PREMIUM CALCULATION FOR DEDUCTIBLE POLICIES WITH AN AGGREGATE LIMIT

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Abstract

In Industrial Fire insurance an aggregate limit for the amount retained by the policyholder under a deductible policy has been agreed upon more frequently in recent times. This agreement is equivalent to a stop-loss cover on the retained loss amount. For the Poisson-lognormal model the corresponding stop-loss net premium is calculated using various methods (normal power, translated gamma, various discretisations) and the methods are compared. Finally, the influence of the model parameters is examined and it is demonstrated how a variety of parameter value combinations can be reduced to only a few rating curves.

Keywords

Deductibles, aggregate limit, stop-loss premium, Industrial Fire insurance, Poisson-lognormal model.

1. Introduction

On a number of markets the practice of adding an aggregate limit to an Industrial Fire insurance policy with a deductible has increased in recent times. An aggregate limit means that the maximum accumulated amount of losses to be retained by the policyholder is limited for each year; the insurer then takes over payment should this maximum be exceeded. The advantage for the policyholder is quite obvious: the risk retained under the deductible is limited, not only in terms of each loss event but also on an annual basis. A policy with a deductible but no aggregate limit, however, may lead to an unexpectedly high retained aggregate loss amount if the policyholder is confronted with an accumulation of loss events. For the insurer, the calculation of deductible rebates, difficult enough as it is, becomes even more complicated. With the aggregate limit, the policyholder is granted in addition a stop-loss cover on his retained losses, which leads to a reduction in the normal deductible rebate. If the size of a loss is independent of the number of losses, the normal deductible rebate depends solely on the distribution of the loss amounts, whereas when an aggregate limit is established, the distribution of the annual number of losses has to be considered too. Moreover, the risk of fluctuation, which in connection with deductibles works against the insurer anyway (cf. Sterk (1979), Mack (1980, 1983)), is increased even further by an aggregate limit.

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In theory, there is no problem involved in developing a formula for the reduction in the deductible rebate resulting from the aggregate limit: the formula for the stop-loss net premium (i.e., the loss expectancy of the stop-loss cover) may be applied without further ado to the distribution of the aggregate retained losses. But the computation itself is a problem, as it is a well-known fact that only in rare cases a closed analytical expression can be given for the distribution of aggregate losses. In Industrial Fire insurance there is the additional problem that the distributions of loss frequency and loss amounts are not known precisely enough (at least for the individual risk to be rated), and rough estimates for some parameters of these distributions are the best we have. It is therefore necessary to use model assumptions that are flexible and cover a broad spectrum of realistic possibilities.

This paper follows the assumption that the distribution of the annual number of losses is Poisson and that the distribution of the loss amounts is lognormal.

It is widely accepted that the Poisson distribution is realistic for the number of losses in Industrial Fire portfolios. Also the validity of the lognormal model for the loss amounts has been demonstrated on several occasions in the past (e.g., BENCKERT (1962), FERRARA (1971), STRAUSS (1975)), and in the field of Industrial Fire in particular.

Generally these distributions cannot immediately be transferred to single risks due to the influence of a big fire on the loss distributions. But a policy for which an aggregate limit is agreed is usually so large that it can be considered as a small portfolio. Therefore the application of the Poisson-lognormal model seems to be an acceptable approximation.

The information available in insurance practice on the loss distribution of the risk to be rated consists for the most part of only the net premium and no more. Therefore in order to estimate the two parameters of the lognormal distribution, additional information is necessary. In this paper it is assumed that the normal deductible rebate is also known, i.e., the reduction in the loss expectancy due to the deductible without the aggregate limit being taken into account. But the calculation of deductible rebates will not be discussed in any further detail as this is dealt with excellently in STERK (1979, p. 180ff). Should the normal deductible rebate not be known, then use can be made of the results of BENCKERT (1962), FERRARA (1971) and STRAUSS (1975), where for one of the two parameters a relatively small range of values was established that is independent of the monetary unit and thus of currency, inflation, etc.

If the mean loss amount, the net premium for full insurance cover, the deductible amount and the corresponding deductible rebate are known, the parameters of the Poisson-lognormal model are determined in full (mean number of losses = net premium/mean loss amount). And in practice these figures are on hand as a rule or they can at least be estimated with a sufficient degree of accuracy by the underwriter. With these figures, the distribution of aggregate losses is determined for the policyholder's retained amount under the deductible before accounting for the aggregate limit. Then for the calculation of the stop-loss net premium, defined by the aggregate limit on this aggregate retained loss, three different ways of approximating the stop-loss net premium are used:

- The "normal power" and "translated gamma" methods based on an analytical approximation of the distribution of aggregate losses. These procedures are extremely simple to handle and require no programming. Up to now, however, little is known of the quality of the results in such cases as the one here with a rather low mean number of losses.
- The method of approximating the loss amount distribution by means of very simple discrete distributions (one, two and three point distributions) for which the stop-loss net premium can be calculated explicitly and simply. Due to the limitation of the amount of each loss by the deductible these methods turn out to give excellent results.
- The recursive procedure for arithmetic distributions, first described by PANJER (1980); the required discretisation follows the "matching moments" method developed by GERBER (1982). This procedure produces results that may be as exact as required depending on the degree of discretisation.

The aim of these comparative calculations is not only to check the quality of these procedures, but first and foremost to find a procedure which is as simple as possible and which at the same time produces acceptable exactness. In addition, the final section investigates the influence of each of the model parameters and suggests a procedure for reducing the large number of possible combinations to a few special cases in order to derive simple rating rules for underwriters.

2. PROBLEM AND NOTATIONS

Let the following data be known for a given risk:

b = net premium (expected value of the aggregate losses) for full insurance cover

c = mean loss amount per loss event

a = deductible amount

$r(a)$ = (net) deductible rebate = reduction in the net premium resulting from the deductible, $0 \leq r(a) \leq 1$

z = annual aggregate limit for the accumulated retained losses under the deductible; z is often expressed as a multiple $z = ka$ of the deductible amount, e.g., $k = 3$.

The expected value of the aggregate retained losses under the deductible before accounting for the aggregate limit is then given by $r(a)b$. The problem is to find the expected value $r(a, z)b$ of the aggregate retained losses considering the aggregate limit z. With $1 - [r(a, z)/r(a)]$ we thus obtain the proportion by which the deductible rebate $r(a)$ is to be reduced as a result of the additional aggregate limit.

The following random variables are considered:

X = loss amount per loss event

N = number of losses per year (assumed to be independent of X)

X_a = retained loss amount (per loss event) under deductible a

$$= \begin{cases} X & \text{if} \quad X \leq a \\ a & \text{if} \quad X > a \end{cases}$$

S_a = aggregate retained losses (per year) under deductible a

$$= \begin{cases} 0 & \text{if } N = 0 \\ \sum_{i=1}^{N} (X_a)_i & \text{if } N > 0 \end{cases}$$

where $(X_a)_i$ denotes the retained amount of the ith loss

$S_{a,z}$ = aggregate retained losses (per year) under deductible a and aggregate limit z

$$= \begin{cases} S_a & \text{if} \quad S_a \leq z \\ z & \text{if} \quad S_a > z. \end{cases}$$

With the given data, the following relationships exist

$$b = E(N)E(X)$$

$$c = E(X)$$

$$r(a)b = E(S_a) = E(N)E(X_a)$$

$$r(a) = \frac{E(S_a)}{b} = \frac{E(X_a)}{E(X)}.$$

Then $E(S_{a,z}) = r(a, z)b$ is to be calculated under the assumption that N is subject to a Poisson distribution and X to a lognormal distribution, i.e., (with Φ denoting the standard normal distribution function)

$$F(x) = p(X \leq x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right) \quad \text{for } 0 < x < \infty,$$

$$f(x) = F'(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left(-\tfrac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right),$$

$$p(N = i) = \frac{\lambda^i}{i!} e^{-\lambda} \quad \text{for } i = 0, 1, 2, \ldots.$$

The parameter λ is given by

$$\lambda = E(N) = \frac{b}{c},$$

and the parameters μ and σ can be deduced from the values for a, $r(a)$ and c with the aid of the formulae

$$c = E(X) = \exp(\mu + \tfrac{1}{2}\sigma^2),$$

$$r(a) = \frac{1}{E(X)}\left\{\int_0^a xf(x)\,dx + a(1-F(a))\right\}$$

$$= \Phi\left(\frac{\ln a - \mu}{\sigma} - \sigma\right) + \frac{a}{c}\left(1 - \Phi\left(\frac{\ln a - \mu}{\sigma}\right)\right)$$

(cf. STERK 1979, p. 234). For this purpose it is convenient to introduce

$$t = \frac{a}{c}.$$

Then the equation for $r(a)$ can be rewritten as

$$r(a) = \Phi\left(\frac{\ln t}{\sigma} - \frac{\sigma}{2}\right) + t\left(1 - \Phi\left(\frac{\ln t}{\sigma} + \frac{\sigma}{2}\right)\right).$$

This equation has a unique solution σ (given t and $r(a)$) because the right-hand side is a strictly increasing function of σ. Parameter μ is thus replaced with t. Besides z we now have to work with the three model parameters t, σ and λ.

Should it happen that the deductible rebate $r(a)$ is not known, it may be possible to choose the parameter value of σ from the interval [2, 2.5] in accordance with the results of BENCKERT (1962), FERRARA (1971) and STRAUSS (1975).

If $E(S_a - z)^+$ denotes the stop-loss net premium with priority (stop-loss attachment point) z on the aggregate retained losses S_a, i.e.,

$$E(S_a - z)^+ = E(S_a) - E(S_{a,z}),$$

the required reduction in the deductible rebate comes to

$$1 - \frac{r(a, z)}{r(a)} = \frac{E(S_a - z)^+}{E(S_a)}.$$

This expression, i.e., the stop-loss net premium measured as a fraction of the mean aggregate retained losses without an aggregate limit, will be called "relative stop-loss net premium" in the following discussion. Similarly, the value

$$k = \frac{z}{a}$$

i.e., the priority expressed as a multiple of the deductible amount, is referred to as "relative priority".

The curve of values of the relative stop-loss net premium as a function of the relative priority is called "stop-loss curve"; it begins at point (0; 1), is degressively and strictly decreasing (convex) and runs to point (∞; 0). The "relevant area" is that part of the curve in which the relative stop-loss net premium amounts to between 50% and 5% as in practice the majority of cases occur in this range.

3. CALCULATION METHODS

As the distribution of the aggregate retained losses S_a, which is required for an exact calculation of the stop-loss net premium, cannot be given in closed form, various approximation methods have been developed in actuarial literature. Several of these methods have been applied in the problem here. As most of these methods are well-known, no further details are given. This applies to the following methods:

1. Normal power method, see BEARD, PENTIKÄINEN and PESONEN (1968/1977, p. 43ff); BERGER (1972); KAUPPI and OJANTAKANEN (1969); PESONEN (1969). More precisely, the NP2 method was used here, i.e., the changed variable was calculated from a quadratic equation.
2. Translated gamma method, see BOHMAN and ESSCHER (1963/1964); SEAL (1977); BOWERS, GERBER, HICKMAN, JONES and NESBITT (1982). In the expression for the stop-loss net premium (cf. SEAL 1977, p. 215) the incomplete gamma function occurs.
3. Recursive calculation of the stop-loss net premium by means of an arithmetic discretisation of the loss amount distribution, see GERBER (1982) who uses the recursive procedure of PANJER (1980). In the problem here the discretisation method called "matching moments" was used where the probability weights for the discretised variable are calculated in such a way that within adjacent pairs of intervals the first two moments for the discretised loss amount are equal to those of X_a according to the lognormal distribution. As an obvious extension of the recursion formula stated by PANJER and GERBER, the occurrence of losses of amount 0 for the discretised distribution was explicitly admitted as this proved to be suitable due to the skewness of the distribution of X_a in order to avoid negative probabilities.

With the normal power and the translated gamma method an estimation of the approximation error is not possible. But for the method with a discretisation of the loss amount distribution an upper bound for the approximation error can be developed using a metric introduced by GERBER (1980) (Chapter 7.3):

$$\max_{z\geq 0} |E(S_a-z)^+ - E(\tilde{S}_a-z)^+| \leq \lambda \max_{0\leq x\leq a} |E(X_a-x)^+ - E(\tilde{X}_a-x)^+|$$
$$= \lambda \max_{0\leq x\leq a} \left| \int_x^a (\tilde{F}(y)-F(y))\, dy \right|,$$

where the symbol $\tilde{}$ refers to the discrete approximating distribution.

If the discretised loss amount distribution only has one or two atoms, the distribution of the aggregate losses and thus the stop-loss net premium can generally be calculated very easily without recursion formula. On account of the finite range $(0, a]$ of the retained loss amount X_a it does not seem unreasonable to approximate the distribution of X_a by such a one-point or two-point distribution. Indeed it will be shown that this method in the problem here leads to astoundingly good results. For this method too, the above formula for the error bound holds true. For the choice of the atoms of the approximating loss amount distribution there are several possibilities (one-point: lower bound, upper bound, third approach; two-point: 1st, 2nd, 3rd possibility), details of which are given in the appendix.

4. COMPARISON OF CALCULATION METHODS

In principle, the matching moments discretisation is the most accurate method of calculating as the accuracy can theoretically be improved as far as desired by raising the number $n+1$ of discretisation points. It is possible that numerical problems will arise when the value of n reaches a certain size, but here it was not necessary to go so far as a better approximation was already arrived at for a relatively small n than with the other methods. Table 1 shows a typical example.

TABLE 1
RELATIVE STOP-LOSS NET PREMIUM (%) USING VARIOUS METHODS
(PARAMETERS $\sigma=2$, $t=1$, $\lambda=3$)

Method	$k=1.0$	$k=1.5$	$k=2.0$	$k=2.5$
Matching moments, $n=100$	32.573	16.375	7.4675	3.2266
$n=30$	32.571	16.373	7.4663	3.2259
$n=10$	32.552	16.350	7.4558	3.2187
Normal power	33.4	16.9	7.97	3.56
Translated gamma	32.1	15.9	7.44	3.33
Two-point, 1st possibility	33.4	16.1	8.03	3.218
2nd possibility	32.0	16.9	7.05	3.41
3rd possibility	32.52	16.37	7.452	3.244
One-point, lower bound	21	6	1.4	0.2
upper bound	35	23	9.6	5.8
third approach	33.5	14.8	7.30	2.97

With the matching moments method for $n=100$ the maximum error amounts to less than $\pm 0.05\%$ according to the inequality in Section 3, i.e., the exact value e.g. for $k=1.0$ is between 32.523% and 32.623%. The normal power method generally overestimated the stop-loss net premium; in all the parameter combinations examined ($\sigma=2$, $t=0.1$, 0.3, 1.0, 3.0, 10.0, $\lambda=1$, 3, 10, 30), only for $\lambda=1$ and $t\leq 1$ was there a small area where this was not the case. Where the aggregate loss distribution was very skewed ($t=10$, $\lambda=1$) the normal power method overestimated the stop-loss net premium in the relevant area by more than one half of the true value in some cases. If the aggregate loss distribution is practically a normal distribution ($\lambda=30$, $t\leq 3$; here the skewness is <0.5) the normal power method, like the translated gamma and the two-point too, produces a very good approximation to 3 decimal places. The relative error however increases with higher priorities (i.e., with a lower stop-loss net premium).

The values produced by the translated gamma method were above and below the exact values in all of the parameter combinations examined, i.e., the translated gamma stop-loss curve intersects the exact stop-loss curve, and in the cases examined more than once. A very good approximation is produced of course near to the intersections. But the accuracy of the translated gamma method is not at every point better than that of the normal power method.

The stop-loss curves of each of the two-point methods also meet the exact curve usually more than once. As shown in Table 2, the third two-point method seems to provide the best approximation apart from the matching moments method. Sometimes, however, this method can produce a small range of values, where the deviation is greater than in the normal power method.

Of the three one-point methods only the third approach produces acceptable results especially if the skewness of the aggregate loss distribution is small (e.g., <0.5). This method may underestimate the true result. The other two methods should be considered as being the simplest way of providing lower and upper bounds rather than being approximations.

Table 2 is an attempt to compare the accuracy of the various methods. For this the values of the methods per priority were put in order of accuracy; the method with the value nearest to the matching moments value was given the order number 1, going down to order number 8 for the method with the value which was furthest removed. Then for each method the mean order number was calculated for a larger number of priorities, which were chosen equidistant in the relevant area.

TABLE 2

MEAN ORDER NUMBER OF THE VARIOUS METHODS IN TERMS OF ACCURACY

Parameters ($\sigma = 2$ throughout)	Method (see key below) N.P.	T.G.	TP1	TP2	TP3	OPL	OPU	OP3
$t = 0.1$, $\lambda = 3$	4.1	3.9	3.1	2.9	1.8	8.0	7.0	5.2
$\lambda = 10$	5.1	4.2	2.5	2.5	1.6	8.0	7.0	5.2
$t = 1.0$, $\lambda = 3$	4.2	3.0	3.9	3.5	1.6	8.0	7.0	4.8
$\lambda = 10$	5.0	3.6	3.3	2.5	1.5	8.0	7.0	5.0
$t = 10$, $\lambda = 3$	3.7	2.5	4.2	3.6	1.8	7.0	8.0	5.2
$\lambda = 10$	3.9	2.5	4.5	3.9	1.7	7.5	7.5	4.6

Key: N.P. = normal power; T.G. = translated gamma; TPi = two-point ith possibility; OPL = one-point, lower bound; OPU = one-point, upper bound; OP3 = one-point, third approach.

The results in this table cannot however be simply transferred to other parameter combinations. For $t = 1$ and $\lambda = 30$, for example, procedure TP2 has a lower mean order number than TP3; here however, all the methods are exact to three decimal places. For $\lambda = 1$ the value according to normal power in the relevant area is sometimes higher than the upper bound given by OPU.

Finally in this connection certain computing problems must be mentioned too. The possible occurrence of negative probabilities in the matching moments method has already been pointed out by GERBER (1982). But these negative probabilities do not seem to have any distorting influence on the stop-loss net premiums calculated with them. With the two-point methods it is possible that when the Poisson probabilities for high priorities are calculated, an underflow will occur, meaning that values are produced that are too small to be expressed in the computer. As these are summands and not very many either, they can be given a value of zero without having any noticeable effect on the accuracy but in general an appropriate instruction should be included in the computer program to avoid an abnormal program termination. An overflow in the translated gamma method occurred for $\lambda = 30$ and $t = 0.1$ or 0.3 in the calculation of the incomplete gamma function, i.e., values were produced which were so large they could not be expressed in the computer. This error can only be avoided by means of applying special techniques in calculating the incomplete gamma function ratio (see KHAMIS and RUDERT 1965). Difficulties in calculating were encountered in all the methods apart from the normal power method—and, of course, the one-point methods. As the normal power method produces results that are nearly always on the safe side and as the safety loading increases relative to the decrease in the stop-loss net premium as it should, this method can be generally recommended, especially if the results have to be produced quickly and without any programming.

5. DEPENDENCY OF THE RESULTS ON THE PARAMETERS

In practice it is recommended that underwriters are given simple rating tables or curves so that they do not have to consult the actuarial department each time a policy with an aggregate limit comes up. In view of the dependency of the stop-loss net premium on three parameters (σ, λ, t) it does not seem possible to provide a calculation model of the kind mentioned. Surprisingly enough however, it is possible to eliminate all three parameters to a large extent if a slight reduction in accuracy is acceptable. In view of the uncertainty of the parameter values pertaining to any one risk, this loss in accuracy can be ignored.

At first it is not automatically clear what influence the variation of one single parameter will have on the relative stop-loss net premium where the other parameters and the relative priority remain constant, as the incorporation of relative values may produce different results to those produced by absolute values. In the case where parameter λ increases, the mean number of losses increases too while the priority (both relative and absolute) remains unchanged. It is therefore obvious that the stop-loss net premium increases overproportionally and leads to an increase in the relative stop-loss net premium. In case of variation of the parameter σ or t it is best to observe the shape of the density function of the amount of retained loss X_a. As t increases, the proportion $E(X_a)/a$ decreases too, meaning that the distribution of X_a is skewed more and more to the right. If σ, λ and k are constant, therefore, the absolute priority ka will increase in relation to the mean aggregate retained losses $E(S_a) = \lambda E(X_a)$ so that the relative stop-loss net premium decreases. The same applies when parameter σ is raised:

a higher σ means a more skewed lognormal distribution with a higher expected value $E(X)$. If t is constant, the deductible $a = tE(X)$ increases too as that the distribution of X_a becomes more skewed. If λ is constant, the absolute priority therefore becomes greater in relation to $E(S_a)$ so that the relative stop-loss net premium decreases.

On account of this, with a constant λ for two pairs of parameters (σ, t), $(\tilde{\sigma}, \tilde{t})$, a similar stop-loss net premium is to be expected if the same proportion $E(X_a)/a = r(a)/t$ is produced in both cases, i.e., if

$$\frac{1}{t}\Phi\left(\frac{\ln t}{\sigma}-\frac{\sigma}{2}\right)-\Phi\left(\frac{\ln t}{\sigma}+\frac{\sigma}{2}\right)=\frac{1}{\tilde{t}}\Phi\left(\frac{\ln \tilde{t}}{\tilde{\sigma}}-\frac{\tilde{\sigma}}{2}\right)-\Phi\left(\frac{\ln \tilde{t}}{\tilde{\sigma}}+\frac{\tilde{\sigma}}{2}\right).$$

Table 3 shows that this is in fact the case. Here the value of t for the various values of σ is selected in such a way that the equation above holds with $\tilde{\sigma} = 2$ and $\tilde{t} = 1$.

TABLE 3

COMPENSATING A VARIATION OF t WITH A VARIATION OF λ

Parameter Values		Relative Stop-Loss Net Premium (%) for $\lambda = 3$			
σ	t	$k=1$	$k=1.5$	$k=2$	$k=2.5$
1.6	1.70	31.4	15.2	6.75	2.79
1.8	1.33	32.0	15.8	7.14	3.02
2.0	1.00	32.6	16.4	7.47	3.23
2.2	0.72	33.0	16.8	7.72	3.39
2.4	0.50	33.3	17.2	7.94	3.55

This being so, it is possible to transpose parameter values $\sigma \neq 2$ to the case $\sigma = 2$ by an appropriate alteration of the parameter t without any essential change in the stop-loss curve. In this way parameter σ is practically eliminated.

If σ is constant a similar situation arises for the influence of variations of the parameters t and λ. Table 4 shows that an increase of t can be compensated by an appropriate increase of λ so that the stop-loss curve remains almost unchanged.

Therefore, parameter t can be eliminated by an appropriate correction of the value of λ. Finally, with σ and t constant and a given value for the relative

TABLE 4

COMPENSATING A VARIATION OF σ WITH A VARIATION OF t

Parameter Values		Relative Stop-Loss Net Premium (%) for $\sigma = 2$				
t	λ	$k=1$	$k=1.5$	$k=2$	$k=2.5$	$k=3$
0.1	2.4	52.0	35.4	21.9	13.6	8.03
0.3	3.4	52.6	35.4	22.3	13.5	7.68
1.0	6	53.4	35.7	22.6	13.4	7.59
3.0	12	54.2	36.1	22.6	13.2	7.32
10.0	31	56.0	37.5	23.1	13.2	7.01

stop-loss net premium, there is an almost linear connection between λ and that value of relative priority which leads to the given stop-loss net premium for this λ:

Example for $\sigma = 2$, $t = 1$:

Parameter Value of λ	1	3	10	30
Relative priority k corresponding to 10% stop-loss net premium	1.09	1.83	3.96	9.74
Straight line $0.31\lambda + 0.81$	1.12	1.74	3.91	10.11
Relative priority k corresponding to 30% stop-loss net premium	0.69	1.06	2.54	6.83
Straight line $0.21\lambda + 0.46$	0.67	1.09	2.56	6.76

This makes it possible to derive from the stop-loss curves for 2 values of the parameter λ the curves for the other values of λ approximately by means of interpolation or extrapolation.

To sum up then, we may say: the stop-loss curves resulting from the lognormal distribution by a deductible with an aggregate limit have very similar shapes for the relevant parameter values of σ, t and λ, and with the aid of appropriate parameter transformations they can be approximately interchanged. It is therefore possible to represent the effect of an aggregate limit on the expected losses in such a way that it can be determined using only a few curves or tables without any great reduction in accuracy.

APPENDIX

Calculation of the Stop-Loss Net Premium by Simple Discrete Approximations of the Distribution of Loss Amounts with One-Point or Two-Point Distributions

This appendix will deal with one-point and two-point distributions as well as a special three-point distribution for the loss amounts, i.e., distributions that allow for only one, two or three different loss amounts. For such distributions, the distribution of the aggregate losses and thus the stop-loss net premium can be calculated exactly without great difficulty. On account of the finite range $(0, a]$ of the loss amount X_a it does not seem unreasonable to approximate the distribution of X_a by such a distribution.

In the following, we shall frequently be needing the first three moments about zero of the retained loss X_a; for $i = 1, 2, 3 \ldots$ we have:

$$E(X_a)^i = \int_0^a x^i \, dF(x) + a^i(1 - F(a))$$
$$= \exp\left(i\mu + \tfrac{1}{2}i^2\sigma^2\right)\Phi\left(\frac{\ln a - \mu - i\sigma^2}{\sigma}\right) + a^i\left(1 - \Phi\left(\frac{\ln a - \mu}{\sigma}\right)\right).$$

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Here again it is convenient to replace parameter μ with t by writing

$$a = tc = tE(X) = t \exp(\mu + \tfrac{1}{2}\sigma^2)$$

for the mean loss amount. This results in

$$E(X_a)^i = c^i\left\{\exp(\tfrac{1}{2}i(i-1)\sigma^2)\Phi\left(\frac{\ln t}{\sigma} - (i-\tfrac{1}{2})\sigma\right) + t^i\left(1 - \Phi\left(\frac{\ln t}{\sigma} + \tfrac{1}{2}\sigma\right)\right)\right\}.$$

The values

$$h_i = c^{-i}E(X_a)^i$$

now only depend on $t = a/c$ and σ. Note that $h_1 = r(a)$ applies. In the following discussion it will always be assumed that the values t, σ and λ are known and therefore h_1, h_2, h_3 too.

A1. *Approximation by Means of One-Point Distributions*

The most simple way of approximating is to work only with a constant loss amount $\theta = E(X_a)$, i.e. to approximate the aggregate retained losses S_a by means of θN. According to a theorem of BÜHLMANN, GAGLIARDI, GERBER and STRAUB (1977), this results in a lower bound for the stop-loss net premium for each priority z, i.e.,

$$E(S_a - z)^+ \geq E(\theta N - z)^+ = \theta E\left(N - \frac{z}{\theta}\right)^+.$$

Another approximation stemming from BENKTANDER only uses losses of the (maximum) amount a. So that the expected value of the aggregate losses remains unchanged, the mean number of losses must be reduced mechanically to

$$\lambda^* = \frac{\theta}{a}\lambda.$$

If N^* denotes the Poisson variable belonging to λ^* then S_a will be approximated by aN^* and we have

$$E(S_a - z)^+ \leq E(aN^* - z)^+ = aE\left(N^* - \frac{z}{a}\right)^+,$$

i.e., an upper bound for the stop-loss net premium is obtained. This also results from the theorem of BÜHLMANN, GAGLIARDI, GERBER and STRAUB (1977). Here the fact is employed that the distribution of aggregate losses based on the number of losses N^* and the constant loss amount a is identical with the distribution of aggregate losses which results from the number of losses N and the loss amount "a with probability θ/a or 0 with probability $1 - \theta/a$". This is a special feature of the Poisson distribution.

In these cases, the explicit calculation of the stop-loss net premium is quite simple:

$$E(N-u)^{+} = \sum_{i>u} (i-u)p(N=i)$$

$$= \sum_{i>u} i\frac{\lambda^{i}}{i!}e^{-\lambda} - up(N>u)$$

$$= \lambda \sum_{i>u} \frac{\lambda^{i-1}}{(i-1)!}e^{-\lambda} - up(N>u)$$

$$= \lambda p(N=[u]) + (\lambda - u)p(N>u),$$

$[u]$ denoting the integer part of u.

In the special problem here, with a priority of $z = ktc$, this produces for the stop-loss net premium $E(S_a - z)^{+}$

- a lower bound

$$ch_1\{\lambda p(N=[v]) - (v-\lambda)(1-p(N \leqslant v))\}$$

with $v = kt/h_1$

- and an upper bound

$$ch_1\{\lambda p(N^*=[k]) - (v-\lambda)(1-p(N^* \leqslant k))\}$$

with $v = kt/h_1$, N^* being Poisson distributed with parameter $(h_1/t)\lambda$.

In a third approach, due to BENKTANDER (1974), the distribution of the aggregate losses S_a is directly approximated by the distribution of $\zeta\tilde{N}$ where $\tilde{N}$ is Poisson distributed with parameter $E(\tilde{N}) = \tilde{\lambda}$ and the values of ζ, $\tilde{\lambda}$ are determined by the equations

$$E(\zeta\tilde{N}) = \zeta\tilde{\lambda} = E(S_a),$$

$$\text{Var}(\zeta\tilde{N}) = \zeta^2\tilde{\lambda} = \text{Var}(S_a).$$

This yields

$$\zeta = \frac{\text{Var}(S_a)}{E(S_a)} = \frac{E(X_a)^2}{E(X_a)}$$

$$\tilde{\lambda} = \frac{(E(S_a))^2}{\text{Var}(S_a)} = \frac{(E(X_a))^2}{E(X_a)^2}\lambda.$$

For the stop-loss net premium with priority z we then get the approximation

$$E(S_a - z)^{+} \approx E(\zeta\tilde{N} - z)^{+} = \zeta E\left(\tilde{N} - \frac{z}{\zeta}\right)^{+}.$$

In the problem to be solved here this leads to (with priority $z = ktc$)

$$ch_1\{\lambda p(\tilde{N}=[\tilde{v}]) - (v-\lambda)(1-p(\tilde{N} \leqslant \tilde{v}))\}$$

with $v = kt/h_1$, $\tilde{v} = kth_1/h_2$ and $\tilde{N}$ being Poisson distributed with parameter $(h_1^2/h_2)\lambda$.

It is easy to see that generally $\tilde{\lambda} \leq \lambda$ and $\theta \leq \zeta \leq a$ holds and using again the theorem of BÜHLMANN, GAGLIARDI, GERBER and STRAUB (1977) it can be shown that the stop-loss net premium according to this third approach is between the bounds defined above, i.e.,

$$E(\theta N - z)^+ \leq E(\zeta \tilde{N} - z)^+ \leq E(aN^* - z)^+.$$

A2. *Approximation by Means of Two-Point Distributions*

Calculating the stop-loss net premium with priority z is still a quite simple matter for a distribution of loss amounts that only provides for two different loss amounts $x < y$ with probability p for x and $q = 1 - p$ for y: if W_{xy} denotes the corresponding variable of aggregate losses with λ-Poisson distributed number of losses N, then the aggregate losses, in the case of exactly $N = j$ losses, i of which have an amount y, come to

$$W_{xy} = iy + (j-i)x = i(y-x) + jx, \qquad i = 0, 1, \ldots, j,$$

with probability

$$\binom{j}{i} q^i p^{j-i} \frac{\lambda^j}{j!} e^{-\lambda}.$$

The constraint $W_{xy} \leq z$ is equivalent to $i \leq (z - jx)/(y - x)$. As $[z/x]$ losses may occur for $W_{xy} \leq z$ at the most, this leads to

$$E(z - W_{xy})^+ = \sum_{j < z/x} \frac{\lambda^j}{j!} e^{-\lambda} \sum_{i \leq w} \binom{j}{i} p^{j-i} q^i (z - jx - i(y-x))$$

with $w = \min(j, (z - jx)/(y - x))$.

This can be worked out on a programmable pocket calculator with 10 memory registers; the size of the factorials does not constitute a problem either as long as the corresponding summands are calculated recursively. Finally the stop-loss net premium is given by

$$\begin{aligned} E(W_{xy} - z)^+ &= E(W_{xy}) - z + E(z - W_{xy})^+ \\ &= \lambda(px + qy) - z + E(z - W_{xy})^+. \end{aligned}$$

There are several ways of approximating the retained loss amount X_a by means of a two-point distribution:

1st possibility:
The distribution of X_a is produced by truncating the distribution of X at the point a, i.e., the distribution of X_a always has a point mass amounting to $1 - F(a)$ at point a. Particularly where low deductibles are concerned, the obvious way is therefore to choose the two-point distribution in such a way that $y = a$. Then the other point x and its point mass p are selected so that the first two moments are

equal to those of X_a, i.e.:

$$px + (1-p)a = E(X_a)$$
$$px^2 + (1-p)a^2 = E(X_a)^2.$$

This leads to

$$p = \frac{(t-h_1)^2}{t^2 - 2th_1 + h_2}$$
$$q = 1-p$$
$$x = \frac{c}{p}(h_1 - qt)$$
$$y = tc.$$

2nd possibility:
If a value for y other than a is admitted, the two-point distribution can be selected in such a way that the first three moments are equal to those of X_a, that is

$$px + (1-p)y = E(X_a)$$
$$px^2 + (1-p)y^2 = E(X_a)^2$$
$$px^3 + (1-p)y^3 = E(X_a)^3.$$

If ξ denotes the skewness of X_a, i.e.,

$$\xi = (h_3 - h_1(3h_2 - 2h_1^2))(h_2 - h_1^2)^{-3/2},$$

then we get

$$p = \frac{1}{2} + \frac{\xi}{2\sqrt{4+\xi^2}}$$
$$q = 1-p$$
$$x = c\left(h_1 - \sqrt{\frac{q}{p}(h_2 - h_1^2)}\right)$$
$$y = c\left(h_1 + \sqrt{\frac{p}{q}(h_2 - h_1^2)}\right).$$

For the corresponding aggregate losses W_{xy}, the first three moments are equal to those of the aggregate retained losses S_a, as is the case too with the normal power and translated gamma methods.

3rd possibility (special three-point distribution):
If a procedure is desired whereby a point mass of $y = a$ is retained as in the first possibility and at the same time the first three moments of X_a are considered as in the second possibility, then this is feasible, similiarly as in the upper bound one-point distribution, if a point mass at loss amount 0 is added and the Poisson

parameter is adjusted accordingly. More precisely a three-point distribution is adjusted with points 0, x, a and point masses u, v, w so that

$$u+v+w=1$$

$$vx+wa=E(X_a)$$

$$vx^2+wa^2=E(X_a)^2$$

$$vx^3+wa^3=E(X_a)^3.$$

This yields

$$w=\frac{h_1h_3-h_2^2}{(h_1t^2-2h_2t+h_3)t}$$

$$v=\frac{(h_1-wt)^2}{h_2-wt^2}$$

$$u=1-v-w$$

$$x=\frac{c}{v}(h_1-wt).$$

In the case of a λ-Poisson distributed number of losses, the corresponding aggregate loss distribution does not change if the loss amount 0 and its point mass u are omitted and the other point masses v and w are raised accordingly and the parameter λ reduced, i.e.,

$$p=\frac{v}{v+w}$$

$$q=\frac{w}{v+w}=1-p$$

$$\lambda^*=\lambda(v+w).$$

The corresponding stop-loss net premium can therefore be calculated using the two-point distribution given by p, q, x and $y=a=tc$; in this case the reduced mean number of losses λ^* is to be used instead of λ.

Further possibilities:
If a value for y is admitted in the 3rd possibility other than $y=a$, it is possible to have the same first four moments of the special three point distribution as those of X_a. Another possibility is to break the interval $[0, a]$ into two intervals and to apply each of the first two one-point methods to each of these intervals. In this way, two-point distributions are produced which give improved upper and lower bounds for the stop-loss net premium.

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