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ON VALUATION AND RISK MANAGEMENT AT THE INTERFACE OF INSURANCE AND FINANCE

By T. Møller

ABSTRACT

This paper reviews methods for hedging and valuation of insurance claims with an inherent financial risk, with special emphasis on quadratic hedging approaches and indifference pricing principles and their applications in insurance. It thus addresses aspects of the interplay between finance and insurance, an area which has gained considerable attention during the past years, in practice as well as in theory. Products combining insurance risk and financial risk have gained considerable market shares. Special attention is paid to unit-linked life insurance contracts, and it is demonstrated how these contracts can be valued and hedged by using traditional methods as well as more recent methods from incomplete financial markets such as risk-minimisation, mean-variance hedging, super-replication and indifference pricing with mean-variance utility functions.

KEYWORDS

Actuarial Valuation Principles; Financial Risk; Hedging; Incomplete Market; Indifference Pricing; Unit-Linked Contracts; Financial Stop-Loss Contracts

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1. Introduction

During the past years, new insurance products that combine elements of insurance risk and financial risk have appeared; examples are unit-linked life insurance contracts, catastrophe insurance futures and bonds, and integrated risk-management solutions. This paper describes some of these new products in detail, and discusses how they can be valued and hedged. This discussion includes a review of some recent theoretical results from the interface of insurance and finance.

Focus will be on specific developments involving methods for hedging and valuation of risk in incomplete financial markets, and the aim is not to give a complete overview of the area. Our aim is to demonstrate how the combined insurance and financial risk inherent in many insurance and

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reinsurance liabilities can be viewed and handled as general contingent claims, which cannot be hedged perfectly by trading in traditional financial assets. Therefore, these insurance liabilities cannot be priced by no-arbitrage argument alone, and this leaves some intrinsic risk to the insurance company, which must choose some subjective criterion for valuation (pricing) and hedging (risk management) of its liabilities.

We review several possible approaches to hedging and valuation in incomplete markets, including super-hedging, risk-minimisation, mean-variance hedging and utility indifference pricing under mean-variance utility functions. Each criterion can be viewed as one possible 'approach to risk', and leads to a description of how this risk may be measured and controlled. We discuss the advantages and disadvantages of the various approaches in general and for specific applications. As a continuing example, the paper investigates how the risk in a portfolio of unit-linked life insurance contracts may be analysed by applying each of the mentioned methods. With a unit-linked life insurance contract, benefits are linked to the development of a stock index or a specific fund. This analysis leads to new insights into the nature of the combined risk of these contracts. The results obtained are compared with what we could call an actuarial approach, proposed by Brennan & Schwartz (1979a,b), that combines traditional law of large number considerations and financial mathematics. Some of the approaches from incomplete markets actually lead to prices that coincide with the ones determined by the principle suggested by Brennan & Schwartz (1979a,b), whereas other principles will lead to alternative prices. We give some explanations of this phenomenon.

The notion of *risk* is used in several different contexts in both the actuarial and the financial literature; often it is simply used vaguely, describing the fact that there is some uncertainty, for example in *mortality risk* known from insurance and *credit risk* known from finance. However, the notion also appears in various more specific concepts. Examples are *insurance risk process*, which is typically defined as the accumulated premiums minus claims in an insurance portfolio, and *risk-minimisation*, which is a theory from mathematical finance that can be used for determining hedging strategies.

1.1 *Insurance Background*

The two fields of insurance and finance started as separate areas. At its very origin, the theory of insurance was mainly concerned with the computation of premiums for life insurance contracts. An overview of the early history of life insurance can be found in Braun (1937), and, according to this exposition, the first known social welfare programmes with elements of life insurance are the Roman *Collegia*, which date back at least to AD 133. The first primitive mortality tables were published in 1662 by John Graunt (1620-1674), who worked with only seven different age groups. The first mortality table, where the expected number of survivors from year to year is given, is due to the astronomer Edmund Halley (1656-1742). These tables allowed for more precise predictions about portfolios of independent lives, and were essential for the computation of premiums for various life insurance contracts. In his book on the evaluation of annuities on life from 1725, Abraham de Moivre (1667-1754) suggested methods for the evaluation of life insurance contracts, combining interest and mortality under very simple assumptions about the mortality.

In 1738, Daniel Bernoulli (1700-1782) argued that risks, i.e. uncertain payoffs, should not be measured by their expectations, and hence laid the foundation for modern utility theory. Using examples related to gambling, he explained that the preferences of an individual may depend on his economic situation, and, more specifically, that in some situations it could be reasonable for a poorer individual to prefer one uncertain future payment to another (more) uncertain payment with a larger expected value, whereas a wealthier person would prefer the payment with the largest expected value. This observation was also of importance for insurance in general, since it explained, for example, why individuals may accept to buy insurance contracts at a price which exceeds the expected value of the payment from the contract.

1.2 *Financial Background*

Bachelier (1900) proposed to describe fluctuations in the price of a stock by a Brownian motion, by assuming that the change in the value of the stock in a time interval of length h was normally distributed with mean αh and variance $\sigma^2 h$ and that changes in non-overlapping intervals were stochastically independent. Samuelson (1965) advocated a framework where the stock price was modelled by a geometric Brownian motion, i.e. the exponential function of a Brownian motion, which had the advantage that it did not generate negative prices. Within this framework, and assuming in addition that money could be deposited in a savings account, Black & Scholes (1973) and Merton (1973) introduced the idea that options on stocks should really be priced such that no sure profits could arise from composing portfolios of long and short positions in the underlying stock and in the option itself. Assuming that the option price was a function of time and the current value of the stock, they obtained the celebrated Black-Scholes formula for European call options. This pricing formula has the (at first glance) surprising feature that it does not involve the expected return of the underlying stock. Cox, Ross & Rubinstein (1979) investigated a simple discrete time model, where the change in the value of the stock between two trading times can attain two different values only. In that setting, they derived option prices and obtained the pricing formulae of Black, Scholes and Merton as limiting cases, by letting the length of the time intervals between trading times tend to 0. Building on concepts and ideas in Harrison & Kreps

(1979) for discrete time models, Harrison & Pliska (1981) gave a mathematical theory for the pricing of options under continuous trading, and clarified the role of martingale theory in the pricing of options and its connection to key concepts such as absence of arbitrage and completeness.

1.3 *Interplay between Insurance and Finance*

The emergence of products combining financial and insurance risk (e.g. so-called unit-linked insurance contracts, various catastrophe futures and options and financial stop-loss reinsurance contracts) has forced the two fields of insurance and finance to search for combinations and unification of methodologies and basic principles. A survey of aspects of the growing interplay between the two fields is given in Embrechts (2000), who mentions institutional issues such as the increasing collaboration between insurance companies and banks (e.g. the construction of so-called 'financial supermarkets') and the deregulation of insurance markets, as two further important aspects.

The present paper is organised as follows. Section 2 gives an overview of valuation techniques in life and non-life insurance, and Section 3 introduces the main concepts related to financial valuation principles. In Section 4 some specific examples of interplay between the two fields of finance and insurance are mentioned. Section 5.1 studies applications in insurance of various hedging criteria, including risk-minimisation, mean-variance hedging and super-replication. Section 5.2 reviews results on indifference pricing with mean-variance utility functions of insurance contracts, and presents some new results on actuarial premium calculation principles adapted to financial models. Finally, Section 5.3 gives indifference prices for a portfolio of unit-linked life insurance contracts, and compares these results analytically and numerically with the prices obtained using other methods.

2. Classical Valuation of Insurance Contracts

Traditionally, actuarial theory is divided into life insurance mathematics and non-life insurance mathematics. In addition to historical aspects, there are fundamental differences between the two areas, for example in respect of the time horizon of the individual contract (for life insurance extending up to 50 years, whereas for non-life insurance typically limited to one year). These are reflected e.g. in the principles that are applied for the calculation of premiums. In this section, we review some notions and key concepts of life and non-life insurance, placing focus on the valuation techniques used there.

2.1 *Life Insurance*

We recall some classical and basic concepts from life insurance; some recent introductory expositions to the area are Gerber (1986) and Norberg (2000).

Consider a portfolio of n lives aged y, say, to be insured at time 0 with i.i.d. remaining life times $T_1, \ldots, T_n$, and assume that there exists a continuous function (called the hazard rate function) μ_{y+t} such that the survival probability is of the form ${}_tp_y = P(T_1 > t) = \exp(-\int_0^t \mu_{y+u}\,du)$. A *pure endowment* contract with sum insured K and term T stipulates that the amount K (the insurance benefit) is payable at time T contingent on survival of the policyholder. Assume that the contract is paid by a single premium κ, say, at time 0. Assume, furthermore, that the seller of the contract (the insurance company) invests the premium κ in some asset which pays a rate of return $r = (r_t)_{0 \le t \le T}$ during $[0, T]$. For the ith policyholder, the obligation of the insurance company is now given by the *present value*:

$$H_i = 1_{\{T_i > T\}} K e^{-\int_0^T r_t\,dt} \tag{2.1}$$

which is obtained by discounting the amount payable at T, $1_{\{T_i > T\}}K$, using the rate of return r. Note that (2.1) is a random variable. The fundamental *principle of equivalence* now states that the premiums should be chosen such that the present values of premiums and benefits balance, on average. If we assume, in addition, that r is stochastically independent of the remaining life times, the principle of equivalence states that:

$$\kappa = \mathrm{E}[H_i] = {}_Tp_y K \mathrm{E}[e^{-\int_0^T r_t\,dt}] \tag{2.2}$$

for the single premium case. Since life insurance portfolios are often very large, this principle can be partly justified by using the law of large numbers. Indeed, as the size n of the portfolio is increased, the relative number of survivors $\frac{1}{n}\sum_{i=1}^n 1_{\{T_i > T\}}$ converges a.s. towards the probability ${}_Tp_y$ of survival to T by the strong law of large numbers, since the lifetimes $T_1, \ldots, T_n$ are stochastically independent. Thus, for n sufficiently large, the actual number of survivors $\sum_{i=1}^n 1_{\{T_i > T\}}$ will be 'approximately' equal to the expected number $n\,{}_Tp_y$. Accumulating the premiums $n\kappa$ with interest now leads to:

$$n\kappa e^{\int_0^T r_t\,dt} = n\,{}_Tp_y K \mathrm{E}\left[e^{-\int_0^T r_t\,dt}\right] e^{\int_0^T r_t\,dt} \approx \sum_{i=1}^n 1_{\{T_i > T\}} K \mathrm{E}\left[e^{-\int_0^T r_t\,dt}\right] e^{\int_0^T r_t\,dt}. \tag{2.3}$$

In particular, when r is non-random, the expression on the right is equal to the amount to be paid to the policyholders. So, in the case of a deterministic rate of return, the principle of equivalence is justified directly by use of the law of large numbers, which essentially guarantees that the actual number of survivors is 'close' to the expected number.

The problem becomes much more delicate in the more realistic situation where r is a stochastic process, and it follows immediately from (2.3) that the simple accumulation of the premium κ will not, in general, generate the amount to be paid, since $e^{-\int_0^T r_t\,dt}$ may differ considerably from its expected value. One way of dealing with this problem is to replace the 'true' rate of return process r in (2.2) with some deterministic rate of return process r', which is such that the single premium $n\kappa$ accumulated by the true rate of return r is larger than K times the expected number of survivors with a large probability. The excess (if any) should then be added to the amount paid to the policyholder and is known as the *bonus*; see e.g. Ramlau-Hansen (1991) and Norberg (1999) and references therein. However, this approach really raises the problem of whether it is reasonable to assume the existence of *any* deterministic and strictly positive r' which, over a very long time horizon, has the property that it will be larger than the actual return on investments with a very large probability. In particular, this is an extremely relevant discussion when one thinks of the historically low interest rates observed in the late 1990s. An alternative to this approach is therefore to replace r by the so-called short rate of interest, and then to replace the last term in (2.2) by the price on the financial market of a financial asset which pays one unit at time T, a so-called *zero coupon bond*; see Persson (1998).

2.2 *Non-Life Insurance*

In comparison to the valuation principles in life insurance, discounting plays a much less prominent role in the classical non-life insurance premium calculation principles; see e.g. Bühlmann (1970) and Gerber (1979) for standard textbooks on the mathematics of these principles. This difference can be partly explained by the relatively short time horizons of most non-life insurance contracts, which typically change from year to year.

Let H denote some claim payable at a fixed time T, say. A premium calculation principle is a mapping which assigns to each claim a number, called the premium. One class of classical actuarial valuation principles applied in non-life insurance can be directly and somewhat pragmatically motivated from the law of large numbers. These principles prescribe charging a premium $\tilde{u}(H)$ which is equal to the expected value $\mathrm{E}[H]$ of the claim augmented by some amount $A(H)$, the so-called *safety-loading*, i.e.:

$$\tilde{u}(H) = \mathrm{E}[H] + A(H). \tag{2.4}$$

The most important examples of such premium calculation principles are: $A(H) = 0$ (*the net premium principle* or *the principle of equivalence*), $A(H) = a\mathrm{E}[H]$ (*the expected value principle*), $A(H) = a(\mathrm{Var}[H])^{1/2}$ (*the standard deviation principle*), $A(H) = a\mathrm{Var}[H]$ (*the variance principle*) and $A(H) = a\mathrm{E}[((H - \mathrm{E}[H])^+)^2]$ (*the semi-variance principle*). Of these, the standard deviation principle seems to be the most widely used principle in practice. Bühlmann (1970) mentions the fact that it is linear up to scaling as one possible explanation for its popularity, but judges its theoretical properties to be inferior to those of the variance principle.

Another interesting class of premium calculation principles consists of the so-called *zero increase expected utility principles*, which are derived as follows. Let u be a utility function, i.e. $u'(x) \geq 0$ and $u''(x) \leq 0$ for any $x \in \mathbb{R}$, and let V_0 denote the insurer's initial capital at time 0 (possibly random, e.g. depending on the result of other business). The zero (increase expected) utility premium of H under u and initial capital V_0 is the solution $\tilde{u}(H)$ to the equation:

$$\mathrm{E}[u(V_0 + \tilde{u}(H) - H)] = \mathrm{E}[u(V_0)] \tag{2.5}$$

which states that the expected utility of the final wealth $V_0 + \tilde{u}(H) - H$ from selling the claim H at the premium $\tilde{u}(H)$ should equal the expected utility of V_0; the latter may be interpreted as the wealth associated with not selling the claim H. The zero utility premium defined by (2.5) is often also called the *fair premium*, since selling the claim leaves the expected utility unaffected, i.e. it leads neither to an increase nor a decrease in expected utility. The most prominent example is probably the so-called *exponential principle*, which is obtained for the exponential utility function $u(x) = \frac{1}{a}(1 - e^{-ax})$. In particular, when V_0 is constant, P-a.s., the solution to (2.5), does not depend on V_0 and is given by:

$$\tilde{u}(H) = \frac{1}{a}\log\left(\mathrm{E}[e^{aH}]\right).$$

Another frequently used utility function is the quadratic utility function, which is defined by $u(x) = x - \frac{x^2}{2s}$, $x \leq s$, and $u(x) = \frac{s}{2}$ for $x > s$. For a more complete survey of utility functions in insurance (and finance), see e.g. Gerber & Pafumi (1998).

An alternative principle is the so-called *Esscher principle*, which states that:

$$\tilde{u}(H) = \frac{\mathrm{E}[He^{aH}]}{\mathrm{E}[e^{aH}]}. \tag{2.6}$$

This principle basically amounts to an exponential scaling of the claim H.

Other premium calculation principles worth mentioning are generalisations of the so-called *maximal loss principle*. For $\varepsilon \in [0, 1]$ and $p \in [0, 1]$, the (generalised) $(1-\varepsilon)$-*percentile principle* states that the premium should be computed as:

$$\tilde{u}(H) = p\mathrm{E}[H] + (1-p)F^{-1}(1-\varepsilon)$$

where F is the distribution function of H and F^{-1} is its generalised inverse, i.e. $F^{-1}(y) = \inf\{x | F(x) \geq y\}$. Thus, the premium is a weighted average of the expected value of H and the $(1-\varepsilon)$-percentile of the distribution of H. In particular, the maximal loss principle is obtained for $\varepsilon = 0$ and $p = 0$.

For a detailed investigation of the above mentioned principles and several other premium calculation principles, see e.g. Goovaerts, De Vylder & Haezendonck (1984) and Heilmann (1987).

3. Financial Valuation Principles

We recall some basic notation and concepts from financial mathematics. Standard textbooks are Duffie (1996) and Lamberton & Lapeyre (1996); see also Hull (1997) for an exposition including some more institutional aspects. Let T denote a fixed finite time horizon and consider a financial market consisting of two traded assets, a stock and a savings account with price processes $S = (S_t)_{0 \leq t \leq T}$ and $B = (B_t)_{0 \leq t \leq T}$, respectively, which are defined on some probability space $(\Omega, \mathcal{F}, P)$, and introduce the discounted price processes $X = S/B$ and $X^0 = B/B \equiv 1$. In this setting, a *trading strategy* (or dynamical portfolio strategy) is a two-dimensional process $\varphi = (\vartheta_t, \eta_t)_{0 \leq t \leq T}$ satisfying certain integrability conditions (which will be indicated later), and where ϑ is predictable and η is adapted with respect to some filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ which describes the evolution of available information. The pair $\varphi_t = (\vartheta_t, \eta_t)$ is the *portfolio* held at time t, that is, ϑ_t is the number of shares of the stock held at t and η_t is the discounted amount invested in the savings account. Thus, the discounted value at time t of φ_t is given by $V_t(\varphi) = \vartheta_t X_t + \eta_t$. A strategy φ is said to be *self-financing* if:

$$V_t(\varphi) = V_0(\varphi) + \int_0^t \vartheta_s \, dX_s. \tag{3.1}$$

Here, $V_0(\varphi)$ can be interpreted as the amount invested at time 0 and $\int_0^t \vartheta_s \, dX_s$ as the accumulated trading gains generated by φ up to and including time t. Thus, for a self-financing strategy φ, the current value of the portfolio φ_t at time t is exactly the initially invested amount plus trading gains, so that no inflow or outflow of capital has taken place during $(0, t]$. A contract (or *claim*) specifying the discounted ($\mathcal{F}_T$-measurable) payoff H at time T is said to be *attainable* if there exists a self-financing strategy φ such that $V_T(\varphi) = H$ a.s., that is, if H coincides with the terminal value of a self-financing strategy. Thus, a claim is attainable if and only if it can be represented as a constant H_0 plus a stochastic integral with respect to the discounted stock price process:

$$H = H_0 + \int_0^T \vartheta_s^H \, dX_s. \tag{3.2}$$

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The initial investment $V_0(\varphi) = H_0$ needed for this perfect replication of H is also called the unique no-arbitrage price of H. To see that any other price will lead to an arbitrage possibility, i.e. to a risk-free gain, suppose that the price of H at time 0 is given by $H_0 + \epsilon$, where $\epsilon > 0$. A risk-free gain of ϵ can now be generated in the following way:

— Sell the claim H at time 0 and receive $H_0 + \epsilon$. Thus, at time T we have to pay H to the buyer of the claim.

— Invest H_0 via the self-financing strategy $\varphi = (\vartheta, \eta)$ defined by taking $\vartheta_t = \vartheta_t^H$ and by choosing η_t such that (3.1) is satisfied, i.e.

$$\eta_t = V_0(\varphi) + \int_0^t \vartheta_s \, dX_s - \vartheta_t X_t.$$

— The value of the portfolio φ_T at time T is now exactly equal to H, see (3.1) and (3.2), which is to be paid to the buyer of the contract.

The net result of these transactions is the gain ϵ. (If the price of H is $H_0 + \epsilon$, with $\epsilon < 0$, the gain $-\epsilon$ can be obtained by buying H and using the hedging strategy $-\varphi$.) The argument illustrates how the amount H_0 received at time 0 can be transformed into the amount H at time T by following a self-financing strategy, so that H_0 is indeed the only reasonable price.

A financial market is said to be *complete* if all claims are attainable. One example of a complete market with continuous trading is the so-called Black-Scholes model, which consists of two assets: a stock whose price process is described by a geometric Brownian motion; and a savings account which pays a deterministic and constant rate of return. An example with discrete time trading is the Cox-Ross-Rubinstein model described above, which is also known as the binomial model. One important feature of complete markets admitting no arbitrage possibilities is the existence of a unique risk-neutral measure. A risk-neutral measure is a probability measure Q which is equivalent to P and which is such that X is a (local) Q-martingale. (Recall that two probability measures P and Q are said to be equivalent if they have the same null sets, i.e. if they assign probability 0 to the same events. This means that the probability of an event A is 0 under P if and only if the probability of A is 0 under Q, i.e. $\forall A \in \mathcal{F}: P(A) = 0 \Leftrightarrow Q(A) = 0$.) From the general theory of stochastic calculus, it follows that $\int \vartheta^H dX$ is also a local Q-martingale under certain conditions on ϑ^H. Furthermore, if ϑ^H is sufficiently integrable for $\int \vartheta^H dX$ to be a true Q-martingale, then it follows from (3.2) that the no-arbitrage price of H is $H_0 = \mathrm{E}_Q[H]$, since in this case $\mathrm{E}_Q[\int_0^T \vartheta^H dX] = 0$.

If there exist claims that are not attainable, i.e. claims which do not allow a representation of the form (3.2), and hence cannot be replicated by means of any self-financing trading strategy, then the market is said to be *incomplete*; in this case there are infinitely many risk-neutral measures. The completeness property is often lost when we move on to more general models than the ones described above. In the discrete time case, incompleteness occurs already if we replace the binomial model with a *trinomial* model, i.e. a model where the change in the value of the stock between two trading times can attain three different values. An example of an incomplete model under continuous trading is obtained by adding to the geometric Brownian motion a Poisson-driven jump component, say. Another class of examples of incomplete markets consists of models where claims are allowed to depend on more uncertainty than the one generated by the financial market. Pricing of non-attainable claims is far more delicate than the pricing of attainable claims, and typically requires a description of the preferences of the buyers and sellers. In the following we list some different approaches to pricing and hedging in incomplete markets.

3.1 *Super-Replication*

One approach to pricing in incomplete markets is super-replication; see e.g. El Karoui & Quenez (1995). For a given contingent claim H, this approach essentially consists in finding the smallest number V_0^*, say, such that there exists a self-financing strategy $\tilde{\varphi}$ with $V_0(\tilde{\varphi}) = V_0^*$ and

$$V_T(\tilde{\varphi}) \geq H, \ \ P\text{-a.s.}$$

By charging the price V_0^* and applying the strategy $\tilde{\varphi}$, the hedger can generate an amount which exceeds the needed amount H, P-a.s. Thus, the main advantage of this approach is that it leaves no risk to the hedger, since, after an initial investment, no additional capital is needed in order to pay the amount H to the buyer of the contract.

3.2 *A (Marginal) Utility Approach*

An alternative is to derive fair prices from some utility function describing the preferences of the buyers and sellers; see Davis (1997) and references therein. Using a marginal utility argument, Davis (1997) defines the fair price of a claim H as the price which makes investors indifferent between investing 'a little of their funds' in the contract and not investing in this contract. More precisely, let u be a utility function, c the investor's initial capital at time 0, p the price charged at time 0 per unit of some claim H, z the amount invested in H, and introduce:

$$W(z, p, c) = \sup_{\vartheta} \mathrm{E}\left[u\left(c - z + \int_0^T \vartheta_u \, dX_u + \frac{z}{p} H\right)\right]$$

where the supremum is taken over all strategies ϑ from some suitable space of processes. The number $W(z, p, c)$ is the maximum obtainable expected utility for an investor with initial capital c who invests in z/p units of the risk H. The fair price $\tilde{u}(H; c)$ of H is then defined as the solution $\tilde{p}$ to the equation:

$$\frac{\partial}{\partial z} W(0, p, c) = 0$$

provided that the relevant quantities exist. One possible disadvantage of this approach is that it focuses on a small fraction of the risk, and hence partly leaves open the choice of hedging strategy for (say) one unit of the risk.

3.3 *Quadratic Approaches*

A third class of approaches for pricing and hedging in incomplete markets consists of the so-called quadratic methods; see e.g. Schweizer (2001a) for a survey. This class of approaches can be divided into (local) risk-minimisation approaches, proposed by Föllmer & Sondermann (1986) for the case where X is a martingale and generalised to semi-martingales by Schweizer (1988, 1991), and mean-variance hedging approaches, proposed by Bouleau & Lamberton (1989) and Duffie & Richardson (1991). With mean-variance hedging approaches, the main idea is essentially to 'approximate' the claim H as closely as possible by the terminal value of a self-financing strategy using a quadratic criterion. More precisely, this amounts to finding a self-financing strategy $\varphi^* = (\vartheta^*, \eta^*)$ which minimises:

$$\mathrm{E}\big[(H - V_T(\varphi))^2\big] = \|H - V_T(\varphi)\|^2_{L^2(P)} \tag{3.3}$$

over all self-financing strategies φ, i.e. a strategy which approximates H in the L^2-sense. By (3.1), this strategy is completely determined by the pair $(V_0(\varphi^*), \vartheta^*)$, so that the solution to the problem of minimising (3.3) is obtained in principle by projecting the random variable H in $L^2(P)$ on the subspace spanned by $\mathbb{R}$ and random variables of the form $\int_0^T \vartheta\, dX$. The optimal initial capital $V_0(\varphi^*)$ is often called the *approximation price* for H, and the optimal strategy is the *mean-variance hedging strategy*.

Let us now turn to the criterion of risk-minimisation. For any (not necessarily self-financing) strategy $\varphi = (\vartheta, \eta)$, we define the *cost process* by:

$$C_t(\varphi) = V_t(\varphi) - \int_0^t \vartheta_s\, dX_s. \tag{3.4}$$

This process keeps track of the hedger's accumulated costs associated with φ. At any time t, it is the current value $V_t(\varphi)$ of the strategy reduced by trading gains $\int_0^t \vartheta\, dX$. In particular, it follows by inserting (3.1) in (3.4) that the cost process of a self-financing strategy is P-a.s. constant. In contrast to

(3.3), Föllmer & Sondermann (1986) proposed to drop the restriction to self-financing strategies, but insisted on keeping the condition $V_T(\varphi) = H$. With their terminology, a strategy φ is now said to be *risk-minimising* (for H) if $V_T(\varphi) = H$ and if it minimises at any time t the conditional expected squared remaining costs:

$$R_t(\varphi) = \mathrm{E}\big[(C_T(\varphi) - C_t(\varphi))^2 \big| \mathcal{F}_t\big].$$

This optimality criterion amounts to keeping the fluctuations in the cost process as small as possible under the condition $V_T(\varphi) = H$. In particular, Föllmer & Sondermann (1986) proved that the cost process of a risk-minimising strategy is a martingale.

3.4 *Quantile Hedging and Shortfall Risk Minimisation*

One possibly undesirable feature of the quadratic approaches is the fact that they punish losses and gains equally. An alternative is to use *quantile hedging*, see Föllmer & Leukert (1999), where the objective is to hedge the claim with a certain probability. Another alternative is the criterion of minimising the expected shortfall risk, i.e. expected losses from hedging, which has been proposed by Föllmer & Leukert (2000) and Cvitanić (2000). They introduce a *loss function* $l : [0, \infty) \mapsto [0, \infty)$, which is taken to be an increasing convex function with $l(0) = 0$, and consider the problem of minimising:

$$\mathrm{E}\big[l\big((H - V_T(\varphi))^+\big)\big] \tag{3.5}$$

over the class of self-financing hedging strategies. Typical loss functions are power functions $l(x) = x^p$, $p \geq 1$, and, in this case, (3.5) is related to minimising the so-called lower partial moments.

4. Interplay between Insurance and Finance

This section mentions some specific areas of the interplay between finance and insurance.

4.1 *Unit-Linked Insurance Contracts*

Unit-linked insurance contracts seem to have been introduced for the first time in the Netherlands in the early 1950s; in the United States of America the first unit-linked insurance contracts were offered around 1954, and in the United Kingdom unit-linked contracts appeared for the first time in 1957. We refer to Turner (1971) for an overview of the early history of unit-linked life insurance products. For a treatment of some institutional aspects of unit-

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linked insurance contracts, see also Squires (1986). The contracts are also called *equity-linked* or *equity-based* insurance contracts, and in the U.S.A. they are known as *variable life* insurance contracts. A unit-linked life insurance contract differs from the traditional life insurance contracts described in Section 2.1 in that benefits (and sometimes also premiums) depend explicitly on the development of some stock index or the value of some (more or less) specified portfolio. This construction allows for great flexibility as compared with traditional life insurance products, in that the policyholder is offered the opportunity of deciding how his or her premiums are to be invested. Today, issuers of unit-linked life insurance contracts typically offer a variety of investment possibilities that include e.g. worldwide or country specific indices, and reference portfolios with specific investment profiles, e.g. investments in companies from certain branches or regions, or organisations with certain ethical codes.

Denote by S_t the value of the stock index at time t. In the following, we shall refer to the entire development of the stock by simply writing S. As in Section 2.1, consider a portfolio consisting of n policyholders with remaining life times $T_1, \ldots, T_n$. Assume, for simplicity, that they all buy the same form of unit-linked pure endowment contract at time 0 and that the life times are stochastically independent of the development on the financial market. The contracts specify the payment of some (non-negative) amount $f(S)$ to the policyholder at time T if he or she is still alive at this time; f is a function which prescribes some dependence on the development of the stock price. Thus, the present value at time 0 of the insurance company's liability towards the n policyholders is:

$$H = \sum_{i=1}^{n} 1_{\{T_i > T\}} f(S) e^{-\int_0^T r_u \, du} \tag{4.1}$$

where we have discounted the payment by the short rate of interest r. For example, the amount paid could be a function of the terminal value of the stock only, that is:

$$f(S) = S_T \tag{4.2}$$

or the terminal value guaranteed against falling short of some prefixed amount K:

$$f(S) = \max(S_T, K). \tag{4.3}$$

The contract (4.2) is known as a *pure unit-linked* contract and (4.3) is called *unit-linked with guarantee* (the guaranteed benefit is K). However, f could also be a more complex function of the process S, for example a guaranteed annual return is given by:

$$f(S) = K \cdot \prod_{j=1}^{T} \max\left(1 + \frac{S_j - S_{j-1}}{S_{j-1}}, 1 + \delta_j\right).$$

Here, the fraction $(S_j - S_{j-1})/S_{j-1}$ is the return in year j on the asset S and δ_j is the guaranteed return in year j. At time 0 the amount payable at time T is guaranteed against falling short of $K \cdot \prod_1^T (1 + \delta_j)$, but the guarantee goes beyond this 'worst case scenario'.

Unit-linked contracts have been analysed by actuaries since the late 1960s; see e.g. Turner (1969), Kahn (1971) and Wilkie (1978); the two last mentioned give simulation studies for an insurance company administering portfolios of unit-linked insurance contracts. Using modern theories of financial mathematics, Brennan & Schwartz (1979a,b) proposed new valuation principles and investment strategies for unit-linked insurance contracts with so-called asset value guarantees (minimum guarantees). Their principles essentially consisted in combining traditional (law of large numbers) arguments from life insurance with the methods of Black & Scholes (1973) and Merton (1973). By appealing to the law of large numbers, Brennan & Schwartz (1979a,b) first replaced the uncertain courses of the insured lives by their expected values, so that the actual insurance claims, including mortality risk as well as financial risk, were replaced by modified claims, which only contained financial uncertainty. More precisely, instead of considering the claim (4.1) they looked at:

$$H' = n \, {}_T p_y f(S) e^{-\int_0^T r_u \, du}. \tag{4.4}$$

(Recall the notation ${}_T p_y = P(T_1 > T)$ introduced in Section 2.1.) These modified claims were then recognised as essentially being options (with a very long maturity, though) which could, in principle, be priced and hedged using the basic principles of (modern) financial mathematics due to Black & Scholes (1973) and Merton (1973). For the pure unit-linked contract (4.2), the claim (4.4) is proportional to the terminal value of the stock S_T, and hence can be hedged by a buy-and-hold strategy which consists of buying $n \, {}_T p_y$ units of the stock at time 0 and holding these until T. Thus, in the case of no guarantee, the unique no-arbitrage price of H' is simply $n \, {}_T p_y S_0$. Consequently, one possible fair premium for each policyholder is ${}_T p_y S_0$, the probability of survival to T times the value at time 0 of the stock index. Now consider the contract with benefit $f(S) = \max(S_T, K) = (S_T - K)^+ + K$. In this case, pricing of (4.4) involves the pricing of a European call option. In the general case, we see that this principle suggests the premium $n \, {}_T p_y V_0^f$, where V_0^f is the price at time 0 of the purely financial contract which pays

$f(S)$ at time T. More recently, the problem of pricing unit-linked life insurance contracts (under constant interest rates) has been addressed by Delbaen (1986), Bacinello & Ortu (1993a) and Aase & Persson (1994), among others, who combined the martingale approach of Harrison & Kreps (1979) and Harrison & Pliska (1981) with law of large numbers arguments. Whereas all the above mentioned papers assumed a constant interest rate, Bacinello & Ortu (1993b), Nielsen & Sandmann (1995) and Bacinello & Persson (1998), among others, generalised existing results to the case of stochastic interest rates.

In contrast to earlier approaches, Aase & Persson (1994) worked with continuous survival probabilities (i.e. with death benefits that are payable immediately upon the death of the policyholder, and not at the end of the year, as would be implied by discrete time survival probabilities) and suggested investment strategies for unit-linked insurance contracts by methods similar to the ones proposed by Brennan & Schwartz (1979a,b) for discrete time survival probabilities. In contrast to Brennan & Schwartz (1979a,b), who considered a 'large' portfolio of policyholders and therefore worked with 'deterministic mortality', Aase & Persson (1994) considered a portfolio consisting of one policyholder only. However, in all the above papers, the uncertain courses of the insured lives were replaced at an early point with the expected courses in order to allow an application of standard financial valuation techniques for complete markets. The resulting strategies, therefore, did not account for the mortality uncertainty within a portfolio of unit-linked life insurance contracts, and the approach thus leaves open the question of how to quantify and manage the combined actuarial and financial risk inherent in these contracts. In particular, it leaves open the question to which extent this combined risk can be hedged on the financial markets.

It is now natural to ask the question: "Is the assumption of diversifiable mortality risk essential for the derivation of prices for unit-linked insurance contracts?" Or alternatively: "Can (the same) prices and hedging strategies be derived by using alternative approaches which do not involve limiting arguments for the size of the insurance portfolio?" These questions are answered in Section 5, where we give examples of incomplete market approaches that lead to the same prices as the ones suggested by Brennan & Schwartz (1979a,b), as well as examples of approaches that lead to alternative prices.

4.2 *Other Insurance Derivatives*

This section describes some further specific products that have appeared in practice and that combine traditional insurance risk and financial derivatives. The best known examples are probably catastrophe futures, catastrophe-linked bonds, financial stop-loss contracts and stop-loss contracts with a barrier. These new products are really genuine combinations of financial

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derivatives and insurance products, and they are known as *insurance derivatives*. The emergence of such products has been serving as a catalyst for breaking down borders between traditional reinsurance and finance and has opened up the possibility of rethinking fundamental principles of reinsurance and investment. This development presents a challenge to direct insurers and reinsurers as well as to financial institutions in general.

4.2.1 *Catastrophe insurance (CAT) futures*

In the 1980s and early 1990s several severe catastrophes impaired the capacity of reinsurers offering traditional catastrophe covers, and this situation caused an increase in reinsurance premiums. In 1992 the so-called catastrophe insurance (CAT) futures and options on CAT futures were introduced. These instruments standardised catastrophe insurance risk and transformed it into tradeable securities, thus providing a new tool for insurers seeking cover against catastrophe risk. This securitisation was modified in 1995, but the underlying idea essentially remained the same. For an introduction to CAT futures, see e.g. Cummins & Geman (1995) and references therein. An overview of securitisation of catastrophe insurance risk and an analysis of some of the problems associated with securitisation can be found in Tilley (1997).

The basic idea is the following. Consider losses occurring in a specific area and caused by certain well defined catastrophic events, e.g. hurricanes with a certain wind speed or earthquakes of a certain magnitude. Clearly, different insurers will be subject to different exposures from such risks as a consequence of differences in the composition of their insurance portfolios, and with traditional reinsurance contracts, each company would purchase its own insurance covers against risk. Assume now that a number of (suitably chosen) insurance companies report premiums and claims related to the pre-specified type of catastrophes (during certain pre-specified periods) to some central office. Based on the reports, this office constructs a *loss index* $L = (L_t)_{0 \leq t \leq T}$, which is taken to be the underlying process for a futures price process. More precisely, this means that the index L is being reported regularly to the public and that a futures price process $F = (F_t)_{0 \leq t \leq T}$ is constructed by fixing the terminal value $F_T = \min(2, L_T/\kappa)$, where κ are the accumulated premiums for the reporting companies and T is some fixed finite time horizon. Insurance and reinsurance companies, as well as other investors, can now buy and sell this standardised catastrophe risk by purchasing and issuing options on this index on some stock exchange. For example, the *call spread* $H = (F_T - K_1)^+ - (F_T - K_2)^+$, $0 \leq K_1 \leq K_2 \leq 2$, provides cover for relative losses (i.e. the ratio of losses over premiums) in the interval $[K_1, K_2]$. The main advantage of this construction lies in the standardisation and securitisation of the catastrophic risk, which serves to transform the risk related to individual insurance companies into one (common) quantity. Thus, this transformed risk may be more attractive and conceivable to a group of investors which extends beyond traditional reinsurance companies, since it is relatively close in nature to existing financial derivatives. By attracting sellers from a wider group of agents than just the traditional reinsurance companies, these instruments increase the financial capacity of the reinsurance market. On the other hand, the disadvantage for direct insurers buying this contract is that their own relative losses may differ considerably from the average relative losses of the reporting companies. Thus, for a direct insurer, the cover from the call spread on the CAT futures index will typically not correspond exactly to the actual loss experienced by this company.

4.2.2 *Catastrophe-linked bonds*

Individual insurance companies can also choose to securitise part of their insurance risk directly, for example by issuing bonds that are linked to insurance losses from certain insurance portfolios. One example of such an arrangement is the so-called *Winterthur Insurance Convertible Bond*, also called WinCAT bond. This bond, which was introduced by Winterthur in 1997, is described and analysed in Schmock (1999); see also Gisler & Frost (1999). With this three-year bond, investors receive annual coupons as long as certain catastrophic events related to one of Winterthur's own insurance portfolios have not occurred. Thus, the investors receive a return from the bond, which exceeds the market interest rate as long as no catastrophe has occurred, and a lower return in the case of a catastrophe. The difference between the return under no catastrophe and the interest rate on the market is essentially a premium that Winterthur pays investors for 'putting their money at risk'; similarly, the low return in connection with a catastrophic event essentially implies that the investors have covered part of Winterthur's losses. In Cox & Pedersen (2000) catastrophe bonds are priced within a discrete time model via some equilibrium considerations.

This type of product has the advantage over, for example, options on the CAT futures index, in that it provides a much more tailor-made cover for the issuer in that the trigger events that knock out the coupons are directly linked to the company's own insurance portfolio and not to some standardised index. The disadvantage is that there may be considerable costs associated with the selling of such bonds, and that the seller will have to convince buyers that they are only subject to a minimal moral hazard and credit risk.

4.2.3 *Financial stop-loss contracts*

Whereas CAT futures and catastrophe-linked bonds are aimed at a larger group of investors, new reinsurance contracts that combine elements of insurance and financial derivatives have also been introduced by traditional reinsurers. In Swiss Re (1998) several new contracts are described under the title 'Integrated Risk Management Solutions'. One example is the so-called

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financial stop-loss contract, which promises to pay at some fixed time T the amount:

$$H = (U_T + Y_T - K)^+ \tag{4.5}$$

where U_T is the aggregate claim amount during $[0, T]$ on some insurance portfolio, Y_T is some financial loss and K is some retention limit. For $Y_T \equiv 0$ P-a.s., the contract is just a traditional stop-loss contract; however, the loss Y_T could, for example, be a put option on some underlying stock index S, that is $Y_T = (c - S_T)^+$, or it could simply be the loss associated with holding one unit of this index, that is, $Y_T = S_0 - S_T$. The financial stop-loss contracts provide a coverage, not only for large losses due to fluctuations within the insurance portfolio (insurance risk), but also for adverse development of the financial markets (financial risk). In practice, reinsurance companies would typically sell spreads of the form $(U_T + Y_T - K_1)^+ - (U_T + Y_T - K_2)^+$, where $0 \leq K_1 \leq K_2$, which covers the $(K_1, K_2]$ layer of the losses $U_T + Y_T$.

The main idea behind the insurance contract (4.5) is that it provides cover for the insurer's *total risk*, i.e. the combined insurance risk from the insurance portfolio and the financial risk from the financial portfolio. With a traditional stop-loss contract, the reinsurer would cover insurance losses exceeding the level K. However, the financial stop-loss contract is designed so that the cover is only paid provided that the insurance loss augmented by the financial loss exceeds this level. Thus, a large financial gain $-Y_T$ may compensate for large insurance losses, and, in this situation, the buyer does not really need additional compensation from the reinsurer. This feature is illustrated by Figure 1(a), where the area above the solid line represents pairs (Y_T, U_T) of financial losses Y_T and accumulated insurance claims U_T that

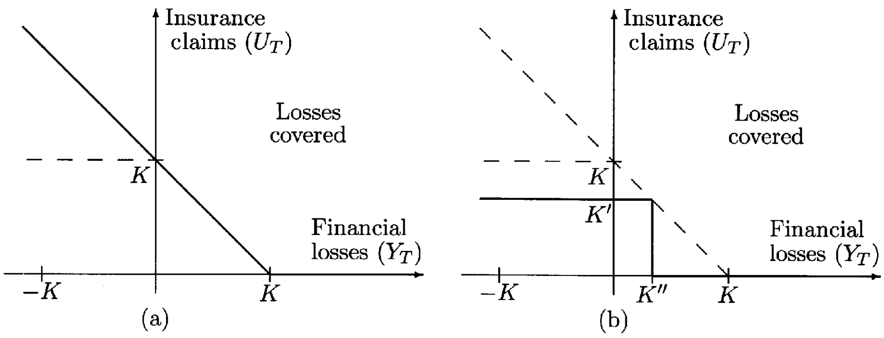

Figure 1. Regions of cover under the financial stop-loss contract with retention K (figure (a)) and under the combination of a traditional stop-loss contract $(U_T - K')^+$ and a call option $(Y_T - K'')^+$ (figure (b))

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generate a payment from the reinsurer. In the area between the solid line and the dashed line are pairs (Y_T, U_T), where (large) insurance claims U_T are partly compensated by financial gains $-Y_T$. The problem of pricing these contracts is a challenge to both actuaries and financial mathematicians. This fact is, for example, underscored by the following quotation from Swiss Re (1998, p15): "..., the risk-neutral valuation technique traditionally used for the pricing of financial derivatives cannot be applied directly but needs to be adjusted and complemented by actuarial methods."

The contract (4.5) should be compared to the alternative of buying a traditional stop-loss contract with retention level K' paying $(U_T - K')^+$ and a traditional financial derivative which pays $(Y_T - K'')^+$; the constants K' and K'' could, for example, be chosen such that $K' + K'' = K$. It follows already from the inequality:

$$(U_T + Y_T - K)^+ \leq (U_T - K')^+ + (Y_T - K'')^+ \tag{4.6}$$

which is satisfied provided that $K' + K'' \leq K$, that the cover from the financial stop-loss contract is dominated by combinations of a traditional stop-loss contract on U_T and a call option on Y_T. The region of cover under the stop-loss contract and the call option is depicted in Figure 1(b) as the area above the solid lines. This figure shows that the region is indeed larger than the corresponding region under the financial stop-loss contract. In particular, it follows that the insurer will receive compensation from the reinsurer also in the situation where very large gains have arisen from investments. Thus, with the traditional instruments, the insurer has actually bought too much insurance cover; the financial stop-loss contract suits the needs of the insurer better.

Finally, we emphasise that the inequality (4.6) indeed indicates that the premium for the financial stop-loss contract should be dominated by the sum of the price on the financial market of $(Y_T - K'')^+$ and the reinsurers' premium for $(U_T - K')^+$. However, the difference may be relatively small, since financial stop-loss contracts have only appeared recently and since they are only bought and sold in very limited amounts. Another important point is that, whereas the call option is sold on the financial market, the (financial and traditional) stop-loss contracts are agreements between a reinsurer and an insurer, and such contracts are typically not traded on stock exchanges. Therefore, it is not, in general, possible to make statements like 'by no-arbitrage arguments', etc. about insurance premiums; see also the discussion on the difference between actuarial and financial valuation principles in Embrechts (2000).

4.3 *Combining Theories for Financial and Actuarial Valuation*

One fundamental difference between the financial valuation techniques, or, more precisely, pricing by no-arbitrage, and the classical actuarial

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valuation principles reviewed above is that the financial valuation principles are formulated within a framework which includes the possibility of trading certain assets, whereas several of the classical actuarial valuation principles are based on more or less *ad hoc* considerations involving the law of large numbers. While the financial valuation principles are based on *dynamic* trading, many decision problems in insurance, for example concerning the choice of optimal reinsurance plans and premiums, were traditionally analysed taking a *static* view. Several attempts have been made to bring together elements of the two theories, and this whole area is still very much 'under construction'. We do not aim at giving a complete overview of this process, but rather at focusing on some specific developments.

4.3.1 *Dynamic reinsurance markets (from financial to actuarial valuation principles)*

Several authors have studied dynamic reinsurance markets in a continuous time framework using no-arbitrage conditions; see, for example, Sondermann (1991), Delbaen & Haezendonck (1989) and de Waegenaere & Delbaen (1992). For an equilibrium analysis of dynamic reinsurance markets, see e.g. Aase (1993) and references therein. The main idea underlying the above mentioned papers is to allow for dynamic rebalancing of proportional reinsurance covers. They all assume that some process related to an *insurance risk process* (accumulated premiums minus claims) of some insurance business is tradeable and that positions can be rebalanced continuously. For example, this could mean that reinsurers can change at any time (continuously) the amount of insurance business that they have accepted. Thus, the insurance risk process can essentially be viewed as a traded security, and this already imposes no-arbitrage bounds on premiums for other (traditional) reinsurance contracts such as stop-loss contracts.

Let us review the main results obtained by Sondermann (1991) and Delbaen & Haezendonck (1989) in more detail. As in the previous section, let U_t be the accumulated claims during $[0, t]$ in some insurance business. Let, furthermore, $p = (p_t)_{0 \leq t \leq T}$ be a predictable process related to the premiums on this business, and define a new process X by:

$$X_t = U_t + p_t. \tag{4.7}$$

Sondermann (1991) takes $-p_t$ to be the premiums paid during $[0, t]$, so that $-X_t$ is, in fact, identical to the insurance risk process. Thus, one can think of X_t as the value at time t of an account where claims are added and premiums subtracted as they occur. In particular, in the special case where premiums are paid continuously at a fixed rate κ, $p_t = -\kappa t$. Reinsurers can now participate in the risk by trading the asset X, i.e. by holding a position in the asset with price process X. Sondermann (1991) points out that, in this setting of a dynamic market for proportional reinsurance contracts, traditional reinsurance contracts such as stop-loss contracts can be viewed as contingent claims, and that these claims should be priced so that no arbitrage possibilities arise. Delbaen & Haezendonck (1989) take p_t to be the premium at which the direct insurer can sell the remaining risk $U_T - U_t$ on the reinsurance market. Thus, in their framework X_t represents the insurer's liabilities at time t. In the special case where the direct insurer receives continuously paid premiums at rate κ and, provided that this premium is identical to the one charged by the reinsurers, we obtain $p_t = \kappa(T - t)$, so that p_t in this situation differs from Sondermann's choice only by the constant κT. Delbaen & Haezendonck (1989) assume that U is a compound Poisson process, i.e. $U_t = \sum_{i=1}^{N_t} Z_i$, where N is a Poisson process and $Z_1, Z_2, \ldots$ is a sequence of i.i.d. non-negative random variables which are independent of N. They then focus on the set of equivalent measures Q which are such that U is also a Q-compound Poisson process. For each such measure Q, a predictable premium process p is obtained by requiring that X be a Q-martingale. This procedure is partly motivated by no-arbitrage considerations (assuming, in addition, that all amounts have been discounted with the interest rate on the market), since this guarantees that no arbitrage possibilities arise from trading in X. In this way, Delbaen & Haezendonck (1989) recover several traditional actuarial valuation principles on a certain subspace of claims from no-arbitrage considerations, namely the expected value principle, the variance principle and the Esscher principle. A more detailed account of the results of Delbaen & Haezendonck (1989) is also given by Embrechts (2000).

In Steffensen (2000, 2001) general life insurance contracts are studied within a securitisation framework which covers both classical and unit-linked life insurance contracts. More precisely, it is assumed that there exist certain traded assets whose price processes are affected by some underlying insurance risk, for example the number of deaths within a portfolio of insured lives. Within this setup, which also opens for a systematic treatment of bonus in life insurance, Steffensen (2000) defines the reserve as the market price of future payments and derives generalised versions of *Thiele's differential equation* under various assumptions about the structure of the payments.

4.3.2 *From actuarial to financial valuation principles*

Gerber & Shiu (1996), among others, consider the situation where the logarithm of the stock price process is a Levy process, i.e. a process with independent and stationary increments. For example, this class of processes includes the geometric Brownian motion and the geometric (shifted) compound Poisson process. Within this setting, they demonstrate how the Esscher transform (2.6) can be used in the pricing of options. They give a very simple option pricing formula which involves Esscher transforms and which, for a European call option, indeed specialises to the well-known

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Black-Scholes formula in the case of a geometric Brownian motion. Furthermore, they demonstrate how this pricing formula can be derived via a simple utility indifference argument in the case of a power utility function $u(x) = \frac{x^{1-a}}{1-a}$ with parameter $a > 0$. This way, they give a candidate for a martingale measure that could be used for pricing in incomplete markets also; they call the resulting martingale measure the *risk-neutral Esscher measure*. For further results on the relation between Esscher transforms, utility theory and equilibrium theory, see Bühlmann (1980, 1984) and references in Delbaen & Haezendonck (1989). A treatment of some of the mathematical aspects associated with Esscher transforms for stochastic processes can be found in Bühlmann *et al.* (1997).

In Schweizer (2001b), the starting points are the traditional standard deviation and variance principles, which are of the form (2.4). These principles are taken as measures of riskiness, which assign to each claim a premium. It is then argued that the measures can equivalently be viewed as measures of preferences which operate on the insurer's terminal wealth by simply changing the sign on the loading factor. This way, Schweizer (2001b) obtains functionals which assign a number to each outcome of the insurer's final wealth, and one can think of this number as the expected value of the insurer's utility of this wealth. For the standard deviation principle, the corresponding functional is given by:

$$u(Y) = \mathrm{E}[Y] - a(\mathrm{Var}[Y])^{1/2}. \tag{4.8}$$

(Dana (1999) refers to a functional of this form as a *mean variance utility function*.) These new functionals are then embedded in a financial framework where the insurer can trade certain assets. Via an indifference argument, Schweizer (2001b) derives financial counterparts of the actuarial standard deviation and variance principles. More precisely, the fair premium π is defined as the unique solution to:

$$\sup_{\varphi} u(c + \pi + V_T(\varphi) - H) = \sup_{\varphi} u(c + V_T(\varphi)) \tag{4.9}$$

where the suprema are taken over self-financing strategies with initial value 0 satisfying certain integrability conditions. Here, the term $c + \pi + V_T(\varphi) - H$ is the insurer's wealth at time T from selling the claim H at the price π. It is given by the initial capital c augmented by the premium π and trading gains $V_T(\varphi)$ and reduced by the claim H payable at time T. Similarly, $c + V_T(\varphi)$ is the insurer's wealth at T from not selling the claim and investing according to the strategy φ. The insurer is now said to be indifferent between selling H and not selling H, if the maximum obtainable utilities in the two scenarios are identical, that is, if (4.9) is satisfied. This leads to new financial valuation principles which resemble their actuarial counterparts, in that they consist of an expectation plus some safety loading. However, for the financial valuation principles, the expected value is now computed under a specific martingale measure $\tilde{P}$ known as the *variance optimal martingale measure*. This martingale measure is the risk-neutral measure whose Radon-Nikodym derivative with respect to P has the smallest variance, i.e. it minimises $\text{Var}[\frac{dQ}{dP}]$ among all risk-neutral measures Q. Moreover, $\tilde{P}$ has the special property that the Radon-Nikodym derivative can be represented as a constant plus a stochastic integral, that is:

$$\frac{d\tilde{P}}{dP} = \tilde{Z}_0 + \int_0^T \tilde{\zeta}_t \, dX_t.$$

Furthermore, the loading factor is now a function of the variance of the so-called non-hedgeable part of the claim H, which, in general, is smaller than the variance of H, and which can be quite difficult to determine. These new financial valuations are in accordance with no-arbitrage pricing for attainable claims, and thus they provide alternative approaches for the valuation of options and other derivatives in incomplete markets.

One undesirable feature with this approach is that the variance optimal measure is, in general, only a signed measure and not necessarily a true probability measure. In particular, this property has the very unfortunate consequence that the financial principles may assign a negative value to a positive claim! However, if the discounted price processes of the traded assets are continuous, then the variance optimal martingale measure is indeed a true probability measure which is equivalent to the true probability measure P. For more details, see e.g. Schweizer (2001a).

5. Hedging and Indifference Pricing in Insurance

In this section we mention some further results for insurance claims that combine financial and insurance risk. Section 5.1 reviews existing applications to insurance of the theory of risk-minimisation with special emphasis on hedging (and pricing) of unit-linked insurance contracts, and Section 5.2 is related to the financial variance and standard deviation principles of Schweizer (2001b).

5.1 *Hedging Unit-Linked Insurance Contracts*

5.1.1 *Risk-minimisation*

In Møller (1998) risk-minimising hedging strategies were determined for a portfolio of unit-linked pure endowment contracts using the theory of risk-minimisation due to Föllmer & Sondermann (1986). An introduction to the problem of the pricing and hedging of unit-linked insurance contracts can also be found in Møller (2001a), where various approaches for hedging and pricing in incomplete markets are discussed in a discrete time model framework. This opens for a simple comparison of the techniques proposed by Brennan & Schwartz (1979a,b) to the ones suggested by risk-minimisation and super-replication, respectively. In contrast to the approaches of Brennan & Schwartz (1979a,b) and Aase & Persson (1994), Møller (1998, 2001a) did not average away the mortality risk (the uncertainty associated with not knowing the number of survivors), but analysed the insurance contracts as contingent claims in an incomplete market. Consequently, the resulting strategies reflect, and react to, the financial risk as well as the insurance risk. In particular, it is clearly visible from these strategies how an insurer applying the risk-minimising hedging strategy is currently adapting his portfolio of stocks and his deposit on the savings account to the actual development within the portfolio of insured lives.

As an example, consider unit-linked pure endowment contracts of the form (4.1) for n policyholders aged y with i.i.d. remaining life times $T_1, \ldots, T_n$, and assume that the amount payable upon survival to T, $f(S)$, is attainable in the sense that:

$$f(S)e^{-\int_0^T r_u du} = H_0^f + \int_0^T \vartheta_u^f \, dX_u \tag{5.1}$$

see Section 3 for more motivation. Thus, the discounted no-arbitrage price at time t of the claim $f(S)$ is given by:

$$V_t^f = H_0^f + \int_0^t \vartheta_u^f \, dX_u.$$

Denote the number of deaths up to time t by $N_t = \sum_{i=1}^n 1_{\{T_i \leq t\}}$, so that the current number of survivors at t is $(n - N_t)$. The filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is defined as:

$$\mathcal{F}_t = \sigma\{(N_u, S_u, B_u),\ u \leq t\}$$

where $B_t = \exp(\int_0^t r_u du)$. This has the usual interpretation. The insurance company is observing the process N as well as the price processes (S, B). Under the natural assumption of independence between the remaining life times and the financial risk, the arguments used in Møller (1998) show the following:

Theorem 1. Assume that P is a martingale measure. The unique risk-minimising strategy $\varphi^* = (\vartheta^*, \eta^*)$ for (4.1) is given by:

$$\vartheta_t^* = (n - N_{t-})\,{}_{T-t}p_{y+t}\,\vartheta_t^f$$
$$\eta_t^* = (n - N_t)\,{}_{T-t}p_{y+t}\,V_t^f - \vartheta_t^*\,X_t.$$

This strategy prescribes holding a number of stocks at time t which is equal to the dynamic hedge ϑ_t^f for $f(S)$ multiplied by the current expected number of survivors just before time t, and the amount invested in the savings account η_t^* is chosen such that at any time t:

$$V_t(\varphi^*) = (n - N_t)\,{}_{T-t}p_{y+t}\,V_t^f.$$

In particular, the value at time 0 is $V_0(\varphi^*) = n\,{}_Tp_y\,V_0^f$, which coincides with the price suggested by Brennan & Schwartz (1979a,b): the expected number of survivors to T multiplied by the no-arbitrage price at time 0 of the amount $f(S)$ payable upon survival to T; see also the discussion in Section 4.1. The strategy in the theorem is *not* self-financing, and its cost process is given by:

$$C_t(\varphi^*) = n\,{}_Tp_y\,V_0^f - \int_0^t V_u^f\,{}_{T-u}p_{y+u}\,dM_u$$

where M is a martingale defined by:

$$dM_t = dN_t - (n - N_{t-})\,\mu_{y+t}\,dt$$

and where μ_{y+t} is the mortality intensity (or the hazard rate); see Section 2.1. Using the results of Föllmer & Sondermann (1986), Møller (1998) also derived measures for the part of the total risk in the unit-linked contracts that cannot be hedged away by trading on financial markets only, the so-called *intrinsic risk*. It is given by:

$$R_0(\varphi^*) = \mathrm{E}[(C_T(\varphi^*) - C_0(\varphi^*))^2] = n\,{}_Tp_y \int_0^T \mathrm{E}[(V_u^f)^2]\,{}_{T-u}p_{y+u}\,\mu_{y+u}\,du$$

see Møller (1998, Theorem 4.2). Furthermore, it was shown that this intrinsic risk could actually be completely eliminated by including in addition a dynamic reinsurance market. More precisely, it was assumed that the insurer could trade continuously, in addition to the stock and the savings account, a third asset with a price process which was, at any time, equal to the prospective reserve associated with a pure endowment insurance with sum insured 1. In this way, the insurance risk was essentially transformed into a traded asset or a security. In the model considered there, this additional asset was indeed sufficient to restore completeness, leading to unique prices and self-financing investment strategies.

The theory of risk-minimisation, introduced by Föllmer & Sondermann (1986), focuses on the problem of hedging a contingent claim payable at a fixed time. However, insurance contracts often generate genuine payment streams where amounts are paid out over time. For example, with a so-called life annuity, payments are due yearly, say, from a certain time and as long as the policyholder is still alive. Similarly, life insurance contracts, in general, are often paid by periodic premiums, e.g. premiums paid at the beginning of each year as long as the policyholder is still alive. In Møller (2001c), general payment streams are incorporated into the theory of risk-minimisation, thus providing a framework which allows for the analysis of (insurance) payment processes. This modified framework is applied to the analysis of general unit-linked life insurance contracts, where the state of the policy is described by a Markov jump process with a finite state space. This generalises previous results obtained in Møller (1998).

5.1.2 *Mean-variance hedging*

In the situation where P is not a martingale measure, that is when the discounted stock price process $X = S/B$ is not a martingale under P, we can instead determine the mean-variance hedging strategy for H and the so-called approximation price, cf. Section 3. In the present situation, we have, under certain technical conditions on the process X, the following result; see Møller (2002a, Section 7):

Theorem 2. The mean-variance hedging strategy $\tilde{\varphi} = (\tilde{\vartheta}, \tilde{\eta})$ for (4.1) is given by:

$$\tilde{\vartheta}_t = (n - N_{t-})\, {}_{T-t}p_{y+t}\, \vartheta_t^f + \tilde{\zeta}_t \int_0^t V_u^f\, \tilde{Z}_u^{-1}\, {}_{T-u}p_{y+u}\, dM_u$$

$$\tilde{\eta}_t = V_0(\tilde{\varphi}) + \int_0^t \tilde{\vartheta}_u\, dX_u - \tilde{\vartheta}_t X_t$$

where the processes $\tilde{\zeta}$ and $\tilde{Z}$ are determined by the variance optimal martingale measure. The approximation price is $V_0(\tilde{\varphi}) = n\, {}_Tp_y\, V_0^f$.

The number of stocks $\tilde{\vartheta}_t$ held with the mean-variance hedging strategy consists of two terms. The first term is exactly the risk-minimising strategy of Theorem 1. The second term is a correction term, which is driven by the martingale M introduced above, and which depends on the entire past development within the portfolio of insured lives. Moreover, this second term is related to the variance optimal martingale $\tilde{P}$; see the last paragraph of Section 4.3, via the terms $\tilde{\zeta}$ and $\tilde{Z}_t = \tilde{\mathrm{E}}[\frac{d\tilde{P}}{dP}|\mathcal{F}_t]$. For more details on the variance optimal martingale measure, see e.g. Schweizer (2001a). In addition, we see that the approximation price is equal to the price proposed by Brennan & Schwartz (1979a,b).

5.1.3 *Super-Replication*

For comparison, we derive the super-replicating strategy for the unit-linked contract (4.1). With super-replication, the objective is essentially to determine the self-financing strategy among the ones whose terminal values dominate the claim H, i.e. $V_T(\varphi) \geq H$, which requires the smallest initial investment. As shown by El Karoui & Quenez (1995), this strategy is closely related to the process:

$$\overline{V}_t = \operatorname*{ess\,sup}_Q \mathrm{E}_Q[H|\mathcal{F}_t] \tag{5.2}$$

where the supremum is taken over all equivalent martingale measures. In the present situation we obtain:

Lemma 1. For the unit-linked pure endowment contract (4.1) the process (5.2) is given by $\overline{V}_t = (n - N_t)\, V_t^f$ and it admits the decomposition:

$$\overline{V}_t = n\, V_0^f + \int_0^t (n - N_{u-})\, \vartheta_u^f\, dX_u - \int_0^t V_u^f\, dN_u. \tag{5.3}$$

Idea of proof. (This result is similar to El Karoui & Quenez, 1995, Example 3.4.2.) We first show that $\overline{V}_t = (n - N_t)\, V_t^f$. To see '$\leq$', note that $N_T \geq N_t$ which implies that $(n - N_T) \leq (n - N_t)$. Thus, for any martingale measure Q:

$$\begin{aligned} &\mathrm{E}_Q[(n - N_T) f(S) e^{-\int_0^T r_u\,du} | \mathcal{F}_t] \\ &\leq (n - N_t)\mathrm{E}_Q[f(S) e^{-\int_0^T r_u\,du} | \mathcal{F}_t] \\ &= (n - N_t)\, V_t^f. \end{aligned}$$

To see that $\overline{V}_t \geq (n - N_t)\, V_t^f$, consider for $h > -1$ the martingale measure $Q^{(h)}$ such that the remaining life times $T_1, \ldots, T_n$ are i.i.d. with mortality intensity $(1 + h)\mu_{y+t}$ and independent of (S, B); see Møller (1998, Section 2) for a construction of this measure. Thus, by the independence between the two sources of risk:

$$\mathrm{E}_{Q^{(h)}}[(n - N_T) f(S) e^{-\int_0^T r_u\,du} | \mathcal{F}_t] = (n - N_t)\, {}_{T-t}p_{y+t}^{(h)}\, V_t^f$$

where the survival probability under $Q^{(h)}$ is given by:

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$$_{T-t}p^{(h)}_{y+t} = Q^{(h)}(T_1 > T | T_1 > t) = \exp\left(-(1+h)\int_t^T \mu_{y+u}\, du\right)$$

and where we have used:

$$\mathrm{E}_{Q^{(h)}}[f(S)e^{-\int_0^T r_u\, du}|\mathcal{F}_t] = V_t^f$$

for any $h > -1$. Since $\lim_{h\searrow -1} {}_{T-t}p^{(h)}_{y+t} = 1$, we obtain $\overline{V}_t \geq (n - N_t)\, V_t^f$. The decomposition (5.3) finally follows by applying the product rule to $(n - N_t)\, V_t^f$.

Using the decomposition (5.3) with $t = T$, we see that:

$$n\, V_0^f + \int_0^T (n - N_{u-})\, \vartheta_u^f\, dX_u = H + \int_0^T V_u^f\, dN_u.$$

Here, the two terms on the left represent the value at T of a self-financing strategy with initial value $n\, V_0^f$ and with $(n - N_{u-})\, \vartheta_u^f$ stocks held at time $0 \leq u \leq T$. This value exceeds H by the amount $\int_0^T V_u^f\, dN_u$, which is non-negative, since $V_u^f \geq 0$ and since N is increasing. We can, in fact, currently withdraw the amount $\int_0^t V_u^f\, dN_u$ from the strategy and still ensure that the terminal value exceeds H. We summarise this result in the following:

Theorem 3. The super-replicating strategy $\hat{\varphi} = (\hat{\vartheta}, \hat{\eta})$ for (4.1) is determined by:

$$\hat{\vartheta}_t = (n - N_{t-})\, \vartheta_t^f$$
$$\hat{\eta}_t = V_0(\hat{\varphi}) + \int_0^t \hat{\vartheta}_u\, dX_u - \hat{\vartheta}_t X_t - \int_0^t V_u^f\, dN_u$$

and $V_0(\hat{\varphi}) = n\, V_0^f$.

Thus, the super-replicating strategy requires an initial investment at time 0 of the amount $n\, V_0^f$. Comparing with the results obtained in Theorems 1 and 2, we see that this corresponds to using a survival probability of 1! Thus, the super-hedging price for the unit-linked contract is identical to the price for the purely financial contract specifying the payoff $nf(S)$ at time T. This result clearly indicates that super-hedging is not the right approach for the pricing of unit-linked contracts in the present framework. However, the result can still be used as an upper bound for reasonable prices. The number of stocks held at t is exactly the current number of survivors multiplied with the hedge ϑ_t^f for $f(S)$, which also differs from the risk-minimising and mean-variance strategies in that no survival probability is involved. If a policyholder dies during the infinitesimal interval $(t, t + dt]$, then $dN_t = 1$, which implies that the discounted deposit on the savings $\hat{\eta}_t$ is being reduced by the amount V_t^f, i.e. that the amount V_t^f can be withdrawn from the strategy.

5.2 *On Transformations of Actuarial Valuation Principles*

This section reviews some results on the financial variance and standard deviation principles of Schweizer (2001b), mentioned in Section 4.3. Instead of using the indifference principle applied there, we present two apparently *ad hoc* ways of modifying the classical principles. These results are closely related to an alternative and more direct characterisation of the financial standard deviation principle given in Møller (2001b), which does not involve an indifference argument. For this purpose, it suffices to consider a standard Black-Scholes market. There are two traded assets S and B with $S_0 = B_0 = 1$ and dynamics:

$$dB_t = rB_t dt$$
$$dS_t = \alpha S_t dt + \sigma S_t dW_t.$$

These processes are defined on some probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, where T is a fixed finite time horizon, W is a standard Brownian motion with respect to $\mathbb{F}$, and r, α and σ are known constants. Consider in addition some insurance (risk) process $U = (U_t)_{t\in[0,T]}$, which, for example, could be defined by $U_t = \sum_{i=1}^{N_t} Z_i$, where N is a homogeneous Poisson process with intensity λ and $Z_1, Z_2, \ldots$ is a sequence of i.i.d. random variables representing claim amounts. Alternatively, U_t could be the number of deaths up to time t within a portfolio of n insured lives. We assume for simplicity that U and W are independent under P (or, equivalently, that U and S are independent) and that $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^U$, i.e. the filtration is taken to be the (P-augmentation of the) natural filtration of (W, U).

With the present construction, one can trade the two assets S and B via trading strategies that are adapted to the filtration generated by S and U. This means that investors can base investment strategies on observed prices as well as on the observed insurance claims. Note, however, that the process U is not related to any traded assets, so that this risk cannot be eliminated by trading on the market. In this setting, we will consider the problem of assigning premiums to insurance contracts that depend on both sources of risk. More precisely, this means that we consider contingent claims H payable at time T which are $\mathcal{F}_T^{U,W}$-measurable. Non-trivial examples are a financial stop-loss contract or a unit-linked life insurance contract described above.

Define an equivalent measure Q via:

$$\frac{dQ}{dP} = L_T = \exp\left(-\nu W_T - \frac{1}{2}\nu^2 T\right)$$

where $\nu = \frac{\alpha - r}{\sigma}$. This measure has the following properties:
— The discounted price process $X = S/B$ is a Q-martingale.
— U is not affected by the change of measure.
— S and U are independent under Q.

Thus, Q is an equivalent martingale measure, so that the model is free of arbitrage. To see that the last property is satisfied, it must be verified that for any $A \in \mathcal{F}_T^W$ and $B \in \mathcal{F}_T^U$ we have: $Q(A \cap B) = Q(A)Q(B)$. By using the definition of the measure Q and by exploiting the independence between the $\mathcal{F}_T^W$-measurable random variable L_T and $\mathcal{F}_T^U$ we get:

$$Q(A \cap B) = \mathrm{E}_Q[1_{A \cap B}] = \mathrm{E}[L_T 1_A 1_B] = \mathrm{E}[L_T 1_A]\mathrm{E}[1_B] = Q(A)Q(B).$$

(Here and throughout we use the notation 'E' and 'Var' for 'E_P' and 'Var_P'.) Finally, we point out that Q is just one possible martingale measure. Whereas the change of measure from P to Q does not affect the process U, one can also consider equivalent martingale measures which change the distribution of U; see e.g. Delbaen & Haezendonck (1989) for the situation where U is a compound Poisson process and Møller (1998) for the case where U counts the number of deaths within a portfolio of insured lives. Since this can be done without affecting the stock S, it follows that there are infinitely many martingale measures in the current model, i.e. the model is incomplete.

According to the classical actuarial standard deviation principle, the premium for a contract specifying a discounted payoff H at T is:

$$\tilde{u}(H) = \mathrm{E}[H] + a(\mathrm{Var}[H])^{1/2} \tag{5.4}$$

cf. Section 2.2 (here we apply the principle on the discounted payoff, thus deviating slightly from tradition). Clearly, it would not make sense to apply this principle directly to (say) a European call option with discounted payoff $H = e^{-rT}(S_T - K)^+$, since this contract is attainable and hence can be priced uniquely by no-arbitrage arguments alone. One idea is therefore to modify (or transform) the principle slightly, so as to get a principle that, on the one hand still resembles the standard deviation principle, and on the other hand is consistent with absence of arbitrage, in the sense that the premium of an attainable claim equals the unique no-arbitrage price. We shall look at two simple ways of modifying the standard deviation principle directly.

5.2.1 *Modified standard deviation principle 1*

Consider the following modified premium principle:

$$\pi_1(H) = \mathrm{E}_Q[H] + a\big(\mathrm{Var}\big[\mathrm{E}_Q\big[H\big|\mathcal{F}_T^U\big]\big]\big)^{1/2}. \qquad (5.5)$$

It is not difficult to show that this principle has the properties:

(1) For $\gamma \geq 0$, $\pi_1(\gamma H) = \gamma\pi_1(H)$; i.e. the principle allows for scaling of the claim. (This property is called *positive homogeneity* in the literature, cf. Goovaerts, De Vylder & Haezendonck, 1984.)

(2) For any $H \sim \mathcal{F}_T^W$:

$$\pi_1(H) = \mathrm{E}_Q[H]$$

i.e. for any purely financial contract, the premium under (5.5) is equal to the unique no-arbitrage price.

(3) For any $H \sim \mathcal{F}_T^{W,U}$:

$$\pi_1(H) = \tilde{u}(\mathrm{E}_Q[H|\mathcal{F}_T^U]).$$

i.e. this modified standard deviation principle corresponds to applying the traditional standard deviation principle to the no-arbitrage price of *H conditional* on the insurance uncertainty $\mathcal{F}_T^U$.

Property 1 follows immediately from the definition (5.5). To see that Property 2 is satisfied, consider a claim H which only depends on the uncertainty from the financial market, so that H is $\mathcal{F}_T^W$-measurable. We can again exploit the independence between U and W under Q and well-known properties for conditional expected values to obtain that $\mathrm{E}_Q[H|\mathcal{F}_T^U] = \mathrm{E}_Q[H]$ for such H. Since the variance of a constant is 0, the loading term in (5.5) vanishes, and this shows that Property 2 is indeed satisfied. In particular, this ensures that the premium for a European call option on S coincides with its unique no-arbitrage price.

Finally, the last property follows by verifying that $\mathrm{E}_Q[H] = \mathrm{E}\mathrm{E}_Q[H|\mathcal{F}_T^U]$, which, in turn, follows from the rule of iterated expectation under Q and the definition of Q:

$$\begin{aligned}\mathrm{E}_Q[H] &= \mathrm{E}_Q[\mathrm{E}_Q[H|\mathcal{F}_T^U]] = \mathrm{E}[L_T \mathrm{E}_Q[H|\mathcal{F}_T^U]] \\ &= \mathrm{E}[L_T]\,\mathrm{E}[\mathrm{E}_Q[H|\mathcal{F}_T^U]] = \mathrm{E}[\mathrm{E}_Q[H|\mathcal{F}_T^U]]\end{aligned}$$

where the third equality follows from the independence between L_T and U, and the last equality follows since $E[L_T] = 1$.

5.2.2 *Modified standard deviation principle 2*

As an alternative, consider the following modification of the classical standard-deviation principle:

$$\pi_2(H) = \mathrm{E}_Q[H] + a\big(\mathrm{E}\big[\mathrm{Var}\big[H\big|\mathcal{F}_T^W\big]\big]\big)^{1/2}. \tag{5.6}$$

Similarly to principle π_1, one can show that π_2 is positively homogeneous, i.e. it satisfies Property 1. Property 2 is also satisfied for π_2, since for any $H \sim \mathcal{F}_T^W$ we have that $\mathrm{Var}[H|\mathcal{F}_T^W] = 0$. The principle π_2 does not satisfy Property 3, but we can instead give the following intuitive characterisation:

(4) For any claim H, there exists a self-financing strategy φ with initial value 0 such that:

$$\pi_2(H) = \tilde{u}(H - V_T(\varphi))$$

where $V_T(\varphi)$ is the terminal value of the strategy φ.

Thus, π_2 simply amounts to applying the traditional standard deviation principle to the claim H *reduced* by the terminal value of a certain self-financing strategy which requires 0 initial investment.

To see that Property 4 is satisfied, consider the (artificial) claim:

$$H' = \mathrm{E}[H|\mathcal{F}_T^W] - \mathrm{E}_Q[H].$$

Since by Møller (2002b, Proposition 3.11), $\mathrm{E}[H|\mathcal{F}_T^W] = \mathrm{E}_Q[H|\mathcal{F}_T^W]$, we find that $\mathrm{E}_Q[H'] = 0$. Furthermore, since H' is $\mathcal{F}_T^W$-measurable, there exists a self-financing strategy φ which replicates H', i.e. $V_T(\varphi) = H'$. Moreover, it follows e.g. from the fact that $\mathrm{E}_Q[H'] = 0$ and the results reviewed in Section 3 that this self-financing strategy requires no initial investment (initial capital 0). To see that Property 4 is satisfied, we only need to compute $\mathrm{E}[H - V_T(\varphi)]$ and $\mathrm{Var}[H - V_T(\varphi)]$ and check that these will correspond to the terms appearing in (5.6):

$$\mathrm{E}[H - V_T(\varphi)] = \mathrm{E}[H - (\mathrm{E}[H|\mathcal{F}_T^W] - \mathrm{E}_Q[H])] = \mathrm{E}_Q[H]$$
$$\mathrm{Var}[H - V_T(\varphi)] = \mathrm{Var}[H - \mathrm{E}[H|\mathcal{F}_T^W]] = \mathrm{E}[\mathrm{Var}[H|\mathcal{F}_T^W]]$$

where the last equality follows by using standard rules for conditional variances. The idea of applying the original standard deviation principle to the claim reduced by the terminal value of a self-financing strategy with initial capital 0 is pursued further in Møller (2001b). More precisely, it is shown that one can give an equivalent definition of the financial standard deviation principle of Schweizer (2001b) by defining a premium principle via:

$$\pi_3(H) = \inf_{\varphi} \tilde{u}(H - V_T(\varphi))$$

where the infimum is taken over all self-financing strategies φ with initial value 0 that in addition satisfy some integrability conditions.

5.2.3 *Indifference pricing under a change of filtration*

In Møller (2002a,b) the properties of the financial variance and standard deviation principles of Schweizer (2001b) are studied further. In particular, focus is on the dependence of the fair premiums (also called indifference prices) on the amount of information available to the insurer, that is, on the choice of filtration. Via a comparison result for mean-variance hedging errors in different filtrations, a natural ordering of the fair premiums is obtained. More precisely, it is shown in Møller (2002b) that more actuarial information leads to lower premiums, and this difference is characterised further. The results allow for derivation of relatively simple upper and lower bounds for the fair premiums of reinsurance contracts under the assumption of independence between the traded assets and the insurance risk involved. An upper bound is obtained by allowing the hedger to adapt trading strategies to the information from the financial market only, and the lower bound corresponds to the artificial situation where the actuarial uncertainty $\mathcal{F}_T^U$ is revealed immediately after the signing of the contract. These bounds are, in fact, closely related to the above mentioned *ad hoc* modifications of the classical standard deviation principles; for comparison, we quote the result from Møller (2002b) here:

Theorem 4. Assume that $a^2 \geq \mathrm{Var}[L_T]$. For the standard deviation principle, the upper bound for the fair premium is:

$$\pi_{max}(H) = \mathrm{E}_Q[H] + a_1 \left(\mathrm{E}[\mathrm{Var}[H \mid \mathcal{F}_T^W]]\right)^{1/2} \tag{5.7}$$

and the lower bound is given by:

$$\pi_{min}(H) = \mathrm{E}_Q[H] + a_2 \left(\mathrm{Var}[\mathrm{E}_Q[H \mid \mathcal{F}_T^U]]\right)^{1/2} \tag{5.8}$$

where:

$$a_1 = a\sqrt{1 - \frac{\mathrm{Var}[L_T]}{a^2}} \qquad a_2 = \frac{a_1}{\sqrt{\mathrm{E}[L_T^2]}}.$$

Note that in the present model the upper and lower bounds in the theorem differ from the modified principles (5.5) and (5.6) only via the safety loading parameters a_1 and a_2. The fair premiums of the above theorem are only valid

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provided that the condition $a^2 \geq \mathrm{Var}[L_T]$ is satisfied. Since $\mathrm{Var}[L_T] = e^{\nu^2 T} - 1$, this means that the so-called *market price of risk* $\nu = \frac{\alpha - r}{\sigma}$ has to be small compared to the safety-loading parameter a. If $\alpha = r$ then $L_T = 1$, so that $P = Q$. In this special situation, $a_1 = a_2 = a$, so that the bounds are actually identical to the above mentioned modified principles. Some applications related to insurance of these results can be found in Møller (2002a), where fair premiums and optimal trading strategies are determined under various scenarios corresponding to different amounts of information, for example for unit-linked insurance contracts and financial stop-loss contracts.

5.3 *Hedging Unit-Linked Insurance Contracts (continued)*

5.3.1 *Indifference pricing*

As described in the previous section, the indifference price depends on the amount of information available, i.e. on the choice of filtration. In Møller (2002a), indifference prices and optimal investment strategies are computed under different filtrations for the unit-linked pure endowment contract (4.1), which can also be written as:

$$H = (n - N_T) f(S)\, e^{-\int_0^T r_u\, du}. \tag{5.9}$$

Here, Theorem 4 gives upper and lower bounds for the fair premiums in the standard Black-Scholes model. For the contract (5.9), we see that:

$$\begin{aligned} \mathrm{E}[\mathrm{Var}[H \mid \mathcal{F}_T^W]] &= \mathrm{Var}[(n - N_T)]\mathrm{E}\left[\left(f(S)e^{-\int_0^T r_u\, du}\right)^2\right] \\ &= n\,{}_T p_y\,(1 - {}_T p_y)\mathrm{E}\left[\left(f(S)e^{-\int_0^T r_u\, du}\right)^2\right] \end{aligned}$$

since the lifetimes are assumed to be i.i.d., so that $\mathrm{Var}[(n - N_T)] = n\,{}_T p_y\,(1 - {}_T p_y)$, and where we have also used the notation of Section 5.1. Similarly, the term appearing in the safety-loading of the lower bound is (with $U = N$):

$$\mathrm{Var}[\mathrm{E}_Q[H \mid \mathcal{F}_T^U]] = \left(V_0^f\right)^2 n\,{}_T p_y\,(1 - {}_T p_y).$$

Thus, the upper bound of Theorem 4 is:

$$\pi_{max}(H) = n\,{}_T p_y\, V_0^f + a_1\, n^{1/2}\left({}_T p_y\,(1 - {}_T p_y)\right)^{1/2}\left(\mathrm{E}\left[\left(f(S)e^{-\int_0^T r_u\, du}\right)^2\right]\right)^{1/2} \tag{5.10}$$

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and the lower bound is:

$$\pi_{min}(H) = n_T p_y V_0^f + a_2 n^{1/2} \big({}_T p_y (1 - {}_T p_y)\big)^{1/2} V_0^f. \tag{5.11}$$

In Møller (2002a) optimal investment strategies associated with the two bounds are derived for the unit-linked contract by considering various filtrations. These strategies are, in fact, closely linked to the mean-variance strategy determined in Section 5.1. If we apply the filtration $\mathbb{F}$ introduced in Section 5.1, the optimal investment strategy becomes:

$$\vartheta_t^* = (n - N_{t-})_{T-t} p_{y+t} \vartheta_t^f + \tilde{\zeta}_t \int_0^t \tilde{Z}_u^{-1} V_u^f \,{}_{T-u} p_{y+u} \, dM_u + \sqrt{\mathrm{Var}[N^H]} \frac{\tilde{Z}_t \lambda_t}{a_1} \tag{5.12}$$

where $\lambda_t = \frac{\alpha - r}{\sigma^2 X_t}$. The term $\mathrm{Var}[N^H]$ is the variance of the part of the liability which cannot be hedged away in the financial market. This term was determined and evaluated numerically in Møller (2001b); it is given by:

$$N^H = -\tilde{Z}_T \int_0^T \frac{V_u^f}{\tilde{Z}_u} \,{}_{T-u} p_{y+u} \, dM_u. \tag{5.13}$$

It was shown in Møller (2002a) that:

$$\frac{\mathrm{Var}[\mathrm{E}_Q[H \mid \mathcal{F}_T^U]]}{\mathrm{E}[L_T^2]} \le \mathrm{Var}[N^H] \le \mathrm{E}[\mathrm{Var}[H \mid \mathcal{F}_T^W]]$$

which gives simple bounds for the variance of the non-hedgeable part of the unit-linked contract. This quantity is also closely related to the indifference prices, since the indifference price corresponding to the filtration $\mathbb{F}$ is given by:

$$n \,{}_T p_y V_0^f + a_1 \sqrt{\mathrm{Var}[N^H]}. \tag{5.14}$$

The first term in the optimal strategy (5.12) is recognised as the risk-minimising strategy of Theorem 1, and the two first terms correspond to the mean-variance strategy of Theorem 2. In particular, the second term provides an adjustment of the number of stocks which depends on the number of survivors. If the current number of survivors is larger than the expected number, then the optimal number of stocks held under the mean-variance principle will typically exceed the one determined under the criterion of risk-minimisation. The optimal strategy under the indifference pricing principle deviates from the mean-variance strategy by an additional correction term, which is proportional to $\sqrt{\mathrm{Var}[N^H]}$.

Table 1. Formulae for the various prices for the unit-linked pure endowment contract

Method	Price
Brennan/Schwartz approach	$n\,{}_Tp_y\,V_0^f$
Risk-minimisation	$n\,{}_Tp_y\,V_0^f$
Mean-variance hedging	$n\,{}_Tp_y\,V_0^f$
Super-hedging	$n\,V_0^f$
Indifference price, upper bound	$n\,{}_Tp_y\,V_0^f + a_1\big(\mathrm{E}[\mathrm{Var}[H \mid \mathcal{F}_T^W]]\big)^{1/2}$
Indifference price, lower bound	$n\,{}_Tp_y\,V_0^f + a_2\big(\mathrm{Var}[\mathrm{E}_Q[H \mid \mathcal{F}_T^U]]\big)^{1/2}$

5.3.2 *A comparison of the pricing formulae*

To get an overview of the approaches discussed above, we list in Table 1 the various formulae derived for the price of the unit-linked pure endowment. As in Section 5.1, V_0^f is the price at time 0 of the contract that pays $f(S)$ at time T.

The table shows how the prices computed under the quadratic approaches of risk-minimisation and mean-variance hedging coincide with the price suggested by Brennan & Schwartz (1979a,b). In all three cases, the price is determined as the expected number of survivors $n\,{}_Tp_y$ multiplied with the price V_0^f of the amount $f(S)$ payable upon survival to T. Thus, the pricing principle obtained by assuming that mortality risk is diversifiable can equivalently be derived via a quadratic approach. This property can be explained by the fact that a quadratic approach punishes gains and losses equally, which, in particular, means that untraded risk such as mortality risk will be valued by its expected value. In addition, we mention that, even though the prices for the unit-linked contracts are the same under the criterion of risk-minimisation and mean-variance hedging, the investment strategies under the two approaches actually differ, compare Theorem 1 and 2.

In contrast, the approach of super-hedging requires that the insurer has sufficient capital, even in the extreme situation where all the policyholders survive. This requirement leads to a price given as the *maximum* number of survivors n times the price of $f(S)$. Thus, the price does not involve the survival probability ${}_Tp_y$. However, one can compare this price further with the other prices by noting that it corresponds to using a survival probability of one! The indifference prices under the standard deviation principle consist of two terms: the first term is equal to the price suggested by Brennan/Schwartz; and the second term is a loading term which is related to the part of the risk which cannot be hedged away in the financial market. Note that the loading terms in the prices (5.10) and (5.11) are proportional to $\sqrt{n}$, whereas the first terms (the Brennan/Schwartz prices) are proportional to n. This implies that the price per policyholder converges to the Brennan/Schwartz price for one policyholder ${}_Tp_y\,V_0^f$, when the size n of the portfolio is increased.

5.3.3 *A numerical comparison of prices*

We finally present a small numerical example in order to illustrate the differences between the principles further. The numbers are essentially taken from Møller (2001b), and we refer to this reference for more details. We consider an insurance portfolio consisting of $n = 100$ policyholders with i.i.d. lifetimes and hazard rate function:

$$\mu_{y+t} = 0.0005 + 0.000075858 \cdot 1.09144^{y+t} \qquad t \geq 0. \tag{5.15}$$

Moreover, we take $y = 45$ and $T = 15$, which gives the survival probability ${}_{T}p_y = 0.8796$. The financial market is modelled by the standard Black-Scholes model described in Section 5.2 with parameters $\sigma = 0.25$, $\alpha = 0.10$ and $r = 0.06$. We analyse a portfolio of unit-linked pure endowment contracts with $f(S_T) = \max(S_T, K)$, where we take $K = 0$ (no guarantee) and $K = e^{rT}$ (guarantee corresponding to risk free interest rate). The option price V_0^f can now be computed via the Black-Scholes formula by using that $\max(S_T, K) = (S_T - K)^+ + K$. For $K = e^{rT}$, we get $V_0^f = 1.3718$. In order to be able to compute the indifference prices of Theorem 4 we need that $a^2 \geq \text{Var}[L_T]$. With the notation of Section 5.2, $\text{Var}[L_T] = e^{\nu^2 T} - 1 = 0.4618$, so that indifference prices are only well defined for $a \geq 0.6796$. If we take $a = 1$, the safety loading parameter in (5.14) and Theorem 4 becomes:

$$a_1 = a\sqrt{1 - \frac{\text{Var}[L_T]}{a^2}} = \sqrt{1 - 0.4618} = 0.733.$$

Moreover, we see that a_1 approaches 0 when a converges to 0.6796, which implies that the two bounds on the indifference price converge to the Brennan/Schwartz price. From Table 1 in Møller (2001b), we have that when $K = e^{rT}$, $\text{Var}[N^H] = 100 \cdot 0.460 = 46.0$, and when $K = 0$, $\text{Var}[N^H] = 100 \cdot 0.415 = 41.5$. A few examples of the relation between a, a_1, the loading $a_1\sqrt{\text{Var}[N^H]}$ and the indifference price (5.14) are listed in Table 2 for the situation $K = e^{rT}$.

In Table 3 we have compared the indifference prices with guarantee

Table 2. Indifference prices as a function of the safety loading a for $K = e^{rT}$

Safety-loading parameter, a	0.68	0.70	0.80	1	2
New loading parameter, a_1	0.02	0.17	0.42	0.73	1.88
Loading, $a_1\sqrt{\text{Var}[N^H]}$	0.17	1.14	2.86	4.97	12.75
Indifference price	120.83	121.80	123.52	125.63	133.41

Table 3. Prices for the portfolio of $n = 100$ unit-linked pure endowment contracts with and without guarantee

Method	$K = 0$	$K = e^{rT}$
Brennan/Schwartz	87.96	120.66
Super-hedging	100.00	137.18
Indifference price ($a = 0.70$)	89.04	121.80
Indifference price ($a = 0.80$)	90.68	123.52
Indifference price ($a = 1$)	92.68	125.63
Indifference price ($a = 2$)	100.07	133.41

$K = e^{rT}$ and no guarantee, respectively, with the super-hedging price and the Brennan/Schwartz prices. These numbers can be reconstructed from the numbers given above. We note that for the case of no guarantee ($K = 0$), the price computed with safety-loading parameter $a = 2$ leads to a price which exceeds the super-hedging price by 0.07. This example illustrates an undesirable property of the indifference pricing principle based on the standard deviation principle; it might lead to prices which are larger than the super-hedging price. The same phenomenon will occur for the guarantee $K = e^{rT}$ for sufficiently big values of a. Thus, one should be careful when applying this indifference principle for general contracts and check whether prices exceed the super-hedging price.

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