



A proof of the linear Arithmetic Fundamental Lemma for GL_4

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Abstract. Let K/F be an unramified quadratic extension of a non-Archimedean local field. In a previous work [1], we proved a formula for the intersection number on Lubin–Tate spaces. The main result of this article is an algorithm for computation of this formula in certain special cases. As an application, we prove the linear Arithmetic Fundamental Lemma for GL_4 with the unit element in the spherical Hecke Algebra.

1 Introduction

1.1 Motivation

In this paper, we give an algorithm to compute intersection numbers of CM cycles in Lubin–Tate spaces in some special cases by following an explicit formula in [1]. Our goal is to identify these intersection numbers with the values of the first derivative of certain orbital integrals. This identity is known as the linear Arithmetic Fundamental Lemma (linear AFL) conjecture, and an application of our algorithm is to prove the conjecture for GL_4 . As we noted in the introduction part of [1], the global motivation for the linear AFL arises from a generalization of the arithmetic Gan–Gross–Prasad conjectures proposed by Zhang [2].

1.2 The linear AFL

We call the two sides of the linear AFL identity the arithmetic-geometric side and the analytic side, respectively. We briefly describe the objects appearing on the two sides. Let F be a non-Archimedean local field with ring of integers \mathcal{O}_F . Let π be a uniformizer of \mathcal{O}_F and denote the residue field by $\mathbb{F}_q \cong \mathcal{O}_F/\pi$. On the arithmetic-geometric side, we consider a one-dimensional formal \mathcal{O}_F -module \mathcal{G}_F of height $2h$ over $\overline{\mathbb{F}}_q$. Let K/F be an unramified quadratic extension. Choose two embeddings

$$(1.1) \quad \begin{aligned} \varphi_1 : \mathcal{O}_K &\longrightarrow \text{End}(\mathcal{G}_F), \\ \varphi_2 : \mathcal{O}_K &\longrightarrow \text{End}(\mathcal{G}_F). \end{aligned}$$

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Each embedding φ_i gives rise to a special cycle $Z(\varphi_i)$ on the Lubin–Tate space $\mathcal{M}_{\mathcal{G}_F}$ of \mathcal{G}_F . The quantity of the arithmetic-geometric side is the intersection number

$$\text{Int}(Z(\varphi_1), Z(\varphi_2))$$

of these two cycles.

On the analytic side, we consider two embeddings of \mathcal{O}_F -algebras

$$\begin{aligned} \tau_1 &: \mathcal{O}_F \times \mathcal{O}_F \longrightarrow \text{Mat}_{2h}(\mathcal{O}_F), \\ \tau_2 &: \mathcal{O}_F \times \mathcal{O}_F \longrightarrow \text{Mat}_{2h}(\mathcal{O}_F). \end{aligned}$$

Abbreviate the symbol GL_{2h} by \mathbf{G}_{2h} . Let $C(\tau_i) \subset \mathbf{G}_{2h}(F)$ be the centralizer of the image of τ_i for each $i = 1, 2$. We fix an element $g \in \mathbf{G}_{2h}(F)$ such that

$$(1.2) \quad \tau_2(x) = g^{-1}\tau_1(x)g \quad \text{for any } x \in \mathcal{O}_F \times \mathcal{O}_F.$$

Using an isomorphism $C(\tau_i) \cong \mathbf{G}_h(F) \times \mathbf{G}_h(F)$, we can write any $x \in C(\tau_i)$ as $x = (x_1, x_2)$ for $x_1, x_2 \in \mathbf{G}_h(F)$. Moreover, define

$$|x| := \left| \det(x_1^{-1}x_2) \right|_F \quad \eta_{K/F}(x) := \eta_{K/F}(\det(x_1x_2))$$

where $\eta_{K/F}$ is the quadratic character of K/F . Let $f : \mathbf{G}_{2h}(F) \rightarrow \mathbb{R}$ be an arbitrary smooth test function with compact support. We associate τ_1, τ_2 with the following relative orbital integral defined by

$$(1.3) \quad \text{Orb}_{\tau_1, \tau_2}(f, s) := \int_{C(\tau_1) \cap C(\tau_2) \backslash C(\tau_1) \times C(\tau_2)} f(u_1^{-1}gu_2)\eta_{E/L}(u_2)|u_1u_2|^s du_1 du_2$$

where we view $C(\tau_1) \cap C(\tau_2)$ as a subgroup of $C(\tau_1) \times C(\tau_2)$ via the diagonal embedding. The linear AFL conjecture states that

$$(1.4) \quad \pm (2 \ln q)^{-1} \left. \frac{d}{ds} \right|_{s=0} \text{Orb}_{\tau_1, \tau_2}(\mathbb{1}_{\mathbf{G}_{2h}(\mathcal{O}_F)}, s) = \text{Int}(Z(\varphi_1), Z(\varphi_2))$$

is a valid equation when (φ_1, φ_2) matches with (τ_1, τ_2) and the sign \pm is chosen so that the quantity is positive. By definition, (φ_1, φ_2) matches with (τ_1, τ_2) , if there is an isomorphism $\text{End}(\mathcal{G}_F) \otimes_{\mathcal{O}_F} C \rightarrow \text{Mat}_{2h}(F) \otimes_{\mathcal{O}_F} C$ between C -algebras for the algebraic closure C of F that makes the following diagram commute for $i = 1, 2$.

$$\begin{array}{ccc} (F \times F) \otimes_{\mathcal{O}_F} C & \xrightarrow{\cong} & K \otimes_{\mathcal{O}_F} C \\ \tau_i \otimes \text{id}_C \downarrow & & \downarrow \varphi_i \otimes \text{id}_C \\ \text{Mat}_{2h}(F) \otimes_{\mathcal{O}_F} C & \longrightarrow & \text{End}(\mathcal{G}_F) \otimes_{\mathcal{O}_F} C \end{array}$$

The identity (1.4) is conjectured to hold in more general settings if we replace f in the analytic side by an arbitrary spherical Hecke function on $\mathbf{G}_{2h}(F)$ and correspondingly replace $Z(\varphi_2)$ in the arithmetic-geometric side by $h_{f,*}Z(\varphi_2)$ via the Hecke correspondence $h_f : \mathcal{M}_{\mathcal{G}_F} \leftarrow \Gamma \rightarrow \mathcal{M}_{\mathcal{G}_F}$ defined by f . In this article, we only study the case when the test function is $\mathbb{1}_{\mathbf{G}_{2h}(\mathcal{O}_F)}$.

1.3 Classification of double structures

In the linear AFL, our parameter is a pair of embeddings from a quadratic étale algebra K to a central simple algebra $D = \text{End}(\mathcal{G}_F) \otimes_{\mathcal{O}_F} F$ over F . We call it a double K -structure on \mathcal{G}_F because this pair gives \mathcal{G}_F two K -actions through self-quasi-isogenies. We call (φ_1, φ_2) an *integral* double K -structure if $\varphi_i(\mathcal{O}_K) \subset \mathcal{O}_D$ for $i = 1, 2$. Two double K -structures (φ_1, φ_2) and (φ'_1, φ'_2) on \mathcal{G}_F are called *isogenous* if there is a self-quasi-isogeny $\gamma : \mathcal{G}_F \rightarrow \mathcal{G}_F$ carrying one double structure to another

$$\gamma \circ \varphi_i(x) = \varphi'_i(x) \circ \gamma \quad \text{for any } x \in K \text{ and } i = 1, 2.$$

In other words, this means that the pairs (φ_1, φ_2) and (φ'_1, φ'_2) are conjugate in D .

Note that both sides of the linear AFL depend only on the isogeny class of the corresponding double structures. For any quadratic étale algebra K , let $\zeta \in K$ such that $\zeta \notin F$. Let ζ^σ be its conjugate. Using the element ζ , we define the invariant polynomial for a double structure $\varphi_i : K \rightarrow D (i = 1, 2)$ to be the characteristic polynomial of

$$\frac{(\varphi_1(\zeta) - \varphi_2(\zeta^\sigma))^2}{(\zeta - \zeta^\sigma)^2} \in C(\varphi_1) \cap C(\varphi_2)$$

as an element of $C(\varphi_1)$, which is a central simple algebra over K . Clearly, this element does not depend on the choice of ζ .

1.4 Main results of the paper

Our formula in [1] simplifies the arithmetic-geometric side and reduces the conjectural linear AFL to the following identity.

Conjecture 1 *Let $f : \mathbf{G}_{2h}(F) \rightarrow \mathbb{R}$ be a spherical Hecke function, and $\gamma = (\tau_1, \tau_2)$ is a double $F \times F$ -structure on $\mathbf{G}_{2h}(F)$. Suppose that (τ_1, τ_2) matches to a double K -structure on a division algebra D of invariant $\frac{1}{2h}$. Let P_γ be the invariant polynomial of (τ_1, τ_2) . Let $\alpha : K \rightarrow \mathbf{G}_{2h}(F)$ be a map of F -algebras, P_g the invariant polynomial of the double K -structure $(\alpha, g^{-1} \circ \alpha \circ g)$ on $\mathbf{G}_{2h}(F)$ for any $g \in \mathbf{G}_{2h}(F)$. Then we have*

$$(1.5) \quad \pm(2 \ln q)^{-1} \frac{d}{ds} \Big|_{s=0} \text{Orb}_{\tau_1, \tau_2}(f, s) = \frac{\epsilon_{F, 2h}}{\epsilon_{K, h}^2} \int_{\mathbf{G}_{2h}(F)} f(g) |\text{Res}(P_\gamma, P_g)|_F^{-1} dg$$

where constants $\epsilon_{F, 2h}$ and $\epsilon_{K, h}$ are densities of invertible matrices in $\text{Mat}_{2h}(\mathcal{O}_F)$ and $\text{Mat}_h(\mathcal{O}_K)$. The symbol Res represents the resultant of two polynomials.

Our main result is a computational method for calculating the arithmetic-geometric side for $f = \mathbb{1}_{\mathbf{G}_{2h}(\mathcal{O}_F)}$. As an application, we proved the identity (1.5) for $h = 2$.

Theorem 1.1 *The equation (1.5) holds for $h = 2$, $f(g) = \mathbb{1}_{\mathbf{G}_{2h}(\mathcal{O}_F)}(g)$.*

For higher h , both sides of (1.5) are computable when we impose the following condition:

- (*) The valuation $\mathbf{v}_F(P_\gamma(1))$ is odd and coprime to h .

In this paper, our algorithm allows us to compute all intersection numbers for higher h in the case of (*). There is also an inductive formula for orbital integrals, but it seems too complicated to be practically useful. In particular, we have not succeeded identifying the inductive formulas for the two sides, except for some lower rank cases.

Now we give more details of our computational methods. The computation for the arithmetic-geometric side is described as follows. We see that the integrand in (1.5) is invariant under the action of $G_h(K)$. Then we only need to compute the intersection number by integrating certain function over the homogeneous space $\mathfrak{S}_h(F) = G_{2h}(F)/G_h(K)$. Then, we divide $\mathfrak{S}_h(F)$ into a disjoint union of subsets with two properties. Firstly, each subset is invariant under the action of $G_h(\mathcal{O}_K)$. Secondly, when we have condition (*), our integrand is a constant on each subset. This method gives us an inductive formula for computing the intersection number. Finally, we prove the $h = 2$ case of the linear AFL by comparing the result of computation at the end of Sections 7 and 6. Our method for the analytic side in Section 7 is counting lattices, which is approachable when $h = 2$, since there is an easy classification of \mathcal{O}_F -subalgebras of a quadratic field extension K over F .

This paper starts with Section 2 to discuss double structures, which are parameters in the linear AFL identity. The constructions and lemmas in Section 2 will be used repeatedly in our computation in the analytic side (Section 7) and the arithmetic-geometric side (Sections 3–6). Section 5 gives a complete list of inductive formulae to compute the arithmetic-geometric side with condition (*). The calculation for $h = 2$ case is done in Section 6. Sections 3 and 4 are preparations for Section 5. The reader may skip those two sections if they are willing to accept the formula (4.1).

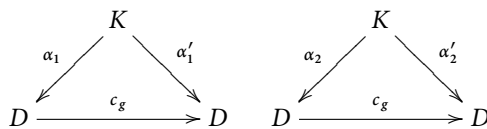
2 Double structures

In this paper, a double structure on an object means two different actions of a quadratic extension of F .

Definition 2.1 Let K be a quadratic etale algebra, $\mathcal{O}_D = \text{End}(\mathcal{G}_F)$ and $D = \text{End}(\mathcal{G}_F) \otimes_{\mathcal{O}_F} F$, a central simple algebra over F . A double K -structure on \mathcal{G}_F is a pair of embeddings of F -algebras

$$(2.1) \quad \begin{aligned} \alpha_1 &: K \longrightarrow D, \\ \alpha_2 &: K \longrightarrow D. \end{aligned}$$

Two double structures (α_1, α_2) and (α'_1, α'_2) are called isogenous if there is an inner automorphism $c_g : D \longrightarrow D$ induced by an element $g \in D$ such that the following diagrams commute



where $c_g(x) = g^{-1}xg$. The double structure is called integral if furthermore $\alpha_i(\mathcal{O}_K) \subset \mathcal{O}_D$ for $i = 1, 2$. Two integral double structures are called isomorphic if one can take $g \in \mathcal{O}_D^\times$ for c_g in the above diagram.

We call this a double K -structure on \mathcal{G}_F because this pair gives \mathcal{G}_F two K -actions through quasi-isogenies. From now on, we will fix an element $\zeta \in K$ with

$$\zeta \notin F.$$

It is clear that the image $\alpha_i(K) \subset D$ is determined by $\alpha_i(\zeta)$. In the rest of this paper, we denote the image of α_i by $K_i \subset D$. Moreover, let

$$D_{K_i} \subset D, D_{K_i}^\times \subset D^\times, \mathcal{O}_{D_{K_i}} \subset \mathcal{O}_D, \mathcal{O}_{D_{K_i}}^\times \subset \mathcal{O}_D^\times$$

be centralizers of $\alpha_i(\zeta)$, respectively, for $i = 1, 2$.

The goal for this section is to establish a general theory, where we allow \mathcal{G}_F to be an arbitrary π -divisible group and we do not impose further conditions on D and K . When we apply our general construction in this Section to Sections 4, 5, and 7, we all specialize to the case where K/F is an unramified extension, $\mathcal{G}_F = (F/\mathcal{O}_F)^{2h}$ and

$$\begin{aligned} D &= \text{Mat}_{2h}(F) & D^\times &= \mathbf{G}_{2h}(F) & \mathcal{O}_D &= \text{Mat}_{2h}(\mathcal{O}_F) & \mathcal{O}_D^\times &= \mathbf{G}_{2h}(\mathcal{O}_F); \\ D_{K_i} &= \text{Mat}_h(K_i) & D_{K_i}^\times &= \mathbf{G}_h(K_i) & \mathcal{O}_{D_{K_i}} &= \text{Mat}_h(\mathcal{O}_{K_i}) & \mathcal{O}_{D_{K_i}}^\times &= \mathbf{G}_h(\mathcal{O}_{K_i}) \end{aligned}$$

for $i = 1, 2$.

Two double K -structures (φ_1, φ_2) and (φ'_1, φ'_2) on \mathcal{G}_F are called *isogenous* if there is a self-quasi-isogeny $\phi : \mathcal{G}_F \rightarrow \mathcal{G}_F$ carrying one double structure to another, in other words

$$\phi \circ \varphi_i(x) = \varphi'_i(x) \circ \phi \quad \text{for any } x \in K \text{ and } i = 1, 2.$$

This implies (φ_1, φ_2) and (φ'_1, φ'_2) are conjugate in D .

In this section, we will construct the homogeneous space $\mathfrak{S}_h(F)$ in (2.4), then attach an invariant polynomial to every point on $\mathfrak{S}_h(F)$ and to every isogeny class of double structures.

Definition 2.2 Let (α_1, α_2) be a double structure, its *interior angle bisector* is defined by

$$i_{\alpha_1, \alpha_2} := \frac{\alpha_2(\zeta) - \alpha_1(\zeta^\sigma)}{\zeta - \zeta^\sigma}.$$

and its *exterior angle bisector* is defined by

$$e_{\alpha_1, \alpha_2} := \frac{\alpha_2(\zeta) - \alpha_1(\zeta)}{\zeta - \zeta^\sigma}.$$

Remark 2.3 The above definition does not depend on the choice of $\zeta \in K$. The names interior and exterior angle bisector come from the case of double structures of the Hamilton quaternion algebra, where $\mathfrak{S}_2(\mathbb{R})$ is isomorphic to a sphere S^2 . Then, α_1 and α_2 correspond to two points on it. Joining those two points with the center of the sphere, we get an angle, and e_{α_1, α_2} and i_{α_1, α_2} are exactly located at the exterior and interior angle bisector, respectively. Their conjugate action is given by reflections by those bisectors.

Proposition 2.4 *The exterior and interior bisectors satisfy the following identities. For any $x \in K$,*

$$i_{\alpha_1, \alpha_2} \circ \alpha_1(x) = \alpha_2(x) \circ i_{\alpha_1, \alpha_2}, \quad i_{\alpha_1, \alpha_2} \circ \alpha_2(x) = \alpha_1(x) \circ i_{\alpha_1, \alpha_2},$$

$$e_{\alpha_1, \alpha_2} \circ \alpha_1(x) = \alpha_2(x^\sigma) \circ e_{\alpha_1, \alpha_2}, \quad e_{\alpha_1, \alpha_2} \circ \alpha_2(x) = \alpha_1(x^\sigma) \circ e_{\alpha_1, \alpha_2}$$

and

$$(2.2) \quad i_{\alpha_1, \alpha_2} \circ e_{\alpha_1, \alpha_2} = -e_{\alpha_1, \alpha_2} \circ i_{\alpha_1, \alpha_2}, \quad (i_{\alpha_1, \alpha_2} \pm e_{\alpha_1, \alpha_2})^2 = 1.$$

Furthermore, we have the Pythagorean theorem

$$(2.3) \quad i_{\alpha_1, \alpha_2}^2 + e_{\alpha_1, \alpha_2}^2 = 1.$$

Proof We have

$$(\alpha_1(x) - \alpha_2(x^\sigma))\alpha_2(x) = \alpha_1(x)\alpha_2(x) - \alpha_2(x^\sigma x).$$

Note $x^\sigma x \in F$, so $\alpha_2(x^\sigma x) = \alpha_1(x^\sigma x)$, this implies that the above equation turns into

$$\alpha_1(x)\alpha_2(x) - \alpha_1(x^\sigma x) = \alpha_1(x)(\alpha_2(x) - \alpha_1(x^\sigma))$$

Now since $x^\sigma + x \in F$, so $\alpha_2(x + x^\sigma) = \alpha_1(x + x^\sigma)$, we obtain

$$\alpha_1(x)(\alpha_1(x) - \alpha_2(x^\sigma)).$$

Since $i_{\alpha_1, \alpha_2} = \frac{\alpha_1(x) - \alpha_2(x^\sigma)}{x - x^\sigma}$, we proved $i_{\alpha_1, \alpha_2} \circ \alpha_2(x) = \alpha_1(x) \circ i_{\alpha_1, \alpha_2}$. The proof of the other identities is similar. The equation (2.2) is followed by direct calculation, and it is easy to verify $i_{\alpha_1, \alpha_2} \circ e_{\alpha_1, \alpha_2} = -e_{\alpha_1, \alpha_2} \circ i_{\alpha_1, \alpha_2}$. To prove $(i_{\alpha_1, \alpha_2} - e_{\alpha_1, \alpha_2})^2 = 1$, simply note that

$$(i_{\alpha_1, \alpha_2} - e_{\alpha_1, \alpha_2})^2 = \left(-\frac{\alpha_2(\zeta - \zeta^\sigma)}{\zeta - \zeta^\sigma} \right)^2 = 1.$$

By expanding the left expression, we also have

$$1 = (i_{\alpha_1, \alpha_2} - e_{\alpha_1, \alpha_2})^2 = i_{\alpha_1, \alpha_2}^2 + e_{\alpha_1, \alpha_2}^2 - i_{\alpha_1, \alpha_2} \circ e_{\alpha_1, \alpha_2} - e_{\alpha_1, \alpha_2} \circ i_{\alpha_1, \alpha_2} = i_{\alpha_1, \alpha_2}^2 + e_{\alpha_1, \alpha_2}^2.$$

So we also have $1 = i_{\alpha_1, \alpha_2}^2 + e_{\alpha_1, \alpha_2}^2 = (i_{\alpha_1, \alpha_2} + e_{\alpha_1, \alpha_2})^2$. We proved this proposition. ■

Definition 2.5 The normalized centralizer of a double structure (α_1, α_2) is defined by

$$i_{\alpha_1, \alpha_2}^2 = \frac{(\alpha_1(\zeta) - \alpha_2(\zeta^\sigma))^2}{(\zeta - \zeta^\sigma)^2} \in D_{K_1} \cap D_{K_2}.$$

Proof This element should be in D , we prove it is in the subset $D_{K_1} \cap D_{K_2}$. Since the expression is symmetric for α_1 and α_2 , we only need to show this element commutes with $\alpha_1(\zeta)$. By the property of the interior angle bisector reflector,

$$i_{\alpha_1, \alpha_2} \circ i_{\alpha_1, \alpha_2} \circ \alpha_1(\zeta) = i_{\alpha_1, \alpha_2} \circ \alpha_2(\zeta) \circ i_{\alpha_1, \alpha_2} = \alpha_1(\zeta) \circ i_{\alpha_1, \alpha_2} \circ i_{\alpha_1, \alpha_2}$$

We have proved $i_{\alpha_1, \alpha_2}^2 \in D_{K_1} \cap D_{K_2}$. ■

Definition 2.6 The invariant polynomial of a double structure (α_1, α_2) is the characteristic polynomial of its normalized centralizer i_{α_1, α_2}^2 , as an element of the central simple algebra D_{K_1} over K .

2.1 Parameter space of double structures

In this section, we define the homogeneous space $\mathfrak{S}_h(F)$ and study the isogeny class of double structures. Given any two double structures (α_1, α_2) and (α'_1, α'_2) , we may find an element $\varphi \in D^\times$ such that $\alpha_1 = \varphi\alpha'_1\varphi^{-1}$ because all embeddings $K \rightarrow D$ are conjugate. Without changing its isogeny class, we may replace the pair (α'_1, α'_2) by $(\alpha_1, \varphi_1\alpha'_2\varphi^{-1})$. This implies that every isogeny class of double structures has a representative $(\alpha_1, -)$ with the first structure given by α_1 . Then without loss of generality, we can fix an embedding α_1 and vary the second embedding α_2 . Since the embedding $\alpha_2 : K \rightarrow D$ is uniquely determined by the value of $\alpha_2(\zeta)$, the moduli space of the embeddings $\alpha_2 : K \rightarrow D$ can be represented by

$$(2.4) \quad \mathfrak{S}_h(F) := \{x \in D^\times : x \text{ is conjugate to } \alpha_1(\zeta)\}.$$

In other words, $\mathfrak{S}_h(F)$ is the set of F -points for the conjugacy class of the matrix

$$\begin{pmatrix} \zeta I_h & \\ & \zeta^\sigma I_h \end{pmatrix}$$

over the algebraic closure.

Our fixed α_1 determines a distinguished point

$$x_0 := \alpha_1(\zeta) \in \mathfrak{S}_h(F).$$

In the rest of this paper, we will keep the notation x_0 . We consider the action of D^\times on $\mathfrak{S}_h(F)$ via the conjugation and write

$$g \cdot x := gxg^{-1}$$

for any $g \in D^\times$ and $x \in \mathfrak{S}_h(F)$. The stabilizer of $\alpha_1(\zeta)$ is $D_{K_1}^\times$. Moreover, any two points on $\mathfrak{S}_h(F)$ are conjugate to each other by an element of D^\times . This implies $\mathfrak{S}_h(F)$ as a homogeneous space can be represented by

$$\mathfrak{S}_h(F) \cong D^\times / D_{K_1}^\times.$$

Points on $\mathfrak{S}_h(F)$ parametrize F -embeddings $\alpha : K \rightarrow D$. Then combining with α_1 , it goes through all possible isogeny class of double structures (α_1, α) . Furthermore, (α_1, α) and (α_1, α') are isogenous if and only if α and α' are conjugate by an element of $D_{K_1}^\times$. Therefore, the space of isogeny class of double structures can be described by

$$D_{K_1}^\times \backslash \mathfrak{S}_h(F) \cong D_{K_1}^\times \backslash D^\times / D_{K_1}^\times.$$

Definition 2.7 For any $x \in \mathfrak{S}_h(F)$, by the double structure induced by x , we mean a double structure (α_1, α_2) with $\alpha_2(\zeta) = x$. We also abbreviate i_{α_1, α_2} and e_{α_1, α_2} as i_x and e_x . In particular,

$$i_x = \frac{x - x_0^\sigma}{\zeta - \zeta^\sigma}, \quad e_x = \frac{x - x_0}{\zeta - \zeta^\sigma}.$$

2.2 Polar stereographic coordinate

Suppose $F = \mathbb{R}$, $K = \mathbb{C}$, and $D = \mathbb{H}$, a Hamilton's quaternion algebra. Then the space $\mathfrak{S}_h(F)$ is a two-dimensional sphere. There is a well-known universal polar stereographic coordinate system on $\mathfrak{S}_h(F)^\circ = \mathfrak{S}_h(F) \setminus \{x_0^\sigma\}$ in this case. In this paper, we call it the polar stereographic coordinate for short. This section is a generalization of the polar stereographic coordinate for $\mathfrak{S}_h(F)$ in general settings.

Definition 2.8 Consider D as a left K -vector space via α_1 . Let D_+ and D_- be eigenspaces of right multiplying x_0 of eigenvalue ζ and $-\zeta$, respectively. Suppose g can be decomposed as

$$g = g_+ + g_- \quad \text{for } g_+ \in D_+, \quad g_- \in D_-.$$

The element

$$x_\# := g_+^{-1}g_-$$

is called the x_0^σ -polar stereographic coordinate of $x = g \cdot x_0$.

We need to show this definition is well defined. In other words, we need to show it only depends on x .

Proposition 2.9 Suppose $g = g_+ + g_-$, then

$$x_\# = g_+^{-1}g_- = e_x \circ i_x^{-1} = (x - x_0)(x - x_0^\sigma)^{-1}.$$

Proof Remember $x = \alpha_2(\zeta)$, $x_0 = \alpha_1(\zeta)$ and g is chosen so that $gx = x_0g$. So we have

$$(g_+ + g_-) \circ \alpha_2(\zeta) = \alpha_1(\zeta) \circ (g_+ + g_-)$$

Using $x_0 \circ g_- = g_- \circ \alpha_1^\sigma(\zeta)$, we have

$$g_+(\alpha_1(\zeta) - \alpha_2(\zeta)) = g_-(\alpha_2(\zeta) - \alpha_1^\sigma(\zeta))$$

This implies

$$g_+ \circ e_x = g_- \circ i_x.$$

We proved this proposition. ■

Remark 2.10 The x_0^σ -polar stereographic coordinate can not be defined for elements x such that $x_0^\sigma - x$ is not invertible.

2.3 Invariant polynomials

In [1], we defined the invariant polynomial by a different way than Definition 2.6. In the next proposition, we show those definitions are equivalent.

Proposition 2.11 Let $\alpha_2 = g^{-1}\alpha_1g$ and decompose $g = g_+ + g_-$ according to the decomposition $D = D_+ \oplus D_-$. Then the normalized centralizer of (α_1, α_2) is given by

$$i_{\alpha_1, \alpha_2}^2 = (g_+ + g_-)^{-1}g_+(g_+ - g_-)^{-1}g_+.$$

Proof It suffices to prove this proposition for a Zariski dense subset. Then, we may assume g_+ is invertible. Then $x_\# = g_+^{-1}g_-$ is well defined, and we have

$$(g_+ + g_-)^{-1}g_+(g_+ - g_-)^{-1}g_+ = (1 - x_\#)^{-1}(1 + x_\#)^{-1}.$$

By Proposition 2.9, we have

$$(1 - x_\#)^{-1} = i_x \circ (i_x - e_x)^{-1}, \quad (1 + x_\#)^{-1} = i_x \circ (i_x + e_x)^{-1}.$$

Therefore, using identities in (2.2),

$$(1 - x_\#)^{-1}(1 + x_\#)^{-1} = i_x \circ (i_x - e_x)^{-1} \circ i_x \circ (i_x + e_x)^{-1} = i_x \circ (i_x + e_x)^{-2} \circ i_x = i_x^2.$$

Therefore, the proposition follows. ■

3 Integration in homogeneous spaces

We will compute our intersection number by an integral over $\mathfrak{S}_h(F)$. In this section, we make a preparation by introducing some general theory of integration over homogeneous spaces. This section has three parts. The first two parts consists of some basic definitions. The Section 3.3 is an outline of our main strategy of integration in this paper. The Theorem 3.1 will be used to prove Theorem 4.2 and Proposition 5.11.

3.1 Invariant measure on homogeneous spaces

Let G be an algebraic group over a local field F . We assume that the Lie algebra of G is a finite-dimensional vector space over F . Let S be a G -homogeneous space with a fixed base point $x_0 \in S$. Let $H = \text{Stab } x_0 \subset G$ be the stabilizer of x_0 . Then, we have a canonical isomorphism $S \cong G/H$. In general, S may not have a G -invariant Haar-measure. For example, the projective space $\mathbb{P}_{\mathbb{R}}^1$ is a $GL_2(\mathbb{R})$ -homogeneous space with no $GL_2(\mathbb{R})$ -invariant Haar-measure. It is well-known that the G -invariant measure exists for $S = G/H$ if and only if their modular characters δ_G and δ_H satisfies $\delta_G(h) = \delta_H(h)$ for any $h \in H$. For the rest of the paper, we only consider the case where H is a compact subgroup. Then, any character from H to $\mathbb{R}_{>0}^\times$ is trivial. This fact implies that we have a G -invariant measure on $S \cong G/H$.

Once we have chosen a left Haar-measure dg on G and a Haar-measure dh on H , the Haar-measure ds on S is defined so that for any function $f : G \rightarrow \mathbb{R}$, we have

$$\int_G f(g)dg = \int_S \tilde{f}(s)ds$$

where $\tilde{f}(s) = \int_H f(g_s h)dh$ with $g_s x_0 = s$.

3.2 Standard Haar-measure

For an algebraic group G over \mathcal{O}_F and a smooth G -homogeneous space S , we may choose a Haar-measure on S such that the total volume of $S(\mathcal{O}_F)$ is given by

$$\text{Vol}(S(\mathcal{O}_F)) = \frac{\#S(\mathbb{F}_q)}{q^{\dim(S)}}.$$

The standard Haar-measure on G is defined in the same way.

3.3 Integration by fibration

In this section, we introduce the main computational strategy in our paper. Theorem 3.1 in this section can be used in situations where the following three conditions are satisfied.

Condition 1 (Fibration) There is a subgroup $C \subset G$, a C -invariant subset $S^\circ \subset S$, a C -homogeneous space T , and a C -equivariant surjective map

$$(3.1) \quad p : S^\circ \longrightarrow T.$$

Condition 2 (Fiber Translation) For any $t \in T$, there exists a subgroup $P_t \subset G$ such that each fiber $p^{-1}\{t\}$ is a subset of a P_t -homogeneous space $R_t = P_t \cdot t \subset S$. We denote $R_t^\circ = p^{-1}\{t\}$.

Condition 3 We have $\dim(R_t) + \dim(T) = \dim(S)$. In other words, we require the fiber $p^{-1}\{t\}$ and R_t have the same dimension.

We give an example of a fibration with those three conditions. If we want to project a sphere S^2 to its equator along longitude lines, we can not do so because of the north and south poles. Therefore, we must choose a subset $S^\circ = S^2 \setminus \{\text{poles}\}$ and apply the projection. Let t be a point on the equator. Each fiber $p^{-1}\{t\}$ of the projection is a semi-circle, which is a subset of a full circle $R_t \subset S^2$ passing through two poles and t . The full circle is a homogeneous space for a subgroup $O_2(\mathbb{R}) \cong P_t \subset \text{Aut}(S^2) \cong O_3(\mathbb{R})$. We see the dimension of the fiber $p^{-1}\{t\}$ and R_t are the same.

When we have the above conditions, we may write the integral over S° into the following form

$$(3.2) \quad \int_{S^\circ} f(s) ds = \int_T \int_{R_t^\circ} f(r) |J_t(r)|_F dr dt.$$

Here dr , dt , and ds are standard G, C, P -invariant measures. The main obstacle is computing the Jacobian determinant $J_t(r)$.

We need to introduce more notation before Theorem 3.1. For any vector space V , by $\wedge V$, we mean the highest wedge product of V . It is an one-dimensional vector space. Note that whenever we have an exact sequence

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0,$$

we have $\wedge V = \wedge U \otimes \wedge W$. For any $\lambda_v \in \wedge V$ and $0 \neq \lambda_u \in \wedge U$, by $\frac{\lambda_v}{\lambda_u}$, we mean the unique element $\lambda_w \in \wedge W$ such that $\lambda_v = \lambda_w \otimes \lambda_u$.

Theorem 3.1 Let $r \in S$ and $t = p(r)$ be as in (3.1) and (3.2). Moreover, let $H = \text{Stab } r$ be the stabilizer of $r \in S \cong G/H$. Let $C, P_t \subset G$ be subgroups as we defined above. Let $\mathfrak{g}, \mathfrak{c}, \mathfrak{p}, \mathfrak{h}$ be Lie algebras of G, C, P_t, H , respectively. Let

$$u = \frac{\mathfrak{g}}{\mathfrak{p}} \quad u_c = \frac{\mathfrak{c}}{\mathfrak{c} \cap \mathfrak{p}} \quad u_h = \frac{\mathfrak{h}}{\mathfrak{h} \cap \mathfrak{p}}$$

be their quotient spaces. Let

$$d\mathfrak{g} \in \wedge \mathfrak{g}^\vee \quad d\mathfrak{h} \in \wedge \mathfrak{h}^\vee \quad d\mathfrak{p} \in \wedge \mathfrak{p}^\vee \quad d\mathfrak{q} \in \wedge (\mathfrak{p} \cap \mathfrak{h})^\vee \quad dt \in \wedge u_c^\vee$$

be volume forms corresponding to left Haar-measures on $G, H, P, P \cap H$, and $C/C \cap P$, respectively. Furthermore, let

$$du = \frac{dg}{dp} \in \bigwedge u^\vee \quad du_h = \frac{dh}{dq} \in \bigwedge u_h^\vee.$$

Suppose $\dim(S) = \dim(T) + \dim(P_t \cdot r)$. Then we have $u = u_c \oplus u_h$ and

$$du = J(r)du_h dt.$$

Proof Let $P_t r \subset S$ be the orbit of r under the action of P_t . Then, we have $P_t r \cong P_t/P_t \cap H$. The tangent spaces of the point $r \in S^\circ$ in Pr and in S are naturally isomorphic to $\frac{\mathfrak{g}}{\mathfrak{h}}$ and $\frac{\mathfrak{p}}{\mathfrak{p} \cap \mathfrak{h}}$, respectively. Note that the natural inclusion $P_t/H \cap P_t \subset S$ gives rise to an inclusion of tangent spaces $\frac{\mathfrak{p}}{\mathfrak{p} \cap \mathfrak{h}} \subset \frac{\mathfrak{g}}{\mathfrak{h}}$. Since $p : S^\circ \rightarrow T$ is an open map, the quotient of $\frac{\mathfrak{g}}{\mathfrak{h}}$ by $\frac{\mathfrak{p}}{\mathfrak{p} \cap \mathfrak{h}}$ is naturally isomorphic to the tangent space of t in T . Let T_t be the tangent space of $t \in T$. The map $c \rightarrow T_t$ factors through u_c . Since $p : S^\circ \rightarrow T$ is surjective, the induced map

$$p_{t,*} : u_c \rightarrow T_t$$

is naturally surjective. Given that $\dim(u_c) \leq \dim(S) - \dim(P_t \cdot r) = \dim(T) = \dim(T_t)$, we know that $p_{t,*}$ is an isomorphism. Therefore, we have the following exact sequence

$$(3.3) \quad 0 \rightarrow \frac{\mathfrak{p}}{\mathfrak{p} \cap \mathfrak{h}} \rightarrow \frac{\mathfrak{g}}{\mathfrak{h}} \rightarrow u_c \rightarrow 0.$$

To prove $u = u_c \oplus u_h$, we complete the above sequence into the following exact diagram.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \mathfrak{p} \cap \mathfrak{h} & \longrightarrow & \mathfrak{h} & \longrightarrow & u_h \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \mathfrak{p} & \longrightarrow & \mathfrak{g} & \longrightarrow & u \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \frac{\mathfrak{p}}{\mathfrak{p} \cap \mathfrak{h}} & \longrightarrow & \frac{\mathfrak{g}}{\mathfrak{h}} & \longrightarrow & u_c \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

By the Snake Lemma, we have the following exact sequence

$$0 \rightarrow u_h \rightarrow u \rightarrow u_c \rightarrow 0.$$

The natural inclusion $c \subset g$ induces an inclusion $u_c \subset u$. This implies that the above exact sequence splits. Then, $u = u_c \oplus u_h$. To compute the Jacobian determinant, we

note that $J(r)$ is defined in the following equation

$$J_t(r)drdt = ds.$$

Since we have $ds = \frac{dg}{dh}$, $dr = \frac{dp}{dq}$. We may write

$$J_t(r)\frac{dp}{dq} \cdot dt = \frac{dg}{dh} \in \wedge \left(\frac{g}{h}\right)^\vee.$$

By multiplying $\frac{dh}{dp}$, this equation can be written to

$$J_t(r)\frac{dh}{dq} \cdot dt = \frac{dg}{dp} \in \wedge u^\vee.$$

This implies

$$J_t(r)du_h \cdot dt = du.$$

This completes the proof. ■

4 Parabolic reduction formula

This section provides an integration formula that will be used in Section 5. Let $\mathfrak{S}_h(\mathcal{O}_F) = \mathfrak{S}_h(F) \cap \mathbf{G}_{2h}(\mathcal{O}_F)$, and a , a non-negative integer. For any t , let P_t denote the corresponding invariant polynomial. Denote

$$\mathfrak{S}_{h,a}^{>r}(\mathcal{O}_F) = \left\{ t \in \mathfrak{S}_h(\mathcal{O}_F) : P_t(X) \text{ has exactly } a \text{ factors } (X - \lambda) \text{ with } \mathbf{v}_F\left(1 - \frac{1}{\lambda}\right) > r \right\}.$$

Furthermore, we put $\mathfrak{S}_a^{>r}(\mathcal{O}_F) := \mathfrak{S}_{a,a}^{>r}(\mathcal{O}_F)$ and $\mathfrak{S}_a^{\leq r}(\mathcal{O}_F) := \mathfrak{S}_{a,0}^{>r}(\mathcal{O}_F)$. Let $f : P_h(F) \rightarrow \mathbb{R}$ be a function on the set $P_h(F)$ of degree h monic F -polynomials.

We will show that

$$(4.1) \quad \begin{aligned} & \frac{1}{\epsilon_{K_h}} \int_{\mathfrak{S}_{h,a}^{>r}(\mathcal{O}_F)} f(P_t)dt \\ &= \frac{1}{\epsilon_{K_{h-a}}} \frac{1}{\epsilon_{K_a}} \int_{\mathfrak{S}_a^{>r}(\mathcal{O}_F)} \int_{\mathfrak{S}_{h-a}^{\leq r}(\mathcal{O}_F)} f(P_{t_1}P_{t_2})|\text{Res}(P_{t_1}, P_{t_2})|_F dt_1 dt_2. \end{aligned}$$

This formula is the key for our induction algorithm. The whole section is a proof for this. The reader willing to accept this formula may skip this section.

This section will follow the strategy of integration-by-fibration as we mentioned in Section 3. In Section 4.1, we introduce our basic construction of fibration and check that it satisfies the three conditions in Section 3.3. Then in Section 4.2, we use our constructions in Section 2 to compute the corresponding Jacobian determinant.

4.1 Fibration over Grassmannian

Now we construct a fibration that satisfies the Condition 1. Consider $V = F^{2h}$ as a free K_1 -module of rank h by the K_1 -structure induced by α_1 . Let $\text{Gr}_{h,a}(K)$ be the Grassmannian variety parametrizing n -dimensional K_1 -subspaces in V . For any $x \in$

$\mathfrak{S}_{h,a}^{>r}(\mathcal{O}_F)$, we may decompose the invariant polynomial of x into

$$(4.2) \quad P_x = P_x^{>r} P_x^{\leq r}$$

where $P_x^{>r}$ is the maximal factor of P_x such that all roots λ of $P_x^{>r}$ satisfies $\mathbf{v}_F(1 - \frac{1}{\lambda}) > r$. Let U_x be the image of the operator $P_x^{>r}(i_x^2)$. Then U_x is the maximal invariant subspace of i_x^2 such that all its eigenvectors in U_x have its eigenvalue λ satisfying $\mathbf{v}_F(1 - \frac{1}{\lambda}) > r$. We define a map by

$$(4.3) \quad \begin{aligned} p : \mathfrak{S}_{h,a}^{>r}(\mathcal{O}_F) &\longrightarrow \text{Gr}_{h,a}(K) \\ x &\longmapsto U_x \end{aligned}$$

Clear $\mathfrak{S}_{h,a}^{>r}(\mathcal{O}_F)$ is a $\mathbf{G}_h(\mathcal{O}_{K_1})$ -invariant subset. This map is $\mathbf{G}_h(\mathcal{O}_{K_1})$ -equivariant.

Next we check Condition 2, that for any $U \in \text{Gr}_{h,a}(K)$, each fiber $p^{-1}(U)$ of the map p in (4.3) is a subset of certain homogeneous space of a certain subgroup. In our scenario here, this subgroup is $\mathbf{P}_{F,U}(\mathcal{O}_F) \subset \mathbf{G}_{2h}(\mathcal{O}_F)$, which is the stabilizer of U . The corresponding homogeneous space is a subset of $\mathfrak{S}_h(\mathcal{O}_F)$ defined by

$$\begin{aligned} \mathfrak{F}_U(\mathcal{O}_F) &:= \mathbf{P}_{F,U}(\mathcal{O}_F) \cdot x_0 \\ &= \{x \in \mathfrak{S}_h(\mathcal{O}_F) : x = gx_0g^{-1} \text{ for some } g \in \mathbf{P}_{F,U}(\mathcal{O}_F)\}. \end{aligned}$$

To check the Condition 2, we need to check the fiber is a subset of this homogeneous space.

Lemma 4.1 For any $U \in \text{Gr}_{h,a}(K)$, we have $p^{-1}(U) \subset \mathfrak{F}_U(\mathcal{O}_F)$.

Proof Let $x \in p^{-1}(U)$, we have $i_x^2 U = U$. Since any $x \in \mathfrak{S}_h(F)$ commutes with the element i_x^2 , the subspace xU is also an invariant subspace for i_x^2 . Furthermore, the restriction of i_x^2 on U and xU gives the same eigenvalues. Since U is the maximal invariant subspace with eigenvalue λ of i_x^2 that satisfies $\mathbf{v}_F(1 - \frac{1}{\lambda}) > r$, then we must have

$$xU \subset U.$$

This implies $xU = U$. Similarly, we have $x_0U = U$. This implies

$$(\zeta - \zeta^\sigma)i_x = x - x_0^\sigma \in \mathbf{P}_{F,U}(\mathcal{O}_F).$$

Now

$$x = i_x x_0 i_x^{-1} \in \mathfrak{F}_U(\mathcal{O}_F)$$

and we are done. ■

Now we check the Condition 3. As varieties over \mathcal{O}_F , we have $\dim(\mathfrak{S}_h(\mathcal{O}_F)) = 2h^2$, $\dim(\mathfrak{F}_U(\mathcal{O}_F)) = 2a^2 + 2(h - a)^2 + 2a(h - a)$, $\dim(\text{Gr}_{h,a}(K)) = 2a(h - a)$. Then we have $\dim(\mathfrak{S}_h(\mathcal{O}_F)) = \dim(\mathfrak{F}_U(\mathcal{O}_F)) + \dim(\text{Gr}_{h,a}(K))$.

4.2 Computation of the Jacobian determinant

From now on, we denote the fiber $p^{-1}(U)$ of U by $\mathfrak{F}_U^{>r}(\mathcal{O}_F)$. Let dU (resp. dt' , dt) be the $\mathbf{G}_h(\mathcal{O}_{K_1})$ (resp. $\mathbf{P}_{K,U}(\mathcal{O}_{K_1})$, $\mathbf{G}_{2h}(\mathcal{O}_F)$)-invariant standard Haar-measure on

$\text{Gr}_{h,a}(K)$ (resp. $\mathfrak{B}_U(\mathcal{O}_F)$, $\mathfrak{S}_h(F)$). We can write

$$(4.4) \quad \int_{\mathfrak{S}_{h,a}^{>r}(\mathcal{O}_F)} f(P_t) dt = \int_{\text{Gr}_{h,a}(K)} \int_{\mathfrak{B}_U^{>r}(\mathcal{O}_F)} f(P_{t'}) |J(t)|_F dt' dU.$$

The Jacobian determinant $J(t)$ is defined by

$$dt = J(t) dt' dU.$$

We determine the value of $J(t)$ via the following theorem.

Theorem 4.2 *Let $x \in \mathfrak{S}_{h,a}^{>r}(\mathcal{O}_F)$ be an element whose invariant polynomial P_x decomposes as*

$$P_x = P_x^{>r} P_x^{\leq r}$$

as in (4.2). Then we have

$$|J(x)|_F = |\text{Res}(P_x^{>r}, P_x^{\leq r})|_F.$$

Proof Let $\alpha_2 : K \rightarrow F^{2h}$ be the map with $\alpha_2(\zeta) = x$. Denote the stabilizer of $x \in \mathfrak{S}_h(\mathcal{O}_F)$ by $\mathbf{G}_h(\mathcal{O}_{K_2})$. Let $\mathfrak{g}_{2h}, \mathfrak{g}_{h,1}, \mathfrak{g}_{h,2}, \mathfrak{p}_F, \mathfrak{p}_{K,i}$ be Lie algebras for groups $\mathbf{G}_{2h}(\mathcal{O}_F)$, $\mathbf{G}_h(\mathcal{O}_{K_1})$, $\mathbf{G}_h(\mathcal{O}_{K_2})$, $\mathbf{P}_{F,U}(\mathcal{O}_F)$, and $\mathbf{P}_{K,U}(\mathcal{O}_{K_i})$, respectively for $i = 1, 2$. Denote the standard Haar-measure for them as dg, dk_1, dk_2, dp_F , and dp_i , respectively. For $i = 1, 2$, we denote

$$u_{K,i} := \frac{\mathfrak{g}_{h,i}}{\mathfrak{p}_{K,i}} \quad u_F := \frac{\mathfrak{g}_{2h}}{\mathfrak{p}_F}.$$

Let $du_1 = \frac{dk_1}{dp_1}, du_2 = \frac{dk_2}{dp_2}, du_F = \frac{dg}{dp_F}$. Applying Theorem 3.1, we have

$$(4.5) \quad du_F = J(x) du_1 \cdot du_2 \in \bigwedge \left(\frac{\mathfrak{g}_{2h}}{\mathfrak{p}_F} \right)^\vee.$$

Note that $u_{K,i}$ are tangent spaces of Grassmannian manifolds at the point of U_x . Let

$$V = F^{2h}.$$

Recall that

$$U_x \cong F^{2a}.$$

We have natural isomorphisms

$$u_{K,i} \cong \text{Hom}_{K_i}(U_x, V/U_x) \quad u_F \cong \text{Hom}_F(U_x, V/U_x) \quad i = 1, 2.$$

Before we calculate $J(t)$, let α'_i and α''_i be the induced K_i -structures on U_x and V/U_x . Let $\overline{\alpha''_i}$ be the Galois conjugate of α''_i . Then (α'_1, α'_2) and (α''_1, α''_2) are double structures on U_x and V/U_x , respectively. Let

$$E_+^{\alpha_i} = \text{Hom}_{\alpha'_i, \alpha''_i}(U_x, V/U_x) \quad E_-^{\alpha_i} = \text{Hom}_{\alpha'_i, \overline{\alpha''_i}}(U_x, V/U_x)$$

be subsets of K -linear homomorphisms with K -structure induced by (α'_i, α''_i) and $(\alpha'_i, \bar{\alpha}''_i)$, respectively, for $i = 1, 2$. For any $i = 1, 2$, the following sequence is exact

$$(4.6) \quad 0 \longrightarrow E_+^{\alpha_i} \longrightarrow \text{Hom}_F(U_x, V/U_x) \xrightarrow{T_i := f \mapsto f \circ \alpha'_i(\zeta) - \alpha''_i(\zeta) \circ f} E_-^{\alpha_i} \longrightarrow 0$$

The last map is a surjective map because its right inverse is given by

$$T'_i : E_-^{\alpha_i} \longrightarrow \text{Hom}_F(U_x, V/U_x) \\ f \mapsto f \circ \alpha_i(\zeta - \zeta^\sigma)^{-1} \circ (1 - \sigma).$$

This implies

$$\text{Hom}_F(U_x, V/U_x) = E_+^{\alpha_i} \oplus E_-^{\alpha_i}.$$

Let du_F , du_i , and \overline{du}_i be the standard Haar-measures on $\text{Hom}_F(U_x, V/U_x)$, $E_+^{\alpha_i}$, $E_-^{\alpha_i}$, respectively. By the exact sequence (4.6) and the fact that $T|_{E_-^{\alpha_i}}(f) = f \circ \alpha'_i(\zeta - \zeta^\sigma)$, we have

$$du_F = \det(T_i|_{E_-^{\alpha_i}}) du_i \overline{du}_i = \text{Disc}_{K/F}^{a(h-a)} du_i \overline{du}_i.$$

Now we start our calculation of $J(x)$. On one hand, we consider the following exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_+^{\alpha_2} & \longrightarrow & E_+^{\alpha_1} \oplus E_+^{\alpha_2} & \longrightarrow & E_+^{\alpha_1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E_+^{\alpha_2} & \longrightarrow & \text{Hom}_F(U_x, V/U_x) & \xrightarrow{T_2} & E_-^{\alpha_2} \longrightarrow 0. \end{array}$$

Then we have

$$du_2 \overline{du}_2 = du_F = J(x) du_1 du_2 \implies J(x) = \frac{\overline{du}_2}{du_1}$$

This implies $J(x)$ is the relative determinant

$$J(x) = \det(E_+^{\alpha_1} \longrightarrow \text{Hom}_F(U_x, V/U_x) \longrightarrow E_-^{\alpha_2}) = \det(T|_{E_+^{\alpha_1}}).$$

Here, the map T_2 in the exact sequence is given by the following map

$$T_2 : \text{Hom}_F(U_x, V/U_x) \longrightarrow E_-^{\alpha_2} \\ f \mapsto f \circ \alpha'_2(\zeta) - \alpha''_2(\zeta) \circ f.$$

For any symbol a , we denote L_a for the left composing map $L_a : f \mapsto a \circ f$ and R_a for right composing map $R_a : f \mapsto f \circ a$. Please note that for any symbol a, b, L_a , and R_b always commute (because composition of maps is associative). Consider the following linear operators

$$(4.7) \quad \Theta_- : \text{Hom}_F(U_x, V/U_x) \longrightarrow \text{Hom}_F(U_x, V/U_x) \\ f \mapsto f \circ (\alpha'_1(\zeta) - \alpha'_2(\zeta)) - (\alpha''_1(\zeta) - \alpha''_2(\zeta)) \circ f,$$

$$(4.8) \quad \Theta_+ : \text{Hom}_F(U_x, V/U_x) \longrightarrow \text{Hom}_F(U_x, V/U_x)$$

$$f \longmapsto f \circ (\alpha'_1(\zeta) - \alpha'_2(\zeta)) + (\alpha''_1(\zeta) - \alpha''_2(\zeta)) \circ f,$$

$$(4.9) \quad L''_i \circ R'_e : \text{Hom}_F(U_x, V/U_x) \longrightarrow \text{Hom}_F(U_x, V/U_x)$$

$$f \longmapsto i_{\alpha'_1, \alpha'_2} \circ f \circ e_{\alpha'_1, \alpha'_2},$$

$$(4.10) \quad L''_i \circ L''_e : \text{Hom}_F(U_x, V/U_x) \longrightarrow \text{Hom}_F(U_x, V/U_x)$$

$$f \longmapsto i_{\alpha''_1, \alpha''_2} \circ e_{\alpha''_1, \alpha''_2} \circ f.$$

Then we know $\Theta_-|_{E_+^{\alpha_1}} = T_2|_{E_+^{\alpha_1}}$ and our goal is to compute

$$\det(E_+^{\alpha_1} \xrightarrow{\Theta_-} E_-^{\alpha_1}).$$

Since $U_x \cong K^a$, we have

$$E_+^{\alpha_i} \cong (V/U_x)^a, \quad E_-^{\alpha_i} \cong (V/U_x)^a.$$

Therefore

$$\det(E_+^{\alpha_i} \xrightarrow{L''_i \circ L''_e} E_-^{\alpha_i}) = \det((V/U_x)^a \xrightarrow{L''_i \circ L''_e} (V/U_x)^a)$$

$$= \det(i''_x \circ e''_x)^a.$$

So we proved

$$\det(E_+^{\alpha_1} \xrightarrow{L''_i \circ L''_e} E_-^{\alpha_1}) = \det(E_+^{\alpha_2} \xrightarrow{L''_i \circ L''_e} E_-^{\alpha_2}).$$

Use this identity and the following commutative diagram

$$\begin{array}{ccc} E_+^{\alpha_1} & \xrightarrow{L''_i \circ L''_e} & E_-^{\alpha_1} \\ L''_i \circ R'_e \downarrow & & \downarrow L''_i \circ R'_e \\ E_-^{\alpha_2} & \xrightarrow{L''_i \circ L''_e} & E_+^{\alpha_2} \end{array},$$

we know that

$$\det(E_+^{\alpha_1} \xrightarrow{L''_i \circ R'_e} E_-^{\alpha_2}) = \det(E_-^{\alpha_1} \xrightarrow{L''_i \circ R'_e} E_+^{\alpha_2}).$$

Again using the exact sequence (4.6), we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_+^{\alpha_1} & \longrightarrow & \text{Hom}_F(U_x, V/U_x) & \longrightarrow & E_-^{\alpha_1} \longrightarrow 0 \\ & & L''_i \circ R'_e \downarrow & & L''_i \circ R'_e \downarrow & & L''_i \circ R'_e \downarrow \\ 0 & \longrightarrow & E_+^{\alpha_2} & \longrightarrow & \text{Hom}_F(U_x, V/U_x) & \longrightarrow & E_-^{\alpha_2} \longrightarrow 0. \end{array}$$

This implies $\det(\text{Hom}_F(U_x, V/U_x) \xrightarrow{L''_i \circ R'_e} \text{Hom}_F(U_x, V/U_x))$ equals to

$$\det(E_+^{\alpha_1} \xrightarrow{L''_i \circ R'_e} E_-^{\alpha_2}) \cdot \det(E_-^{\alpha_1} \xrightarrow{L''_i \circ R'_e} E_+^{\alpha_2})$$

$$= \det(E_+^{\alpha_1} \xrightarrow{L''_i \circ R'_e} E_-^{\alpha_2})^2.$$

Since the map $\text{Hom}_F(U_x, V/U_x) \xrightarrow{L'_i \circ R'_e} \text{Hom}_F(U_x, V/U_x)$ does not change when changing indices from 1 to 2, we conclude that

$$\det \left(E_+^{\alpha_1} \xrightarrow{L'_i \circ R'_e} E_-^{\alpha_2} \right) = \det \left(E_-^{\alpha_2} \xrightarrow{L'_i \circ R'_e} E_+^{\alpha_1} \right).$$

This identity and the following commutative diagram

$$\begin{array}{ccc} E_+^{\alpha_1} & \xrightarrow{\Theta_-} & E_-^{\alpha_2} \\ L'_i \circ R'_e \downarrow & & \downarrow L'_i \circ R'_e \\ E_-^{\alpha_2} & \xrightarrow{\Theta_+} & E_+^{\alpha_1} \end{array}$$

imply that

$$\det \left(E_+^{\alpha_1} \xrightarrow{\Theta_-} E_-^{\alpha_2} \right) = \det \left(E_-^{\alpha_2} \xrightarrow{\Theta_+} E_+^{\alpha_1} \right).$$

Since

$$\Theta_+ \circ \Theta_- = (L_{\alpha_1''}(\zeta) - \alpha_2''(\zeta) + R_{\alpha_1'}(\zeta) - \alpha_2'(\zeta)) \circ (L_{\alpha_1''}(\zeta) - \alpha_2''(\zeta) - R_{\alpha_1'}(\zeta) - \alpha_2'(\zeta)),$$

we have

$$\Theta_+ \circ \Theta_- = L_{\alpha_1''}^2(\zeta) - \alpha_2''^2(\zeta) - R_{\alpha_1'}^2(\zeta) - \alpha_2'^2(\zeta).$$

Since the characteristic polynomial of $\frac{(\alpha_1''(x) - \alpha_2''(x))^2}{(x - x^\sigma)^2}$ and $\frac{(\alpha_1'(x) - \alpha_2'(x))^2}{(x - x^\sigma)^2}$ are $P_x^{>r}$ and $P_x^{\leq r}$ respectively, we have

$$\det_K(\Theta_+ \circ \Theta_-|_{E_+^{\alpha_1}}) = \text{Res}(P_x^{>r}, P_x^{\leq r}) \text{Disc}_{K/F}^{a(h-a)}.$$

Therefore, $|\det(E_+^{\alpha_1} \xrightarrow{\Theta_-} E_-^{\alpha_2})|_F$ equals

$$\sqrt{|\det_K(\Theta_+ \circ \Theta_-|_{E_+^{\alpha_1}})|_K} = \sqrt{|\text{Res}(P_x^{>r}, P_x^{\leq r})|_K} = |\text{Res}(P_x^{>r}, P_x^{\leq r})|_F |\text{Disc}_{K/F}^{a(h-a)}|_F.$$

This completes the proof. ■

By Theorem 4.2, we can write (4.4) as

$$\begin{aligned} \int_{\mathfrak{S}_{h,a}^{>r}(\mathcal{O}_F)} f(P_t) dt &= \int_{\text{Gr}_{h,a}(K)} \int_{\mathfrak{S}_U^{>r}(\mathcal{O}_F)} f(P_t) |\text{Res}(P_t^{>r}, P_t^{\leq r})|_F dt dU \\ (4.11) \qquad \qquad \qquad &= \text{Vol}(\text{Gr}_{h,a}(K)) \int_{\mathfrak{S}_U^{>r}(\mathcal{O}_F)} f(P_t) |\text{Res}(P_t^{>r}, P_t^{\leq r})|_F dt. \end{aligned}$$

Since

$$\text{Vol}(\text{Gr}_{h,a}(K)) = \frac{\epsilon_{K_h}}{\epsilon_{K_m} \epsilon_{K_n}},$$

we have

$$(4.12) \quad \frac{1}{\epsilon_{K_h}} \int_{\mathfrak{S}_{h,a}^{>r}(\mathcal{O}_F)} f(P_t) dt = \frac{1}{\epsilon_{K_m} \epsilon_{K_n}} \int_{\mathfrak{S}_U^{>r}(\mathcal{O}_F)} f(P_t) |\text{Res}(P_t^{>r}, P_t^{\leq r})|_F dt.$$

Let $\mathbb{1}_{\mathfrak{P}_U^{>r}(\mathcal{O}_F)}$ be the characteristic function of the subset $\mathfrak{P}_U^{>r}(\mathcal{O}_F) \subset \mathfrak{P}_U(\mathcal{O}_F)$. We can write

$$\int_{\mathfrak{P}_U^{>r}(\mathcal{O}_F)} f(P_t) |\text{Res}(P_t^{>r}, P_t^{\leq r})|_F dt = \int_{\mathfrak{P}_U(\mathcal{O}_F)} \mathbb{1}_{\mathfrak{P}_U^{>r}(\mathcal{O}_F)}(t) f(P_t) |\text{Res}(P_t^{>r}, P_t^{\leq r})|_F dt.$$

Consider a fibration

$$\begin{aligned} \mathfrak{P}_U(\mathcal{O}_F) &\longrightarrow \mathfrak{S}_a(\mathcal{O}_F) \times \mathfrak{S}_{h-a}(\mathcal{O}_F) \\ (\alpha_1, \alpha_2) &\longmapsto ((\alpha'_1, \alpha'_2), (\alpha''_1, \alpha''_2)). \end{aligned}$$

This is an $\mathbf{P}_{F,U}(\mathcal{O}_F)$ -equivariant map and both spaces are $\mathbf{P}_{F,U}(\mathcal{O}_F)$ -homogeneous spaces. Each fiber is a homogeneous space of its unipotent subgroup

$$\mathbf{U}_{2m,2n}(\mathcal{O}_F) = \{g \in \mathbf{P}_{F,U}(\mathcal{O}_F) : g|_U = \text{id}_U, g|_{V/U} = \text{id}_{V/U}\}.$$

Let dt_1, dt_2 , and du be standard Haar-measures on $\mathfrak{S}_a(\mathcal{O}_F)$, $\mathfrak{S}_{h-a}(\mathcal{O}_F)$ and $\mathbf{U}_{2m,2n}(\mathcal{O}_F)$ respectively. We may write the integral

$$\begin{aligned} &\int_{\mathfrak{P}_U(\mathcal{O}_F)} \mathbb{1}_{\mathfrak{P}_U^{>r}(\mathcal{O}_F)}(t) f(P_t) |\text{Res}(P_t^{>r}, P_t^{\leq r})|_F dt \\ &= \int_{\mathfrak{S}_a(\mathcal{O}_F)} \int_{\mathfrak{S}_{h-a}(\mathcal{O}_F)} \int_{\mathbf{U}_{2m,2n}(\mathcal{O}_F)} \mathbb{1}_{\mathfrak{P}_U^{>r}(\mathcal{O}_F)}(t) f(P_t) |\text{Res}(P_t^{>r}, P_t^{\leq r})|_F du dt_1 dt_2 \\ &= \int_{\mathbf{U}_{2m,2n}(\mathcal{O}_F)} du \int_{\mathfrak{S}_a^{>r}(\mathcal{O}_F)} \int_{\mathfrak{S}_{h-a}^{\leq r}(\mathcal{O}_F)} f(P_{t_1} P_{t_2}) |\text{Res}(P_{t_1}, P_{t_2})|_F dt_1 dt_2. \end{aligned}$$

Since $\text{Vol}(\mathbf{U}_{2m,2n}(\mathcal{O}_F)) = 1$, we have

(4.13)

$$\frac{1}{\epsilon_{K_h}} \int_{\mathfrak{S}_{h,a}^{>r}(F)} f(P_t) dt = \frac{1}{\epsilon_{K_{h-a}}} \frac{1}{\epsilon_{K_a}} \int_{\mathfrak{S}_a^{>r}(\mathcal{O}_F)} \int_{\mathfrak{S}_{h-a}^{\leq r}(\mathcal{O}_F)} f(P_{t_1} P_{t_2}) |\text{Res}(P_{t_1}, P_{t_2})|_F dt_1 dt_2.$$

5 Inductive formulae for the intersection number

In this section, we introduce the inductive formulae for the intersection number $\text{Int}(\gamma)$, where the input double structure $\gamma = (\varphi_1, \varphi_2)$ satisfies the following condition.

- (*) The valuation $\mathbf{v}_F(P_\gamma(1))$ for the invariant polynomial at 1 is odd and coprime to h .

Then, let $r = \frac{-\mathbf{v}_F(P_\gamma(1))}{h}$, which also equals to $\mathbf{v}_F(\gamma_\#^2)$ where $\gamma_\#$ is the polar stereographic coordinate of (φ_1, φ_2) . This section is divided into three parts.

In the first part, we simplify the intersection formula by defining the following integrals.

(5.1)

$$A_r[a, k] = \frac{1}{\epsilon_{K,a}} \int_{\mathfrak{S}_a^{>r}(\mathcal{O}_F)} |P_x(1)|_F^{-k} dx,$$

(5.2)

$$B_0[c] = \frac{1}{\epsilon_{K,c}} \int_{\mathfrak{S}_c^{\leq 0}(\mathcal{O}_F)} dx,$$

$$(5.3) \quad C_r[b, c] = \frac{1}{\epsilon_{K,b}} \int_{\mathfrak{E}_b^{\leq r}(\mathcal{O}_F) \cap \mathfrak{E}_b^{>0}(\mathcal{O}_F)} |P_x(1)|_F^{-b-c} dx.$$

We prove that under the condition (*), the number $\text{Int}(\gamma)$ can be written into the following form

$$(5.4) \quad \text{Int}(\gamma) = \sum_{a+b+c=h} |P_\gamma(1)|_F^{-a} A_r[a, 0] C_r[b, c] B_0[c].$$

In particular, in the situation of (*), the intersection number only depends on r . To simplify our notation in the rest of the paper, we will denote it by $N(r) := \text{Int}(\gamma)$.

The second part will introduce the inductive formulae for computing $A_r[a, 0]$, $C_r[b, c]$, $B_0[c]$. Specifically, we will show

$$(5.5) \quad C_r[n, m] = A_0[n, n - m] - \sum_{i=0}^{n-1} C_r[i, m] A_r[n - i, n - m] \text{ for } m \geq n,$$

$$(5.6) \quad B_0[a] = \prod_{i=1}^a \frac{1 - q^{1-2i}}{1 - q^{-2i}} - \sum_{i=1}^a A_0[i, 0] B_0[a - i].$$

The formulae (5.2), (5.3), and (5.4) imply that the computation of $\text{Int}(\gamma)$ can be reduced to $A_r[a, 0]$.

In the third part of this section, we will use polar stereographic coordinates to prove the following formula. Define

$$(5.7) \quad a_0[n, n - m] := q^{-2m} \prod_{\substack{i=0 \\ i \neq m}}^{n-1} \frac{1}{1 - q^{-2(m-i)}},$$

and define recursively

$$(5.8) \quad a_r[n, n - m] := a_0[n, n - m] - \sum_{i=1}^m C_r[i, m] a_r[n - i, n - m] \text{ for } 0 \leq m < n.$$

Then, we claim

$$(5.9) \quad A_r[n, n - m] = \sum_{i=0}^{n-1} a_r[n, n - i] \frac{q^{2(i-m)\lfloor \frac{nr}{2} \rfloor}}{1 - q^{2(i-m)}} \text{ for } m \geq n.$$

All the above formulae are sufficient for calculating the intersection number $N(r) = \text{Int}(\gamma)$ in the case of (*). The application for $h = 2$ case is introduced in Section 6. The reader willing to accept these formulae may skip the rest of the section.

5.1 Simplification of the intersection formula

In this section, our goal is to prove the formula (5.4). For any integer h , let $\epsilon_{K,h}$ and $\epsilon_{F,2h}$ be the volume of $\mathbf{G}_h(\mathcal{O}_K)$ and $\mathbf{G}_{2h}(\mathcal{O}_F)$ respectively with the standard Haar-measure(see Section §3). It is well-known that

$$\epsilon_{F,2h} = \prod_{i=1}^{2h} (1 - q^{-i}) \quad \epsilon_{K,h} = \prod_{i=1}^h (1 - q^{-2i}).$$

Suppose dg is the normalized Haar-measure on $\mathbf{G}_{2h}(\mathcal{O}_F)$, then the standard Haar-measure is $\epsilon_{F,2h}dg$. From now on, we will use standard Haar-measures for the rest of our discussion. The intersection formula is written with the standard Haar-measure dg by

$$(5.10) \quad \text{Int}(\gamma) = \frac{1}{\epsilon_{K,h}^2} \int_{\mathbf{G}_{2h}(\mathcal{O}_F)} \left| \text{Res}(P_\gamma, P_g) \right|_F^{-1} dg.$$

The integrand only depends on P_g . Let $\mathfrak{S}_h(\mathcal{O}_F) = \mathfrak{S}_h(F) \cap \mathbf{G}_{2h}(\mathcal{O}_F)$. There is an isomorphism

$$\mathfrak{S}_h(\mathcal{O}_F) \cong \mathbf{G}_{2h}(\mathcal{O}_F) / \mathbf{G}_h(\mathcal{O}_{K_1}).$$

We notice that for any $k_1, k_2 \in \mathbf{G}_h(K_1)$, we have

$$P_{k_1 g k_2} = P_g.$$

To simplify notation, we fix the following function throughout the whole section

$$f(P_t) = |\text{Res}(P_t, P_\gamma)|_F.$$

Then, we can write $\text{Int}(\gamma)$ as

$$(5.11) \quad \begin{aligned} \frac{1}{\epsilon_{K,h}^2} \int_{\mathbf{G}_{2h}(\mathcal{O}_F)} f(P_g) dg &= \frac{1}{\epsilon_{K,h}^2} \int_{\mathfrak{S}_h(\mathcal{O}_F)} \int_{\mathbf{G}_h(\mathcal{O}_{K_1})} f(P_{kt}) dk dt \\ &= \frac{1}{\epsilon_{K,h}^2} \int_{\mathfrak{S}_h(\mathcal{O}_F)} f(P_t) dt \cdot \int_{\mathbf{G}_h(\mathcal{O}_{K_1})} dk \\ &= \frac{1}{\epsilon_{K,h}} \int_{\mathfrak{S}_h(\mathcal{O}_F)} f(P_t) dt. \end{aligned}$$

Since we have

$$\mathfrak{S}_h(\mathcal{O}_F) = \prod_{a=0}^h \mathfrak{S}_{h,a}^{>r}(\mathcal{O}_F) = \prod_{a=0}^h \prod_{c=0}^a \mathfrak{S}_{h,a}^{>r}(\mathcal{O}_F) \cap \mathfrak{S}_{h,c}^{\leq 0}(\mathcal{O}_F),$$

we can write the integral (5.11) as

$$\text{Int}(\gamma) = \frac{1}{\epsilon_{K,h}} \int_{\mathfrak{S}_h(\mathcal{O}_F)} f(P_t) dt = \sum_{a+b+c=h} I(a, b, c)$$

where $I(a, b, c)$ equals to

$$I(a, b, c) = \frac{1}{\epsilon_{K,h}} \int_{\mathfrak{S}_{h,a}^{>r}(\mathcal{O}_F) \cap \mathfrak{S}_{h,c}^{\leq 0}(\mathcal{O}_F)} f(P_t) dt.$$

For any three polynomials $P_1, P_2, \text{ and } P_3$, by $\text{Res}(P_1, P_2, P_3)$, we mean the product $\text{Res}(P_1, P_2 P_3) \text{Res}(P_2, P_3)$. Using the reduction formula in (4.13), we can simplify

$I(a, b, c)$ to

$$\begin{aligned}
 (5.12) \quad & \frac{1}{\epsilon_{K,h}} \int_{\mathfrak{S}_{h,a}^{>r}(\mathcal{O}_F) \cap \mathfrak{S}_{h,c}^{\leq 0}(\mathcal{O}_F)} f(P_t) dt \\
 &= \frac{1}{\epsilon_{K,a+b} \epsilon_{K,c}} \int_{\mathfrak{S}_c^{\leq 0}(\mathcal{O}_F)} \int_{\mathfrak{S}_{a+b,a}^{>r}(\mathcal{O}_F) \cap \mathfrak{S}_{a+b}^{>0}(\mathcal{O}_F)} f(P_{t_1} P_{t_2}) |Res(P_{t_1}, P_{t_2})|_F dt_1 dt_2 dt_3 \\
 &= \frac{1}{\epsilon_{K,a} \epsilon_{K,b} \epsilon_{K,c}} \int_{\mathfrak{S}_c^{\leq 0}(\mathcal{O}_F)} \int_{\mathfrak{S}_b^{\leq r}(\mathcal{O}_F) \cap \mathfrak{S}_b^{>0}(\mathcal{O}_F)} \\
 &\quad \times \int_{\mathfrak{S}_a^{>r}(\mathcal{O}_F)} f(P_{t_1} P_{t_2} P_{t_3}) |Res(P_{t_1}, P_{t_2}, P_{t_3})|_F dt_1 dt_2 dt_3 \\
 &= \frac{1}{\epsilon_{K,a} \epsilon_{K,b} \epsilon_{K,c}} \int_{\mathfrak{S}_c^{\leq 0}(\mathcal{O}_F)} \int_{\mathfrak{S}_b^{\leq r}(\mathcal{O}_F) \cap \mathfrak{S}_b^{>0}(\mathcal{O}_F)} \\
 &\quad \times \int_{\mathfrak{S}_a^{>r}(\mathcal{O}_F)} \frac{|Res(P_{t_1}, P_{t_2}, P_{t_3})|_F}{|Res(P_\gamma, P_{t_1} P_{t_2} P_{t_3})|_F} dt_1 dt_2 dt_3.
 \end{aligned}$$

Then the equation (5.4) is obvious if we can prove the following proposition.

Proposition 5.1 *Suppose γ satisfies the condition (*) and $t_1 \in \mathfrak{S}_a^{>r}(\mathcal{O}_F)$, $t_2 \in \mathfrak{S}_b^{\leq r}(\mathcal{O}_F) \cap \mathfrak{S}_b^{>0}(\mathcal{O}_F)$, $t_3 \in \mathfrak{S}_c^{\leq 0}(\mathcal{O}_F)$, then*

$$\frac{|Res(P_{t_1}, P_{t_2}, P_{t_3})|_F}{|Res(P_\gamma, P_{t_1} P_{t_2} P_{t_3})|_F} = |P_\gamma^{-a}(1) P_{t_2}^{-b-c}(1)|_F.$$

The rest of this section is devoted to proving this proposition.

Definition 5.2 (In this section only) we say that an F -coefficient polynomial P_y dominates another F -coefficient polynomial P_x if any root λ of P_x and μ of P_y have the property

$$\mathbf{v}_F(1 - \lambda^{-1}) > \mathbf{v}_F(1 - \mu^{-1}).$$

Our proof will use the following lemma.

Lemma 5.3 *Suppose P_x, P_y are two F -coefficient polynomials. Let $a = \deg(P_x)$ and $b = \deg(P_y)$. If P_y dominates P_x , then we have*

$$|Res(P_x, P_y)|_F = |P_y^a(1) P_x^b(0)|_F.$$

Proof Let $\lambda_1, \dots, \lambda_a$ be roots of P_x , and μ_1, \dots, μ_b roots of P_y . Then by definition,

$$|Res(P_x, P_y)|_F = \prod_{i=1}^a \prod_{j=1}^b |\lambda_i - \mu_j|_F.$$

Since $P_x(0) = \prod_{i=1}^a \lambda_i$ and $P_y(0) = \prod_{j=1}^b \mu_j$, we can write

$$|Res(P_x, P_y)|_F = |P_x(0)^b P_y(0)^a|_F \prod_{i=1}^a \prod_{j=1}^b \left| \frac{1}{\lambda_i} - \frac{1}{\mu_j} \right|_F.$$

The assumption $\mathbf{v}_F(1 - \lambda^{-1}) > \mathbf{v}_F(1 - \mu^{-1})$ implies

$$\left| \frac{1}{\lambda_i} - \frac{1}{\mu_j} \right|_F = \left| \left(1 - \frac{1}{\lambda_i} \right) - \left(1 - \frac{1}{\mu_j} \right) \right|_F = \left| 1 - \frac{1}{\mu_j} \right|_F.$$

Therefore, we have

$$|\text{Res}(P_x, P_y)|_F = |P_x(0)^b P_y(0)^a|_F \left| 1 - \frac{1}{\mu_j} \right|_F^a = |P_x(0)^b P_y(0)^a|_F \left| \frac{P_y(1)}{P_y(0)} \right|_F^a.$$

This lemma follows. ■

Lemma 5.4 For any $t \in \mathfrak{S}_h(\mathcal{O}_F)$, let λ and μ be roots of P_t and P_y , respectively. If P_y satisfies (*), then $1 - \lambda^{-1}$ and $1 - \mu^{-1}$ have different valuation. In other words, either P_t dominates P_y or P_y dominates P_t .

Proof For any $t \in \mathfrak{S}_h(\mathcal{O}_F)$, we only need to show that any λ with $\mathbf{v}_F(1 - \lambda^{-1}) = \frac{d}{h}$ is not a root of P_t for any odd integer d coprime to h . Indeed, if P_t has such a root and d is coprime to h , then all roots λ of P_t have $\mathbf{v}_F(1 - \lambda^{-1}) = \frac{d}{h}$. On the other hand, P_t is the characteristic polynomial of i_t^2 as an element of $\mathbf{G}_h(K)$. Let $\lambda_1, \dots, \lambda_h$ be the roots of P_t . Then,

$$\prod_{i=1}^h (1 - \lambda_i^{-1}) = \det_K(1 - i_t^{-2}).$$

Using (2.3) and (2.2) we know

$$1 - i_t^{-2} = (i_t^2 - 1) \circ i_t^{-2} = -e_t^2 \circ i_t^{-2} = (e_t \circ i_t^{-1})^2 = t_\#^2.$$

Then $\mathbf{v}_F(\det_K(t_\#^2)) = d$ is an odd integer. However, we have $t_\#x_0 = x_0^\sigma t_\#$. Let $\sigma \in \mathbf{G}_{2h}(\mathcal{O}_F)$ be the linear transformation of Galois conjugation on K_1^h . Then $\sigma x_0 = x_0^\sigma \sigma$. So $t_\# \sigma$ commutes with x_0 . Therefore, $t_\# \sigma \in \mathbf{G}_h(K_1)$ and $\mathbf{v}_K(\det(t_\#)^2) = \mathbf{v}_K(\det(t_\# \sigma)^2) = 2\mathbf{v}_K(\det(t_\# \sigma))$ is an even number. This is a contradiction. ■

Lemma 5.5 Let P_x be an invariant polynomial for an integral double structure. If all roots λ of P_x satisfy $\mathbf{v}_F(1 - \lambda^{-1}) > 0$, we have $|P_x(0)|_F = 1$. If all roots λ of P_x satisfy $\mathbf{v}_F(1 - \lambda^{-1}) \leq 0$, we have $|P_x(1)|_F = 1$.

Proof Let λ be any root of P_x . If we have $\mathbf{v}_F(1 - \frac{1}{\lambda}) > 0$, then we must have $|\lambda|_F = 1$. Therefore, $P_x(0)$ is the product of all eigenvalues. So $|P_x(0)|_F = 1$.

If $\mathbf{v}_F(1 - \frac{1}{\lambda}) \leq 0$, so $\mathbf{v}_F(\lambda - 1) \leq \mathbf{v}_F(\lambda)$, then by the triangle inequality, we have

$$\mathbf{v}_F(1 - \lambda) \leq 1$$

for any eigenvalue λ of P_x . This implies

$$|P_x(1)|_F \geq 1.$$

Since P_x is the characteristic polynomial of $i_x^2 \in \mathcal{O}_{D_K}$, the value $P_x(1)$ is the determinant of

$$1 - i_x^2 = e_x^2 \in \mathcal{O}_{D_{K_1}}$$

by using (2.3). So $|P_x(1)|_F \leq 1$. This completes the proof. ■

Now we are ready to prove Proposition 5.1.

Proof of Proposition 5.1 Since we have

$$t_1 \in \mathfrak{S}_a^{>r}(\mathcal{O}_F), \quad t_2 \in \mathfrak{S}_b^{\leq r}(\mathcal{O}_F) \cap \mathfrak{S}_b^{>0}(\mathcal{O}_F), \quad t_3 \in \mathfrak{S}_c^{\leq 0}(\mathcal{O}_F),$$

P_{t_3} dominates P_{t_2} . By Lemma 5.4, we have P_{t_2} dominates P_γ . Moreover, P_γ dominates P_{t_1} . By Lemma 5.3, this implies

$$|\text{Res}(P_{t_1}, P_{t_2}, P_{t_3})|_F = |P_{t_1}(0)^{b+c} P_{t_2}(0)^c P_{t_2}(1)^a P_{t_3}(1)^{a+b}|_F.$$

Similarly,

$$|\text{Res}(P_\gamma, P_{t_1} P_{t_2} P_{t_3})|_F = |P_\gamma(0)^{b+c} P_\gamma(1)^a P_{t_1}(0)^h P_{t_2}(1)^h P_{t_3}(1)^h|_F.$$

By Lemma 5.5, we have $|P_{t_1}(0)|_F = |P_{t_2}(0)|_F = |P_\gamma(0)|_F = |P_{t_3}(1)|_F = 1$. Therefore

$$\frac{|\text{Res}(P_{t_1}, P_{t_2}, P_{t_3})|_F}{|\text{Res}(P_\gamma, P_{t_1} P_{t_2} P_{t_3})|_F} = |P_\gamma^{-a}(1) P_{t_2}^{-b-c}(1)|_F$$

as desired. ■

Combining these lemmas, we have obtained a proof of Proposition 5.1.

5.2 Recursion formula

Our next goal is to calculate the integrals defined in (5.1), (5.2), and (5.3). We establish the equation (5.6) in Proposition 5.6 and equation (5.5) in Corollary 5.8.

Proposition 5.6 We have

$$(5.13) \quad \sum_{i=0}^a A_0[i, 0] B_0[a - i] = \frac{\epsilon_F}{\epsilon_{K,a}^2} = \prod_{i=1}^a \frac{1 - q^{1-2i}}{1 - q^{-2i}}.$$

Proof On one hand, by applying our formula (4.13), we have

$$\begin{aligned} \frac{1}{\epsilon_{K,a}} \int_{\mathfrak{S}_a(\mathcal{O}_F)} dt &= \sum_{i=0}^a \frac{1}{\epsilon_{K,a} \epsilon_{K,h-a}} \int_{\mathfrak{S}_{a-i}^{\leq 0}(\mathcal{O}_F)} \int_{\mathfrak{S}_i^{>0}(\mathcal{O}_F)} |\text{Res}(P_{t'}, P_{t''})|_F dt' dt'' \\ &= \sum_{i=0}^a \frac{1}{\epsilon_{K,a} \epsilon_{K,h-a}} \int_{\mathfrak{S}_{a-i}^{\leq 0}(\mathcal{O}_F)} \int_{\mathfrak{S}_i^{>0}(\mathcal{O}_F)} |P_{t'}(0)^{a-i} P_{t''}(1)^i|_F dt' dt''. \end{aligned}$$

By Lemma 5.5, we have $|P_{t'}(0)^{a-i} P_{t''}(1)^i|_F = 1$ for $t' \in \mathfrak{S}_i^{>0}(\mathcal{O}_F)$ and $t'' \in \mathfrak{S}_{a-i}^{\leq 0}(\mathcal{O}_F)$. Then, the above integral can be simplified as

$$\sum_{i=0}^a \frac{1}{\epsilon_{K,a} \epsilon_{K,h-a}} \int_{\mathfrak{S}_{a-i}^{\leq 0}(\mathcal{O}_F)} \int_{\mathfrak{S}_i^{>0}(\mathcal{O}_F)} dt' dt'' = \sum_{i=0}^a A_0[i, 0] B_0[a - i].$$

On the other hand, $\text{Vol}(\mathfrak{S}_a(\mathcal{O}_F)) = \frac{\text{Vol}(\mathbf{G}_{2a}(\mathcal{O}_F))}{\text{Vol}(\mathbf{G}_a(\mathcal{O}_K))} = \frac{\epsilon_F}{\epsilon_{K,a}}$. Then,

$$\frac{1}{\epsilon_{K,a}} \int_{\mathfrak{S}_a(\mathcal{O}_F)} dt = \frac{\epsilon_F}{\epsilon_{K,a}^2}$$

as desired. ■

To compute $A_r[a, k]$, we consider a more general integral

$$A_r(a, X) := \frac{1}{\epsilon_{K,a}} \int_{\mathfrak{E}_a^{>r}(\mathcal{O}_F)} X^{\frac{v_F(p_X(1))}{2}} dx.$$

Besides, we define

$$C_r(b, X) := \frac{1}{\epsilon_{K,b}} \int_{\mathfrak{E}_b^{\leq r}(\mathcal{O}_F) \cap \mathfrak{E}_b^{>0}(\mathcal{O}_F)} |P_X(1)|_F^{-b} X^{\frac{v_F(p_X(1))}{2}} dx.$$

Then, we can write

$$A_r[a, i] = A_r(a, X)|_{X=q^{2i}} \quad C_r[a, i] = C_r(a, X)|_{X=q^{2i}}.$$

Lemma 5.7 *We have*

$$(5.14) \quad A_0(a, X) = \sum_{i=0}^a A_r(i, X) C_r(a - i, q^{-2a} X).$$

Proof Clearly,

$$\begin{aligned} A_0(a, X) &= \int_{\mathfrak{E}_a^{>0}(\mathcal{O}_F)} X^{\frac{v_F(p_X(1))}{2}} dx = \sum_{i=0}^a \int_{\mathfrak{E}_{a-i}^{>0}(\mathcal{O}_F) \cap \mathfrak{E}_{a-i}^{\leq r}(\mathcal{O}_F)} \\ &\quad \times \int_{\mathfrak{E}_i^{>r}(\mathcal{O}_F)} X^{\frac{v_F(p_{X'}(1))}{2} + \frac{v_F(p_{X''}(1))}{2}} |\text{Res}(P_{X'}, P_{X''})|_F dx' dx'' \\ &= \sum_{i=0}^a \int_{\mathfrak{E}_{a-i}^{>0}(\mathcal{O}_F) \cap \mathfrak{E}_{a-i}^{\leq r}(\mathcal{O}_F)} \\ &\quad \times \int_{\mathfrak{E}_i^{>r}(\mathcal{O}_F)} X^{\frac{v_F(p_{X'}(1))}{2} + \frac{v_F(p_{X''}(1))}{2}} |P_{X'}(0)^{a-i} P_{X''}(1)^i|_F dx' dx'' \\ &= \sum_{i=0}^a \int_{\mathfrak{E}_{a-i}^{>0}(\mathcal{O}_F) \cap \mathfrak{E}_{a-i}^{\leq r}(\mathcal{O}_F)} |P_{X''}(1)|_F^{i-a} |P_{X'}(1)|_F^a X^{\frac{v_F(p_{X''}(1))}{2}} dx'' \\ &\quad \times \int_{\mathfrak{E}_i^{>r}(\mathcal{O}_F)} X^{\frac{v_F(p_{X'}(1))}{2}} dx' \\ &= \sum_{i=0}^a \int_{\mathfrak{E}_{a-i}^{>0}(\mathcal{O}_F) \cap \mathfrak{E}_{a-i}^{\leq r}(\mathcal{O}_F)} |P_{X''}(1)|_F^{i-a} (q^{-2a} X)^{\frac{v_F(p_{X''}(1))}{2}} dx'' \\ &\quad \times \int_{\mathfrak{E}_i^{>r}(\mathcal{O}_F)} X^{\frac{v_F(p_{X'}(1))}{2}} dx' \\ &= \sum_{i=0}^a C_r(a - i, q^{-2a} X) A_r(i, X). \end{aligned}$$

This completes the proof. ■

Corollary 5.8 *For all $m \geq n$, we have*

$$A_0[n, n - m] = \sum_{i=0}^n C_r[i, m] A_r[n - i, n - m]$$

Proof By Lemma 5.7, we have $A_0(n, X) = \sum_{i=0}^n A_r(n - i, X) C_r(i, q^{-2n} X)$. The proof follows by setting $X = q^{2n-2m}$. ■

Therefore, we proved equations (5.5) and (5.6).

5.3 Computation for $A_r[n, m]$

In previous steps, we have essentially reduced everything to the computation of $A_r(i, X)$. In this section, we will consider the core $A_r(i, X)$. We will use the polar stereographic coordinate for our computation. This section is divided into three parts. In the first section, we introduce the integration over $\mathfrak{S}_h(F)$ using polar stereographic coordinate. In the second section, we use the polar stereographic coordinate to compute $A_0(i, X)$. We will see $A_0(i, X)$ is a rational function. In the last section, we give the formula for $A_r(i, X)$.

5.3.1 Polar Stereographic Coordinates

The polar stereographic coordinate is given by a map

$$\begin{aligned} \bullet_{\#} : \mathfrak{S}_h^{\circ}(F) &\longrightarrow \mathcal{H}_h(F) \\ x &\longmapsto x_{\#} := e_x \circ i_x^{-1} = (x - x_0)(x - x_0^{\sigma})^{-1} \end{aligned}$$

where $\mathfrak{S}_h^{\circ}(F)$ is the Zariski dense open subset given by

$$\mathfrak{S}_h^{\circ}(F) = \{x \in \mathfrak{S}_h(F) : x - x_0^{\sigma} \text{ is invertible}\}$$

and $\mathcal{H}_h(F)$ is the subspace of $\text{Mat}_{2h}(F)$ given by

$$\mathcal{H}_h(F) = \{x_{\#} \in \text{Mat}_{2h}(F) : x_{\#}x_0 - x_0^{\sigma}x_{\#} = 0\}.$$

We call this map the polar stereographic projection from the pole x_0^{σ} . Note that the set $\mathcal{H}_h(F)$ is the set of all semi- K_1 -linear endomorphisms of F^{2h} with the K_1 -structure induced by α_1 .

Our goal for this section is to prove the following formula.

Proposition 5.9 *Let $dx_{\#}$ be the standard additive Haar-measure for $\mathcal{H}_h(F)$ and dx , the standard Haar-measure for $\mathfrak{S}_h(\mathcal{O}_F)$. For any function $f : \mathcal{H}_h(F) \rightarrow \mathbb{R}$, we have*

$$\int_{\mathfrak{S}_h(F)} f(x_{\#}) dx = \int_{\mathcal{H}_h(F)} f(x_{\#}) |P_x(0)|_F^{2h} dx_{\#}.$$

Proof We consider both $\mathfrak{S}_h(F)$ and $\mathcal{H}_h(F)$ as subvarieties of the ambient variety $\text{Mat}_{2h}(F)$. For any $x \in \mathfrak{S}_h(\mathcal{O}_F)$, let $T_{\text{Mat}_{2h}(\mathcal{O}_F), x}$, $T_{\mathfrak{S}_h(F), x}$ be tangent spaces at x for $\text{Mat}_{2h}(F)$ and $\mathfrak{S}_h(F)$. Let $T_{\text{Mat}_{2h}(F), x_{\#}}$, $T_{\mathcal{H}_h(F), x_{\#}}$ be tangent spaces at $x_{\#}$ for $\text{Mat}_{2h}(F)$ and $\mathcal{H}_h(F)$.

Note that $di_x = de_x = \frac{dx}{\zeta - \zeta^{\sigma}}$, from the following one-form calculation:

$$\begin{aligned} dx_{\#} &= d(e_x \circ i_x^{-1}) \\ &= e_x \circ i_x^{-1}(di_x)i_x^{-1} + (de_x)i_x^{-1} \\ (5.15) \quad &= (e_x \circ i_x^{-1} + 1)(di_x)i_x^{-1} \\ &= (e_x + i_x) \circ i_x^{-2} \circ i_x(di_x)i_x^{-1}, \end{aligned}$$

we deduce that the induced map at the tangent space is given by the following map (we also draw the ambient space in the picture).

$$\begin{array}{ccc}
 T_{\mathfrak{S}_h(F),x} & \xrightarrow{\quad\quad\quad} & T_{\mathcal{H}_h(F),x_\#} \\
 \downarrow & & \downarrow \\
 T_{\text{Mat}_{2h}(\mathcal{O}_F),x} & \xrightarrow{\quad\quad\quad} & T_{\text{Mat}_{2h}(F),x_\#} \\
 \\
 X & \longmapsto & (e_x + i_x) \circ i_x^{-2} \circ i_x X i_x^{-1}
 \end{array}$$

Note that we can factorize this map into the following form

$$\begin{array}{ccccc}
 T_{\mathfrak{S}_h(F),x} & \xrightarrow{\quad\quad\quad} & T_{\mathfrak{S}_h(F),x_0} & \xrightarrow{\quad\quad\quad} & T_{\mathcal{H}_h(F),x_\#} \\
 \downarrow & & \downarrow & & \downarrow \\
 T_{\text{Mat}_{2h}(\mathcal{O}_F),x} & \xrightarrow{\quad\quad\quad} & T_{\text{Mat}_{2h}(\mathcal{O}_F),x_0} & \xrightarrow{\quad\quad\quad} & T_{\text{Mat}_{2h}(F),x_\#} \\
 \\
 X & \longmapsto & i_x X i_x^{-1} & \longmapsto & (e_x + i_x) \circ i_x^{-2} \circ i_x X i_x^{-1} \\
 \\
 Y & \longmapsto & & \longmapsto & (e_x + i_x) \circ i_x^{-2} Y
 \end{array}$$

Here the first map is induced by the following action of i_x

$$\begin{array}{ccc}
 i_x \cdot : \mathfrak{S}_h(F) & \longrightarrow & \mathfrak{S}_h(F) \\
 y & \longmapsto & i_x y i_x^{-1}
 \end{array}$$

Since $i_x \in \mathbf{G}_{2h}(F)$ and the Haar measure on $\mathfrak{S}_h(F)$ is $\mathbf{G}_{2h}(F)$ -invariant, the relative determinant of the first map $X \mapsto i_x X i_x^{-1}$ is 1. This implies the relative determinant for $T_{\mathfrak{S}_h(F),x} \rightarrow T_{\mathcal{H}_h(F),x_\#}$ is equal to the relative determinant of $T_{\mathfrak{S}_h(F),x_0} \rightarrow T_{\mathcal{H}_h(F),x_\#}$ with the map $Y \mapsto (e_x + i_x) \circ i_x^{-2} Y$. Note that if we identify $T_{\text{Mat}_{2h}(\mathcal{O}_F),x_0}$ with $T_{\text{Mat}_{2h}(F),x_\#}$ by additive translation, $T_{\mathfrak{S}_h(F),x_0}$ and $T_{\mathcal{H}_h(F),x_\#}$ are the same subspaces (intuitively, the projection plane for the polar-stereographic-projection and the tangent space of the opposite point of the pole are parallel to each other). Since $T_{\mathfrak{S}_h(F),x_0}$ is isomorphic to $\text{Mat}_h(K_1)$, the relative determinant for $T_{\mathfrak{S}_h(F),x} \rightarrow T_{\mathcal{H}_h(F),x_\#}$ equals

$$\det_{K_1}((e_x + i_x) \circ i_x^{-2})^h.$$

Note that $(e_x + i_x)^2 = 1$ implies $|\det_K(e_x + i_x)|_K = 1$, so

$$|\det_{K_1}((e_x + i_x) \circ i_x^{-2})^h|_K = |\det_{K_1}(i_x^2)|_{K_1}^{-h} = |P_x(0)|_{K_1}^{-h} = |P_x(0)|_F^{-2h}.$$

Now let dx and $dx_\#$ be standard Haar-measures for $\mathfrak{S}_h(F)$ and $\mathcal{H}_h(F)$, we have

$$|P_x(0)|_F^{-2h} dx = dx_\#.$$

This yields the formula that for any function $f : \mathcal{H}_h(F) \rightarrow \mathbb{R}$,

$$\int_{\mathfrak{E}_h(F)} f(x_\#) dx = \int_{\mathcal{H}_h(F)} f(x_\#) |P_x(0)|_F^{2h} dx_\#.$$

This completes the proof. ■

5.3.2 Computation of $A_0(i, X)$

In this section, we will prove the following formula.

Theorem 5.10 *We have*

$$A_0(a, X) = q^{-2a} X \prod_{i=1}^a \frac{1}{1 - q^{-2i} X}.$$

We briefly introduce our method. Firstly, we split $\mathcal{H}_h(F)$ into a disjoint union

$$\mathcal{H}_h(F) = \bigsqcup_{a=0}^h \mathcal{H}_{h,a}^{>0}(\mathcal{O}_F)$$

of the following subsets

$$\mathcal{H}_{h,a}^{>0}(\mathcal{O}_F) := \{x_\# \in \mathcal{H}_h(F) : \text{there is exactly } a \text{ many eigenvalues of } x_\# \text{ with the positive valuation}\}.$$

Then, we will prove the following proposition.

Proposition 5.11 *For any h, a , we have*

$$\int_{\mathcal{H}_{h,a}^{>0}(\mathcal{O}_F)} X^{\text{vk}(x_\#)} dx_\# = A_0(a, X).$$

Suppose this proposition has been proved, then we know

$$\int_{\mathcal{H}_h(F)} X^{\text{vk}(x_\#)} dx_\# = \sum_{i=0}^h \int_{\mathcal{H}_{h,i}^{>0}(\mathcal{O}_F)} X^{\text{vk}(x_\#)} dx_\# = \sum_{i=0}^h A_0(i, X).$$

This implies

$$A_0(a, X) = \int_{\mathcal{H}_a(\mathcal{O}_F)} X^{\text{vk}(x_\#)} dx_\# - \int_{\mathcal{H}_{a-1}(\mathcal{O}_F)} X^{\text{vk}(x_\#)} dx_\#.$$

Suppose, we know the following formula.

Proposition 5.12 *We have*

$$\frac{1}{\epsilon_{K,a}} \int_{\mathcal{H}(\mathcal{O}_F)} X^{\text{vk}(x_\#)} dx_\# = \prod_{i=1}^a \frac{1}{1 - q^{-2i} X}.$$

Then, we just finish the proof for Theorem 5.10 by direct computation

$$A_0(a, X) = \prod_{i=1}^a \frac{1}{1 - q^{-2i} X} - \prod_{i=1}^{a-1} \frac{1}{1 - q^{-2i} X} = q^{-2a} X \prod_{i=1}^a \frac{1}{1 - q^{-2i} X}.$$

We will prove Proposition 5.12 first, then prove Proposition 5.11.

Proof of Proposition 5.12 Consider the Galois conjugation map (F-linear)

$$\begin{aligned} \sigma : K_1^h &\longrightarrow K_1^h \\ x &\longmapsto \bar{x}. \end{aligned}$$

Clearly, we have $\sigma \in \mathcal{H}_h(F)$ and $\det(\sigma) = (-1)^h$. Then we consider an isomorphism

$$\begin{aligned} l(\sigma) : \mathcal{H}(\mathcal{O}_F) &\longrightarrow \text{Mat}_a(\mathcal{O}_K) \\ x &\longmapsto \sigma \circ x. \end{aligned}$$

This implies

$$\frac{1}{\epsilon_{K,h}} \int_{\mathcal{H}_a(\mathcal{O}_F)} X^{\mathbf{v}_K(x)} dx = \frac{1}{\epsilon_{K,h}} \int_{\text{Mat}_a(\mathcal{O}_K)} X^{\mathbf{v}_K(x)} dx.$$

Denote the above integral by $F(a, X)$. Let dg be the standard Haar-measure on $\mathbf{G}_a(K)$. Then, we have $dx = |g|_K^a dg$. We may write $F(a, X)$ into

$$F(a, X) = \frac{1}{\epsilon_{K,h}} \int_{\mathbf{G}_a(K) \cap \text{Mat}_a(\mathcal{O}_K)} X^{\mathbf{v}_K(g)} |g|_K^a dg.$$

Let $\Gamma \subset \mathbf{G}_a(K)$ be the subgroup of upper triangular matrices. By Iwasawa decomposition, we can write $\mathbf{G}_a(K) = \Gamma \mathbf{G}_a(\mathcal{O}_K)$. Then, we can view $\mathbf{G}_a(K)$ as a homogeneous space with the left $\Gamma \times \mathbf{G}_a(\mathcal{O}_K)$ -action given by

$$\begin{aligned} (\Gamma \times \mathbf{G}_a(\mathcal{O}_K)) \cdot \mathbf{G}_a(K) &\longrightarrow \mathbf{G}_a(K) \\ (\gamma, k) \cdot g &\longmapsto \gamma g k^{-1}. \end{aligned}$$

The stabilizer of each point is isomorphic to a compact subgroup $\mathbf{G}_a(\mathcal{O}_K) \cap \Gamma$. Then, $\mathbf{G}_a(K)$ has an $\Gamma \times \mathbf{G}_a(\mathcal{O}_K)$ -invariant Haar-measure. Since this measure has to be unique, this measure coincide with the Haar-measure of $\mathbf{G}_a(K)$. We choose the identity matrix $I_h \in \mathbf{G}_a(K)$ as our base point of $\mathbf{G}_a(K)$. Then, for any $g \in \mathbf{G}_a(K)$, we can write $g = pt$ for some $p \in \Gamma$ and $t \in \mathbf{G}_a(\mathcal{O}_K)$. Therefore, we have

$$\int_{\mathbf{G}_a(K)} \mathbb{1}_{\text{Mat}_a(\mathcal{O}_K)}(g) X^{\mathbf{v}_K(g)} |g|_K^a dg = \frac{\int_{\Gamma \times \mathbf{G}_a(\mathcal{O}_K)} \mathbb{1}_{\text{Mat}_a(\mathcal{O}_K)}(pt) X^{\mathbf{v}_K(pt)} |pt|_K^a dp dt}{\int_{\Gamma \cap \mathbf{G}_a(\mathcal{O}_K)} \mathbb{1}_{\text{Mat}_a(\mathcal{O}_K)}(g) X^{\mathbf{v}_K(g)} |g|_K^a dg}.$$

Note that the integrand $\mathbb{1}_{\text{Mat}_a(\mathcal{O}_K)}(g) X^{\mathbf{v}_K(g)} |g|_K^a$ is equal to 1 for $g \in \mathbf{G}_a(\mathcal{O}_K)$, we have

$$\begin{aligned} &\int_{\mathbf{G}_a(K)} \mathbb{1}_{\text{Mat}_a(\mathcal{O}_K)}(g) X^{\mathbf{v}_K(g)} |g|_K^a dg \\ &= \frac{\text{Vol}(\mathbf{G}_a(\mathcal{O}_K))}{\text{Vol}(\mathbf{G}_a(\mathcal{O}_K) \cap \Gamma)} \int_{\Gamma} \mathbb{1}_{\text{Mat}_a(\mathcal{O}_K)}(p) X^{\mathbf{v}_K(p)} |p|_K^a dp \end{aligned}$$

Since $\text{Vol}(\mathbf{G}_a(\mathcal{O}_K) \cap \Gamma) = \epsilon_{K,1}^a$ and $\text{Vol}(\mathbf{G}_a(\mathcal{O}_K)) = \epsilon_{K,a}$, we can write

$$F(a, X) = \frac{1}{\epsilon_{K,1}^a} \int_{\Gamma} \mathbb{1}_{\text{Mat}_a(\mathcal{O}_K)}(p) X^{\mathbf{v}_K(p)} |p|_K^a dp.$$

We remind the reader that dp is the standard left-Haar-measure of P . Let $\Lambda \subset \Gamma$ be the subgroup of diagonal matrices, $U \subset \Gamma$ the subgroup of unipotent matrices. We take the decomposition $p = \delta u$ such that $\delta \in \Lambda$ and $u \in U$. Note that $\det(u) = 1$ for all $u \in U$.

We can write

$$\begin{aligned}
 F(a, X) &= \frac{1}{\epsilon_{K,1}^a} \int_{\Lambda} \int_U \mathbb{1}[\delta u \in \text{Mat}_a(\mathcal{O}_K)] |\delta u|^a X^{\mathbf{v}_K(\delta u)} \, du \, d\delta \\
 &= \frac{1}{\epsilon_{K,1}^a} \int_{\Lambda} f(\delta) |\delta|^a X^{\mathbf{v}_K(\delta)} \, d\delta
 \end{aligned}$$

where

$$f(\delta) = \int_U \mathbb{1}[\delta u \in \text{Mat}_a(\mathcal{O}_K)] \, du.$$

Here $\mathbb{1}[\bullet] : \{\text{True}, \text{False}\} \rightarrow \{0, 1\}$ is the map such that $\mathbb{1}[\text{True}] = 1$ and $\mathbb{1}[\text{False}] = 0$. Let $\delta_{11}, \dots, \delta_{aa} \in K$ be diagonal entries of δ and u_{ij} the entry of u in i th row and j th column. Then, $\delta u \in \text{Mat}_a(\mathcal{O}_K)$ is equivalent to that $u_{ij} \in \delta_{ii}^{-1} \mathcal{O}_K$. We thus have

$$f(\delta) = \prod_{i=1}^a \prod_{j=i+1}^a \int_K \mathbb{1}[u_{ij} \in \delta_{ii}^{-1} \mathcal{O}_K] \, du_{ij} = \prod_{i=1}^a \prod_{j=i+1}^a |\delta_{ii}|_K^{-1} = \prod_{i=1}^a |\delta_{ii}|_K^{i-a}.$$

Therefore,

$$F(a, X) = \frac{1}{\epsilon_{K,1}^a} \int_{\Lambda \cap \text{Mat}_a(\mathcal{O}_K)} \prod_{i=1}^a |\delta_{ii}|_K^{i-a} |\delta_{ii}|_K^a X^{\mathbf{v}_K(\delta_{ii})} \, d\delta.$$

This equals

$$F(a, X) = \prod_{i=1}^a \frac{1}{\epsilon_{K,1}} \int_{\mathcal{O}_K} (q^{-2i} X)^{\mathbf{v}_K(\delta_{ii})} \, d\delta_{ii} = \prod_{i=1}^a \frac{1}{1 - q^{-2i} X}$$

as desired. ■

Our next goal is to prove Proposition 5.11. We will use the method in Section 3 for our calculation.

Proof of Proposition 5.11 This proof follows the strategy in Section 3.3. We will adapt the conditions to our situation.

Condition 1 We choose a subgroup C and set up the C -equivariant fibration: For any $t \in \mathcal{H}_{h,a}^{>0}(\mathcal{O}_F)$, we can decompose the characteristic polynomial $P(X)$ of t as $P(X) = P_0(X)P_{>0}(X)$ such that $P_0(0) \in \mathcal{O}_F^\times$ and $P_{>0}(X) \equiv X^a$ modulo π . Then, let $U_t = \ker(P_{>0}(t))$, which is the maximal invariant subspace such that all eigenvectors of t on U_t have eigenvalue λ with $\mathbf{v}_F(\lambda) > 0$. Using this way, we have defined a map

$$\begin{aligned}
 (5.16) \quad p : \mathcal{H}_{h,a}^{>0}(\mathcal{O}_F) &\longrightarrow \text{Gr}_{h,a}(K) \\
 x &\longmapsto U_x.
 \end{aligned}$$

Let $\mathcal{H}_h^\circ(F) \subset \mathcal{H}_h(F)$ be the subset of invertible matrices. Then, $\mathcal{H}_h^\circ(F)$ is a left-homogeneous space for the group $\mathbf{G}_a(K) \times \mathbf{G}_a(K)$ with the action given by

$$\begin{aligned}
 (\mathbf{G}_a(K) \times \mathbf{G}_a(K)) \cdot \mathcal{H}_h^\circ(F) &\longrightarrow \mathcal{H}_h^\circ(F) \\
 (k_1, k_2) \cdot x &\longmapsto k_1 x k_2^{-1}.
 \end{aligned}$$

Furthermore, we choose our subgroup C as the following

$$\mathbf{G}_a^\Delta(\mathcal{O}_K) := \{(x, x) : x \in \mathbf{G}_a(\mathcal{O}_K)\}.$$

Then the surjective map in (5.16) is a $\mathbf{G}_a^\Delta(\mathcal{O}_K)$ -equivariant map when we consider $\text{Gr}_{h,a}(K)$ as a $\mathbf{G}_a^\Delta(\mathcal{O}_K)$ -homogeneous space.

Condition 2 We choose a subgroup P such that each fiber is a subset of a P -homogeneous space: We denote each fiber $p^{-1}(U)$ by $\mathcal{P}_U^{>0}(\mathcal{O}_F)$. Then, $\mathcal{P}_U^{>0}(\mathcal{O}_F)$ is a subset of

$$\mathcal{P}_U(\mathcal{O}_F) = \{x \in \mathcal{H}_h(F) : xU \subset U\}.$$

Clearly, $\mathcal{P}_U(\mathcal{O}_F)$ is a homogeneous space of

$$P = \mathbf{P}_{K,U}(\mathcal{O}_{K_2}) \times \mathbf{P}_{K,U}(\mathcal{O}_{K_2})$$

where, $\mathbf{P}_{K,U}(\mathcal{O}_{K_2})$ is the stabilizer of U_t .

Condition 3 Clearly, we have

$$\dim(\mathcal{H}_h^\circ(F)) = \dim(\mathcal{P}_U^{>0}(\mathcal{O}_F)) + \dim(\text{Gr}_{h,a}(K)).$$

Notation Denote the stabilizer of $t \in \mathcal{H}(\mathcal{O}_F)$ by $\mathbf{G}_a^t(\mathcal{O}_K)$. We have

$$\mathbf{G}_a^t(\mathcal{O}_K) := \{(x, t^{-1}xt) : x \in \mathbf{G}_a(\mathcal{O}_K)\}.$$

Let \mathfrak{g}_a and $\mathfrak{p}_{U,K}$ be the Lie-algebra of $\mathbf{G}_a(\mathcal{O}_K)$ and $\mathbf{P}_{K,U}(\mathcal{O}_{K_2})$ respectively. Let $\mathfrak{g}_a^\Delta, \mathfrak{g}_a^t(\mathcal{O}_K) \subset \mathfrak{g}_a \times \mathfrak{g}_a$ be sub-Lie-algebras corresponding to $\mathbf{G}_a^\Delta(\mathcal{O}_K), \mathbf{G}_a^t(\mathcal{O}_K) \subset \mathbf{G}_a(\mathcal{O}_K) \times \mathbf{G}_a(\mathcal{O}_K)$. Let $\mathfrak{p}_{U,K}^\Delta = \mathfrak{g}_a^\Delta \cap \mathfrak{p}_{U,K} \times \mathfrak{p}_{U,K}$ and $\mathfrak{p}_{U,K}^t = \mathfrak{g}_a^t(\mathcal{O}_K) \cap \mathfrak{p}_{U,K} \times \mathfrak{p}_{U,K}$. Let $\mathfrak{u}_K = \mathfrak{g}_a / \mathfrak{p}_{U,K}, \mathfrak{u}_K^\Delta = \mathfrak{g}_a^\Delta / \mathfrak{p}_{U,K}^\Delta$ and $\mathfrak{u}_K^t = \mathfrak{g}_a^t(\mathcal{O}_K) / \mathfrak{p}_{U,K}^t$. Let du, du^Δ , and du^t be corresponding Haar-measures for $\mathfrak{u}_K \times \mathfrak{u}_K, \mathfrak{u}_K^\Delta$, and \mathfrak{u}_K^t respectively.

By Theorem 3.1, we have

$$\mathfrak{u}_K \times \mathfrak{u}_K = \mathfrak{u}_K^\Delta \oplus \mathfrak{u}_K^t.$$

Let $J(t) \in F$ be the element such that

$$du = J(t)du^\Delta du^t.$$

Theorem 3.1 then implies that we have

$$\int_{\mathcal{H}_h(F)} \mathbb{1}_{\mathcal{H}_h^\circ(\mathcal{O}_F)}(g) X^{\mathbf{v}_K(g)} dg = \int_{\text{Gr}_{h,a}(K)} \int_{\mathcal{P}_U(\mathcal{O}_F)} \mathbb{1}_{\mathcal{P}_U^{>0}(\mathcal{O}_F)}(t) X^{\mathbf{v}_K(t)} |J(g)|_F dt dx.$$

Now, we will calculate $|J(t)|_F$. Let t', t'' be induced linear operators of t on W and V/W . We have a natural isomorphism

$$\mathfrak{u}_K \cong \text{Hom}(W, V/W).$$

Then, the isomorphism $\mathfrak{u}_K^\Delta \oplus \mathfrak{u}_K^t \longrightarrow \mathfrak{u}_K \times \mathfrak{u}_K$ is given by

$$\begin{aligned} s : \text{Hom}(W, V/W) \times \text{Hom}(W, V/W) &\longrightarrow \text{Hom}(W, V/W) \times \text{Hom}(W, V/W) \\ (x, y) &\longmapsto (x + y, x + t''^{-1}yt'). \end{aligned}$$

Then, we use $J(t)$ for the determinant of this map. Note that the valuation of all eigenvalues of t' is larger than 0. And the valuation of all eigenvalues of t'' is zero. This implies that all valuations of eigenvalues of s is 0. This implies that $|J(t)|_F = 1$.

Therefore, we have

$$\int_{\mathcal{H}_h(F)} \mathbb{1}_{\mathcal{H}_{h,a}^{>0}(\mathcal{O}_F)}(g) X^{\mathbf{v}_K(g)} dg = \int_{\text{Gr}_{h,a}(K)} \int_{\mathcal{P}_U(\mathcal{O}_F)} \mathbb{1}_{\mathcal{P}_U^{>0}(\mathcal{O}_F)}(t) X^{\mathbf{v}_K(t)} dt dx.$$

Since the volume of $\text{Gr}_{h,a}(K)$ is $\frac{\epsilon_{K,h}}{\epsilon_{K,a}\epsilon_{K,h-a}}$, we have

$$\frac{1}{\epsilon_{K,h}} \int_{\mathcal{H}_h(F)} \mathbb{1}_{\mathcal{H}_{h,a}^{>0}(\mathcal{O}_F)}(g) X^{\mathbf{v}_K(g)} dg = \frac{1}{\epsilon_{K,a}\epsilon_{K,h-a}} \int_{\mathcal{P}_U(\mathcal{O}_F)} \mathbb{1}_{\mathcal{P}_U^{>0}(\mathcal{O}_F)}(t) X^{\mathbf{v}_K(t)} dt.$$

Furthermore, let $\mathcal{U}_U(\mathcal{O}_F) = \{t \in \mathbf{G}_h(\mathcal{O}_K) : t|_V = \text{id}, t_{W/V} = \text{id}\}$. After we choose a lifting $l : W/U \rightarrow W$, we can write every element $t \in \mathcal{P}_U^{>0}(\mathcal{O}_F)$ by $t_1 t_2 u$ where $u \in \mathcal{U}_U(\mathcal{O}_F)$ and W and $l(W/U)$ are invariant subspaces of t_1 and t_2 , such that t_1 acts trivially on $l(W/U)$ and t_2 acts trivially on W . This implies that $t_1 \in \mathcal{H}_{h,a}^{>0}(\mathcal{O}_F)$ and $t_2 \in \mathcal{H}_{h,h-a}^{>0}(\mathcal{O}_F)$. Therefore, we have a decomposition

$$\mathcal{P}_U^{>0}(\mathcal{O}_F) = \mathcal{H}_{h,a}^{>0}(\mathcal{O}_F) \times \mathcal{H}_{h,h-a}^{>0}(\mathcal{O}_F) \times \mathcal{U}_U(\mathcal{O}_F).$$

Then, we can write

$$\begin{aligned} & \frac{1}{\epsilon_{K,a}\epsilon_{K,h-a}} \int_{\mathcal{P}_U(\mathcal{O}_F)} \mathbb{1}_{\mathcal{P}_U^{>0}(\mathcal{O}_F)}(t) X^{\mathbf{v}_K(t)} dt \\ &= \frac{1}{\epsilon_{K,a}\epsilon_{K,h-a}} \int_{\mathcal{H}_{h,a}^{>0}(\mathcal{O}_F)} \int_{\mathcal{H}_{h,h-a}^{>0}(\mathcal{O}_F)} \int_{\mathcal{U}_U(\mathcal{O}_F)} X^{\mathbf{v}_K(t_1 t_2 u)} dt_1 dt_2 du \\ &= \frac{1}{\epsilon_{K,h-a}} \int_{\mathcal{H}_{h,h-a}^{>0}(\mathcal{O}_F)} dt_2 \cdot \frac{1}{\epsilon_{K,a}} \int_{\mathcal{H}_{h,a}^{>0}(\mathcal{O}_F)} X^{\mathbf{v}_K(t_1)} dt_1. \end{aligned}$$

By Proposition 5.12, we have

$$\frac{1}{\epsilon_{K,h-a}} \int_{\mathcal{H}_{h,h-a}^{>0}(\mathcal{O}_F)} dt_2 = \prod_{i=1}^a \frac{1}{1 - q^{-2i} X} \Big|_{X=0} = 1.$$

Therefore,

$$\frac{1}{\epsilon_{K,h}} \int_{\mathcal{H}_{h,a}^{>0}(\mathcal{O}_F)} X^{\mathbf{v}_K(g)} dg = \frac{1}{\epsilon_{K,a}} \int_{\mathcal{H}_{h,a}^{>0}(\mathcal{O}_F)} X^{\mathbf{v}_K(t)} dt.$$

Our final goal is to prove this quantity equals to $A_0(a, X)$. Indeed, since

$$x_{\#}^2 = e_x \circ i_x^{-1} \circ e_x \circ i_x^{-1} = -e_x^2 \circ i_x^{-2} = 1 - i_x^{-2},$$

$1 - \frac{1}{\lambda}$ is an eigenvalue of $x_{\#}^2$ for any root λ of P_x . This implies $x_{\#} \in \mathcal{H}_a^{>0}(\mathcal{O}_F)$ if and only if $x \in \mathcal{E}_a^{>0}(\mathcal{O}_F)$. Furthermore, by Lemma 5.5, $|P_x(0)|_F = 1$. Then,

$$\begin{aligned} \frac{1}{\epsilon_{K,a}} \int_{\mathcal{H}_a^{>0}(\mathcal{O}_F)} X^{\mathbf{v}_K(t)} dt &= \frac{1}{\epsilon_{K,a}} \int_{\mathcal{H}_a^{>0}(\mathcal{O}_F)} X^{\mathbf{v}_K(t)} |P_x(0)|_F^{2h} dt \\ &= \frac{1}{\epsilon_{K,a}} \int_{\mathcal{E}_a^{>0}(\mathcal{O}_F)} X^{\mathbf{v}_K(t)} dt \\ &= A_0(a, X) \end{aligned}$$

as desired. ■

5.3.3 Computation of $A_r(i, X)$

Now we are able to compute $A_r(i, X)$ by using complex analysis strategies.

Lemma 5.13 For any n, r , the function $A_r(n, X)$ is a rational function with poles at $X = q^2, q^4, \dots, q^{2n}$. Let $a_r[n, m]$ be the residue of $-q^{-2m}A_r(n, X)$ at $X = q^{2m}$. Then, we have

$$(5.17) \quad A_r(n, X) = \sum_{i=1}^n \frac{a_r[n, i](q^{-2i}X)^{\lfloor \frac{nr}{2} \rfloor}}{1 - q^{-2i}X}.$$

Here $\lceil r \rceil$ means the smallest integer larger than r . In other words, $\lceil r \rceil = n \iff r \in [n - 1, n)$.

Proof Remember that the degree of a rational function is the order of its pole at the infinity (for example, let $P(x), Q(x)$ be two polynomials. The degree of $\frac{P(x)}{Q(x)}$ is $\deg(P(x)) - \deg(Q(x))$). By Lemma 5.7, we have

$$A_r(n, X) = A_0(n, X) - \sum_{i=0}^{n-1} A_r(i, X)C_r(n - i, q^{-2n}X).$$

Firstly, we claim that the degree of $A(a, r, X)$ is at most $\lfloor \frac{nr}{2} \rfloor - 1$. We prove it by induction. When $n = 0$, the degree of $A_r(n, X) = 1$ is indeed $0 = \lfloor 0 \rfloor - 1$. Now we assume the induction hypothesis that the degree of $A(i, r, X)$ is at most $\lfloor \frac{ir}{2} \rfloor - 1$ for all $i < n$. By Lemma 5.10, the degree of $A_0(n, X)$ is at most 0. For each summand $A_r(i, X)C_r(n - i, q^{-2n}X)$, the degree of $C_r(n - i, q^{-2n}X)$ is at most $\lfloor \frac{(n-i)r}{2} \rfloor$, where the symbol $\lfloor r \rfloor$ means the largest integer no larger than r . By induction hypothesis, the degree of $A_r(i, X)$ is at most $\lfloor \frac{ir}{2} \rfloor - 1$. Therefore, the degree of each summand $A_r(i, X)C_r(n - i, q^{-2n}X)$ is at most

$$\left\lfloor \frac{(n-i)r}{2} \right\rfloor + \left\lfloor \frac{ir}{2} \right\rfloor - 1 \leq \left\lfloor \frac{nr}{2} \right\rfloor - 1.$$

This proves our claim. Now let $P(X)$ be a rational function such that

$$A_r(n, X) = \sum_{i=1}^n \frac{a(n, r, q^{2i})(q^{-2i}X)^{\lfloor \frac{nr}{2} \rfloor}}{1 - q^{-2i}X} + P(X).$$

Then, $P(X)$ has no poles except the infinity. This implies $P(X)$ is a polynomial. Our claim implies that the degree of $P(X)$ is at most $\lfloor \frac{nr}{2} \rfloor - 1$. Since by definition of $A_r(n, X)$, the coefficient for X^j must be 0 for any $j < \frac{nr}{2}$, this proves $P(X) = 0$, which proves this lemma. ■

Corollary 5.14 We have

$$(5.18) \quad q^{2m-2n} \prod_{\substack{i=1 \\ i \neq m}}^n \frac{1}{1 - q^{2m-2i}} = \sum_{i=m}^n a_r[i, m]C_r[n - i, 2m - 2n].$$

Proof Using Lemma 5.13 and Theorem 5.10, we can write (5.14) as

$$q^{-2n} X \prod_{i=1}^n \frac{1}{1 - q^{-2i} X} = \sum_{i=0}^n \sum_{j=1}^i \frac{a_r[i, j](q^{-2j} X)^{\lfloor \frac{nr}{2} \rfloor}}{1 - q^{-2j} X} C_r(n - i, q^{-2n} X).$$

Multiplying the above equation by $1 - q^{-2m} X$, we can write this equation into

$$q^{-2n} X \prod_{i=1, i \neq m}^n \frac{1}{1 - q^{-2i} X} = \sum_{i=0}^n a_r[i, m](q^{-2m} X)^{\lfloor \frac{nr}{2} \rfloor} C_r(n - i, q^{-2n} X) + (1 - q^{-2m} X)o(X)$$

where $o(X)$ refers to a function without poles at $X = q^{2m}$. This Corollary follows by evaluating this equation at $X = q^{2m}$. ■

6 Computation of the intersection number for $h = 2$

In this section, we use our algorithm developed in Section 5 to compute the arithmetic geometric side of the linear AFL for the case of $h = 2$. Our result is listed at the end of this section. Those results are written in a form comparable with the analytic side computed in Section 7.

By Theorem 5.10, we have

$$(6.1) \quad a_0[n, n - m] = q^{-2m} \prod_{\substack{i=0 \\ i \neq m}}^{n-1} \frac{1}{1 - q^{-2(m-i)}}.$$

Furthermore, by (5.8), we have $a_0[n, n] = a_r[n, n]$ for any n . Firstly, we have

$$a_r[1, 1] = a_0[1, 1] = 1.$$

Therefore, by the equation (5.9), we have

$$A_r[1, 1 - m] = \frac{q^{-2m \lfloor \frac{r}{2} \rfloor}}{1 - q^{-2m}}.$$

Plugging in $m = -1$ and $m = 1$, we have

$$(6.2) \quad A_r[1, 2] = \frac{-q^{-2+2 \lfloor \frac{r}{2} \rfloor}}{1 - q^{-2}} \quad A_r[1, 0] = \frac{q^{-2 \lfloor \frac{r}{2} \rfloor}}{1 - q^{-2}} \quad A_0[1, 0] = \frac{q^{-2}}{1 - q^{-2}}.$$

Note that $C_r[0, m] = 1$ for any m . Using (5.5), we have

$$C_r[1, m] = A_0[1, 1 - m] - A_r[1, 1 - m] = \frac{q^{-2m}}{1 - q^{-2m}} - \frac{q^{-2m \lfloor \frac{r}{2} \rfloor}}{1 - q^{-2m}}.$$

Evaluating this expression at $m = 0$ and $m = 1$, we have

$$(6.3) \quad C_r[1, 1] = \frac{q^{-2} - q^{-2 \lfloor \frac{r}{2} \rfloor}}{1 - q^{-2}}, \quad C_r[1, 0] = \left\lfloor \frac{r}{2} \right\rfloor - 1, \quad C_r[1, -1] = \frac{q^{2 \lfloor \frac{r}{2} \rfloor - 2} - 1}{1 - q^{-2}}.$$

Continue the same process. By (5.8) we have

$$a_r[2, 2] = a_0[2, 2] = \frac{-q^{-2}}{1 - q^{-2}}.$$

Using (6.1), we see

$$a_0[2, 1] = \frac{q^{-2}}{1 - q^{-2}}.$$

By (5.8), we have

$$a_r[2, 1] = a_0[2, 1] - C_r[1, 1]a_r[1, 1] = \frac{q^{-2[\frac{r}{2}]}}{1 - q^{-2}}.$$

Again by applying (5.9), we have

$$(6.4) \quad A_r[2, 2 - m] = \frac{-q^{-2-2m[r]}}{(1 - q^{-2})(1 - q^{-2m})} + \frac{q^{-2[\frac{r}{2}] + 2(1-m)[r]}}{(1 - q^{-2})(1 - q^{-2(m-1)})}.$$

Let $m = 2$, we have

$$(6.5) \quad A_r[2, 0] = \frac{-q^{-2-4[r]}}{(1 - q^{-2})(1 - q^{-4})} + \frac{q^{-2[\frac{r}{2}] - 2[r]}}{(1 - q^{-2})^2}.$$

By Formula (5.5), we obtain

$$(6.6) \quad C_r[2, m] = A_0[2, 2 - m] - A_r[2, 2 - m] - C_r[1, m]A_r[1, 2 - m].$$

From (6.3) and (6.2), we know $C_r[1, 0]A[1, 2] = (1 - [\frac{r}{2}]) \frac{q^{-2+2[\frac{r}{2}]}}{1 - q^{-2}}$, therefore, moving this term to the left and evaluating (6.6) at $m = 0$, we have

$$(6.7) \quad C_r[2, 0] + \left(1 - \left[\frac{r}{2}\right]\right) \frac{q^{-2+2[\frac{r}{2}]}}{1 - q^{-2}} = \frac{q^{-2-2m[r]} - q^{-2-2m}}{(1 - q^{-2})(1 - q^{-2m})} \Big|_{m=0} + \frac{q^{-2-2[\frac{r}{2}] + 2[r]} - q^{-2}}{(1 - q^{-2})^2}.$$

Applying L'Hospital rule, we have

$$C_r[2, 0] = \frac{q^{-2-2[\frac{r}{2}] + 2[r]} - q^{-4}}{(1 - q^{-2})^2} + \frac{\left([\frac{r}{2}] - 1\right)q^{2[\frac{r}{2}] - 2} - [r]q^{-2}}{1 - q^{-2}}.$$

Now we calculate $B_0[1]$ and $B_0[2]$. By formula (5.6),

$$B_0[1] = \frac{1 - q^{-1}}{1 - q^{-2}} - A_0[1, 0] = \frac{1 - q^{-1} - q^{-2}}{1 - q^{-2}}.$$

By formula (5.13), we have

$$B_0[2] = \frac{(1 - q^{-1})(1 - q^{-3})}{(1 - q^{-2})(1 - q^{-4})} - B_0[1]A_0[1, 0] - A_0[2, 0]$$

and

$$B_0[2] = \frac{q^{-3} - q^{-2}}{(1 - q^{-2})^2} + \frac{q^{-6} - q^{-3} + 1 - q^{-1} + q^{-4}}{(1 - q^{-2})(1 - q^{-4})}.$$

Since $C_r[1, 0] = \left\lfloor \frac{r}{2} \right\rfloor - 1$, we have

$$B_0[1] + C_r[1, 0] = \frac{-q^{-1}}{1 - q^{-2}} + \left\lfloor \frac{r}{2} \right\rfloor$$

whence

$$B_0[2] + C_r[1, -1]B_0[1] + C_r[2, 0]$$

equals

$$\begin{aligned} & \frac{q^{-2-2\lfloor \frac{r}{2} \rfloor + 2[r]} (1 - q^{-2})^2} + \left\lfloor \frac{r}{2} \right\rfloor \frac{q^{2\lfloor \frac{r}{2} \rfloor - 2}}{1 - q^{-2}} - \frac{q^{2\lfloor \frac{r}{2} \rfloor - 2}}{1 - q^{-2}} - [r] \frac{q^{-2}}{1 - q^{-2}} - \frac{q^{-6}}{(1 - q^{-2})(1 - q^{-4})} \\ & + (q^{2\lfloor \frac{r}{2} \rfloor - 2} - q^{-2}) \frac{(1 - q^{-1} - q^{-2})}{(1 - q^{-2})^2}. \end{aligned}$$

By our formula (5.4), the intersection number equals

$$\begin{aligned} N(r) = & q^{4r} A_0[2, 0] + q^{2r} A_0[1, 0](B_0[1] + C_r[1, 0]) + (B_0[2] \\ & + C_r[1, -1]B_0[1] + C_r[2, 0]). \end{aligned}$$

This equals

$$\begin{aligned} & \frac{q^{-2-2\lfloor \frac{r}{2} \rfloor + 2[r]} (1 - q^{-2})^2} + \left(\left\lfloor \frac{r}{2} \right\rfloor - 1 \right) \frac{q^{2\lfloor \frac{r}{2} \rfloor - 2}}{1 - q^{-2}} - [r] \frac{q^{-2}}{1 - q^{-2}} - \frac{q^{-6}}{(1 - q^{-2})(1 - q^{-4})} \\ & + (q^{2\lfloor \frac{r}{2} \rfloor - 2} - q^{-2}) \frac{(1 - q^{-1} - q^{-2})}{(1 - q^{-2})^2} + \frac{\left\lfloor \frac{r}{2} \right\rfloor q^{-2\lfloor \frac{r}{2} \rfloor + 2r}}{1 - q^{-2}} - \frac{q^{-1-2\lfloor \frac{r}{2} \rfloor + 2r}}{(1 - q^{-2})^2} \\ & + \frac{-q^{-2-4[r] + 4r}}{(1 - q^{-2})(1 - q^{-4})} + \frac{q^{-2\lfloor \frac{r}{2} \rfloor - 2[r] + 4r}}{(1 - q^{-2})^2}. \end{aligned}$$

Since we have $2r - 2\lfloor r \rfloor = -1$, the intersection formula is simplified to

$$\begin{aligned} (6.8) \quad N(r) = & \frac{q^{-2-2\lfloor \frac{r}{2} \rfloor + 2[r]} - q^{-4}}{(1 - q^{-2})^2} + \left(\left\lfloor \frac{r}{2} \right\rfloor - 1 \right) \frac{q^{2\lfloor \frac{r}{2} \rfloor - 2}}{1 - q^{-2}} - [r] \frac{q^{-2}}{1 - q^{-2}} \\ & + (q^{2\lfloor \frac{r}{2} \rfloor - 2} - q^{-2}) \frac{(1 - q^{-1} - q^{-2})}{(1 - q^{-2})^2} + \frac{\left\lfloor \frac{r}{2} \right\rfloor q^{-2\lfloor \frac{r}{2} \rfloor + 2r}}{1 - q^{-2}}. \end{aligned}$$

We found $N(\frac{1}{2}) = 1$ and $N(\frac{3}{2}) = q + 2$. Furthermore, we compute

$$\begin{aligned} (6.9) \quad N(r + 2) - N(r) = & \frac{q^{-2\lfloor \frac{r}{2} \rfloor + 2[r]} (1 - q^{-2})^2} + \left(\left\lfloor \frac{r}{2} \right\rfloor - 1 \right) q^{2\lfloor \frac{r}{2} \rfloor} + \frac{q^{2\lfloor \frac{r}{2} \rfloor}}{1 - q^{-2}} - 2 \frac{q^{-2}}{1 - q^{-2}} \\ & + q^{2\lfloor \frac{r}{2} \rfloor} \frac{(1 - q^{-1} - q^{-2})}{1 - q^{-2}} + \left\lfloor \frac{r}{2} \right\rfloor q^{-2\lfloor \frac{r}{2} \rfloor + 2r + 2} + \frac{q^{-2\lfloor \frac{r}{2} \rfloor + 2r + 2}}{1 - q^{-2}}. \end{aligned}$$

Note that $2 \lceil r \rceil = 2r + 1$. By simplifying this equation, we may write $N(r + 2) - N(r)$ as

$$(6.10) \quad q^{-2\lceil \frac{r}{2} \rceil + 2\lceil r \rceil + 1} \left(\frac{1}{1 - q^{-1}} + \left\lceil \frac{r}{2} \right\rceil \right) + q^{2\lceil \frac{r}{2} \rceil} \left(\left\lceil \frac{r}{2} \right\rceil - \frac{1}{1 - q^{-1}} \right) + \frac{2(1 - q^{2\lceil \frac{r}{2} \rceil + 2})}{1 - q^2}.$$

In order to compare this result with orbital integrals, we will rewrite this expression to another form in the rest of the section. Please note that our goal here is preparing a result for comparison with the analytic side rather than simplifying the expression. We do this part of computation only after we know the result of the analytic side. For $a = 0$ or $a = 1$, we have an identity

$$(6.11) \quad (q^{-a+1} + q^a)a + \frac{q^{-a} - q^a}{1 - q^{-1}} = 0.$$

For any $r \in \frac{1}{2}\mathbb{Z}$, we have

$$2 \left\lceil \frac{r}{2} \right\rceil - \lceil r \rceil = 1 \quad \text{or} \quad 0.$$

Then, let $a = 2 \lceil \frac{r}{2} \rceil - \lceil r \rceil$ in the equation (6.11), we could write

$$q^{\lceil r \rceil} \left(q^{-2\lceil \frac{r}{2} \rceil + \lceil r \rceil + 1} + q^{2\lceil \frac{r}{2} \rceil - \lceil r \rceil} \right) \left(2 \left\lceil \frac{r}{2} \right\rceil - \lceil r \rceil \right) + q^{\lceil r \rceil} \frac{q^{-2\lceil \frac{r}{2} \rceil + \lceil r \rceil} - q^{2\lceil \frac{r}{2} \rceil - \lceil r \rceil}}{1 - q^{-1}} = 0.$$

In other words,

$$(6.12) \quad q^{-2\lceil \frac{r}{2} \rceil + 2\lceil r \rceil + 1} \left(2 \left\lceil \frac{r}{2} \right\rceil - \lceil r \rceil \right) + q^{2\lceil \frac{r}{2} \rceil} \left(2 \left\lceil \frac{r}{2} \right\rceil - \lceil r \rceil \right) + q^{\lceil r \rceil} \frac{q^{-2\lceil \frac{r}{2} \rceil + \lceil r \rceil} - q^{2\lceil \frac{r}{2} \rceil - \lceil r \rceil}}{1 - q^{-1}} = 0.$$

Now by computing the difference of the equation (6.10) and the equation (6.12), we can write $N(r + 2) - N(r)$ into

$$q^{-2\lceil \frac{r}{2} \rceil + 2\lceil r \rceil + 1} \left(\frac{1}{1 - q^{-1}} - \left\lceil \frac{r}{2} \right\rceil + \lceil r \rceil \right) + q^{2\lceil \frac{r}{2} \rceil} \left(\left\lceil \frac{r}{2} \right\rceil - \left\lceil \frac{r}{2} \right\rceil - \frac{1}{1 - q^{-1}} \right) + \frac{2(1 - q^{2\lceil \frac{r}{2} \rceil + 2})}{1 - q^2} - q^{\lceil r \rceil} \frac{q^{-2\lceil \frac{r}{2} \rceil + \lceil r \rceil} - q^{2\lceil \frac{r}{2} \rceil - \lceil r \rceil}}{1 - q^{-1}}.$$

Simplifying this expression, we obtain

$$(6.13) \quad N(r + 2) - N(r) = 2 \frac{1 - q^{2\lceil \frac{r}{2} \rceil}}{1 - q^2} + \left(\lceil r \rceil + 2 - \left\lceil \frac{r}{2} \right\rceil \right) q^{2\lceil \frac{r}{2} \rceil} + \left(\lceil r \rceil - \left\lceil \frac{r}{2} \right\rceil + 1 \right) q^{2\lceil r \rceil - 2\lceil \frac{r}{2} \rceil + 1}.$$

Note that if $2r$ is an odd number, then $\lceil r \rceil = r + \frac{1}{2}$. If $2r \equiv 1 \pmod 4$, we have $\lceil \frac{r}{2} \rceil = \frac{r}{2} + \frac{3}{4}$. If $2r \equiv 3 \pmod 4$, we have $\lceil \frac{r}{2} \rceil = \frac{r}{2} + \frac{1}{4}$. This implies

$$\begin{cases} N(r+2) - N(r) = 2 \frac{1 - q^{r+\frac{3}{2}}}{1 - q^2} + \left(\frac{r}{2} + \frac{7}{4}\right) q^{r+\frac{3}{2}} + \left(\frac{r}{2} + \frac{3}{4}\right) q^{r+\frac{1}{2}} & \text{when } 2r \equiv 1 \pmod 4 \\ N(r+2) - N(r) = 2 \frac{1 - q^{r+\frac{1}{2}}}{1 - q^2} + \left(\frac{r}{2} + \frac{9}{4}\right) q^{r+\frac{1}{2}} + \left(\frac{r}{2} + \frac{5}{4}\right) q^{r+\frac{3}{2}} & \text{when } 2r \equiv 3 \pmod 4 \end{cases}$$

These expressions determine the value of $N(r)$ completely with the initial condition

$$\begin{cases} N\left(\frac{1}{2}\right) = 1, \\ N\left(\frac{3}{2}\right) = q + 2. \end{cases}$$

7 Computation of orbital integrals for $h = 2$

In this section, we will finish our proof of the linear AFL in $h = 2$ case by computing the Analytic side of the linear AFL conjecture. Our test function is the unit of the spherical Hecke algebra. This section have five parts. In Sections 7.1–7.3, we briefly describe the combinatorial method for general h . Along the way, we provide a combinatorial picture of orbital integrals. Section 7.4 describes the orbits that occur in the linear AFL. In Section 7.5, we specialize to the computation to the case $h = 2$.

7.1 Definition of orbital integrals

In this section, we will define the relative orbital integral in (7.1). We introduce two important operators in (7.2) and (7.3). These two operators will play a central role in our calculation.

Now we prepare materials to define the relative orbital integral with respect to two \mathcal{O}_F -algebra embeddings

$$\begin{aligned} \tau_1 &: \mathcal{O}_F \times \mathcal{O}_F \longrightarrow \text{Mat}_{2h}(\mathcal{O}_F), \\ \tau_2 &: \mathcal{O}_F \times \mathcal{O}_F \longrightarrow \text{Mat}_{2h}(\mathcal{O}_F). \end{aligned}$$

Let $g \in \mathbf{G}_{2h}(\mathcal{O}_F)$ be an element such that $\tau_2(x) = g\tau_1(x)g^{-1}$ for any $x \in \mathcal{O}_F \times \mathcal{O}_F$. Let $C(\tau_i) \subset \mathbf{G}_{2h}(F)$ be centralizers of τ_i . It is clear that $C(\tau_i) \cong \mathbf{G}_h(F) \times \mathbf{G}_h(F)$ for $i = 1, 2$. For $x \in C(\tau_1)$, we write $x = (x_1, x_2)$ for $x_i \in \mathbf{G}_h(F)$. Moreover, define

$$|x| := |\det(x_1^{-1}x_2)|_F \quad \eta_{K/F}(x) := \eta_{K/F}(\det(x_1x_2))$$

where $\eta_{K/F}$ is the quadratic character of K/F . Let $\mathbb{1}_{\mathbf{G}_{2h}(\mathcal{O}_F)}$ be the characteristic function of $\mathbf{G}_{2h}(\mathcal{O}_F)$. The relative orbital integral is defined by

$$(7.1) \quad \begin{aligned} &\text{Orb}_{\tau_1, \tau_2}(\mathbb{1}_{\mathbf{G}_{2h}(\mathcal{O}_F)}, s) \\ &:= \int_{C(\tau_1) \cap C(\tau_2) \backslash C(\tau_1) \times C(\tau_1)} \mathbb{1}_{\mathbf{G}_{2h}(\mathcal{O}_F)}(u_1^{-1}gu_2) \eta_{E/L}(u_2) |u_1u_2|^s du_1 du_2. \end{aligned}$$

where the Haar-measure on $C(\tau_1)$ and $C(\tau_1) \cap C(\tau_2)$ is normalized by $C(\tau_1) \cap \mathbf{G}_{2h}(\mathcal{O}_F)$ and $C(\tau_1) \cap C(\tau_2) \cap \mathbf{G}_{2h}(\mathcal{O}_F)$ respectively. In the case that the invariant polynomial of (τ_1, τ_2) has distinct roots, the F algebra

$$L_{\tau_1, \tau_2} := \{l \in \text{Mat}_{2h}(F) : l\tau_i(x) = \tau_i(x)l \text{ for any } i = 1, 2 \text{ and } x \in \mathcal{O}_F \times \mathcal{O}_F\}$$

is a commutative etale algebra over F and one has $L \cap \mathbf{G}_{2h}(F) \cong C(\tau_1) \cap C(\tau_2)$.

Now, we introduce two important operators. Let

$$\zeta = (a, b) \in \mathcal{O}_F^\times \oplus \mathcal{O}_F^\times$$

be a generator. Recall Definition 2.2. Let $\mathbf{w}_{\tau_1, \tau_2}$ be an element in L_{τ_1, τ_2} defined by

$$(7.2) \quad \mathbf{w}_{\tau_1, \tau_2} := i_{\tau_1, \tau_2}^2 \in L_{\tau_1, \tau_2}.$$

Let $\mathbf{z}_{\tau_1, \tau_2}$ be an element in $\mathbf{G}_{2h}(F)$ defined by

$$(7.3) \quad \mathbf{z}_{\tau_1, \tau_2} := i_{\tau_1, \tau_2} \circ e_{\tau_1, \tau_2} \in \mathbf{G}_{2h}(F).$$

The trace of $\mathbf{z}_{\tau_1, \tau_2}$ is zero because $i_{\tau_1, \tau_2} \circ \mathbf{z}_{\tau_1, \tau_2} \circ i_{\tau_1, \tau_2}^{-1} = -\mathbf{z}_{\tau_1, \tau_2}$. In this section, we abbreviate those symbols by \mathbf{z} and \mathbf{w} .

7.2 Orbital integral and lattice counting

In this section, we give a combinatorial formula for orbital integrals in terms of lattices. A lattice $\Lambda \subset F^{2h}$ is an \mathcal{O}_F -submodule such that $\Lambda \otimes_{\mathcal{O}_F} F \cong F^{2h}$. Our orbital integral (7.1) has a natural interpretation of counting lattices. From now, we fix the embedding τ_1 such that

$$\tau_1(\zeta) = \begin{pmatrix} aI_h & \\ & bI_h \end{pmatrix}.$$

We fix a lattice

$$\Lambda_0 = \mathcal{O}_F^{2h} \subset F^{2h}.$$

To translate the orbital integral into an object-counting problem, we study the integrand of (7.2),

$$\mathbb{1}_{\mathbf{G}_{2h}(\mathcal{O}_F)}(u_1^{-1}gu_2)\eta_{E/L}(u_2) |u_1u_2|^s.$$

We will discuss $\mathbb{1}_{\mathbf{G}_{2h}(\mathcal{O}_F)}(u_1^{-1}gu_2)$ first. Then, study $\eta_{E/L}(u_2) |u_1u_2|^s$.

Note that the integrand of the orbital integral (7.1) does not vanish only if

$$\mathbb{1}_{\mathbf{G}_{2h}(\mathcal{O}_F)}(u_1^{-1}gu_2) \neq 0,$$

which is equivalent to $gu_2\Lambda_0 = u_1\Lambda_0$. Furthermore, it is straightforward to verify $\Lambda = gu_2\Lambda_0 = u_1\Lambda_0$ if and only if Λ is closed under both actions of $\tau_1(\zeta)$ and $\tau_2(\zeta)$. Finally, since the integrand is considered under the equivalence of $C(\tau_1) \cap C(\tau_2) = L^\times$, the object-counting process is considering elements of the following subset

$$\mathcal{L}_{\tau_1, \tau_2} := \{ \Lambda \subset F^{2h} \text{ lattice, } \tau_i(\zeta)\Lambda = \Lambda \text{ for } i = 1, 2 \} / L^\times.$$

Here, two lattices Λ_1, Λ_2 are equivalent if and only if $\Lambda_1 = l\Lambda_2$ for some $l \in L^\times$.

Now we explain the term $\eta_{E/L}(u_2) |u_1 u_2|^s$. Let $\Lambda \subset F^{2h}$ be a lattice with $\tau_1(\zeta)\Lambda = \Lambda$, there is a direct sum decomposition of Λ corresponding to eigenspaces of $\tau_1(\zeta)$ by

$$\Lambda = \Lambda_+ \oplus \Lambda_-$$

where

$$\begin{aligned} \Lambda_+ &= \text{Im} \left(\Lambda \xrightarrow{\tau_1(\zeta)-a} \Lambda \right) = \ker \left(\Lambda \xrightarrow{\tau_1(\zeta)-b} \Lambda \right), \\ \Lambda_- &= \text{Im} \left(\Lambda \xrightarrow{\tau_1(\zeta)-b} \Lambda \right) = \ker \left(\Lambda \xrightarrow{\tau_1(\zeta)-a} \Lambda \right). \end{aligned}$$

The action of \mathbf{w} preserves components whence $\mathbf{w}(\Lambda_+) = \Lambda_+$ and $\mathbf{w}(\Lambda_-) = \Lambda_-$. We denote their induced maps by $\mathbf{w}_+ : \Lambda_+ \rightarrow \Lambda_+$, $\mathbf{w}_- : \Lambda_- \rightarrow \Lambda_-$. The action of \mathbf{z} interchanges components. In other words, $\mathbf{z}(\Lambda_+) = \Lambda_-$ and $\mathbf{z}(\Lambda_-) = \Lambda_+$. We denote their induced maps by $\mathbf{z}_- : \Lambda_+ \rightarrow \Lambda_-$, $\mathbf{z}_+ : \Lambda_- \rightarrow \Lambda_+$. Finally, for each representative Λ of a lattice class in $\mathcal{L}_{\tau_1, \tau_2}$, we can associate to it an order $R_\Lambda \subset L$ defined by

$$R_\Lambda = \{l \in L : l\Lambda \subset \Lambda\}.$$

This definition does not depend on the choice of representatives.

With above constructions, we can write orbital integrals in a combinatorial way as in the following theorem.

Theorem 7.1 Assume $\Lambda_{0-} = \mathbf{z}\Lambda_{0+}$, we have

$$(7.4) \quad \text{Orb}_{\tau_1, \tau_2}(\mathbb{1}_{\mathbf{G}_{2h}(\mathcal{O}_F)}, s) = \sum_{[\Lambda] \in \mathcal{L}_{\tau_1, \tau_2}} [\mathcal{O}_L^\times : R_\Lambda^\times] (-q^{ks})^{\text{length}(\Lambda_+/\mathbf{z}\Lambda_-)}.$$

with $k = 2$ if $i_{\tau_1, \tau_2} \in \mathcal{O}_L^\times$ or $k = 0$ if $e_{\tau_1, \tau_2} \in \mathcal{O}_L^\times$.

Proof Consider a subset $\mathcal{U}_{\tau_1, \tau_2} \subset \mathbf{G}_{2h}(F)/\mathbf{G}_{2h}(\mathcal{O}_F)$ defined by

$$\mathcal{U}_{\tau_1, \tau_2} = \{u \in \mathbf{G}_{2h}(F)/\mathbf{G}_{2h}(\mathcal{O}_F) : \tau_i(\zeta)u\Lambda_0 = u\Lambda_0 \text{ for } i = 1, 2\}.$$

Since $\tau_1(\zeta)u_1\Lambda_0 = u_1\Lambda_0$, and $\tau_2(\zeta)gu_2\Lambda_0 = gu_2\Lambda_0$ (because $g\tau_1(\zeta)g^{-1} = \tau_2(\zeta)$), we have

$$\mathbb{1}_{\mathbf{G}_{2h}(\mathcal{O}_F)}(u_1^{-1}gu_2) \neq 0 \implies gu_2\Lambda_0 = u_1\Lambda_0 \implies u_1 \in \mathcal{U}_{\tau_1, \tau_2}.$$

This implies that we can write the orbital integral into

$$\int_{\substack{L^\times \setminus \mathcal{U}_{\tau_1, \tau_2} \\ u_1\Lambda_0 = gu_2\Lambda_0}} |u_1 u_2|^s \eta_{K/F}(u_2) du_1 du_2 = \sum_{\substack{u_1 \in \mathcal{U}_{\tau_1, \tau_2} \\ u_1\Lambda_0 = gu_2\Lambda_0}} [\mathcal{O}_L^\times : \text{Stab}_{L^\times}(u)] |u_1 u_2|^s \eta_{K/F}(u_2).$$

Note that here we can omit $u_1\Lambda_0 = gu_2\Lambda_0$ because $x_0 u_1\Lambda_0 = u_1\Lambda_0$ implies there is an $u_2 \in \mathcal{U}_{\tau_1, \tau_2}$ with $u_1\Lambda_0 = gu_2\Lambda_0$. Furthermore, this u_2 is unique. Then, we can write the orbital integral directly into

$$(7.5) \quad \sum_{u_1 \in \mathcal{L}_{\tau_1, \tau_2}} [\mathcal{O}_L^\times : \text{Stab}_{L^\times}(u)] |u_1 u_2|^s \eta_{K/F}(u_2) \quad \text{with } gu_2\Lambda_0 = u_1\Lambda_0.$$

Let $\Lambda = u_1\Lambda_0$. Next we will prove

$$(7.6) \quad |u_1 u_2|^s \eta_{K/F}(u_2) = (-q^{ks})^{\text{length}(\Lambda_+/\mathbf{z}\Lambda_-)}$$

with $k = 2$ if $i_{\tau_1, \tau_2} \in \mathcal{O}_L^\times$ or $k = 0$ if $e_{\tau_1, \tau_2} \in \mathcal{O}_L^\times$. For $i = 1, 2$, denote $\Lambda_i = u_i \Lambda_0$. Decompose $\Lambda_i = \Lambda_{i+} \oplus \Lambda_{i-}$ by eigenspaces of $\tau_i(\zeta)$. Note that

$$(7.7) \quad |u_i| = |\det(u_{i+}^{-1} u_{i-})|_F = \frac{[\Lambda_{i-} : \Lambda_{0-}]}{[\Lambda_{i+} : \Lambda_{0+}]} = \frac{[\Lambda_{i-} : \mathbf{z}\Lambda_{0+}]}{[\mathbf{z}\Lambda_{i+} : \mathbf{z}\Lambda_{0+}]} = [\Lambda_{i-} : \mathbf{z}\Lambda_{i+}].$$

Since $\tau_2(\zeta), \tau_1(\zeta) \in \text{Mat}_{2h}(\mathcal{O}_F)$, we have $i_{\tau_1, \tau_2}, e_{\tau_1, \tau_2} \in \text{Mat}_{2h}(\mathcal{O}_F)$.

On one hand, If $\det(i_{\tau_1, \tau_2}) \in \mathcal{O}_F^\times$, we have $i_{\tau_1, \tau_2} \in \mathbf{G}_{2h}(\mathcal{O}_F)$. Since

$$i_{\tau_1, \tau_2} \tau_1(\zeta) = \tau_2(\zeta) i_{\tau_1, \tau_2},$$

we have $i_{\tau_1, \tau_2} \Lambda_{1+} = \Lambda_{2+}$ and $i_{\tau_1, \tau_2} \Lambda_{2+} = \Lambda_{1+}$. Therefore,

$$[\Lambda_{2-} : \mathbf{z}\Lambda_{2+}] = [i_{\tau_1, \tau_2} \Lambda_{2-} : i_{\tau_1, \tau_2} \mathbf{z}\Lambda_{2+}] = [\Lambda_{1-} : \mathbf{z}\Lambda_{1+}].$$

By (7.7), this implies $|u_1| = |u_2| = q^{\text{length}(\Lambda_- : \mathbf{z}\Lambda_+)}$. Furthermore, since

$$(7.8) \quad \eta_{K/F}(u_2) = (-1)^{\text{v}_F(\det(u_{2+}^{-1} u_{2-}))} = (-1)^{\text{length}(\Lambda_{2-} : \mathbf{z}\Lambda_{2+})},$$

we have $|u_1 u_2|^s \eta_{K/F}(u_2) = (-q^{2s})^{\text{length}(\Lambda_- : \mathbf{z}\Lambda_+)}$.

On the other hand, if $\det(e_{\tau_1, \tau_2}) \in \mathcal{O}_F^\times$, we have $e_{\tau_1, \tau_2} \in \mathbf{G}_{2h}(\mathcal{O}_F)$. Since

$$e_{\tau_1, \tau_2} \tau_1(\zeta) = \tau_2^\sigma(\zeta) e_{\tau_1, \tau_2},$$

we have $e_{\tau_1, \tau_2} \Lambda_{1+} = \Lambda_{2-}$ and $e_{\tau_1, \tau_2} \Lambda_{2-} = \Lambda_{1+}$. This implies $|u_2|^{-1} = |u_1| = q^{\text{length}(\Lambda_- : \mathbf{z}\Lambda_+)}$. Furthermore, by (7.8) again, we conclude that $|u_1 u_2|^s \eta_{K/F}(u_2) = (-1)^{\text{length}(\Lambda_- : \mathbf{z}\Lambda_+)}$.

Now after (7.6) has been proved, we could write the orbital integral as

$$\sum_{u \in \mathcal{L}_{\tau_1, \tau_2}} [\mathcal{O}_L^\times : \text{Stab}_{L^\times}(u)] (-q^{ks})^{\text{length}(\Lambda_+ / \mathbf{z}\Lambda_-)}.$$

Since $\text{Stab}_{L^\times}(u)$ is the stabilizer of u as an element of $\mathcal{U}_{\tau_1, \tau_2}$ under the action of L^\times , it is identified with the stabilizer of $\Lambda = u \Lambda_0$ under actions of L^\times . Then $\text{Stab}_{L^\times}(u) = R_\Lambda^\times$. This completes the proof of the theorem. ■

7.3 Lattices and fractional ideals

In this section, we simplify the orbital integral in (7.4) by parametrizing elements of the set $\mathcal{L}_{\tau_1, \tau_2}$ in more details. For each class $[\Lambda] \in \mathcal{L}_{\tau_1, \tau_2}$, each representative $\Lambda \subset F^{2h}$ is a lattice which is closed under actions of $\tau_1(\zeta)$ and $\tau_2(\zeta)$. Then, we can decompose $\Lambda = \Lambda_+ \oplus \Lambda_-$ according to eigenspaces of $\tau_1(\zeta)$. Each of the component Λ_+, Λ_- is a lattice in F^h . We denote their stabilizers in L as

$$R_{\Lambda_+} = \{l \in L : l \Lambda_+ \subset \Lambda_+\},$$

$$R_{\Lambda_-} = \{l \in L : l \Lambda_- \subset \Lambda_-\}.$$

It is clear that the definition of R_{Λ_+} and R_{Λ_-} only depends on the class $[\Lambda]$. We also have $R_\Lambda = R_{\Lambda_+} \cap R_{\Lambda_-}$.

Note that $\Lambda_+ \otimes_{\mathcal{O}_F} F \cong F^{2h}$. A vector $\mathbf{v} \in F^h$ gives rise to an isomorphism $F^h \cong L$, which identifies Λ_+ as a R_{Λ_+} -submodule of L . This is also called a R_{Λ_+} -fractional ideal in L in the literature. Two lattices Λ_{1+} and Λ_{2+} are related by $\Lambda_{2+} = l \Lambda_{1+}$ for an $l \in L$

if and only if $R_{\Lambda_{1+}} = R_{\Lambda_{2+}}$ and they correspond to $R_{\Lambda_{1+}}$ -fractional ideals in the same ideal class. Let \mathcal{C} be the set of all fractional-ideal-classes of L . Let $\mathcal{C}(R) \subset \mathcal{C}$ be the set of R -fractional-ideal-classes.

To describe elements in $\mathcal{L}_{\tau_1, \tau_2}$, we need more definitions. Let

$$S_{[\Lambda_+], [\Lambda_-]} = \{z_- : \Lambda_+ \rightarrow \Lambda_-\} / \cong,$$

where two maps $\Lambda_{1+} \rightarrow \Lambda_{1-}$ and $\Lambda_{2+} \rightarrow \Lambda_{2-}$ are defining the same element in $S_{[\Lambda_+], [\Lambda_-]}$ if and only if there are R_{Λ_+} and R_{Λ_-} -isomorphisms l_+ and l_- such that the following diagram commutes

$$\begin{array}{ccc} \Lambda_{1+} & \xrightarrow{z_{1-}} & \Lambda_{1-} \\ l_+ \downarrow & & \downarrow l_- \\ \Lambda_{2+} & \xrightarrow{z_{2-}} & \Lambda_{2-} \end{array}$$

We have the following description of $\mathcal{L}_{\tau_1, \tau_2}$.

Lemma 7.2 We have an isomorphism

$$\mathcal{L}_{\tau_1, \tau_2} \cong \{([\Lambda_+], [\Lambda_-], z_-) : [\Lambda_+], [\Lambda_-] \in \mathcal{C}, z_- \in S_{[\Lambda_+], [\Lambda_-]}, z_-^2 \Lambda_- \subset z_- \Lambda_+\}.$$

Proof For any $[\Lambda] \in \mathcal{L}_{\tau_1, \tau_2}$, we have $z_- (\Lambda_+) \subset \Lambda_-$. Since we also have $z_+ (\Lambda_-) \subset \Lambda_+$, we have $z_-^2 \Lambda_- = z_- z_+ (\Lambda_-) \subset z_- \Lambda_+$. Therefore, we could have a map

$$\mathcal{L}_{\tau_1, \tau_2} \rightarrow \{([\Lambda_+], [\Lambda_-], z_-) : [\Lambda_+], [\Lambda_-] \in \mathcal{C}, z_- \in S_{[\Lambda_+], [\Lambda_-]}, z_-^2 \Lambda_- \subset z_- \Lambda_+\}.$$

We need to show this map is well defined and that this map is an isomorphism. Let Λ_1, Λ_2 be lattices with $[\Lambda_1] = [\Lambda_2]$. Then, there exists $l \in L$ with $\Lambda_2 = l\Lambda_1$. This implies $\Lambda_{1+} = l\Lambda_{2+}$ and $\Lambda_{1-} = l\Lambda_{2-}$. Then we have $[\Lambda_{1+}] = [\Lambda_{2+}]$ and $[\Lambda_{1-}] = [\Lambda_{2-}]$. Furthermore, since l commutes with z_{1-} , and z_{2-} , we see we defined the same element in $S_{[\Lambda_+], [\Lambda_-]}$. This map is well defined.

On the other hand, we can construct the inverse map and prove the inverse map is well defined. For any $([\Lambda_+], [\Lambda_-], z_-)$, we define an element $[\Lambda] \in \mathcal{L}_{\tau_1, \tau_2}$ by the following steps. First, we embed Λ_+, Λ_- into F^{2h} such that the homomorphism $z_- : \Lambda_+ \rightarrow \Lambda_-$ and $z_+ : \Lambda_- \rightarrow \Lambda_+$ is induced by z . Then, we define $\Lambda = \Lambda_+ \oplus \Lambda_-$.

To prove this map is well-defined, we need to show that after the embedding, if $\Lambda_{2+} = l_+ \Lambda_{1+}$, $\Lambda_{2-} = l_- \Lambda_{1-}$ and $z \circ l_+ = l_- \circ z$, then $\Lambda_2 = l\Lambda_1$ for some $l \in L$. Indeed, let $l = l_+ \oplus l_-$, we have l commutes with z . Since l preserves the eigenspaces of $\tau_1(\zeta)$, then l commutes with $\tau_1(\zeta)$. Similarly l commutes with $\tau_2(\zeta)$. This implies $l \in L$. ■

By this lemma and (7.4), we can write the orbital integral as

$$\begin{aligned} \text{Orb}_{\tau_1, \tau_2}(\mathbb{1}_{G_{2h}(\mathcal{O}_F)}, s) &= \sum_{\substack{R_+ \ni \mathfrak{w} \\ R_- \ni \mathfrak{w}}} \sum_{\substack{\Lambda_+ \in C(R_+) \\ \Lambda_- \in C(R_-)}} \sum_{\substack{z_- \in S_{[\Lambda_+], [\Lambda_-]} \\ z_- \Lambda_+ \supset z_-^2 \Lambda_-}} \\ (7.9) \quad &\times [\mathcal{O}_L^\times : (R_+ \cap R_-)^\times] (-q^{ks})^{\text{length}(\Lambda_- / z_- \Lambda_+)}. \end{aligned}$$

7.4 Double structures for division algebra

Before we calculate the orbital integral, we study the pair (τ_1, τ_2) in more details. By our assumption, it matches to a pair in the division algebra D over F given by

$$\begin{aligned} \varphi_1 &: \mathcal{O}_{K_1} \longrightarrow \mathcal{O}_D \\ \varphi_2 &: \mathcal{O}_{K_2} \longrightarrow \mathcal{O}_D. \end{aligned}$$

By our construction,

$$\begin{aligned} \mathbf{z}_{\varphi_1, \varphi_2} &= e_{\varphi_1, \varphi_2} \circ i_{\varphi_1, \varphi_2} \\ \mathbf{w}_{\varphi_1, \varphi_2} &= i_{\varphi_1, \varphi_2}^2. \end{aligned}$$

In this section, we study the properties of those $\mathbf{z} = \mathbf{z}_{\varphi_1, \varphi_2}$ and $\mathbf{w} = \mathbf{w}_{\varphi_1, \varphi_2}$ that arises from double structures of division algebras.

Lemma 7.3 *Let D be a quaternion algebra over L and $\gamma \in D$. Let $\zeta \in \mathcal{O}_D^\times$ be a trace zero element and $\zeta\gamma = \gamma\zeta^\sigma$. Then $\mathbf{v}_D(\gamma)$ is an odd integer.*

Proof Let ω be the uniformizer of \mathcal{O}_D . Since \mathcal{O}_D/ω is a finite division algebra, it is a field. Therefore, the reduction $\bar{\gamma}$ modulo ω must commute with all elements in \mathcal{O}_D/ω , this implies

$$\gamma\zeta - \zeta\gamma \in \omega\mathcal{O}_D$$

Since $\gamma\zeta^\sigma = \zeta\gamma$, this implies $\gamma \notin \mathcal{O}_D^\times$. Let π be an uniformizer in \mathcal{O}_L . Since D is a quaternion algebra, we have $\mathbf{v}_D(\pi) = 2$. Suppose $\mathbf{v}_D(\gamma)$ is even. Hence $\mathbf{v}_D(\gamma) = \mathbf{v}_D(\pi^m)$ for some m . Then, we have $\pi^{-m}\gamma \in \mathcal{O}_D^\times$ and $\pi^{-m}\gamma\zeta = \zeta^\sigma\pi^{-m}\gamma$, contradiction. ■

From now to the rest of the paper, we use γ to represent the fixed double structure (ϕ_1, ϕ_2) , and use $\gamma_\#$ to represent $e_{\phi_1, \phi_2} \circ i_{\phi_1, \phi_2}^{-1}$.

Corollary 7.4 *Let $\gamma_\# = e_{\varphi_1, \varphi_2} \circ i_{\varphi_1, \varphi_2}^{-1} = (\varphi_1(\zeta) - \varphi_2(\zeta))(\varphi_1(\zeta) + \varphi_2(\zeta))^{-1}$ be the polar stereographic coordinate of the double structure arises in a division algebra. Then, $\mathbf{v}_L(\gamma_\#^2)$ is an odd integer.*

Proof It is clear that $\gamma_\# \in D_L$. Since D_L is a quaterenion algebra over L , we have

$$\mathbf{v}_L(\gamma_\#^2) = \mathbf{v}_{D_L}(\gamma_\#).$$

Since $\varphi_i(\zeta) \in \mathcal{O}_{D_L}^\times$ and $\gamma_\#\varphi_i(\zeta) = \varphi_i^\sigma(\zeta)\gamma_\#$, by Lemma 7.3, the number $\mathbf{v}_{D_L}(\gamma_\#)$ must be an odd integer. ■

This corollary implies that for any orbit considered in the linear AFL, the number $\mathbf{v}_F(P_\gamma(1))$ is always an odd integer.

7.5 Calculation of orbital integral for $h = 2$

We call a lattice a principal lattice if it corresponds to a principal fractional ideal. In the case of $h = 2$, all lattices $\Lambda \subset F^{2h}$ that are closed under $\tau_2(\zeta)$ and $\tau_1(\zeta)$ have principal lattices Λ_+ and Λ_- as its components. For general h , given an \mathcal{O}_F -order $R \subset L$, the R -

fractional ideal class group is not necessarily trivial. The following proposition is well known.

Proposition 7.5 *Let L/F be a quadratic extension of non-Archimedean fields. Then, any \mathcal{O}_F -lattice $\Lambda \subset L$ is a principal R -fractional ideal in L with R of the form*

$$R = \mathcal{O}_F + \pi^m \mathcal{O}_L$$

for some positive integer m . We denote such a ring as R_m .

In the case of $h = 2$, Proposition 7.5 implies both Λ_+ and Λ_- are principal lattices; we may furthermore write the orbital integral as

$$(7.10) \quad \text{Orb}_{\tau_1, \tau_2}(\mathbb{1}_{\mathbf{G}_{2h}(\mathcal{O}_F)}, s) = \sum_{\substack{R_+ \ni \mathbf{w} \\ R_- \ni \mathbf{w}}} \sum_{\substack{\mathbf{z} \in S_{[R_+], [R_-]} \\ \mathbf{z} \cdot R_+ \supset \mathbf{z}^2 R_-}} [\mathcal{O}_L^\times : (R_+ \cap R_-)^\times] (-q^{ks})^{\text{length}(R_-/\mathbf{z} \cdot R_+)}.$$

Here $[R_+]$ and $[R_-]$ are classes of principal R_+ and R_- lattices.

7.5.1 Calculation of orbital integral

Using the same notation as previous sections, we have $r = \mathbf{v}_F(\gamma_\#^2)$. Then $2r = \mathbf{v}_L(\gamma_\#^2) = \mathbf{v}_{D_L}(\gamma_\#)$. If $i_{\varphi_1, \varphi_2} \notin \mathbf{G}_{2h}(\mathcal{O}_F)$, by Theorem 7.1, we know the orbital integral does not depend on s . Then the derivative is zero. In this section, we only consider the non-trivial case where $i_{\varphi_1, \varphi_2} \in \mathbf{G}_{2h}(\mathcal{O}_F)$. Since, we have $\gamma_\# \varphi_i(\zeta) = \varphi_i^\sigma(\zeta) \gamma_\#$, by Lemma 7.3, we know $2r$ is an odd integer. Since $\gamma_\# = e_{\varphi_1, \varphi_2} \circ i_{\varphi_1, \varphi_2}^{-1}$, $\mathbf{v}_F(i_{\varphi_1, \varphi_2}) = 0$ and $e_{\varphi_1, \varphi_2} \circ i_{\varphi_1, \varphi_2} = \mathbf{z}$, we have

$$\mathbf{v}_F(\mathbf{z}^2) = \mathbf{v}_F(\gamma_\#^2) = 2r.$$

This corresponds to $k = 2$ case in Theorem 7.1. To make our calculations clear we write the orbital integral in the following way

$$(7.11) \quad \text{Orb}_{\tau_1, \tau_2}(\mathbb{1}_{\mathbf{G}_{2h}(\mathcal{O}_F)}, s) = \sum_{\substack{R_+ \ni \mathbf{w} \\ R_- \ni \mathbf{w}}} [\mathcal{O}_L^\times : (R_+ \cap R_-)^\times] I_r(R_+, R_-, s).$$

with

$$(7.12) \quad I_r(R_+, R_-, s) = \sum_{\substack{\mathbf{z} \in S_{[R_+], [R_-]} \\ \Lambda \supset \mathbf{z}^2 R_-}} (-q^{2s})^{\text{length}(R_-/\mathbf{z} \cdot R_+)}.$$

Lemma 7.6 *We have a canonical isomorphism*

$$S_{[R_+], [R_-]} \cong R_- / (R_+ \cup R_-)^\times$$

Proof $S_{[R_+], [R_-]}$ is the set of maps $f : R_+ \rightarrow R_-$, where two such maps f_1, f_2 are equivalent if and only if $g_- \circ f_1 \circ g_+ = f_2$ for some R_+ and R_- -isomorphisms g_+ and g_- . Since all maps are obtained by multiplying by an element, the map f is uniquely determined by $f(1)$. Suppose f_1 and f_2 are equivalent, we have $g_-(1)f_1(1)g_+(1) = f_2(1)$. Therefore, we have $S_{[R_+], [R_-]} \cong R_- / (R_+ \cup R_-)^\times$. ■

To study the structure of $S_{[R_+],[R_-]}$, we consider a valuation map

$$S_{[R_+],[R_-]} \cong R_- / (R_+ \cup R_-)^\times \longrightarrow \mathcal{O}_L / \mathcal{O}_L^\times \cong \mathbb{Z}_{\geq 0}.$$

Let $S_{R_+,R_-}^{\text{even}}, S_{R_+,R_-}^{\text{odd}}$ be the preimage of even and odd integers under this map respectively. In other words,

$$\begin{aligned} S_{R_+,R_-}^{\text{even}} &= \{[\mathbf{z}_-] \in S_{[R_+],[R_-]} : v_L(\mathbf{z}_-(1)) \text{ is even}\}, \\ S_{R_+,R_-}^{\text{odd}} &= \{[\mathbf{z}_-] \in S_{[R_+],[R_-]} : v_L(\mathbf{z}_-(1)) \text{ is odd}\}. \end{aligned}$$

This gives us a partition

$$S_{[R_+],[R_-]} = S_{R_+,R_-}^{\text{even}} \amalg S_{R_+,R_-}^{\text{odd}}.$$

Let

$$\begin{aligned} C^{\text{even}} &= \{(R_+, R_-, \mathbf{z}_-) : \mathbf{z}_- \in S_{R_+,R_-}^{\text{even}}\}, \\ C^{\text{odd}} &= \{(R_+, R_-, \mathbf{z}_-) : \mathbf{z}_- \in S_{R_+,R_-}^{\text{odd}}\}, \end{aligned}$$

and

$$\begin{aligned} C_{\mathbf{z}^2}^{\text{even}} &= \{(R_+, R_-, \mathbf{z}_-) \in C^{\text{even}} : \mathbf{z}^2 R_+ \subset \mathbf{z}_- R_-\}, \\ C_{\mathbf{z}^2}^{\text{odd}} &= \{(R_+, R_-, \mathbf{z}_-) \in C^{\text{odd}} : \mathbf{z}^2 R_+ \subset \mathbf{z}_- R_-\}. \end{aligned}$$

Then, we can write

$$\text{Orb}_{\tau_1, \tau_2}(\mathbb{1}_{\mathbf{G}_{2h}(\mathcal{O}_F)}, s) = \sum_{(R_+, R_-, \mathbf{z}_-) \in C_{\mathbf{z}^2}^{\text{even}} \amalg C_{\mathbf{z}^2}^{\text{odd}}} [\mathcal{O}_L^\times : (R_+ \cap R_-)^\times] (-q^{2s})^{\text{length}(R_- / \mathbf{z}_- R_+)}.$$

Since $v_L(\mathbf{z}^2)$ is odd, and any $(R_+, R_-, \mathbf{z}_-) \in C_{\mathbf{z}^2}^{\text{even}}$ satisfies $\mathbf{z}^2 R_+ \subset \mathbf{z}_- R_-$, there is an isomorphism

$$\begin{aligned} C_{\mathbf{z}^2}^{\text{even}} &\longrightarrow C_{\mathbf{z}^2}^{\text{odd}} \\ \mathbf{z}_- &\longmapsto \mathbf{z}_+ \end{aligned}$$

where the map $\mathbf{z}_+ : R_- \longrightarrow R_+$ is defined by the following commutative diagram

$$\begin{array}{ccc} & R_- & \\ \mathbf{z}_- \nearrow & & \searrow \mathbf{z}_+ \\ R_+ & \xrightarrow{\mathbf{z}^2} & R_+ \end{array}$$

Furthermore, we noticed that

$$\text{length}(R_- / \mathbf{z}_- R_+) + \text{length}(R_+ / \mathbf{z}_+ R_-) = \text{length}(R_+ / \mathbf{z}^2 R_+) = 2r.$$

Therefore, we have

$$\begin{aligned} &\sum_{(R_+, R_-, \mathbf{z}_+) \in C_{\mathbf{z}^2}^{\text{odd}}} [\mathcal{O}_L^\times : (R_+ \cap R_-)^\times] (-q^{2s})^{\text{length}(R_+ / \mathbf{z}_+ R_-)} \\ &= \sum_{(R_+, R_-, \mathbf{z}_-) \in C_{\mathbf{z}^2}^{\text{even}}} [\mathcal{O}_L^\times : (R_+ \cap R_-)^\times] (-q^{2s})^{2r - \text{length}(R_- / \mathbf{z}_- R_+)}. \end{aligned}$$

Using this expression, we can reduce the orbital integral to the following form

$$\text{Orb}_{\tau_1, \tau_2}(\mathbb{1}_{\mathfrak{G}_{2h}(\mathcal{O}_F)}, s) = \sum_{(R_+, R_-, \mathbf{z}_-) \in C_{\mathbf{z}^2}^{\text{even}}} [\mathcal{O}_L^\times : (R_+ \cap R_-)^\times] \times \left((-q^{2s})^{2r - \text{length}(R_- / \mathbf{z}_- R_+)} + (-q^{2s})^{\text{length}(R_- / \mathbf{z}_- R_+)} \right).$$

We can decompose the set C^{even} by the following

Lemma 7.7 *Let*

$$C_{a,b}^{(c)} = \{ (R_+, R_-, \mathbf{z}_-) \in C^{\text{even}} : R_+ = \mathcal{O}_F + \pi^a \mathcal{O}_L, R_- = \mathcal{O}_F + \pi^b \mathcal{O}_L, \mathbf{v}_L(\mathbf{z}_-(1)) = 2(b - c) \}.$$

We have

$$C_{\mathbf{z}^2}^{\text{even}} = \coprod_{\substack{a \geq c; b \geq c \\ a+b-c \leq \frac{2r-1}{2}}} C_{a,b}^{(c)}.$$

Proof We have $C^{\text{even}} = \coprod_{a,b,c \in \mathbb{Z}_{\geq 0}} C_{a,b}^{(c)}$. To prove the lemma, we only need to determine the value of a, b and c so that we have some $x \in R_-$ satisfies $\mathbf{v}_L(x) = 2(b - c)$ and

$$(7.13) \quad \mathcal{O}_F + \pi^b \mathcal{O}_L \supset x(\mathcal{O}_F + \pi^a \mathcal{O}_L) \supset \mathbf{z}^2(\mathcal{O}_F + \pi^b \mathcal{O}_L).$$

Now, we will prove the lemma by considering the inclusion $\mathcal{O}_F + \pi^b \mathcal{O}_L \supset x(\mathcal{O}_F + \pi^a \mathcal{O}_L)$ and $x(\mathcal{O}_F + \pi^a \mathcal{O}_L) \supset \mathbf{z}^2(\mathcal{O}_F + \pi^b \mathcal{O}_L)$ one by one.

Firstly, by considering all possible elements with odd valuations of each subset, it is clear that $2b \leq 2a + (2b - 2c)$ is equivalent to the existence of an element $x \in R_-$ with $\mathbf{v}_L(x) = 2(b - c)$ and

$$\mathcal{O}_F + \pi^b \mathcal{O}_L \supset x(\mathcal{O}_F + \pi^a \mathcal{O}_L).$$

Secondly, since $\mathbf{v}_L(\mathbf{z}^2) = 2r$ is odd, we have $x(\mathcal{O}_F + \pi^a \mathcal{O}_L) \subset \mathbf{z}^2(\mathcal{O}_F + \pi^b \mathcal{O}_L)$ if and only if $(2b - 2c) + 2a \leq 2r - 1$. This implies the relation (7.13) holds only for values $a, b, c \in \mathbb{Z}_{\geq 0}$ with $a \geq c, b \geq c, a + b - c \leq \frac{2r-1}{2}$. ■

Lemma 7.8 *We have*

$$\sum_{(R_+, R_-, \mathbf{z}_-) \in C_{a,b}^{(c)}} [\mathcal{O}_L^\times : (R_+ \cap R_-)^\times] = q^{a+b-\max\{0,c\}}.$$

Proof We have $[\mathcal{O}_L^\times : (R_+ \cap R_-)^\times] = q^{\max\{a,b\}}$ for all element in $C_{a,b}^{(c)}$. Furthermore, we note that there is an isomorphism

$$\begin{aligned} C_{a,b}^{(c)} &\longrightarrow R_-^\times \setminus (\pi^{a-c} \mathcal{O}_L^\times \cap R_-) / R_+^\times \\ ([R_+], [R_-], \mathbf{z}_-) &\longmapsto \mathbf{z}_-(1) \end{aligned}$$

where the representative map is $\mathbf{z}_- : R_+ \longrightarrow R_-$. Since the action of R_+ and R_- commute, we have either $R_+ \supset R_-$ or $R_+ \subset R_-$. Hence,

$$C_{a,b}^{(c)} \cong (\pi^{a-c} \mathcal{O}_L^\times \cap R_-) / R_+ \cup R_-$$

The cardinality of $C_{a,b}^{(c)}$ is the volume of the above set. Then,

$$\#C_{a,b}^{(c)} = \frac{[\mathcal{O}_L^\times : (R_+ \cup R_-)^\times]}{[\pi^{a-c}\mathcal{O}_L^\times : \pi^{a-c}\mathcal{O}_L^\times \cap R_-]} = \frac{q^{\min\{a,b\}}}{q^{\min\{b-(b-c),0\}}}.$$

This completes the proof of the lemma. ■

Besides, it is clear that for any $(R_+, R_-, \mathbf{z}_-) \in C_{a,b}^{(c)}$, we have $\text{length}(R_-/\mathbf{z}_-R_+) = a + b - 2c$. Using Lemmas 7.8–7.6, we can write the orbital integral $\text{Orb}_{\tau_1, \tau_2}(\mathbb{1}_{\mathbf{G}_{2h}(\mathcal{O}_F)}, s)$ into the following form

$$\sum_{l=0}^{\frac{2r-1}{2}} q^l \left(\sum_{c=-\frac{2r-1}{2}+l}^{-1} \sum_{\substack{a+b=l \\ a,b \geq 0}} ((-q^{2s})^{l-2c} + (-q^{2s})^{2r-l+2c}) + \sum_{c=0}^l \sum_{\substack{a+b=l+c \\ a,b \geq c}} ((-q^{2s})^{l-c} + (-q^{2s})^{2r-l+c}) \right).$$

Now by taking the derivative at $s = 0$, we can write

$$\begin{aligned} & \left. \frac{d}{ds} \right|_{s=0} \text{Orb}_{\tau_1, \tau_2}(\mathbb{1}_{\mathbf{G}_{2h}(\mathcal{O}_F)}, s) \\ &= \sum_{l=0}^{\frac{2r-1}{2}} q^l \left(\sum_{c=-\frac{2r-1}{2}+l}^{-1} \sum_{\substack{a+b=l \\ a,b \geq 0}} (-1)^l (2(l-2c) - 2r) + \sum_{c=0}^l \sum_{\substack{a+b=l+c \\ a,b \geq c}} (-1)^{l-c} (2(l-c) - 2r) \right) \\ &= \sum_{l=0}^{\frac{2r-1}{2}} q^l \left(\sum_{c=-\frac{2r-1}{2}+l}^{-1} (-1)^l (l+1)(2(l-2c) - 2r) + \sum_{c=0}^l (-1)^{l-c} (l-c+1)(2(l-c) - 2r) \right) \\ &= \sum_{l=0}^{\frac{2r-1}{2}} q^l \left((-1)^l (l+1) \left(\frac{2r-1}{2} - l \right) + \left((-1)^l \left(l^2 + 2l + \frac{1}{2} - rl - \frac{3r}{2} \right) - \left(\frac{1}{2} + \frac{r}{2} \right) \right) \right) \\ &= \sum_{l=0}^{\frac{2r-1}{2}} \left((-1)^l \left(\frac{l}{2} - \frac{r}{2} \right) - \left(\frac{1}{2} + \frac{r}{2} \right) \right) q^l. \end{aligned}$$

Use $N'(r)$ to denote the value for the above expression. We have

$$\begin{aligned} N'(r+2) - N'(r) &= - \sum_{l=0}^{\frac{2r-1}{2}} \left((-1)^l + 1 \right) q^l + \left((-1)^{r+\frac{1}{2}} \left(-\frac{3}{4} \right) - \left(\frac{1}{2} + \frac{r+2}{2} \right) \right) q^{r+\frac{1}{2}} \\ &\quad + \left((-1)^{r+\frac{3}{2}} \left(-\frac{1}{4} \right) - \left(\frac{1}{2} + \frac{r+2}{2} \right) \right) q^{r+\frac{3}{2}}. \end{aligned}$$

If $2r \equiv 1 \pmod{4}$, $r + \frac{1}{2}$ odd, and we have

$$\begin{aligned} N'(r+2) - N'(r) &= -2 \sum_{l=0}^{\frac{r-1}{4}} q^{2l} - \left(\frac{r}{2} + \frac{3}{4} \right) q^{r+\frac{1}{2}} - \left(\frac{r}{2} + \frac{7}{4} \right) q^{r+\frac{3}{2}} \\ &= - \left(\frac{1 - q^{r+\frac{3}{2}}}{1 - q^2} + \left(\frac{r}{2} + \frac{7}{4} \right) q^{r+\frac{3}{2}} + \left(\frac{r}{2} + \frac{3}{4} \right) q^{r+\frac{1}{2}} \right). \end{aligned}$$

And if $2r \equiv 3 \pmod 4$, $r + \frac{1}{2}$ even, we have

$$\begin{aligned} N'(r+2) - N'(r) &= -2 \sum_{l=0}^{\frac{r-3}{4}} q^{2l} - \left(\frac{r}{2} + \frac{9}{4}\right) q^{r+\frac{1}{2}} - \left(\frac{r}{2} + \frac{5}{4}\right) q^{r+\frac{3}{2}} \\ &= -\left(2 \frac{1 - q^{r+\frac{1}{2}}}{1 - q^2} + \left(\frac{r}{2} + \frac{9}{4}\right) q^{r+\frac{1}{2}} + \left(\frac{r}{2} + \frac{5}{4}\right) q^{r+\frac{3}{2}}\right). \end{aligned}$$

It also easy to verify $N(\frac{1}{2}) = 1 = -N'(\frac{1}{2})$ and $N(\frac{3}{2}) = q + 2 = -N'(\frac{3}{2})$. This proves

$$N(r) = N'(r).$$

We finished the proof of the linear AFL of the case $h = 2$ for the identity test function.

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