Bull. Aust. Math. Soc. **109** (2024), 254–264
doi:10.1017/S0004972723000898

THE MORDELL–LANG CONJECTURE FOR SEMIABELIAN VARIETIES DEFINED OVER FIELDS OF POSITIVE CHARACTERISTIC

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(Received 11 April 2023; accepted 2 August 2023; first published online 8 September 2023)

Abstract

Let G be a semiabelian variety defined over an algebraically closed field K of prime characteristic. We describe the intersection of a subvariety X of G with a finitely generated subgroup of $G(K)$.

2020 *Mathematics subject classification*: primary 11G10; secondary 14G17.

Keywords and phrases: semiabelian varieties, finite fields, Mordell–Lang conjecture.

1. Introduction

The purpose of this note is to prove a variant of the Mordell–Lang conjecture for semiabelian varieties defined over fields of positive characteristic. More precisely, let G be a semiabelian variety defined over an algebraically closed field K, that is, there exists a short exact sequence of algebraic groups defined over K:

$$1 \longrightarrow \mathbb{G}_m^N \longrightarrow G \longrightarrow A \longrightarrow 1,$$

where $N \geq 0$ is an integer and A is an abelian variety. Assuming K has characteristic $p > 0$, then for any subvariety $X \subseteq G$ defined over K and any finitely generated subgroup $\Gamma \subset G(K)$, we describe the intersection $X(K) \cap \Gamma$. In particular, we fix an error in the paper [2] of the first author where a simplified form of such a result was claimed in the case when G is defined over a finite subfield of K; we present several examples showing that the intersection $X(K) \cap \Gamma$ involves the more general F-sets appearing in Definition 1.5.

1.1. General background. The Mordell–Lang conjecture for semiabelian varieties G defined over fields of characteristic 0 predicts that the intersection of a subvariety $X \subseteq G$ with a finitely generated subgroup Γ of G is a finite union of cosets of subgroups of Γ. This conjecture was proven by Laurent [4] in the case of tori, by Faltings [1]

The first author is supported by a Discovery NSERC grant, while the second author is supported by an NSFC Grant (No. 12271007).

in the case of abelian varieties and by Vojta [7] in the general case of semiabelian varieties. In particular, their results show that if X is an irreducible subvariety of G which intersects a finitely generated group in a Zariski dense subset, then X must be a translate of a semiabelian subvariety of G.

The picture for positive characteristic fields K is more complicated due to the existence of the Frobenius endomorphism for varieties defined over finite fields; in particular, it is no longer true that only translates of semiabelian subvarieties of G have the property that they intersect a finitely generated subgroup of G in a Zariski dense subset. Hrushovski [3] obtained the right shape for the irreducible subvarieties X whose intersection with a finitely generated subgroup Γ is Zariski dense.

THEOREM 1.1 (Hrushovski [3]). *Let G be a semiabelian variety defined over an algebraically closed field K of characteristic p. Let $\Gamma \subset G(K)$ be a finitely generated subgroup and let $X \subseteq G$ be an irreducible subvariety with the property that $X(K) \cap \Gamma$ is Zariski dense in X. Then there exists $\gamma \in G(K)$, a semiabelian subvariety $G_0 \subseteq G$ defined over K, a semiabelian variety H along with a subvariety $X_0 \subseteq H$ both defined over a finite subfield $\mathbb{F}_q$ of K, and a surjective group homomorphism $h : G_0 \longrightarrow H$ such that $X = \gamma + h^{-1}(X_0)$.*

However, [3] left open the description of the actual intersection between the subvariety X and the group Γ. Next, we will address exactly this issue.

1.2. The case of semiabelian varieties defined over finite fields and of finitely generated subgroups invariant under the Frobenius endomorphism. Essentially, Hrushovski's result (see Theorem 1.1) reduced the description of the intersection $X(K) \cap \Gamma$ to the case when the ambient semiabelian variety is defined over a finite field. Moosa and Scanlon [5, 6] addressed precisely this problem under an additional assumption on the subgroup Γ; to state their main result, we introduce some notation.

DEFINITION 1.2. For a semiabelian variety G defined over a finite subfield $\mathbb{F}_q$ of an algebraically closed field K of characteristic p, we define a *groupless F-set* to be any subset of $G(K)$ of the following form:

$$\Big\{\alpha_0 + \sum_{i=1}^{r} F^{kn_i}(\alpha_i) : n_i \in \mathbb{N}\Big\}, \tag{1.1}$$

where $r \geq 0, \alpha_0, \alpha_1, \ldots, \alpha_r \in G(K)$ and $k \in \mathbb{N}$, while F is the Frobenius endomorphism of G corresponding to the finite field $\mathbb{F}_q$.

For any finitely generated subgroup $\Gamma \subset G(K)$, we define a *groupless F-set in Γ* as a groupless F-set contained in Γ. Also, an *F-set in Γ* is any set of the form $S + B$, where S is a groupless F-set in Γ and B is a subgroup of Γ. (For any two subsets B and C of G, as always, $C + B$ is simply the set of all $c + b$ where $b \in B$ and $c \in C$.)

https://doi.org/10.1017/S0004972723000898 Published online by Cambridge University Press

REMARK 1.3. In [6, Theorem B], Moosa and Scanlon allowed for the possibility that a groupless F-set involves sums of F-orbits as in (1.1) of the form

$$\alpha_0 + \sum_{i=1}^{r} F^{k_i n_i}(\alpha_i) \quad \text{(as the } n_i \text{ vary in } \mathbb{N}), \tag{1.2}$$

for given, but potentially distinct, positive integers k_i. However, each F-set from (1.2) is a union of finitely many F-sets as in Definition 1.2 (simply by working with k as the least common multiple of $k_1, \ldots, k_r$).

THEOREM 1.4 (Moosa–Scanlon [6]). *Let G be a semiabelian variety defined over a finite subfield $\mathbb{F}_q$ of an algebraically closed field K and let $F : G \longrightarrow G$ be the Frobenius endomorphism associated to the finite field $\mathbb{F}_q$. Let $X \subseteq G$ be a subvariety defined over K and let $\Gamma \subset G(K)$ be a finitely generated subgroup. If Γ is invariant under F^ℓ for some $\ell \in \mathbb{N}$, then $X(K) \cap \Gamma$ is a finite union of F-sets in Γ.*

1.3. The case of an arbitrary finitely generated subgroup. It is natural to ask whether the description from Theorem 1.4 of the intersection $X(K) \cap \Gamma$ remains valid when Γ is no longer invariant under a power of the Frobenius endomorphism of G (but we only assume Γ is finitely generated).

One could consider the $\mathbb{Z}[F]$-submodule $\tilde{\Gamma} \subset G(K)$ spanned by Γ. Since F is integral over $\mathbb{Z}$ (seen as a subring of $\mathrm{End}(G)$), then $\tilde{\Gamma}$ is still finitely generated and so, by Moosa–Scanlon's result (see Theorem 1.4), $X(K) \cap \tilde{\Gamma}$ is a finite union of F-sets in $\tilde{\Gamma}$. So, the problem reduces to understanding the intersection of an F-set S in $\tilde{\Gamma}$ with Γ. The first author [2, Theorem 3.1] proved that when S is a *groupless* F-set in $\tilde{\Gamma}$, then its intersection with Γ is a finite union of groupless F-sets in Γ. Also in [2], the first author analysed the intersection with Γ of an arbitrary F-set in $\tilde{\Gamma}$; however, the final assertion from [2, Step 3, page 3842] claiming that the general case of an F-set reduces to the groupless case is not valid, as shown by the constructions in Section 2 (see Examples 2.1 and 2.2 which were found by the second author). Essentially, the error from [2] was to claim that the pullback of a groupless F-set in $\tilde{\Gamma}$ through a group homomorphism restricted to Γ must be an F-set in Γ (as in Definition 1.2). Furthermore, Example 2.3 shows that when Γ is an arbitrary finitely generated subgroup, the intersection $X(K) \cap \Gamma$ can be quite wild; this motivates our Definition 1.5 which yields the right form of the sets appearing in the intersection of a subvariety of G with a finitely generated group.

DEFINITION 1.5. For a semiabelian variety G defined over a finite subfield $\mathbb{F}_q$ of an algebraically closed field K of characteristic p and a finitely generated subgroup $\Gamma \subset G(K)$, we define a *generalised F-set in* Γ to be any subset of Γ of the form

$$(\pi|_\Gamma)^{-1}(S), \tag{1.3}$$

where $\pi : G \longrightarrow H$ is a surjective group homomorphism of semiabelian varieties both defined over a finite subfield of K for which $\dim(\ker(\pi)) > 0$, $\pi|_\Gamma$ is its restriction to the subgroup Γ and $S \subset H(K)$ is a groupless F-set in $\pi(\Gamma)$.

Note that H may be defined over another finite subfield $\mathbb{F}_{q'} \subset K$ and thus the set S from (1.3) is a groupless F-set in $\pi(\Gamma)$ where F stands for the Frobenius endomorphism of H associated to the finite field $\mathbb{F}_{q'}$.

1.4. Our results. Now we can state our main results, first for describing the intersection with a finitely generated group of a subvariety of a semiabelian variety defined over a finite field. We note that even though our results are formulated for semiabelian varieties defined over an algebraically closed field K, one could formulate all of the results working instead with an infinite field K, as observed by the referee.

THEOREM 1.6. *Let G be a semiabelian variety defined over a finite subfield $\mathbb{F}_q$ of an algebraically closed field K of characteristic p. Let $X \subset G$ be a subvariety defined over K and let $\Gamma \subset G(K)$ be a finitely generated subgroup. Then the intersection $X(K) \cap \Gamma$ is a union of finitely many groupless F-sets in Γ along with finitely many generalised F-sets in Γ.*

Our Examples 2.1, 2.2 and 2.3 show that the sets appearing as intersections between a subvariety X of a semiabelian variety G defined over a finite field with a finitely generated subgroup can be quite complicated, well beyond the world of F-sets from Definition 1.2. However, when X is a curve or G is a simple semiabelian variety, then we can show that the intersection $X(K) \cap \Gamma$ is a finite union of F-sets in Γ.

THEOREM 1.7. *Let G be a semiabelian variety defined over a finite subfield of an algebraically closed field K of prime characteristic, let $X \subseteq G$ be a subvariety defined over K and let $\Gamma \subset G(K)$ be a finitely generated subgroup. If either $\dim(X) = 1$ or G is a simple semiabelian variety (that is, either a simple abelian variety or a 1-dimensional torus), then $X(K) \cap \Gamma$ is a finite union of F-sets.*

Next, combining our Theorem 1.6 with Hrushovski's result (see Theorem 1.1), we obtain the description of the intersection of a subvariety of an arbitrary semiabelian variety G defined over a field of prime characteristic with a finitely generated subgroup of G. For this end, we introduce the notion of *pseudo-generalised F-sets*.

DEFINITION 1.8. Let G be a semiabelian variety defined over an algebraically closed field K of characteristic p and let $\Gamma \subset G(K)$ be a finitely generated subgroup. A *pseudo-generalised F-set in* Γ is a set of the form

$$x_0 + (\pi|_{\Gamma_0})^{-1}(S),$$

where $x_0 \in \Gamma$, $G_0 \subseteq G$ is a semiabelian subvariety, $\Gamma_0 = G_0(K) \cap \Gamma$, H is a semiabelian variety defined over a finite subfield $\mathbb{F}_q \subset K$, $\pi : G_0 \longrightarrow H$ is a surjective group homomorphism of semiabelian varieties and $S \subset H(K)$ is a groupless F-set in $\pi(\Gamma_0)$.

REMARK 1.9. In Definition 1.8, if G is defined over a finite subfield of K, then the pseudo-generalised F-sets from Definition 1.8 cover both the groupless F-sets in Γ from Definition 1.2 and also the generalised F-sets in Γ from Definition 1.5, but they are a bit more general than those two types of sets.

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THEOREM 1.10. *Let G be a semiabelian variety defined over an algebraically closed field K of characteristic p, let $X \subseteq G$ be a subvariety and let $\Gamma \subset G(K)$ be a finitely generated group. Then $X(K) \cap \Gamma$ is a finite union of pseudo-generalised F-sets in Γ.*

1.5. Plan for our paper. In Section 2, we introduce three examples which progressively show the complexity of the sets appearing as intersections between a subvariety of a semiabelian variety G with a finitely generated group. Even though in our examples, G is defined over a finite field, each such example can be 'embedded' as isotrivial semiabelian subvarieties of a semiabelian variety defined over an arbitrary field of positive characteristic, thus providing complex examples of pseudo-generalised F-sets. In Section 3, we prove Theorems 1.6 and 1.10. Also, we prove Theorem 1.7 as a consequence of two more precise results (see Propositions 3.1 and 3.2) regarding the structure of the intersection $X(K) \cap \Gamma$ when either X is a curve or G is a simple semiabelian variety.

2. Examples

Our first example already shows that $X(K) \cap \Gamma$ is not always an F-set in Γ (when Γ is not invariant under a power of the Frobenius endomorphism of G).

EXAMPLE 2.1. We let $G = \mathbb{G}_m^2 \times E$, where E is a supersingular elliptic curve defined over $\mathbb{F}_p$; for example, we can take E to be the elliptic curve given by the equation in affine coordinates $y^2 = x^3 + 1$ when $p = 5$, in which case, the square F^2 of the usual Frobenius endomorphism of E corresponding to $\mathbb{F}_5$ equals the multiplication map $[-5]$ on E. We let $C \subset \mathbb{G}_m^2$ be the line given by the equation $x_2 = x_1 + 1$ and then let $X = C \times E$. We let $K = \overline{\mathbb{F}_p(t)}$ and let $P \in E(K)$ be a nontorsion point. Finally, we let $\Gamma \subset G(K)$ be the cyclic group spanned by $Q := (t, t+1, P) \subset G(K)$. Then

$$X(K) \cap \Gamma = \{p^n Q : n \geq 0\}. \tag{2.1}$$

Furthermore, the set from (2.1) cannot be expressed as a groupless F-set; the closest it comes to being an F-set is expressing it as the following slight twist of groupless F-sets. We let $Q_1 := (t, t+1, 0) \in G(K)$ and $Q_2 := (1, 1, P) \in G(K)$, and then the set from (2.1) is the union of the two sets:

$$\{F^{2n}(Q_1) + F^{4n}(Q_2) : n \geq 0\} \quad \text{and} \quad \{F^{2n+1}(Q_1) - F^{4n+2}(Q_2) : n \geq 0\}. \tag{2.2}$$

Now, comparing the sets from (2.2) with the actual (groupless) F-sets, the difference seems quite small and so one might think that perhaps slightly extending the definition of F-sets as in (2.2) would be enough. The main issue in Example 2.1 comes from the fact that the Frobenius endomorphism has 'different weights' on the abelian and affine parts of G. It might seem reasonable to think that allowing different weights in the definition of a groupless F-set by considering sets of the form

$$\left\{ \sum_{i=1}^{r} \sum_{j=1}^{s} F^{k_{i,j} \cdot n_j}(\alpha_j) : n_j \geq 0 \text{ for } j = 1, \ldots, s \right\}$$

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would suffice for describing $X(K) \cap \Gamma$. However, the next example shows that no simple extension of the definition of F-sets would work.

EXAMPLE 2.2. We still work with $G = \mathbb{G}_m^2 \times E$, but this time, the elliptic curve E is ordinary; for example, we could take $p = 5$ and let E be the elliptic curve given by the equation in affine coordinates $y^2 = x^3 + x$. One can check that the Frobenius endomorphism corresponding to $\mathbb{F}_5$ satisfies the integral equation $F^2 - 2F + 5 = 0$ on E. As before, we let $K = \overline{\mathbb{F}_p(t)}$ and we work with the cyclic group Γ spanned by $Q := (t, t+1, P) \in G(K)$ for some nontorsion point $P \in E(K)$. Then letting $X = C \times E$, where $C \subset \mathbb{G}_m^2$ is the line $x_2 = x_1 + 1$, we find

$$X(K) \cap \Gamma = \{p^n Q \colon n \geq 0\}. \tag{2.3}$$

However, one can show that the set from (2.3) *cannot* be split into finitely many sets of the form

$$\Big\{\sum_{i=1}^{r}\sum_{j=1}^{s} F^{k_{i,j}n_j}(Q_j) \colon n_j \geq 0 \text{ for } j = 1, \ldots, s\Big\}, \tag{2.4}$$

for any given $r, s \in \mathbb{N}$ and any choice of nonnegative integers $k_{i,j}$ and any choice of given points $Q_j \in G(K)$. In other words, even the most complex definition of a groupless F-set as in (2.4) would still not cover a possible intersection $X(K) \cap \Gamma$.

Now, Examples 2.1 and 2.2 may still suggest that the intersection $X(K) \cap \Gamma$ could be expressed using more general (groupless) F-sets in which one would allow also the multiplication-by-p map on G playing a similar role to the Frobenius endomorphism. However, the next example shows that $X(K) \cap \Gamma$ may have a very complex structure.

EXAMPLE 2.3. We let A and B be semiabelian varieties defined over a finite subfield $\mathbb{F}_q$ of an algebraically closed field K, let $G = A \times B$ and let F be the corresponding Frobenius endomorphism associated to $\mathbb{F}_q$. We let h be the minimal (monic) polynomial with integer coefficients for which $h(F) = 0$ on B. Depending on the abelian part of the semiabelian variety B, the degree m of the polynomial h may be arbitrarily large.

We let $C \subset B$ be a curve defined over $\mathbb{F}_q$ with trivial stabiliser in B and let $P \in C(K)$ be a nontorsion point; one can even choose C and P so that $C(K)$ intersects the cyclic $\mathbb{Z}[F]$-module Γ_1 spanned by P precisely in the orbit of P under the Frobenius endomorphism F. We also let $Q_1, \ldots, Q_m \in A(K)$ be linearly independent points (note that $A(K) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an infinite dimensional $\mathbb{Q}$-vector space). Then we consider $X := A \times C$ and also consider the group $\Gamma \subset G(K)$ spanned by the points

$$R_1 := (Q_1, P), \quad R_2 := (Q_2, F(P)), \quad R_3 := (Q_3, F^2(P)), \ldots, R_m := (Q_m, F^{m-1}(P)).$$

Then letting $\pi_2 : G \longrightarrow B$ be the projection of $G = A \times B$ on the second coordinate, we have $\pi_2(\Gamma) = \Gamma_1$ because Γ_1 is spanned by the points

$$P, F(P), F^2(P), \ldots, F^{m-1}(P) \in B(K),$$

since Γ_1 is the cyclic $\mathbb{Z}[F]$-module spanned by P and $h(F)(P) = 0$. So, we can find m sequences $\{a_n^{(i)}\}_{n\geq 0}$ of integers (for $i = 0, \ldots, m-1$) such that for any $n \geq 0$,

$$F^n(P) = \sum_{i=0}^{m-1} a_n^{(i)} \cdot F^i(P). \tag{2.5}$$

From (2.5), $X(K) \cap \Gamma$ is the set

$$\left\{ \sum_{i=1}^{m} a_n^{(i-1)} \cdot R_i : n \geq 0 \right\}.$$

So, due to the potential complexity of the coefficients of the polynomial h satisfied by the Frobenius endomorphism (on the semiabelian variety B), the sequences $\{a_n^{(i)}\}_{n\geq 0}$ may be quite complicated.

3. Proofs of our main results

PROOF OF THEOREM 1.6. We proceed by induction on $\dim(X)$; the case when $\dim(X) = 0$ is obvious since then, $X(K) \cap \Gamma$ is a finite set and so each of the groupless F-sets from our intersection are singletons (corresponding to $r = 0$ in (1.1)).

Clearly, it suffices to assume X is irreducible. Also, we may assume $X(K) \cap \Gamma$ is Zariski dense in X since otherwise, we could replace X by the Zariski closure of $X(K) \cap \Gamma$ and use the inductive hypothesis.

We let $U := \mathrm{Stab}_G(X)$ be the stabiliser of X in G. We have two possibilities depending on whether U is finite or not.

Case 1: $\dim(U) > 0$. In this case, we let $\pi_0 : G \longrightarrow G/U$ be the natural group homomorphism; in particular, $G_0 := G/U$ is a semiabelian variety defined over a finite field since U is defined over a finite extension of $\mathbb{F}_q$. We let $\Gamma_0 := \pi_0(\Gamma)$ and $X_0 := \pi_0(X)$.

Since $\dim(U) > 0$, then $\dim(X_0) < \dim(X)$ and so, by the inductive hypothesis, $X_0(K) \cap \Gamma_0$ is a union of finitely many groupless F-sets B_i in Γ_0 along with finitely many generalised F-sets C_i in Γ_0. We have

$$X(K) \cap \Gamma = \pi_0^{-1}(X_0(K) \cap \Gamma_0) \cap \Gamma = (\pi_0|_\Gamma)^{-1}(X_0(K) \cap \Gamma_0). \tag{3.1}$$

Clearly, each $(\pi_0|_\Gamma)^{-1}(B_i)$ is a generalised F-set in Γ as in Definition 1.5. Now, each C_i is a set of the form

$$(f|_{\Gamma_0})^{-1}(S_0) = f^{-1}(S_0) \cap \Gamma_0,$$

where $f : G_0 \longrightarrow H$ is a surjective group homomorphism of semiabelian varieties over K in which $\dim(\ker(f)) > 0$ and H is defined over a finite extension of $\mathbb{F}_q$, and S_0 is a

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groupless F-set in $f(\Gamma_0) \subset H(K)$ as in Definition 1.2. So, using (3.1), along with the fact that

$$\pi_0^{-1}(f^{-1}(S_0) \cap \Gamma_0) \cap \Gamma = (f \circ \pi_0)^{-1}(S_0) \cap \Gamma,$$

shows that $X(K) \cap \Gamma$ has the desired form as in the conclusion of Theorem 1.6.

Case 2: U is finite. In this case, we let $\tilde{\Gamma}$ be the $\mathbb{Z}[F]$-submodule spanned by Γ inside $G(K)$; since F is integral over $\mathbb{Z}$ (inside $\mathrm{End}(G)$), then $\tilde{\Gamma}$ is also a finitely generated subgroup of $G(K)$. According to [6] (see Theorem 1.4),

$$X(K) \cap \tilde{\Gamma} = \bigcup_{i=1}^{\ell} (S_i + \Gamma_i), \tag{3.2}$$

where each $S_i \subset \tilde{\Gamma}$ is a groupless F-set as in Definition 1.2, while each Γ_i is a subgroup of $\tilde{\Gamma}$. Now, since

$$X(K) \cap \Gamma = (X(K) \cap \tilde{\Gamma}) \cap \Gamma, \tag{3.3}$$

it suffices to prove that for each $i = 1, \ldots, \ell$, there exists a subset $A_i \subseteq X(K) \cap \Gamma$ which is a union of finitely many groupless F-sets in Γ along with finitely many generalised F-sets in Γ such that

$$(S_i + \Gamma_i) \cap \Gamma \subseteq A_i; \tag{3.4}$$

then combining (3.2), (3.3) and (3.4), we see that

$$X(K) \cap \Gamma = \bigcup_{i=1}^{\ell} (S_i + \Gamma_i) \cap \Gamma = \bigcup_{i=1}^{\ell} A_i$$

is indeed a finite union of groupless F-sets in Γ along with finitely many generalised F-sets in Γ, as claimed in the conclusion of Theorem 1.6.

To prove the existence of a set A_i (for each $i = 1, \ldots, \ell$) as in (3.4), we deal with two additional cases.

Case 2a: Γ_i is an infinite subgroup. In this case, we let X_i be the Zariski closure of $S_i + \Gamma_i$; clearly, $X_i \subseteq X$. We claim that X_i is a proper closed subvariety of X. Indeed, by construction, $\Gamma_i \subseteq \mathrm{Stab}_G(X_i)$ and since Γ_i is infinite, we cannot have $X_i = X$ because $\mathrm{Stab}_G(X)$ is finite. So, $\dim(X_i) < \dim(X)$ and by our inductive hypothesis, $A_i := X_i(K) \cap \Gamma$ satisfies the conclusion from Theorem 1.6. Therefore,

$$(S_i + \Gamma_i) \cap \Gamma \subseteq A_i,$$

where A_i is a union of finitely many groupless F-sets along with finitely many generalised F-sets, as desired for (3.4).

Case 2b: Γ_i is finite. In this case, letting $s := \#\Gamma_i$, we see that $S_i + \Gamma_i$ is a union of s groupless F-sets as in Definition 1.2. Now, [2, Theorem 3.1] shows that the intersection

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of a groupless F-set with a finitely generated group is itself a finite union of groupless F-sets; so,

$$A_i := (S_i + \Gamma_i) \cap \Gamma$$

is a finite union of groupless F-sets in Γ as desired for (3.4).

This concludes our proof of Theorem 1.6. □

Theorem 1.7 is an immediate corollary of our next two results which provide a more precise form of the intersection between a subvariety X of G with a finitely generated subgroup of $G(K)$ when X is a curve and when G is a simple semiabelian variety.

PROPOSITION 3.1. *Let G be a semiabelian variety defined over a finite subfield of an algebraically closed field K, let $\Gamma \subset G(K)$ be a finitely generated subgroup and let $X \subseteq G$ be an irreducible curve.*

(i) *If* $\dim(\mathrm{Stab}_G(X)) > 0$, *then* $X(K) \cap \Gamma$ *is a coset of a subgroup of* Γ.
(ii) *If* $\mathrm{Stab}_G(X)$ *is finite, then* $X(K) \cap \Gamma$ *is a finite union of groupless F-sets.*

PROOF. The proof of part (i) is immediate since then, $X = \gamma + G_1$ for some point $\gamma \in G(K)$ and some 1-dimensional connected algebraic subgroup $G_1 \subseteq G$. So, the intersection $X(K) \cap \Gamma$ is simply a coset of the subgroup $G_1(K) \cap \Gamma$ of Γ.

Now, we assume $\mathrm{Stab}_G(X)$ is finite. Then we let $\tilde{\Gamma}$ be the $\mathbb{Z}[F]$-submodule of $G(K)$ spanned by Γ. By Theorem 1.4, $\tilde{\Gamma}$ intersects $X(K)$ in a finite union of F-sets S_i in $\tilde{\Gamma}$. However, then at the expense of replacing each S_i with finitely many other F-sets, we may assume that each such F-set is groupless (see also the proof of *Case 2b* in Theorem 1.6). Finally, by another application of [2, Theorem 3.1], each $S_i \cap \Gamma$ is a finite union of groupless F-sets in Γ, as desired. □

PROPOSITION 3.2. *Let G be a simple semiabelian variety (that is, either a simple abelian variety or a* 1*-dimensional torus) defined over a finite subfield of an algebraically closed field K, let $\Gamma \subset G(K)$ be a finitely generated group and let $X \subset G$ be a proper closed subvariety. Then, $X(K) \cap \Gamma$ is a finite union of groupless F-sets in Γ.*

PROOF. First of all, we note that if Γ is a finite group, then clearly $X(K) \cap \Gamma$ is a finite set and thus a finite union of groupless F-sets, as desired.

So, from now on, we assume that Γ is infinite. According to Theorem 1.6, $X(K) \cap \Gamma$ is a finite union of groupless F-sets in Γ along with (possibly) finitely many generalised F-sets in Γ. Now, for any such generalised F-set in Γ (call it S),

$$S = (\pi|_{\Gamma})^{-1}(S_0),$$

where $\pi : G \longrightarrow H$ is a surjective group homomorphism of semiabelian varieties defined over a finite subfield of K, S_0 is a groupless F-set in $\pi(\Gamma) \subset H(K)$ and moreover, $\dim(\ker(\pi)) > 0$. However, since G is a simple semiabelian variety, this means that $\ker(\pi) = G$, that is, H is the trivial group variety and so, S would have to be the entire subgroup Γ. However, then its Zariski closure in G is an infinite algebraic subgroup of G (note that Γ is assumed now to be infinite) and so, once again because G is simple, we would conclude that Γ is Zariski dense in G. However, then because $S = \Gamma$ is contained in X, we would have $X = G$, contradicting the fact that X is a proper closed subvariety of G. Therefore, we have no generalised F-sets in Γ contained in the intersection $X(K) \cap \Gamma$.

This concludes our proof for Proposition 3.2. □

Theorem 1.10 follows easily from our Theorem 1.6 combined with Theorem 1.1.

PROOF OF THEOREM 1.10. Clearly, as argued in the proof of Theorem 1.6, it suffices to prove Theorem 1.10 assuming that X is an irreducible subvariety of G and $X(K) \cap \Gamma$ is Zariski dense in X. Then Theorem 1.1 yields

$$X = \gamma + \pi^{-1}(X_0),$$

where $\pi : G_0 \longrightarrow H$ is a surjective group homomorphism of semiabelian varieties, while G_0 is a semiabelian subvariety of G and $\gamma \in G(K)$; moreover, H and the subvariety $X_0 \subseteq H$ are defined over a finite subfield $\mathbb{F}_q \subset K$. Then for $x \in G(K)$, we have $x \in X(K)$ if and only if '$x - \gamma \in G_0(K)$ and $\pi(x - \gamma) \in X_0(K)$'. We denote $\Gamma_0 = G_0(K) \cap \Gamma$.

Pick $x_0 \in X(K) \cap \Gamma$. Let $g_0 = x_0 - \gamma \in G_0(K)$. We have $x_0 + \Gamma_0 = (\gamma + G_0(K)) \cap \Gamma$. As a result, for any $x \in \Gamma$, we have $x - \gamma \in G_0(K)$ if and only if there exists $\gamma_0 \in \Gamma_0$ such that $x = x_0 + \gamma_0$. Thus, $x - \gamma = g_0 + \gamma_0$ and so, $\pi(x - \gamma) \in X_0(K)$ yields $\pi(\gamma_0) \in -\pi(g_0) + X_0(K)$.

Let $X_0' = -\pi(g_0) + X_0$ which is a subvariety of H. The discussion above implies that $X(K) \cap \Gamma = x_0 + (\pi|_{\Gamma_0})^{-1}(X_0'(K) \cap \pi(\Gamma_0))$. So, considering the subvariety $X_0' \subseteq H$, along with the finitely generated subgroup $\pi(\Gamma_0)$ of $H(K)$, we apply Theorem 1.6 to conclude that the intersection $X_0'(K) \cap \pi(\Gamma_0)$ is a finite union of generalised F-sets in $\pi(\Gamma_0)$ along with finitely many groupless F-sets in $\pi(\Gamma_0)$. However, whether S is a generalised F-set in $\pi(\Gamma_0)$ or a groupless F-set in $\pi(\Gamma_0)$, $x_0 + (\pi|_{\Gamma_0})^{-1}(S)$ will always be a pseudo-generalised F-set in Γ (see also Remark 1.9). This shows that $X(K) \cap \Gamma$ is a finite union of pseudo-generalised F-sets in Γ, as desired. □

Acknowledgements

We thank the anonymous referee for helpful comments and suggestions. The second author is very grateful to his advisor Junyi Xie who introduced this topic to him.

References

[1] G. Faltings, 'The general case of S. Lang's conjecture', in: *Barsotti Symposium in Algebraic Geometry (Abano Terme, 1991)*, Perspectives in Mathematics, 15 (eds. V. Cristante and W. Messing) (Academic Press, San Diego, CA, 1994), 175–182.

[2] D. Ghioca, 'The isotrivial case in the Mordell–Lang theorem', *Trans. Amer. Math. Soc.* **360**(7) (2008), 3839–3856.

[3] E. Hrushovski, 'The Mordell–Lang conjecture for function fields', *J. Amer. Math. Soc.* **9**(3) (1996), 667–690.

[4] M. Laurent, 'Équations diophantiennes exponentielles', *Invent. Math.* **78** (1984), 299–327.

https://doi.org/10.1017/S0004972723000898 Published online by Cambridge University Press

[5] R. Moosa and T. Scanlon, 'The Mordell–Lang conjecture in positive characteristic revisited', in: *Model Theory and Applications*, Quad. Mat., 11 (eds. L. Bélair, P. D'Aquino, D. Marker, M. Otero, F. Point and A. Wilkie) (Aracne, Rome, 2002), 273–296.
[6] R. Moosa and T. Scanlon, 'F-structures and integral points on semiabelian varieties over finite fields', *Amer. J. Math.* **126** (2004), 473–522.
[7] P. Vojta, 'Integral points on subvarieties of semiabelian varieties. I', *Invent. Math.* **126**(1) (1996), 133–181.

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https://doi.org/10.1017/S0004972723000898 Published online by Cambridge University Press