J. Fluid Mech. (2001), *vol.* 442, *pp.* 157–159. Printed in the United Kingdom

Refraction of gravity waves by weak current gradients

By KRISTIAN B. DYSTHE

Department of Mathematics, University of Bergen, Johs. Brunsgt. 12, N-5008 Bergen, Norway

(Received 4 April 2001 and in revised form 20 May 2001)

When deep water surface waves cross an area with variable current, refraction takes place. If the group velocity of the waves is much larger than the current velocity we show that the curvature of a ray, $\varkappa$, is given by the simple formula $\varkappa = \zeta/v_g$. Here ζ is the vertical component of the current vorticity and v_g is the group velocity.

Wave refraction by currents is a classical subject studied for decades (Longuet-Higgins & Stewart 1961; Whitham 1974; Peregrine 1976). That some simple results might have remained undetected for so long seems rather unlikely. However, the result described in the following note may be such a case. It will be demonstrated that the curvature of a ray on a weak current is given by the ratio between the vertical component of the current vorticity and the group velocity.

What caught my interest in the subject was a paper by White & Fornberg (1998) titled: 'On the chance of freak waves at sea'. There they argue that freak waves on the open ocean may be due to the refraction of the gravity waves by the weak and fluctuating ocean currents. With unidirectional incoming waves, they find that currents with r.m.s. values of the order of $10\,\mathrm{cm\,s^{-1}}$ having a horizontal scale of fluctuations $\gtrsim 10\,\mathrm{km}$, give focusing or caustics.

The energy-carrying ocean waves have group velocities, v_g, of the order of $10\,\mathrm{m\,s^{-1}}$, while a typical current velocity, $\boldsymbol{U} = (u, v)$, is of the order of $10\,\mathrm{cm\,s^{-1}}$. Thus the ratio $\epsilon \equiv |\boldsymbol{U}|/v_g$ is very small, a fact that we shall take advantage of in the following. It will also be assumed that we can neglect the vertical shear of the current.

The dispersion relation for gravity waves on deep water is $\omega = \Omega(\boldsymbol{k}, \boldsymbol{x})$ where ω is the frequency, $\boldsymbol{k}$ the wave vector and†

$$\Omega(\boldsymbol{k}, \boldsymbol{x}) = \sqrt{gk} + \boldsymbol{k} \cdot \boldsymbol{U}.$$

We shall assume that the current field can be considered time independent, meaning that a characteristic time for the variation of $\boldsymbol{U}$ is much larger than the time it takes for a gravity wave to traverse a characteristic scale length of the current. This scale length is assumed much longer than the wavelength such that the approximation of geometrical optics is valid.

The canonical equations of geometrical optics can be written

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \frac{\partial \Omega}{\partial \boldsymbol{k}}, \quad \frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = -\frac{\partial \Omega}{\partial \boldsymbol{x}}. \tag{1}$$

Consider a ray $\boldsymbol{x}(s)$, that is a wave group trajectory, where s is the arclength from some initial point $\boldsymbol{x}(0)$. Let $\boldsymbol{t}$ be the unit tangent to the ray and $\widehat{\boldsymbol{k}}$ the unit vector

† It should be pointed out that the result described in the following does not depend on the particular dispersion relation for gravity waves. It is valid for gravity–capillary waves for example.

along $\boldsymbol{k}$. We then have

$$\boldsymbol{t} = \frac{\partial \Omega}{\partial \boldsymbol{k}} \bigg/ \left| \frac{\partial \Omega}{\partial \boldsymbol{k}} \right| = \frac{\boldsymbol{U} + \boldsymbol{v}_g}{|\boldsymbol{U} + \boldsymbol{v}_g|} = \frac{\boldsymbol{u} + \widehat{\boldsymbol{k}}}{|\boldsymbol{u} + \widehat{\boldsymbol{k}}|}, \tag{2}$$

where $\boldsymbol{u} = \boldsymbol{U}/|\boldsymbol{v}_g| = \boldsymbol{U}/v_g$ and $\boldsymbol{v}_g = \frac{1}{2}\sqrt{g/k}\,\widehat{\boldsymbol{k}}$ is the group velocity in the absence of currents.

The right-hand side of (2) is then expanded in $\epsilon \equiv |\boldsymbol{u}| \ll 1$ to give

$$\boldsymbol{t} = \widehat{\boldsymbol{k}} + \boldsymbol{u} - \boldsymbol{u} \cdot \widehat{\boldsymbol{k}}\widehat{\boldsymbol{k}} + O(\epsilon^2) \tag{3}$$

which shows that $\boldsymbol{t}$ and $\widehat{\boldsymbol{k}}$ differ by the vector $\boldsymbol{u} - \boldsymbol{u} \cdot \widehat{\boldsymbol{k}}\widehat{\boldsymbol{k}}$ which is normal to $\widehat{\boldsymbol{k}}$ and of order ϵ. Taking the time derivative of (3) we have

$$\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}t} = \frac{\mathrm{d}\widehat{\boldsymbol{k}}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \cdot \nabla \boldsymbol{u} \cdot (\boldsymbol{I} - \widehat{\boldsymbol{k}}\widehat{\boldsymbol{k}}) + O(\epsilon^2), \tag{4}$$

where $\boldsymbol{I}$ is the unit dyadic. Since we are considering a steady-state situation ($\boldsymbol{u}$ time independent, and $\omega =$ constant) we change from the time t to the arclength s as the free variable. Equation (4) can then be written

$$\varkappa \boldsymbol{n} = \frac{\mathrm{d}\widehat{\boldsymbol{k}}}{\mathrm{d}s} + \boldsymbol{t} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{n}\boldsymbol{n} + O(\epsilon^2), \tag{5}$$

where $\varkappa$ is the ray curvature and $\boldsymbol{n}$ is the unit vector normal to the ray such that $(\boldsymbol{t}, \boldsymbol{n})$ is a right-handed system. The dyadic $(\boldsymbol{I} - \widehat{\boldsymbol{k}}\widehat{\boldsymbol{k}})$ in equation (4), which occurs in an $O(\epsilon)$ term, has been replaced by the dyadic $\boldsymbol{n}\boldsymbol{n}$ which differs from the former to order ϵ (since $\boldsymbol{t}$ and $\widehat{\boldsymbol{k}}$ differ by that amount). Thus this substitution is correct to $O(\epsilon^2)$.

From the second of the canonical equations (1) we have

$$\frac{\mathrm{d}\widehat{\boldsymbol{k}}}{\mathrm{d}t} k + \frac{\mathrm{d}k}{\mathrm{d}t} \widehat{\boldsymbol{k}} = -\nabla \boldsymbol{U} \cdot \widehat{\boldsymbol{k}} k.$$

In the above equation take the scalar product of each side with $\boldsymbol{n}$. The second term on the left-hand side will then be $O(\epsilon^2)$ since $\widehat{\boldsymbol{k}} \cdot \boldsymbol{n}$ and $\mathrm{d}k/\mathrm{d}t$ are both $O(\epsilon)$. Changing from t to s as before we obtain

$$\frac{\mathrm{d}\widehat{\boldsymbol{k}}}{\mathrm{d}s} \cdot \boldsymbol{n} = -\boldsymbol{n} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{t} + O(\epsilon^2). \tag{6}$$

Now combining (5) and (6) we obtain

$$\varkappa = \boldsymbol{t} \cdot (\nabla \boldsymbol{u} - \widetilde{\nabla \boldsymbol{u}}) \cdot \boldsymbol{n} + O(\epsilon^2), \tag{7}$$

where a tilde means the transpose of the dyadic. So if ζ denotes the vertical component of the current vorticity, i.e.

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

we obtain from equation (7) the beautiful result

$$\varkappa = \frac{\zeta}{v_g} + O(\epsilon^2).$$

This has at least two quite surprising aspects. To order ϵ we have that

(i) a potential velocity field gives no refraction;

(ii) at any given point the ray curvature $\varkappa$ is independent of the ray direction.

An example of (i) is given by the irrotational current field $\boldsymbol{U} = (u(x), 0)$. It can easily be shown that, while the waves are refracted by an $O(\epsilon)$ angular variation in $\boldsymbol{k}$ due to the current, the ray paths themselves remain as straight lines, to order ϵ^2.

Note added in proof. It was pointed out by one of the referees (during the publication process of this paper) that a result similar to, and in some respects more general than, mine was published in Landau & Lifschitz (1959, p. 216) as a solved problem.

They consider refraction of a sound wave in three dimensions and get the result (written in my notation)

$$\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s} = -\frac{\boldsymbol{t} \times \nabla \times \boldsymbol{u}}{c}$$

where c is the speed of sound, and it is assumed that $|\boldsymbol{u}|/c \ll 1$. From their result we deduce that the ray-curvature $\varkappa$ is given by

$$\varkappa = \frac{\boldsymbol{b} \cdot \nabla \times \boldsymbol{u}}{c}$$

where $\boldsymbol{b}$ is the *binormal vector* of the ray. Although their result is derived for non-dispersive waves (sound waves) it is not difficult to show (using the results of this note) that their result is indeed valid also for dispersive waves under the substitution $c \to v_g$.

This work was done while the author was visiting Scripps Institution of Oceanography. I am grateful for the hospitality of professor Ken Melville and his colleagues. Support was given by the Norwegian Research Council through the project nr. 139177/431.

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