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ALMOST NONNEGATIVE CURVATURE ON SOME FAKE 6- AND 14-DIMENSIONAL PROJECTIVE SPACES

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Abstract

We apply the lifting theorem of Searle and the second author to put metrics of almost nonnegative curvature on the fake $\mathbb{R}P^6$s of Hirsch and Milnor and on the analogous fake $\mathbb{R}P^{14}$s.

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1. Introduction

One of the great unsolved problems of Riemannian geometry is to determine the structure of collapse with a lower curvature bound. An apparently simpler, but still intractable, problem is to determine which closed manifolds collapse to a point with a lower curvature bound. Such manifolds are called almost nonnegatively curved. Here we construct almost nonnegative curvature on some fake $\mathbb{R}P^6$s and $\mathbb{R}P^{14}$s.

THEOREM 1.1. *The Hirsch–Milnor fake $\mathbb{R}P^6$s and the analogous fake $\mathbb{R}P^{14}$s admit Riemannian metrics that simultaneously have almost nonnegative sectional curvature and positive Ricci curvature.*

REMARK 1.2. By considering cohomogeneity one actions on Brieskorn varieties, Schwachhöfer and Tuschmann observed in [14] that in each odd dimension of the form $4k + 1$, there are at least 4^k oriented diffeomorphism types of homotopy $\mathbb{R}P^{4k+1}$s that admit metrics that simultaneously have positive Ricci curvature and almost nonnegative sectional curvature.

The Hirsch–Milnor fake $\mathbb{R}P^6$s are quotients of free involutions on the images of embeddings ι of the standard 6-sphere, $\mathbb{S}^6$, into some of the Milnor exotic 7-spheres, Σ^7_k [11, 13]. Our proof begins with the observation that the $SO(3)$-actions that Davis constructed on the Σ^7_ks in [5] leave these Hirsch–Milnor S^6s invariant and commute with the Hirsch–Milnor free involution. Next we compare the Hirsch–Milnor/Davis

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$(SO(3)\times\mathbb{Z}_2)$-action on $\iota(S^6)\subset\Sigma^7_k$ with a very similar linear action of $(SO(3)\times\mathbb{Z}_2)$ on $\mathbb{S}^6\subset\mathbb{R}^7$ and apply the following lifting result of Searle and the second author.

THEOREM 1.3 [16, Proposition 8.1 and Theorems B and C]. *Let (M_e, G) and (M_s, G) be smooth, compact, n-dimensional G-manifolds with G a compact Lie group. Suppose that the orbit spaces M_e/G and M_s/G are equivalent and that M_s/G has almost nonnegative curvature. Then M_e admits a G-invariant family of metrics that has almost nonnegative sectional curvature. Moreover, if the principal orbits of (M_e, G) have finite fundamental group and the quotient of the principal orbits of M_s has Ricci curvature ≥ 1, then every metric in the almost nonnegatively curved family on M_e can be chosen to also have positive Ricci curvature.*

We emphasise that to apply Theorem 1.3, M_s/G need not be a Riemannian manifold, but since M_s is compact, M_s/G is an Alexandrov space with curvature bounded from below. The meaning of almost nonnegative curvature for Alexandrov spaces is as follows.

DEFINITION 1.4. We say that a sequence of Alexandrov spaces $\{(X, \mathrm{dist}_\alpha)\}_\alpha$ is almost nonnegatively curved if and only if there is a $D > 0$ so that

$$\sec(X, g_\alpha)\geq -\frac{1}{\alpha}\quad\text{and}\quad \mathrm{Diam}(X, g_\alpha)\leq D,$$

or equivalently, after a rescaling, X collapses to a point with a uniform lower curvature bound.

The following is the precise notion of equivalence of orbit spaces required by the hypotheses of Theorem 1.3.

DEFINITION 1.5. Suppose that G acts on M_e and on M_s. We say that the orbit spaces M_e/G and M_s/G are equivalent if and only if there is a strata-preserving homeomorphism $\Phi : M_e/G \longrightarrow M_s/G$ whose restriction to each stratum is a diffeomorphism with the following property. Let $\pi_s : M_s \longrightarrow M_s/G$ and $\pi_e : M_e \longrightarrow M_s/G$ be the quotient maps. If $\mathcal{S}\subset M_e$ is a stratum, then for any $x_e\in\mathcal{S}$ and any $x_s\in\pi_s^{-1}(\Phi(\pi_e(x_e)))$, the action of G_{x_e} on $\nu(\mathcal{S})_{x_e}$ is linearly equivalent to the action of G_{x_s} on $\nu(\mathcal{S})_{x_s}$. Here G_x is the isotropy subgroup at x and $\nu(\mathcal{S})_x$ is the normal space to $\mathcal{S}$ at x.

To construct the metrics on the fake $\mathbb{R}P^6$s of Theorem 1.1, we apply Theorem 1.3 with $G = (SO(3)\times\mathbb{Z}_2)$. M_e will be the Hirsch–Milnor embedded image of $\mathbb{S}^6$ in Σ^7_k, and M_s will be $\mathbb{S}^6$ with the following $(SO(3)\times\mathbb{Z}_2)$-action. View $\mathbb{S}^6$ as the unit sphere in $\mathbb{H}\oplus\mathrm{Im}\mathbb{H}$, where $\mathbb{H}$ stands for the quaternions, and let $SO(3)\times\mathbb{Z}_2$ act on $\mathbb{S}^6\subset\mathbb{H}\oplus\mathrm{Im}\mathbb{H}$ via

$$\begin{aligned} SO(3)\times\mathbb{Z}_2\times\mathbb{S}^6 &\longrightarrow \mathbb{S}^6\\ (g, \pm, (a, c)) &\longmapsto \pm(g(a), g(c)). \end{aligned}\tag{1.1}$$

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Here the $SO(3)$-action on the $\mathbb{H}$-factor is the direct sum of the standard action of $SO(3)$ on $\mathrm{Im}\mathbb{H}$ with the trivial action on $\mathrm{Re}\mathbb{H}$.

Since quotient maps of isometric group actions preserve lower curvature bounds, $\mathbb{S}^6/(SO(3)\times\mathbb{Z}_2)$ has curvature greater than or equal to 1 [4]. Thus to construct the metrics on the fake $\mathbb{R}P^6$s of Theorem 1.1, it suffices to combine Theorem 1.3 with the following result.

Lemma 1.6. *The orbit space of the Hirsch–Milnor and Davis actions of $SO(3)\times\mathbb{Z}_2$ on $\iota(\mathbb{S}^6)\subset\Sigma_k^7$ is equivalent to the orbit space of the linear action (1.1) on $\mathbb{S}^6$.*

Our metrics on fake $\mathbb{R}P^{14}$s are octonionic analogs of our metrics on fake $\mathbb{R}P^6$s. The analogy begins with Shimada's observation that Milnor's proof of the total spaces of certain $\mathbb{S}^3$-bundles over $\mathbb{S}^4$ being exotic spheres also applies to certain $\mathbb{S}^7$-bundles over $\mathbb{S}^8$ [17]. Davis's construction of the $SO(3)$-actions on Σ_k^7s is based on the fact that $SO(3)$ is the group of automorphisms of $\mathbb{H}$. Exploiting the fact that G_2 is the group of automorphisms of the octonions, $\mathbb{O}$, Davis constructs analogous G_2 actions on Shimada's exotic Σ_k^{15}s. By applying a result of [3], we will see that the Hirsch and Milnor construction of fake $\mathbb{R}P^6$s as quotients of $\iota(\mathbb{S}^6)\subset\Sigma_k^7$ also works to construct fake $\mathbb{R}P^{14}$s as quotients of $\iota(\mathbb{S}^{14})\subset\Sigma_k^{15}$. Thus to construct the fake $\mathbb{R}P^{14}$s of Theorem 1.1, it suffices to show the following.

Lemma 1.7. *The orbit space of the Hirsch–Milnor and Davis actions of $G_2\times\mathbb{Z}_2$ on $\iota(\mathbb{S}^{14})\subset\Sigma_k^{15}$ is equivalent to the orbit space of the following linear action of $G_2\times\mathbb{Z}_2$ on $\mathbb{S}^{14}\subset\mathbb{O}\oplus\mathrm{Im}\mathbb{O}$:*

$$\begin{aligned} G_2\times\mathbb{Z}_2\times\mathbb{S}^{14} &\longrightarrow \mathbb{S}^{14}\\ (g,\pm,(a,c)) &\longmapsto \pm(g(a),g(c)). \end{aligned}\tag{1.2}$$

In Section 2 we review the construction of the Hirsch–Milnor and Davis actions and explain why the Hirsch–Milnor construction works in the octonionic case. In Section 3 we prove Lemmas 1.6 and 1.7 and hence Theorem 1.1. In Section 4 we make some concluding remarks. We refer the reader to [2, page 185] for a description of how G_2 acts as automorphisms of the octonions.

Remark 1.8. Explicit formulas for exotic involutions on $\mathbb{S}^6$ and $\mathbb{S}^{14}$ are given in [1], where it is shown, on pages 13–17, that the corresponding fake $\mathbb{R}P^6$ is diffeomorphic to the Hirsch–Milnor $\mathbb{R}P^6$ that corresponds to Σ_3^7.

2. How to construct exotic real projective spaces

In this section, we review Milnor spheres, the Hirsch–Milnor construction, and the Davis actions. We then explain how the Hirsch–Milnor argument gives fake $\mathbb{R}P^{14}$s.

To construct the Milnor spheres, we write Λ for $\mathbb{H}$ or $\mathbb{O}$ and b for the real dimension of Λ. To get an $\mathbb{S}^{b-1}$-bundle over $\mathbb{S}^b$ with structure group $SO(b)$, $(E_{h,j},p_{h,j})$, we glue two copies of $\Lambda \times \mathbb{S}^{b-1}$ together via

$$\begin{aligned} \Phi_{h,j} &: \Lambda\backslash\{0\} \times \mathbb{S}^{b-1} \longrightarrow \Lambda\backslash\{0\} \times \mathbb{S}^{b-1} \\ \Phi_{h,j} &: (u, q) \longmapsto \Big(\frac{u}{|u|^2}, \Big(\frac{u}{|u|}\Big)^h q \Big(\frac{u}{|u|}\Big)^j\Big). \end{aligned} \tag{2.1}$$

To define the projection $p_{h,j} : E_{h,j} \to \mathbb{S}^b$, we think of $\mathbb{S}^b$ as obtained by gluing together two copies of Λ along $\Lambda\backslash\{0\}$ via $u \mapsto u/|u|^2$. Then $p_{h,j}$ is defined to be the projection to either copy of Λ.

When $h + j = \pm 1$, the smooth function

$$f : (u, q) \mapsto \frac{\mathrm{Re}(q)}{\sqrt{1 + |u|^2}} = \frac{\mathrm{Re}(vr^{-1})}{\sqrt{1 + |v|^2}}$$

is regular except at $(u, q) = (0, \pm 1)$. Hence, $E_{h,j}$ is homeomorphic to $\mathbb{S}^{2b-1}$ if $h + j = \pm 1$, and a Mayer–Vietoris argument shows that $E_{h,j}$ is not homeomorphic to $\mathbb{S}^{2b-1}$ if $h + j \neq \pm 1$. Since $f(0, \pm 1) = \pm 1$, it also follows that $f^{-1}(0)$ is diffeomorphic to $\mathbb{S}^{2b-2}$.

From now on we assume that

$$h + j = 1, \tag{2.2}$$

and we set

$$k = h - j \tag{2.3}$$

so that

$$k = 2h - 1.$$

For simplicity, we will write Σ_k^{2b-1} for $E_{h,j}$ and Φ_k for $\Phi_{h,j}$, and set

$$S_k^{2b-2} \equiv f^{-1}(0).$$

The Hirsch–Milnor construction [11] begins with the observation that the involution

$$\begin{aligned} T &: \Lambda \times \mathbb{S}^{b-1} \longrightarrow \Lambda \times \mathbb{S}^{b-1} \\ T &: (u, q) \longmapsto (u, -q) \end{aligned} \tag{2.4}$$

induces a well-defined free involution of Σ_k^{2b-1}. Moreover, T leaves S_k^{2b-2} invariant. Lemma 3 of [11] says that the quotient of any fixed point free involution on $\mathbb{S}^n$ is homotopy equivalent to $\mathbb{R}P^n$. In particular, all of our spaces

$$P_k^{2b-2} \equiv S_k^{2b-2}/T$$

are homotopy equivalent to $\mathbb{R}P^{2b-2}$. Hirsch and Milnor then show that when $b = 4$, P_k^6 is not diffeomorphic to $\mathbb{R}P^6$, provided Σ_k^7 is an odd element of Θ_7, the group of oriented diffeomorphism classes of differential structures on $\mathbb{S}^7$. According to [6, pages 102–103], there are 16 oriented diffeomorphism classes among the Σ_k^7s and, among these, eight are odd elements of Θ_7.

To understand how this works octonionically, we let Θ_{15} be the group of oriented diffeomorphism classes of differential structures on $\mathbb{S}^{15}$, and we let bP_{16} be the set of

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the elements of Θ_{15} that bound parallelisable manifolds. According to [12], bP_{16} is a cyclic subgroup of Θ_{15} of order 8128 and index 2, and according to [3, Theorem 1.3], Θ_{15} is not cyclic. Thus

$$\Theta_{15} \cong bP_{16} \oplus \mathbb{Z}_2 \cong \mathbb{Z}_{8,128} \oplus \mathbb{Z}_2.$$

According to Wall [18], a homotopy sphere bounds a parallelisable manifold if and only if it bounds a 7-connected manifold. In particular, each of the Σ_k^{15}s is in bP_{16}.

According to [6, pages 101–107], Σ_k^{15} represents an odd element of bP_{16} if and only if $\frac{1}{2}h(h-1)$ is odd, that is, h is congruent to 2 or 3 (mod 4).

The Hirsch–Milnor argument, combined with the fact that $\Theta_{15} \cong bP_{16} \oplus \mathbb{Z}_2$, implies that P_k^{14} is not diffeomorphic to $\mathbb{R}P^{14}$, if Σ_k^{15} is an odd element of bP_{16}.

We let

$$G^\Lambda \equiv \begin{cases} SO(3) & \text{when } \Lambda = \mathbb{H}, \\ G_2 & \text{when } \Lambda = \mathbb{O}. \end{cases}$$

Davis observed that since G^Λ is the automorphism group of Λ, the diagonal action

$$\begin{aligned} G^\Lambda \times \Lambda \times \mathbb{S}^{b-1} &\longrightarrow \Lambda \times \mathbb{S}^{b-1} \\ g(u, v) &= (g(u), g(v)) \end{aligned} \tag{2.5}$$

induces a well-defined G^Λ-action on Σ_k^{2b-1} [5].

Next we observe that the Davis action leaves $S_k^{2b-2} = f^{-1}(0)$ invariant and commutes with T, giving us the $SO(3) \times \mathbb{Z}_2$ actions of Lemma 1.6 and the $G_2 \times \mathbb{Z}_2$ actions of Lemma 1.7.

3. Identifying the orbit spaces

In this section, we prove Lemmas 1.6 and 1.7 simultaneously and hence Theorem 1.1. In Lemma 3.1 we identify the quotient map for the standard G^Λ-action of $\mathbb{S}^{2b-2}$. In Lemma 3.2 we identify the quotient map for the Davis action on S_k^{2b-2}. Then in Key Lemma 3.3, we show that the two G^Λ quotients are the same. It is then a simple matter to identify the two $G^\Lambda \times \mathbb{Z}_2$ quotient spaces with each other.

Lemma 3.1. *Let $\mathbb{S}^{2b-2}$ be the unit sphere in $\Lambda \oplus \mathrm{Im}(\Lambda)$ and let $\langle\, , \rangle$ be the real dot product. The map*

$$\begin{aligned} Q_s : \mathbb{S}^{2b-2} &\longrightarrow Q_s(\mathbb{S}^{2b-2}) \subsetneq \mathbb{R}^3 \\ \begin{pmatrix} a \\ c \end{pmatrix} &\longmapsto (|a|,\ \mathrm{Re}\, a,\ \langle \mathrm{Im}\, a, \mathrm{Im}\, c\rangle) \end{aligned}$$

has the following properties.

(1) *The fibres of Q_s coincide with the orbits of the G^Λ action*

$$\begin{aligned} G^\Lambda \times \mathbb{S}^{2b-2} &\longrightarrow \mathbb{S}^{2b-2} \\ (g, (a, c)) &\longmapsto (g(a), g(c)). \end{aligned}$$

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(2) *The image of* Q_s *is* $Q_s(\mathbb{S}^{2b-2}) =$
$$\Big\{(x,y,z) \mid x \in [0,1]\ y \in [-x,x],\ z \in \Big[-\sqrt{(x^2-y^2)(1-x^2)},\ \sqrt{(x^2-y^2)(1-x^2)}\Big]\Big\}.$$

(3) *The principal orbits are mapped to the interior of* $Q_s(\mathbb{S}^{2b-2})$. *The fixed points are mapped to* $(1,1,0)$ *and* $(1,-1,0)$, *and the other orbits are mapped to* $\partial Q_s(\mathbb{S}^{2b-2})\backslash\{(1,1,0),(1,-1,0)\}$.

Proof. Part (2) follows from the observations that

$$\begin{aligned} |a| &\in [0,1],\\ \operatorname{Re} a &\in [-|a|,|a|],\\ \langle \operatorname{Im} a, \operatorname{Im} c\rangle &\in [-|\operatorname{Im}(a)||\operatorname{Im}(c)|,\ |\operatorname{Im}(a)||\operatorname{Im}(c)|], \end{aligned}$$

and

$$|\operatorname{Im}(a)||\operatorname{Im}(c)| \in [0,\ \sqrt{(|a|^2-\operatorname{Re}(a)^2)(1-|a|^2)}].$$

Since the three quantities $|a|$, $\operatorname{Re} a$, $\langle \operatorname{Im} a, \operatorname{Im} c\rangle$ are invariant under G^Λ, each orbit of G^Λ is contained in a fibre of Q_s.

Conversely, if (a_1,c_1) and (a_2,c_2) satisfy $Q_s(a_1,c_1) = Q_s(a_2,c_2)$, then

$$\begin{aligned} |a_1| &= |a_2|\\ \operatorname{Re}(a_1) &= \operatorname{Re}(a_2), \text{ and}\\ \langle \operatorname{Im} a_1, \operatorname{Im} c_1\rangle &= \langle \operatorname{Im} a_2, \operatorname{Im} c_2\rangle. \end{aligned}$$

Together with $\operatorname{Re}(c_i) = 0$ and $|a_i|^2 + |c_i|^2 = 1$, this gives

$$\begin{aligned} |\operatorname{Im}(a_1)| &= |\operatorname{Im}(a_2)|\\ |\operatorname{Im}(c_1)| &= |\operatorname{Im}(c_2)|. \end{aligned}$$

Since we also have $\langle \operatorname{Im} a_1, \operatorname{Im} c_1\rangle = \langle \operatorname{Im} a_2, \operatorname{Im} c_2\rangle$, it follows that an element of G^Λ carries (a_1,c_1) to (a_2,c_2). This completes the proof of part (1).

To prove part (3), we first note that the orbit of (a,c) is not principal if and only if

$$|\langle \operatorname{Im} a, \operatorname{Im} c\rangle| = |\operatorname{Im}(a)||\operatorname{Im}(c)|,$$

and this is equivalent to $Q_s(a,c) \in \partial Q_s(a,c)$. So the principal orbits are mapped onto the interior of $Q_s(\mathbb{S}^{2b-2})$. On the other hand, the fixed points are $(\pm 1, 0)$ and $Q_s(\pm 1,0) = (1,\pm 1,0)$ as claimed. □

Before proceeding, recall that we view

$$\Sigma_k^{2b-1} = (\Lambda \times \mathbb{S}^{b-1}) \cup_{\Phi_k} (\Lambda \times \mathbb{S}^{b-1}),$$

where Φ_k is determined by (2.1)–(2.3). Combining this with the definition of S_k^{2b-2}, we have that

$$S_k^{2b-2} = U_1 \cup_{\Phi_k} U_2,$$

where

$$\begin{aligned} U_1 &\equiv \{(u,q) \in \Lambda \times \mathbb{S}^{b-1} \mid \operatorname{Re}(q) = 0\},\\ U_2 &\equiv \{(v,r) \in \Lambda \times \mathbb{S}^{b-1} \mid \operatorname{Re}(vr^{-1}) = \operatorname{Re}\bar{v}r = 0\}. \end{aligned}$$

The quotient map of the G^Λ-action on S_k^{2b-2} has the following description.

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Lemma 3.2. *Let* $\phi : R^n \longrightarrow R$ *be given by* $\phi(v) = 1/\sqrt{1+|v|^2}$. *The map*

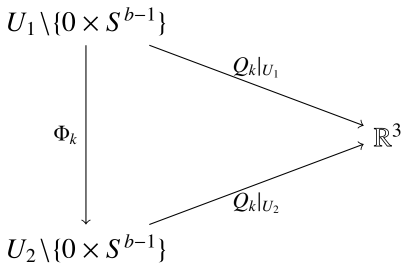
$$Q_k : S_k^{2b-2} \longrightarrow Q_k(S_k^{2b-2}) \subsetneq \mathbb{R}^3$$
$$Q_k|_{U_1}(u,q) = \phi(u)(|u|,\ \operatorname{Re} uq,\ \phi(u)\langle \operatorname{Im} uq, \operatorname{Im} q\rangle).$$
$$Q_k|_{U_2}(v,r) = \phi(v)(|r|,\ \operatorname{Re} r,\ \phi(v)\langle \operatorname{Im} r, \operatorname{Im} \bar{v}r\rangle)$$

is well defined and has fibres that coincide with the orbits of G^Λ.

Proof. To see that Q_k is well defined, we will show that

$$Q_k|_{U_1\setminus\{0\times\mathbb{S}^{b-1}\}} = Q_k|_{U_2\setminus\{0\times\mathbb{S}^{b-1}\}} \circ \Phi_k|_{U_1\setminus\{0\times\mathbb{S}^{b-1}\}}, \tag{3.1}$$

which is equivalent to the commutative diagram



Since

$$\Phi_k(u,q) = \left(\frac{u}{|u|^2}, \left(\frac{u}{|u|}\right)^h q \left(\frac{u}{|u|}\right)^{-(h-1)}\right),$$

where $k = 2h-1$, the left-hand side of (3.1) is

$$Q_k|_{U_2\setminus\{0\times\mathbb{S}^{b-1}\}} \circ \Phi_k|_{U_1\setminus\{0\times\mathbb{S}^{b-1}\}}(u,q) = Q_k\left(\frac{u}{|u|^2}, \frac{u^h q u^{-(h-1)}}{|u|}\right)$$
$$= \phi\left(\frac{u}{|u|^2}\right)\left(\left|\frac{u^h q u^{-(h-1)}}{|u|}\right|,\ \operatorname{Re}\frac{u^h q u^{-(h-1)}}{|u|}, \phi\left(\frac{u}{|u|^2}\right)\left\langle \operatorname{Im}\frac{u^h q u^{-(h-1)}}{|u|}, \operatorname{Im}\frac{\bar{u}}{|u|^2}\frac{u^h q u^{-(h-1)}}{|u|}\right\rangle\right). \tag{3.2}$$

To see that this is equal to $Q_k|_{U_1\setminus\{0\times\mathbb{S}^{b-1}\}}(u,q)$, we will simplify each coordinate separately. Before doing so, we point out that

$$\frac{1}{|u|}\phi\left(\frac{u}{|u|^2}\right) = \frac{1}{|u|}\frac{1}{\sqrt{1+1/|u|^2}} = \frac{1}{\sqrt{|u|^2+1}} = \phi(u). \tag{3.3}$$

So the first coordinate of the right-hand side of (3.2) is

$$\phi\left(\frac{u}{|u|^2}\right)\left|\frac{u^h q u^{-(h-1)}}{|u|}\right| = \phi\left(\frac{u}{|u|^2}\right) = |u|\phi(u),$$

and the second coordinate of the right-hand side of (3.2) is

$$\phi\left(\frac{u}{|u|^2}\right)\operatorname{Re}\frac{u^h q u^{-(h-1)}}{|u|} = \phi\left(\frac{u}{|u|^2}\right)\operatorname{Re}\left(\frac{uq}{|u|}\right) = \frac{1}{|u|}\phi\left(\frac{u}{|u|^2}\right)\operatorname{Re}(uq)$$
$$= \phi(u)\operatorname{Re}(uq), \quad \text{by (3.3)}.$$

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Finally, the third coordinate of the right-hand side of (3.2) is

$$\begin{aligned}
&\phi\Big(\frac{u}{|u|^2}\Big)^2\Big\langle \operatorname{Im}\frac{u^h q u^{-(h-1)}}{|u|}, \operatorname{Im}\frac{\bar{u}}{|u|^2}\frac{u^h q u^{-(h-1)}}{|u|}\Big\rangle \\
&= \phi\Big(\frac{u}{|u|^2}\Big)^2\Big\langle \operatorname{Im}\frac{u^h q u^{-(h-1)}}{|u|}, \operatorname{Im}\frac{u^{h-1} q u^{-(h-1)}}{|u|}\Big\rangle \\
&= \phi\Big(\frac{u}{|u|^2}\Big)^2\frac{1}{|u|^2}\langle \operatorname{Im} u^{h-1}(uq)u^{-(h-1)}, \operatorname{Im} u^{h-1}(q)u^{-(h-1)}\rangle \\
&= \phi(u)^2\langle \operatorname{Im} uq, \operatorname{Im} q\rangle, \quad \text{by (3.3).}
\end{aligned}$$

Combining the previous three displays with (3.2) and the definition of $Q_k|_{U_1}$, we see that $Q_k : S_k^{2b-2} \longrightarrow Q_k(S_k^{2b-2}) \subsetneq \mathbb{R}^3$ is well defined.

To see that $Q_k|_{U_1}$ is constant on each orbit of G^Λ, we use the fact that G^Λ acts by isometries and commutes with conjugation to get

$$\operatorname{Re} g(u)g(q) = \langle g(u), \overline{g(q)}\rangle = \langle g(u), g(\bar{q})\rangle = \langle u, \bar{q}\rangle = \operatorname{Re}(uq).$$

We also have

$$\begin{aligned}
&\langle \operatorname{Im}(g(u)g(q)), \operatorname{Im} g(q)\rangle \\
&= \langle \operatorname{Re}(g(u))\operatorname{Im} g(q) + \operatorname{Re}(g(q))\operatorname{Im} g(u) + \operatorname{Im} g(u)\operatorname{Im} g(q),\ \operatorname{Im} g(q)\rangle \\
&= \langle \operatorname{Re}(u)\operatorname{Im} g(q) + \operatorname{Re}(q)\operatorname{Im} g(u),\ \operatorname{Im} g(q)\rangle \\
&= \langle g(\operatorname{Re}(u)\operatorname{Im}(q) + \operatorname{Re}(q)\operatorname{Im}(u)),\ g(\operatorname{Im}(q))\rangle \\
&= \langle \operatorname{Re}(u)\operatorname{Im}(q) + \operatorname{Re}(q)\operatorname{Im}(u), \operatorname{Im}(q)\rangle \\
&= \langle \operatorname{Re}(u)\operatorname{Im}(q) + \operatorname{Re}(q)\operatorname{Im}(u) + \operatorname{Im} uq\operatorname{Im} q,\ \operatorname{Im}(q)\rangle \\
&= \langle \operatorname{Im}(uq),\ \operatorname{Im} q\rangle.
\end{aligned}$$

Since $|g(u)| = |u|$ and $\phi(gu) = \phi(u)$, it follows that

$$Q_k|_{U_1}\begin{pmatrix} g(u) \\ g(q)\end{pmatrix} = Q_k|_{U_1}\begin{pmatrix} u \\ q\end{pmatrix}.$$

Combining this with

$$Q_k|_{U_2} g\begin{pmatrix} 0 \\ r\end{pmatrix} = (1, \operatorname{Re}(r), 0) = Q_k|_{U_2}\begin{pmatrix} 0 \\ r\end{pmatrix},$$

it follows that Q_k is constant on each orbit of G^Λ.

On the other hand, if

$$Q_k|_{U_1}(u_1, q_1) = Q_k|_{U_1}(u_2, q_2),$$

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then

$$\phi(u_1)|u_1| = \phi(u_2)|u_2|, \tag{3.4}$$
$$\phi(u_1)^2\langle \mathrm{Im}(u_1q_1), q_1\rangle = \phi(u_2)^2\langle \mathrm{Im}(u_2q_2), q_2\rangle, \tag{3.5}$$
$$\phi(u_1)\mathrm{Re}\, u_1q_1 = \phi(u_2)\mathrm{Re}\, u_2q_2. \tag{3.6}$$

Equation (3.4) implies that $|u_1| = |u_2|$ and $\phi(u_1) = \phi(u_2)$. So

$$\begin{aligned}
\mathrm{Re}(u_1) &= \mathrm{Re}(u_1)\langle q_1, q_1\rangle \\
&= \langle(\mathrm{Re}(u_1) + \mathrm{Im}(u_1))q_1, q_1\rangle, \quad \text{since } \mathrm{Re}(q_1) = 0 \\
&= \langle u_1q_1, q_1\rangle \\
&= \langle \mathrm{Im}(u_1q_1), q_1\rangle, \quad \text{since } \mathrm{Re}(q_1) = 0 \\
&= \langle \mathrm{Im}(u_2q_2), q_2\rangle, \quad \text{by (3.5) and the fact that } \phi(u_1) = \phi(u_2) \\
&= \mathrm{Re}(u_2)
\end{aligned}$$

and

$$\begin{aligned}
\langle \mathrm{Im}(u_1), q_1\rangle &= -\langle u_1, \bar{q}_1\rangle, \ \text{since } \mathrm{Re}(q_1) = 0 \\
&= -\mathrm{Re}\, u_1q_1 \\
&= -\mathrm{Re}\, u_2q_2, \quad \text{by (3.6) and the fact that } \phi(u_1) = \phi(u_2) \\
&= -\langle u_2, \bar{q}_2\rangle \\
&= \langle \mathrm{Im}(u_2), q_2\rangle.
\end{aligned}$$

Together with $|u_1| = |u_2|$ and the fact that q_1 and q_2 are imaginary, the previous two displays imply that $\begin{pmatrix} u_1 \\ q_1 \end{pmatrix}$ and $\begin{pmatrix} u_2 \\ q_2 \end{pmatrix}$ are in the same orbit.

Finally, suppose that

$$Q_k|_{U_2}(0, r_1) = Q_k|_{U_2}(0, r_2).$$

Then

$$(1, \mathrm{Re}(r_1), 0) = (1, \mathrm{Re}(r_2), 0).$$

Since we also have that $|r_1| = |r_2| = 1$, it follows that $(0, r_1)$ and $(0, r_2)$ are in the same G^Λ-orbit. □

Key Lemma 3.3. *Let Q_s be as in Lemma 3.1.*

(1) *There is a well-defined surjective map*

$$\tilde{Q}_k : S_k^{2b-2} \to \mathbb{S}^{2b-2}/G^\Lambda$$

whose fibres coincide with the orbits of the G^Λ action on S_k^{2b-2}.

(2) *The orbit types of $p \in S_k^{2b-2}$ and $Q_s^{-1}(\tilde{Q}_k(p))$ coincide.*

(3) *For $p \in S_k^{2b-2}$ and any $q \in Q_s^{-1}(\tilde{Q}_k(p))$, the isotropy representations of G_p^Λ and G_q^Λ are equivalent.*

In particular, $\mathbb{S}^{2b-2}/G^\Lambda$ and S_k^{2b-2}/G^Λ are equivalent orbit spaces.

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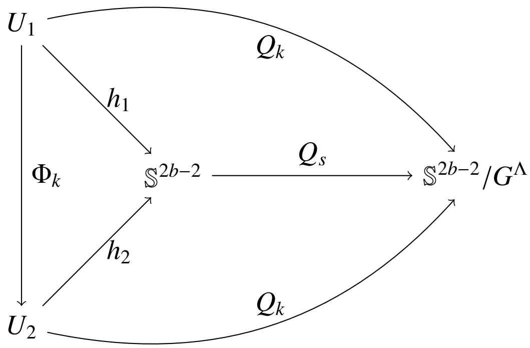
Proof. Motivated by [7, 19], we let $h_1, h_2 : \Lambda \times \mathbb{S}^{b-2} \to \mathbb{S}^{2b-2}$ be given by

$$h_1(u,q) = \begin{pmatrix} uq \\ q \end{pmatrix} \phi(u),$$
$$h_2(v,r) = \begin{pmatrix} r \\ \bar{v}r \end{pmatrix} \phi(v).$$

We claim that Q_s and Q_k are related by

$$Q_k = \begin{cases} Q_s \circ h_1 & \text{on } U_1, \\ Q_s \circ h_2 & \text{on } U_2. \end{cases} \tag{3.7}$$

In other words, the following diagram commutes:



where the map Q_k is not defined everywhere, but only on $U_1 \setminus \{0 \times S^{b-1}\}$ and $U_2 \setminus \{0 \times S^{b-1}\}$. Indeed,

$$\begin{aligned} Q_s \circ h_1(u,q) &= Q_s \begin{pmatrix} uq \\ q \end{pmatrix} \phi(u) \\ &= \phi(u)(|u|, \operatorname{Re} uq,\ \phi(u)\langle \operatorname{Im} uq, \operatorname{Im} q\rangle) \\ &= Q_k(u,q) \end{aligned}$$

and

$$\begin{aligned} Q_s \circ h_2(v,r) &= Q_s \begin{pmatrix} r \\ \bar{v}r \end{pmatrix} \phi(v) \\ &= \phi(v)(|r|, \operatorname{Re}(r), \phi(v)\,\langle \operatorname{Im} r, \operatorname{Im} \bar{v}r\rangle) \\ &= Q_k(v,r), \end{aligned}$$

proving (3.7).

Since $h_1(\Lambda \times \mathbb{S}^{b-2}) \cup h_2(\Lambda \times \mathbb{S}^{b-2}) = \mathbb{S}^{2b-2}$, (3.7) implies that $Q_k(S_k^{2b-2}) = Q_s(\mathbb{S}^{2b-2})$; so setting $\tilde{Q}_k = Q_k$ gives a well-defined surjective map

$$\tilde{Q}_k : S_k^{2b-2} \to \mathbb{S}^{2b-2}/G^\Lambda,$$

and part (1) is proven. Parts (2) and (3) follow from the observation that h_1 and h_2 are G^Λ-equivariant embeddings. □

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Since the antipodal map $A : \mathbb{S}^{2b-2} \longrightarrow \mathbb{S}^{2b-2}$ and the involution

$$T : S_k^{2b-2} \longrightarrow S_k^{2b-2},$$

defined by (2.4), commute with the G^{Λ}-actions (1.1), (1.2) and (2.5), they induce well-defined $\mathbb{Z}_2$-actions on the orbit space

$$Q_s(\mathbb{S}^{2b-2}) = Q_e(S_e^{2b-2})$$
$$= \Big\{(x, y, z) \mid x \in [0, 1],\ y \in [-x, x],\ z \in \Big[-\sqrt{(x^2 - y^2)(1 - x^2)},\ \sqrt{(x^2 - y^2)(1 - x^2)}\Big]\Big\}.$$

A simple calculation shows that the two $\mathbb{Z}_2$-actions on $Q_s(\mathbb{S}^{2b-2})$ coincide and are given by

$$(x, y, z) \mapsto (x, -y, z).$$

Since quotient maps of isometric group actions preserve lower curvature bounds, $\mathbb{S}^{2b-2}/(SO(3) \times \mathbb{Z}_2)$ has curvature greater than or equal to 1 [4]. Therefore, Theorem 1.1 follows from Theorem 1.3 and Key Lemma 3.3.

4. Some closing remarks

In the same paper, Hirsch and Milnor also constructed exotic $\mathbb{R}P^5$s and P_k^5s. The Davis action also descends to the P_k^5s where they commute with an $SO(2)$-action. The combined $SO(2) \times SO(3)$-action on the P_k^5s is by cohomogeneity one. Dearricott and, independently, Grove and Ziller, observed that since these cohomogeneity-one actions have codimension-two singular orbits, [9, Theorem E] implies that they admit invariant metrics of nonnegative curvature.

Octonionically, the Hirsch–Milnor construction yields closed 13-manifolds, P_k^{13}, that are homotopy equivalent to $\mathbb{R}P^{13}$. Their proof that the P_k^5s are not diffeomorphic to $\mathbb{R}P^5$ breaks down, since in contrast to dimension 6, there is an exotic 14-sphere; however, Chenxu He has informed us that some of the P_k^{13}s are in fact exotic (personal communication).

The Davis construction yields a cohomogeneity-one action of $SO(2) \times G_2$ on the P_k^{13}s, only now one of the singular orbits has codimension six. So we cannot apply [9, Theorem E]. Moreover, there are cohomogeneity-one manifolds that do not admit invariant metrics with nonnegative curvature [8, 10]. On the other hand, by the main theorem of [15], every cohomogeneity-one manifold admits an invariant metric with almost nonnegative curvature.

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