}_\lambda(\theta)$ be such that $|\{d_x \cap c \mid c \subset x\}| < \lambda$. Now let r_f be an $\bar{n}$-step extension of p_f, of the form $p_f{}^\frown \vec{v}$, where each $v_i \in Y_i$. Then $r_f \Vdash \dot{d} \cap c = x_f$ for some x_f.

By the construction, if $f \neq g$, then $x_f \neq x_g$. But there are λ-many such f's in $V[G]$ and only $< \lambda$ possibilities of $\dot{d} \cap c$. Contradiction.

This concludes the proof of Theorem 4.1.

The above construction provides a second proof of Theorem 3.9, that it is consistent to have $\mathsf{ITP}(\lambda)$, for λ the double successor of a singular strong limit cardinal, together with club many non $\aleph_1$-internally unbounded models of size less than λ.

§5. Down to $\aleph_{\omega+2}$.

In this section we modify the construction from the previous section to obtain the results for $\lambda = \aleph_{\omega+2}$ and prove the following theorem.

THEOREM 5.1. *Suppose that* $\langle \kappa_n \mid n < \omega \rangle$ *are strong cardinals with limit* κ *and* λ *is super compact cardinal above* κ. *Then there is a forcing extension where* $\kappa = \aleph_\omega$, $\lambda = \aleph_{\omega+2}$ *and we have* $\mathsf{ITP}(\aleph_{\omega+2})$, *together with club many non* $\aleph_1$-*internally unbounded models of size less than* $\aleph_{\omega+2}$.

We will use short extender forcing with interleaved collapses from Section 3 of [4]. And just as in [4], first we have to prepare the ground model as follows. Fix measurable cardinals $\langle \delta_n \mid n < \omega \rangle$, such that for each n, $2^{\kappa_n} < \delta_n < \kappa_{n+1}$ and normal measures U_n on each δ_n. Force with the full support iteration of Levy collapses $\mathrm{Col}(\kappa_n^{+n+4}, < \delta_n)$, and call the resulting model V. Then in V each U_n will give rise to a precipitous ideal I_n, such that forcing with its positive sets is κ_n^{+n+4} -strategically closed. We will use this in place of measurability.

Let $\mathbb{P}$ be the poset defined in Definition 3.1 in Section 3 of [4].

We list some of the key properties of $\mathbb{P}$:

(1) $(\mathbb{P}, \leq, \leq^*)$ has the Prikry property. In particular, for any p and dense open set D, there is $n < \omega$ and $q \leq^* p$, such that every n-step extension of p is in D. As a corollary, for all p, ϕ, there is $q \leq^* p$ deciding ϕ.

(2) From the above it follows that κ and κ^+ is preserved and κ remains a strong limit.

(3) There is a suborder $\rightarrow$ on $\mathbb{P}$, such that $(\mathbb{P}, \leq)$ projects to $(\mathbb{P}, \rightarrow)$, and $(\mathbb{P}, \rightarrow)$ has the λ-chain condition.

(4) Forcing with $(\mathbb{P}, \rightarrow)$ makes $\lambda = \kappa^{++} = 2^\kappa = \aleph_{\omega+2}$.

For proofs of the above, see Section 4 of [5] and also [13]. A theorem in [4] is that the tree property holds in this extension at $\aleph_{\omega+2}$. Here we show the case for ITP.

Lemma 5.2. *$j(\mathbb{P}; \leq)/(\mathbb{P}; \leq)$ has the thin λ-approximation property.*

Proof. We run the same argument as in the previous section. By the Prikry property we still have Lemma 4.5. We will use the same notation as in [4], Definition 3.1: for a condition $p = \langle p_n \mid n < \omega \rangle$, we denote $p_n = (\rho_n, h_{<n}, h_{>n}, f_n)$ for $n < \mathrm{lh}(p)$ and $p_n = (a_n, A_n, S_{<n}, h_{>n}, f_n)$ for $n \geq \mathrm{lh}(p)$.

Then we claim that Lemma 4.7 holds for $\delta = \kappa_k^{+k+5}$. To prove the lemma, we construct the $\leq^*$-decreasing sequence $\langle q_\xi \mid \xi < \delta \rangle$ in $j(\mathbb{P})$ as before, so that in addition to the requirements listed in the proof of Lemma 4.7, we also have:

- for all $i < \mathrm{lh}(p)$, for all ξ, $h^{q_\xi}_{<i} = h^p_{<i}$,
- for all $\mathrm{lh}(p) \leq i \leq k$, for all ξ, $S_i^{q_\xi} = S_i^p$,
- for all $i < k$, for all ξ, $h^{q_\xi}_{>i} = h^p_{>i}$,
- $\langle h^{q_\xi}_{>k} \mid \xi < \delta \rangle$ is decreasing.

We can do the first three items by passing to an unbounded subset of δ if necessary. For the last, we use that the closure is $\kappa_k^{+k+8} > \delta$. Also, since $\delta < \kappa_{k+1}$ for coordinates $i > k$, we have enough closure to make sure the sequence is decreasing. Also, since now we are using short extenders, for $i < k$, here we maintain $\mathrm{dom}(a_i^{p_\xi}) = \mathrm{dom}(a_i^p)$. The rest of the lemma goes as before.

Then in the last section, we construct $\langle x_n, q_n \mid n < \omega \rangle$ in $\mathcal{P}_\lambda(\theta)$ and $\langle p_\sigma \mid \sigma \in \prod_{n \leq k} Y_n, k < \omega \rangle$ in $j(\mathbb{P})$, where each $Y_n \in U_n$ with the additional properties that:

- for all n, for all $i \leq n$, $\mathrm{dom}(a_i^{q_n}) = \mathrm{dom}(a_i^{q_{n+1}})$,
- for all n, for all $\sigma \in \prod_{m<n} Y_m$ and $\xi \in Y_n$,
  - for all $i \leq n$, $\mathrm{dom}(a_i^{p_\sigma}) = \mathrm{dom}(a_i^{p_{\sigma^\frown \xi}})$, $S_i^{p_\sigma} = S_i^{p_{\sigma^\frown \xi}}$,
  - for all $i < n$, $h^{p_\sigma}_{>i} = h^{p_{\sigma^\frown \xi}}_{>i}$.

The U_n's are no longer normal, but all we need is their closure properties to fix the components of the conditions where we do not have sufficient closure. The rest of the argument is the same as in the previous section. ⊣

Now suppose that $\langle d_x \mid x \in \mathbb{P}_\lambda(\theta) \rangle$ is a $\mathbb{P}_\lambda(\theta)$-list in $V[\mathbb{P}; \rightarrow]$. Let $j : V \rightarrow M$ be a θ-supercompact embedding with critical point λ. As in the last section, lift j to obtain an ineffable branch d for this list with $d \in V[j(\mathbb{P}; \rightarrow)]$. In particular, $d \in V[j(\mathbb{P}; \leq)]$, and so by the approximation property, we have that $d \in V[\mathbb{P}; \leq]$. Note that any condition in $\mathbb{P}$ can also be viewed as a condition in $j(\mathbb{P})$.

Now let G be $(\mathbb{P};\to)$-generic. We have to show that $d \in V[G]$. Consider the quotient $(\mathbb{P};\leq)/G$ and let $\dot{d} \in V[G]$ be a $(\mathbb{P};\leq)/G$-name for the branch. For two conditions $p, q \in (\mathbb{P};\leq)/G$ if p and q decide contradictory information about $\dot{d}$, then clearly $p \perp_{(\mathbb{P};\leq)} q$, but also $p \perp_{j(\mathbb{P};\to)} q$, since the branch is in the extension by $j(\mathbb{P};\to)$. But since the projection $j(\mathbb{P}) \to \mathbb{P}$ is the identity on conditions in $\mathbb{P}$, that means that $p \perp_{(\mathbb{P};\to)} q$, which is a contradiction with $p, q \in G$.

It follows that there is no splitting, and so the branch is in $V[G]$.

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REFERENCES

[1] J. Cummings and M. Foreman, *The tree property*. ***Advances in Mathematics***, vol. 133 (1998), no. 1, pp. 1–32.

[2] L. Fontanella, *Strong tree properties for small cardinals*, this Journal, vol. 78 (2013), no. 1, pp. 317–333.

[3] M. Gitik, *Prikry type forcing*, ***Handbook of Set Theory*** (M. Foreman and A. Kanamori, editors), Springer, Dordrecht, 2010, pp. 1351–1447.

[4] ———, *Extender based forcings, fresh sets, and Aronszajn trees*. ***Infinity, Computability, and Metamathematics*** (S. Geschke, B. Loewe, and P. Schlicht, editors), Tributes, vol. 23, College Publications, London, 2014, pp. 183–203.

[5] M. Gitik and S. Unger, *Short extender forcing*, ***Appalachian Set Theory 2006–2012*** (J. Cummings and E. Schimmerling, editors), Cambridge University Press, Cambridge, 2012, pp. 245–264.

[6] W. Mitchell, *Aronszajn trees and the independence of the transfer property*. ***Annals of Mathematical Logic***, vol. 5 (1972), pp. 21–46.

[7] M. Magidor, *Combinatorial characterization of supercompact cardinals*. ***Proceedings of the American Mathematical Society***, vol. 42 (1974), pp. 279–285.

[8] A. Sharon, ***Weak squares, scales, stationary reflection and the failure of SCH***, Ph.D. thesis, Tel Aviv University, 2005.

[9] D. Sinapova and S. Unger, *Modified extender based forcing*, this Journal, vol. 81 (2016), no. 4, pp. 1432–1443.

[10] ———, *The tree proeprty at* $\aleph_{\omega^2+1}$ *and* $\aleph_{\omega^2+1}$, this Journal, vol. 83 (2018), no. 2, pp. 669–682.

[11] E. Specker, *Sur un problème de Sikorski*. ***Colloquium Mathematicum***, vol. 2 (1949), pp. 9–12.

[12] S. Unger, *A model of Cummings and Foreman revisited*. ***Annals of Pure and Applied Logic***, vol. 165 (2014), pp. 1813–1831.

[13] ———, *Gitik's Gap 2 short extender forcing with collapses*, unpublished note, last accessed February 13, 2018.

[14] M. Viale, *Guessing models and generalized Laver diamond*. ***Annals of Pure and Applied Logic***, vol. 163 (2011), pp. 1660–1678.

[15] M. Viale and C. Weiss, *On the consistency strength of the proper forcing axiom*. ***Advances of Mathematics***, vol. 228 (2011), pp. 2672–2687.

[16] C. Weiss, *The combinatorial essene of supercompactness*. ***Annals of Pure and Applied Logic***, vol. 163 (2012), pp. 1710–1717.

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