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CHARACTERS OF COVERING GROUPS OF $SL(n)$

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Abstract We study characters of an n-fold cover $\widetilde{SL}(n,\mathbb{F})$ of $SL(n,\mathbb{F})$ over a non-Archimedean local field. We compute the character of an irreducible representation of $\widetilde{SL}(n,\mathbb{F})$ in terms of the character of an irreducible representation of a cover $\widetilde{GL}(n,\mathbb{F})$ of $GL(n,\mathbb{F})$. We define an analogue of L-packets for $\widetilde{SL}(n,\mathbb{F})$, such that the character of a linear combination of the representations in such a packet is computed in terms of the character of an irreducible representation of $PGL(n,\mathbb{F})$. This is analogous to stable endoscopic lifting for linear groups. We also prove an 'inversion' formula expressing the character of a genuine irreducible representation of $\widetilde{SL}(n,\mathbb{F})$ as a linear combination of virtual characters, each of which is obtained from $PGL(n,\mathbb{F})$.

Keywords: metaplectic group; representation theory; lifting; character

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1. Introduction

Let $\mathbb{G}$ be a reductive linear group defined over a local field $\mathbb{F}$ of characteristic 0, and let $G = \mathbb{G}(\mathbb{F})$. One of the ingredients of the local portion of the Langlands program for G is the study of characters of admissible representations of G. These are used on one side of the trace formula, and provide information about automorphic representations of $\mathbb{G}$ over a global field.

Important examples of automorphic representations involve reductive groups which are not linear, such as the oscillator representation of the metaplectic group, the twofold cover of $Sp(2n,\mathbb{F})$. We refer to a finite central extension $\tilde{G}$ of G which is not itself a linear group as a *nonlinear* group. It would be interesting to understand the representation theory of such groups, and to extend the Langlands program to the study of their automorphic representations.

A representation π of $\tilde{G}$ is said to be *genuine* if it does not factor to any proper quotient of $\tilde{G}$. One approach to the representation theory of $\tilde{G}$ is to relate genuine representations of $\tilde{G}$ to representations of a linear group via character theory. There are a number of examples of this approach. See the references and [**3**] for a survey.

Now assume the cardinality of the nth roots of unity μ_n of $\mathbb{F}$ is n. We consider a certain central extension $\widetilde{SL}(n,\mathbb{F})$ of $SL(n,\mathbb{F})$ by μ_n (cf. § 2).

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Arbitrary covers $\widetilde{GL}(n,\mathbb{F})$ of $GL(n,\mathbb{F})$ have been studied extensively [**5**–**8**]. Flicker, Kazhdan and Patterson relate character theory of $\widetilde{GL}(n,\mathbb{F})$ to that of $GL(n,\mathbb{F})$. The group $\widetilde{SL}(n,\mathbb{F})$ is a subgroup of a corresponding group $\widetilde{GL}(n,\mathbb{F})$, and a natural approach is to study representations of $\widetilde{SL}(n,\mathbb{F})$ by restricting representations of $\widetilde{GL}(n,\mathbb{F})$. The corresponding problem for $SL(n,\mathbb{F})$ and $GL(n,\mathbb{F})$ is quite difficult [**9**, **18**]. For example the case of $n=2$ is the first example of endoscopy and is highly non-trivial [**10**].

Surprisingly the corresponding restriction problem for genuine representations of $\widetilde{SL}(n,\mathbb{F})$ is very easy, and character theory of $\widetilde{SL}(n,\mathbb{F})$ reduces to that of $\widetilde{GL}(n,\mathbb{F})$. Our first step is to write a formula (Theorem 3.3) for the character of an irreducible genuine representation π of $\widetilde{SL}(n,\mathbb{F})$ in terms of the character of an irreducible representation of $\widetilde{GL}(n,\mathbb{F})$ which contains π in its restriction.

We are interested in relating the characters of representations of $\widetilde{SL}(n,\mathbb{F})$ to those of a linear group. This is modelled on the theory of endoscopy for linear groups. So suppose for the moment that $\mathbb{G}$ is a connected reductive algebraic group defined over $\mathbb{F}$, and let $G=\mathbb{G}(\mathbb{F})$. A virtual representation π of G is a formal sum $\sum_{i=1}^{n} a_i\pi_i$ of irreducible representations π_i with integral coefficients. We consider the global character

$$\Theta_\pi=\sum a_i\Theta_{\pi_i}$$

as a conjugation invariant function on the strongly regular semisimple elements of G. It is said to be *stable* if it is invariant under conjugation by $\mathbb{G}(\bar{\mathbb{F}})$ where $\bar{\mathbb{F}}$ is the algebraic closure of $\mathbb{F}$. The stable virtual characters are simpler than general virtual characters, and are basic objects in the theory.

The set of irreducible representations of G is conjecturally the disjoint union of finite sets called L-packets. If $\Pi=\{\pi_1,\dots,\pi_n\}$ is a tempered L-packet (i.e. each π_i is tempered), then $\sum_i\pi_i$ is conjectured to be stable. The goal of endoscopy is to find virtual sums $\sum_i a_i\pi_i$, each of which is computed via 'transfer' or 'lifting' from a stable virtual character on a smaller quasi-split 'endoscopic' group of the same rank. Furthermore we attempt to write each π_i as a linear combination of such lifted characters ('inversion').

If Π is not tempered, then $\sum_i\pi_i$ may not be stable. Arthur has conjectured that in some cases Π may be expanded to a larger 'Arthur packet' which does contain a stable sum [**4**]. Unlike L-packets Arthur packets may contain both tempered and non-tempered representations.

A special case of an endoscopic group is $H=G_{\mathrm{qs}}$, the quasi-split form of G. In this case transfer preserves stability, and the stable virtual characters of G are obtained from the stable virtual characters of G_{qs}.

Flicker, Kazhdan and Patterson have defined a lifting theory modelled on endoscopy for linear groups, conjecturally taking an irreducible unitary representation π of $GL(n,\mathbb{F})$ to an irreducible genuine unitary representation $t_*(\pi)$ of $\widetilde{GL}(n,\mathbb{F})$ or 0 [**6**–**8**]. The character of $t_*(\pi)$ is computed in terms of the character of π. For $GL(n,\mathbb{F})$ all virtual characters are stable, so this theory is analogous to transfer from G_{qs} to G in the linear case.

We are interested in the analogous theory for $\widetilde{SL}(n,\mathbb{F})$. The preceding results for $\widetilde{GL}(n,\mathbb{F})$, together with Theorem 3.3, express the character of an irreducible constituent of $t_*(\pi)$ restricted to $\widetilde{SL}(n,\mathbb{F})$ in terms of characters of $GL(n,\mathbb{F})$. The resulting formula

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does not formally have the properties of lifting. In particular $GL(n,\mathbb{F})$ and $\widetilde{SL}(n,\mathbb{F})$ do not have the same rank.

Our main result is that by taking an appropriate sum of representations of $\widetilde{SL}(n,\mathbb{F})$ we do obtain such a lifting formula, relating the character of the sum to an irreducible character of $PGL(n,\mathbb{F})$. This sum is our analogue of a stable virtual character for $\widetilde{SL}(n,\mathbb{F})$. In the tempered case the set of representations is analogous to an L-packet, and more generally to an Arthur packet. We proceed to describe this sum.

A constituent of $t_*(\pi)$ restricted to $\widetilde{SL}(n,\mathbb{F})$ is determined by a character ν of $\mathbb{F}^*$ for which ν^n equals the central character of π. More precisely, let

$$\widetilde{GL}(n,\mathbb{F})_+ = \{g \in \widetilde{GL}(n,\mathbb{F}) \mid \det(g) \in \mathbb{F}^{*n}\} = \widetilde{SL}(n,\mathbb{F})\tilde{Z}.$$

Here $\tilde{Z}$ is the inverse image of the centre Z of $GL(n,\mathbb{F})$ in $\widetilde{GL}(n,\mathbb{F})_+$, which is also the centre of $\widetilde{GL}(n,\mathbb{F})_+$. The constituents of $t_*(\pi)$ restricted to $\widetilde{GL}(n,\mathbb{F})_+$, equivalently $\widetilde{SL}(n,\mathbb{F})$, are parametrized by their central characters. These in turn are parametrized by characters ν of $\mathbb{F}^*$ with given restriction to $\mathbb{F}^{*n}$. See Proposition 3.1.

We write $L(\pi,\nu)$ for the summand of $t_*(\pi)$ corresponding to ν. This is an irreducible genuine representation of $\widetilde{SL}(n,\mathbb{F})$. For any character α of $\mathbb{F}^*$ we have $L(\pi\alpha^n,\nu\alpha^n) \approx L(\pi,\nu)$; we sum over $\widehat{\mathbb{F}^*}/\widehat{\mathbb{F}^{*n}} \approx \hat{\mu}_n$ and define (cf. Definition 5.7)

$$L_{\mathrm{st}}(\pi,\nu) = \sum_{\alpha\in\hat{\mu}_n} L(\pi\alpha,\nu\alpha).$$

This is our candidate for a 'stable' virtual character of $\widetilde{SL}(n,\mathbb{F})$. Now $\pi\nu^{-1}$ factors to $PGL(n,\mathbb{F})$, and it turns out that the character $\Theta_{L_{\mathrm{st}}(\pi,\nu)}$ of $L_{\mathrm{st}}(\pi,\nu)$ may be computed in terms of the character $\Theta_{\pi\nu^{-1}}$ of $\pi\nu^{-1}$. The main result is Theorem 8.1:

$$\Theta_{L_{\mathrm{st}}(\pi,\nu)}(g) = \sum_{\substack{h\in PGL(n,\mathbb{F})\\ \phi(h)=p(g)}} \Delta_\mu(h,g)\Theta_{\pi\nu^{-1}}(h). \tag{1.1}$$

Here g is a regular semisimple element of $\widetilde{SL}(n,\mathbb{F})$, and we identify the character of a representation with a function on the regular semisimple elements. Also ϕ is the *orbit correspondence* $\phi(g) = \det(g^{-1})g^n \in SL(n,\mathbb{F})$ (see § 6), p is projection from $\widetilde{SL}(n,\mathbb{F})$ to $SL(n,\mathbb{F})$, and $\Delta_\mu(h,g)$ is a *transfer factor* (see § 7). These ingredients are analogous to those of endoscopy for linear groups. Formula (1.1) is analogous to endoscopic lifting from G_{qs} to G, and $L_{\mathrm{st}}(\pi,\nu)$ is analogous to the stable lift of $\pi\nu^{-1}$, although since $\widetilde{SL}(n,\mathbb{F})$ is nonlinear the notion of stable distribution is not defined. If π is tempered, the set $\Pi(\pi,\nu) = \{L(\pi\alpha,\nu\alpha) \mid \alpha \in \hat{\mu}_n\}$ is our analogue of an L-packet for a linear group. In general it is analogous to an Arthur packet.

The group $PGL(n,\mathbb{F})$ is the one predicted by the Hecke algebra isomorphism of [**14**].

An L-packet for $SL(n,\mathbb{F})$ is the set of constituents of the restriction of an irreducible representation of $GL(n,\mathbb{F})$ to $SL(n,\mathbb{F})$ [**9**]. The character of the sum of these representations is stable, i.e. invariant by conjugation by $SL(n,\bar{\mathbb{F}})$, and these sets satisfy other required properties of L-packets. It is interesting to note, however, that $\Pi(\pi,\nu)$ is *not* the set of constituents of the restriction of a representation of $\widetilde{GL}(n,\mathbb{F})$. In particular (see

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the remark following Theorem 8.1) $\Theta_{L_{\rm st}(\pi,\nu)}$ is typically not $\widetilde{GL}(n,\mathbb{F})$ conjugation invariant. It would be interesting to find an intrinsic characterization of the virtual characters $L_{\rm st}(\pi,\nu)$.

We turn now to inversion. By analogy with the linear case we seek to write $L(\pi,\nu)$ as a linear combination of virtual representations, in the span of the elements of $\Pi(\pi,\nu)$, each of which is computed in terms of characters of a linear group. For $\zeta\in\mu_n$ let

$$L_\zeta(\pi,\nu)=\sum_{\alpha\in\hat{\mu}_n}\alpha(\zeta)L(\pi\alpha,\nu\alpha).$$

We obtain an inversion formula (Theorem 9.3),

$$\Theta_{L(\pi,\nu)}(g)=\frac{1}{n}\sum_{\zeta\in\mu_n}L_\zeta(\pi,\nu)=\frac{1}{n}\sum_{\zeta\in\mu_n}\chi^{-1}(z_\zeta)\Theta_{L_{\rm st}(\pi,\nu)}(z_\zeta g).$$

Here χ is the central character of $L(\pi,\nu)$ and z_ζ is an element of $\widetilde{SL}(n,\mathbb{F})$ with image $\zeta I\in SL(n,\mathbb{F})$.

These results all hold as stated for $\mathbb{F}=\mathbb{R}$ and $n=2$, in which case they are equivalent to a special case of [**2**].

Similar results hold for certain other N-fold covers of $\widetilde{SL}(n,\mathbb{F})$. One would not expect the general N-fold cover to be amenable to these methods, as the case $N=1$ makes abundantly clear.

The case of $n=2$, worked out in detail, is the subject of the University of Maryland thesis of Schultz [**15**]. This gives an intrinsic characterization of the local lift of Waldspurger [**19**]. In this case the set Π containing a genuine discrete series representation π consists of two elements π,π' where π' is the 'Waldspurger involution' [**19**] applied to π. This goes back to the Shimura correspondence for modular forms of half-integral weight which is the origin of the theory of nonlinear groups.

1.1. Desiderata

We consider covering groups which fit in an exact sequence,

$$1\to\mu_n\to\tilde{G}\xrightarrow{p}G\to 1,$$

with μ_n central in $\tilde{G}$ (cf. § 2). We write χ_π for the central character of a representation π. We say a representation π of $\tilde{G}$ is genuine if π has a central character χ_π whose restriction to μ_n is injective. If a representation π with a central character is not genuine, then π factors to a representation of a cover of G with kernel a subgroup of μ_n. If $\iota:\mu_n\hookrightarrow\mathbb{C}^*$ is an embedding, we say π is of type ι if $\chi_\pi|_{\mu_n}=\iota$.

An important role is played by the exact sequences

$$1\to\mu_n\to\mathbb{F}^*\xrightarrow{n}\mathbb{F}^{*n}\to 1,\tag{1.2}$$

$$1\to\mathbb{F}^{*n}\xrightarrow{\iota}\mathbb{F}^*\to\mathbb{F}^*/\mathbb{F}^{*n}\to 1\tag{1.3}$$

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and their Pontriagin duals,

$$1 \to \widehat{\mathbb{F}^{*n}} \to \widehat{\mathbb{F}^*} \xrightarrow{\text{res}} \hat{\mu}_n \to 1, \tag{1.4}$$

$$1 \to \widehat{\mathbb{F}^*/\mathbb{F}^{*n}} \to \widehat{\mathbb{F}^*} \xrightarrow{\text{res}} \widehat{\mathbb{F}^{*n}} \to 1. \tag{1.5}$$

Suppose μ_n is in the kernel of a character λ of $\mathbb{F}^*$. Then by (1.4) $\lambda(x) = \mu(x^n)$ for some character μ of $\widehat{\mathbb{F}^{*n}}$, which by (1.5) extends to $\tau \in \widehat{\mathbb{F}^*}$. This gives the following well-known lemma which we use repeatedly.

Lemma 1.1. *Let $\lambda \in \widehat{\mathbb{F}^*}$. Then $\lambda = \mu^n$ for some $\mu \in \widehat{\mathbb{F}^*}$ if and only if $\lambda(\zeta) = 1$ for all $\zeta \in \mu_n$.*

We identify the centre Z of $GL(n, \mathbb{F})$ with $\mathbb{F}^*$ and the central character χ_π of a representation of $GL(n, \mathbb{F})$ with an element of $\widehat{\mathbb{F}^*}$.

For $\alpha \in \widehat{\mathbb{F}^*}$ we write α for the character $\alpha \circ \det$ of $GL(n, \mathbb{F})$, and also for the character $\alpha \circ p$ of $\widetilde{GL}(n, \mathbb{F})$. Note that for π a representation of $GL(n, \mathbb{F})$ (with a central character)

$$\chi_{\pi\alpha} = \chi_\pi \alpha^n. \tag{1.6}$$

We write Θ_π for the global character of a representation π, considered as a function on the set of regular semisimple elements.

2. Group structure

We continue with the notation of § 1. We first define the group $\widetilde{SL}(n, \mathbb{F})$ (cf. [**11**,**12**,**17**]); this is a topological group which fits in an exact sequence,

$$1 \to \mu_n \xrightarrow{i} \widetilde{SL}(n, \mathbb{F}) \xrightarrow{p} SL(n, \mathbb{F}) \to 1, \tag{2.1}$$

with i, p continuous, i closed and p open. The classes of such extensions are parametrized by the group of (bilinear) Steinberg cocycles with values in μ_n. Let $(\cdot, \cdot)_n : \mathbb{F}^* \times \mathbb{F}^* \to \mu_n$ denote the nth norm residue symbol for $\mathbb{F}$. For properties of $(\cdot, \cdot)_n$ see [**16**] and [**7**, § 0.1]. In particular $(\cdot, \cdot)_n$ is a perfect pairing on $\mathbb{F}^*/\mathbb{F}^{*n}$ and gives an isomorphism of $\mathbb{F}^*/\mathbb{F}^{*n}$ with $\widehat{\mathbb{F}^*/\mathbb{F}^{*n}}$. Each Steinberg cocycle is given by $c(x, y) = (x, y)_n^k$ for some k. Write $G[k]$ for the group defined by the cocycle $(x, y)_n^k$. Then $G[k]$ and $G[k']$ are equivalent extensions if and only if $k \equiv k' \bmod (n)$.

The commutator subgroup $G[k]_{\mathrm{c}}$ of $G[k]$ is a covering group of $SL(n, \mathbb{F})$ with kernel a subgroup of μ_n. If $G[k]$ is not perfect, then $G[k] = G[k]_{\mathrm{c}}\mu_n$ and the representations of $G[k]$ of type ι are in bijection with the representations of $G[k]_{\mathrm{c}}$ of type $\iota|_{\mu_n \cap G[k]_{\mathrm{c}}}$. For this reason we assume $G[k]$ is perfect, which holds if and only if $\gcd(k, n) = 1$.

The map $G[k] \ni (g, \zeta) \to (g, \zeta^j) \in G[kj]$ is a homomorphism, and is an isomorphism if $\gcd(j, n) = 1$. In particular if $\gcd(k, n) = 1$ then $G[k]$ is isomorphic to $G[1]$ (although not equivalent as an extension unless $k \equiv 1 \bmod (n)$). We let $\widetilde{SL}(n, \mathbb{F}) = G[1]$. Once and for all we fix an embedding

$$\iota : \mu_n(\mathbb{F}) \hookrightarrow \mathbb{C}^*$$

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and we identify μ_n with its image. Henceforth we assume all genuine representations are of type ι.

The Steinberg cocycle defines a cover $\widetilde{GL}(n,\mathbb{F})$ of $GL(n,\mathbb{F})$ by [**7**], and $\widetilde{SL}(n,\mathbb{F})$ is a subgroup of $\widetilde{GL}(n,\mathbb{F})$ (we are taking $c=0$ in the notation of [**7**]).

We write $c(\cdot,\cdot)$ for the cocycle defining $\widetilde{GL}(n,\mathbb{F})$. Then

$$\widetilde{GL}(n,\mathbb{F}) = \{(g,\zeta) \mid g \in GL(n,\mathbb{F}),\ \zeta \in \mu_n\},$$

with multiplication $(g,\zeta)(g',\zeta') = (gg', \zeta\zeta' c(g,g'))$.

An essential role is played by the commutator. Suppose g and h are commuting elements of $GL(n,\mathbb{F})$. Choose any inverse images $\tilde{g}$, $\tilde{h}$ of g, h in $\widetilde{GL}(n,\mathbb{F})$. Then $\eta = \tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1} \in \mu_n$ is independent of the choices of $\tilde{g}$ and $\tilde{h}$. We write $\{g,h\} = \eta$.

An important property of the commutator is (see [**7**, Proof of Proposition 0.1.1])

$$\{xI, g\} = (x, \det(g))_n^{-1}. \tag{2.2}$$

2.1. Centres

Let

$$GL(n,\mathbb{F})_+ = \{g \in GL(n,\mathbb{F}) \mid \det(g) \in \mathbb{F}^{*n}\} = SL(n,\mathbb{F})Z.$$

Write $\tilde{H}$ for the inverse image in $\widetilde{GL}(n,\mathbb{F})$ of a subgroup H of $GL(n,\mathbb{F})$. The following lemma follows immediately from (2.2) and properties of the norm residue symbol.

Lemma 2.1. *Let $Z_+ = \{xI \mid x \in \mathbb{F}^{*n}\}$.*

(1) *The centre of $\widetilde{GL}(n,\mathbb{F})$ is $\tilde{Z}_+$.*

(2) *The centre of $\widetilde{GL}(n,\mathbb{F})_+$ is $\tilde{Z}$.*

(3) $\mathrm{Cent}_{\widetilde{GL}(n,\mathbb{F})}(\widetilde{GL}(n,\mathbb{F})_+) = \tilde{Z}$ *and* $\mathrm{Cent}_{\widetilde{GL}(n,\mathbb{F})}(\tilde{Z}) = \widetilde{GL}(n,\mathbb{F})_+$.

Thus $\tilde{Z}$ and $\widetilde{GL}(n,\mathbb{F})_+$ form a dual pair in the sense of Howe.

Therefore, $\widetilde{GL}(n,\mathbb{F})_+ = \widetilde{SL}(n,\mathbb{F})\tilde{Z}$, and $\tilde{Z}$ is the centre of $\widetilde{GL}(n,\mathbb{F})_+$. Consequently, an irreducible representation of $\widetilde{GL}(n,\mathbb{F})_+$ restricts to an irreducible representation of $\widetilde{SL}(n,\mathbb{F})$, and every irreducible representation of $\widetilde{SL}(n,\mathbb{F})$ is obtained this way. For many purposes we may replace $\widetilde{SL}(n,\mathbb{F})$ by $\widetilde{GL}(n,\mathbb{F})_+$. This is analogous to the corresponding situation for the linear groups. Note that

$$\frac{\widetilde{GL}(n,\mathbb{F})}{\widetilde{GL}(n,\mathbb{F})_+} \approx \frac{GL(n,\mathbb{F})}{GL(n,\mathbb{F})_+} \approx \frac{\mathbb{F}^*}{\mathbb{F}^{*n}}.$$

The cocycle restricted to Z_+ is trivial so $\tilde{Z}_+ \approx \mathbb{F}^{*n} \times \mu_n$. The cocycle restricted to Z is given by

$$c(xI, yI) = \prod_{i<j} (x,y)_n = (x,y)_n^{n(n-1)/2}.$$

This is equal to 1 if n is odd, or ± 1 if n is even.

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For later use we note there exists a (genuine) character μ of $\tilde{Z}$ satisfying

$$\mu|_{\tilde{Z}_+} = 1 \times \iota. \tag{2.3}$$

In fact we may take

$$\mu(xI,\zeta) = \begin{cases} \zeta, & n \text{ odd}, \\ \gamma(x,\psi)\zeta, & n \text{ even}. \end{cases} \tag{2.4}$$

Here ψ is a non-trivial additive character of $\mathbb{F}$ and $\gamma(x,\psi) \in \{\pm 1, \pm \mathrm{i}\}$ is the Weil index (see [**13**, Appendix]). In particular $\mu^n = 1$ (n odd), and $\mu^{2n} = 1$ (n even). We only use this explicit formula for (7.2).

Given μ, the genuine characters of $\tilde{Z}$ are in bijection with $\widehat{\mathbb{F}^*}$; given $\nu \in \widehat{\mathbb{F}^*}$ let

$$\chi_\nu(z) = \mu(z)\nu(x), \quad z \in \tilde{Z}, \quad p(z) = xI, \tag{2.5}$$

i.e.

$$\chi_\nu(xI,\zeta) = \mu(xI,\zeta)\nu(x) = \mu(xI,1)\zeta\nu(x). \tag{2.6}$$

2.2. Cartan subgroups

We define a Cartan subgroup of $\widetilde{GL}(n,\mathbb{F})$ or $\widetilde{SL}(n,\mathbb{F})$ to be the inverse image of a Cartan subgroup of the corresponding linear group. These groups are in general non-abelian, and an important role is played by their centres. We say an element of a covering group is semisimple (respectively, regular) if its image in the linear group is semisimple (respectively, regular).

Lemma 2.2. *Let T be a Cartan subgroup of $GL(n,\mathbb{F})$ with inverse image $\tilde{T}$ in $\widetilde{GL}(n,\mathbb{F})$.*

(1) *The centre of $\tilde{T}$ is $p^{-1}(T^n)$.*

(2) *The centre of $\tilde{T} \cap \widetilde{SL}(n,\mathbb{F})$ is $p^{-1}(ZT^n \cap SL(n,\mathbb{F}))$.*

Proof. (1) is proved in [**6**, § 3], and (2) follows from this as well. We will sketch another proof of (2) in § 3. □

We say a regular semisimple element $g \in \tilde{T}$ is *relevant* if it is contained in the centre of $\tilde{T}$ [**3**]. It is a basic fact that if π is a genuine representation of $\tilde{G}$, then $\Theta_\pi(g) = 0$ if g is not relevant (see [**5**] and [**3**, Proposition 2.7]).

3. Restriction from $\widetilde{GL}(n,\mathbb{F})$ to $\widetilde{SL}(n,\mathbb{F})$

We compute the character of an irreducible representation of $\widetilde{SL}(n,\mathbb{F})$ or $\widetilde{GL}(n,\mathbb{F})_+$ in terms of a character of $\widetilde{GL}(n,\mathbb{F})$ (Theorem 3.3). The main point is that Clifford theory for restriction of a genuine representation Π of $\widetilde{GL}(n,\mathbb{F})$ to $\widetilde{GL}(n,\mathbb{F})_+$ is very easy: each such representation restricts to a direct sum of $|\mathbb{F}^*/\mathbb{F}^{*n}|$ distinct irreducible representations which are permuted by the action of $\widetilde{GL}(n,\mathbb{F})/\widetilde{GL}(n,\mathbb{F})_+ \approx \mathbb{F}^*/\mathbb{F}^{*n}$. Furthermore, the

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character of each summand may be computed in terms of the character of Π using Fourier inversion on $\tilde{Z}/\tilde{Z}_+ \approx \mathbb{F}^*/\mathbb{F}^{*n}$.

Let π be a genuine representation of $\widetilde{GL}(n,\mathbb{F})_+$. Write $\pi \to \pi^g$ for the action (by conjugation on $\widetilde{GL}(n,\mathbb{F})_+$) of $g \in \widetilde{GL}(n,\mathbb{F})$ on representations of $\widetilde{GL}(n,\mathbb{F})_+$. Assume π has a central character χ_π. We compute χ_{π^g}. Let $z \in \tilde{Z}$ with $p(z) = xI$. Then

$$\begin{aligned}\chi_{\pi^g}(z) &= \chi_\pi(gzg^{-1}) \\ &= \chi_\pi(\{p(g), xI\}z) \\ &= \chi_\pi((x, \det(g))_n z) \quad \text{(by (2.2))} \\ &= \chi_\pi(z)(x, \det(g))_n \quad \text{(since } \pi \text{ is genuine).}\end{aligned} \tag{3.1}$$

By non-degeneracy of the symbol, if $\det(g) \notin \mathbb{F}^{*n}$ there exists x such that $(x, \det(g))_n \neq 1$. Therefore, if $g \notin \widetilde{GL}(n,\mathbb{F})_+$, $\chi_{\pi^g} \neq \chi_\pi$, and *a fortiori* $\pi^g \not\approx \pi$. Note the assumption π is genuine is essential; the corresponding result is false for representations of $\widetilde{GL}(n,\mathbb{F})$ which factor to $GL(n,\mathbb{F})$.

Let

$$\Pi = \mathrm{Ind}_{\widetilde{GL}(n,\mathbb{F})_+}^{\widetilde{GL}(n,\mathbb{F})}(\pi).$$

By (3.1) and Clifford theory $\widetilde{GL}(n,\mathbb{F})/\widetilde{GL}(n,\mathbb{F})_+$ acts simply transitively on the set of constituents of Π restricted to $\widetilde{GL}(n,\mathbb{F})_+$. For each $x \in \mathbb{F}^*/\mathbb{F}^{*n}$ choose $g_x \in \widetilde{GL}(n,\mathbb{F})$ with $\det(g) \equiv x \bmod(\mathbb{F}^{*n})$. Let $\pi^x = \pi^{g_x}$; the isomorphism class of π^x is independent of the choice of g_x. Thus

$$\Pi|_{\widetilde{GL}(n,\mathbb{F})_+} = \sum_{x \in \mathbb{F}^*/\mathbb{F}^{*n}} \pi^x. \tag{3.2}$$

If π' is a constituent of the restriction of Π to $\widetilde{GL}(n,\mathbb{F})_+$, then $\chi_{\pi'}$ (a character of $\tilde{Z}$) restricted to $\tilde{Z}_+$ is equal to χ_Π. The set of extensions of χ_Π to $\tilde{Z}$ is in bijection with $\widehat{\mathbb{F}^*/\mathbb{F}^{*n}}$. By (3.2) the constituents of this restriction are in bijection with $\mathbb{F}^*/\mathbb{F}^{*n}$. This proves the following result.

Proposition 3.1. *Let Π be an irreducible genuine representation of $\widetilde{GL}(n,\mathbb{F})$. Let S be the set of extensions of χ_Π to $\tilde{Z}_+$; this set is in bijection with $\widehat{\mathbb{F}^*/\mathbb{F}^{*n}}$. For $\lambda \in S$ let Π_λ be the λ eigenspace of Π.*

For all λ, Π_λ is an irreducible representation of $\widetilde{GL}(n,\mathbb{F})_+$ and

$$\Pi|_{\widetilde{GL}(n,\mathbb{F})_+} = \sum_{\lambda \in S} \Pi_\lambda.$$

Fix an irreducible constituent π of this restriction. Then

$$\Pi|_{\widetilde{GL}(n,\mathbb{F})_+} = \sum_{x \in \mathbb{F}^*/\mathbb{F}^{*n}} \pi^x$$

and the central character of π^x is $\chi_\pi(\cdot, x)_n$.

Remark 3.2. A similar result holds for $\widetilde{SL}(n,\mathbb{F})$: $\Pi|_{\widetilde{SL}(n,\mathbb{F})} = \sum_x \pi^x$ as above. However, the π^x are not necessarily distinct; in some cases $\pi \approx \pi^x$ (this implies $(x, \zeta)_n = 1$ for all $\zeta \in \mu_n$).

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We strengthen this result using Fourier inversion on $\mathbb{F}^*/\mathbb{F}^{*n}$ to write Θ_π in terms of Θ_Π.

For $z \in \tilde{Z}, z' \in \tilde{Z}_+$, $\chi_\pi(zz')^{-1}\Theta_\Pi(zz'g) = \chi_\pi(z)\Theta_\Pi(zg)$. Thus $\chi_\pi(z)^{-1}\Theta_\Pi(zg)$ is well defined for $z \in \tilde{Z}/\tilde{Z}_+$. We compute

$$\begin{aligned}\sum_{z\in\tilde{Z}/\tilde{Z}_+} \chi_\pi(z)^{-1}\Theta_\Pi(zg) &= \sum_{z\in\tilde{Z}/\tilde{Z}_+} \chi_\pi(z)^{-1} \sum_{x\in\mathbb{F}^*/\mathbb{F}^{*n}} \Theta_{\pi^x}(zg) \quad \text{(by (3.2))}\\ &= \sum_{z\in\tilde{Z}/\tilde{Z}_+} \sum_{x\in\mathbb{F}^*/\mathbb{F}^{*n}} \chi_\pi(z)^{-1}\chi_{\pi^x}(z)\Theta_{\pi^x}(g).\end{aligned}$$

Now $\chi_\pi(z)^{-1}\chi_{\pi^x}(z)$ factors to $\tilde{Z}/\tilde{Z}_+ \approx \mathbb{F}^*/\mathbb{F}^{*n}$, and by orthogonality of characters the right-hand side equals $|\mathbb{F}^*/\mathbb{F}^{*n}|\Theta_\pi(g)$. Explicitly, by (3.1),

$$\chi_\pi(z)^{-1}\chi_{\pi^x}(z) = \chi_\pi(z)^{-1}\chi_\pi(z)(y,x)_n = (y,x)_n,$$

where $p(z) = yI$. As z runs over $\tilde{Z}/\tilde{Z}_+$, y runs over representatives of $\mathbb{F}^*/\mathbb{F}^{*n}$, and this gives

$$\sum_{y\in\mathbb{F}^*/\mathbb{F}^{*n}} \sum_{x\in\mathbb{F}^*/\mathbb{F}^{*n}} (y,x)_n\Theta_{\pi^x} = \left|\frac{\mathbb{F}^*}{\mathbb{F}^{*n}}\right|\Theta_\pi(g),$$

since $(\cdot,\cdot)_n$ is a perfect pairing. This proves the following result.

Theorem 3.3. *Let π be a genuine representation of $\widetilde{GL}(n,\mathbb{F})_+$ with a central character. Let*

$$\Pi = \mathrm{Ind}_{\widetilde{GL}(n,\mathbb{F})_+}^{\widetilde{GL}(n,\mathbb{F})}(\pi).$$

This is a genuine representation of $\widetilde{GL}(n,\mathbb{F})$ and is irreducible if π is irreducible. Assume Θ_π exists. Then for $g \in \widetilde{GL}(n,\mathbb{F})_+$ (a regular semisimple element)

$$\begin{aligned}\Theta_\pi(g) &= \frac{1}{|\mathbb{F}^*/\mathbb{F}^{*n}|} \sum_{z\in\tilde{Z}/\tilde{Z}_+} \chi_\pi(z)^{-1}\Theta_\Pi(zg)\\ &= \frac{1}{|\mathbb{F}^*/\mathbb{F}^{*n}|} \sum_{x\in\mathbb{F}^*/\mathbb{F}^{*n}} \chi_\pi(z_x)^{-1}\Theta_\Pi(z_xg). \qquad (3.3)\end{aligned}$$

In the first sum z runs over any set of coset representatives of $\tilde{Z}/\tilde{Z}_+$. In the second the sum runs over any coset representatives of $\mathbb{F}^*/\mathbb{F}^{*n}$, and for each x, $z_x \in \tilde{Z}$ is any element satisfying $p(z_x) = xI$. Each term is independent of the choices.

Essentially the same result holds for genuine representations of $\widetilde{SL}(n,\mathbb{F})$. Let π be a genuine representation of $\widetilde{SL}(n,\mathbb{F})$, and extend π to a genuine representation π' of $\widetilde{GL}(n,\mathbb{F})_+$ (cf. § 2). Then (3.3) holds for $g \in \widetilde{SL}(n,\mathbb{F})$ and π replaced by π'. Each summand is independent of the choice of π'.

We can now sketch a proof of Lemma 2.2 (2).

Sketch of proof. By the theorem, if π is a genuine representation of $\widetilde{GL}(n,\mathbb{F})_+$ then $\Theta_\pi(g) = 0$ unless $p(zg) \in T^n$ for some z by Lemma 2.2 (1), i.e. $p(g) \in ZT^n$. Conversely,

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if g satisfies this condition then there exists a genuine representation for which $\Theta_\pi(g) \neq 0$, by the property that characters separate points. This proves the result for regular semisimple elements. For general elements apply a continuity argument. Alternatively apply the argument of Theorem 3.1 directly to an irreducible finite-dimensional genuine representation π of $\tilde{T}$, in which case Θ_π is defined for all $g \in \tilde{T}$. □

4. Lifting from $GL(n,\mathbb{F})$ to $\widetilde{GL}(n,\mathbb{F})$

In this section we summarize results on lifting of characters from $GL(n,\mathbb{F})$ to $\widetilde{GL}(n,\mathbb{F})$ (see [**6**–**8**]).

We first define transfer factors in this setting. Recall that the Weyl denominator for $GL(n,\mathbb{F})$ is given by

$$\Delta(g) = \prod_{i<j} \frac{|x_i - x_j|_{\bar{\mathbb{F}}}}{|x_i x_j|_{\bar{\mathbb{F}}}^{1/2}}$$

if g is a regular semisimple element with (distinct) eigenvalues x_i (in an algebraic closure of $\mathbb{F}$).

Definition 4.1. Suppose $h \in GL(n,\mathbb{F})$, $g \in \widetilde{GL}(n,\mathbb{F})$ are regular semisimple elements satisfying $h^n = p(g)$.

Let

$$\tau(h,g) = g s(h)^{-n} u(h). \tag{4.1}$$

Here $u(h) = \pm 1 \in \mu_n$ is defined by [**8**, § 2] (we take $u(h) = 1$ if n is odd), and $s : GL(n,\mathbb{F}) \to \widetilde{GL}(n,\mathbb{F})$ is any section. Note that $p\tau(h,g) = 1$ and we consider $\tau(h,g)$ to be an element of μ_n.

Let

$$\Delta(h,g) = |n^n|_{\mathbb{F}}^{-1/2} \tau(h,g) \frac{\Delta(h)}{\Delta(g)}. \tag{4.2}$$

Let π be a representation of $GL(n,\mathbb{F})$ with central character χ_π satisfying $\chi_\pi(\zeta I) = 1$ for all $\zeta \in \mu_n$. Suppose g is a regular semisimple element of $\widetilde{GL}(n,\mathbb{F})$, so $p(g)$ is contained in a Cartan subgroup T of $GL(n,\mathbb{F})$. Let

$$t_*(\Theta_\pi)(g) = \sum_{\substack{h \in T \\ h^n = p(g)}} \Delta(h,g)\Theta_\pi(h). \tag{4.3}$$

This is a conjugation invariant function on the regular semisimple elements of $\widetilde{GL}(n,\mathbb{F})$.

This is a special case of [**6**, 26.1], and we have written it in a different form. We use the notation of [**6**]. To see that (4.3) agrees with [**6**] first note that in our case the centre $\tilde{Z}_+$ of $\widetilde{GL}(n,\mathbb{F})$ is equal to $s(Z^n)\mu_n$, and it follows that the supplementary choice of $\tilde{\omega}$ of [**6**] is unnecessary. The summand in [**6**] is over

$$\{h \in T \mid h^{*-1} g \in \tilde{Z}_+\}/Z.$$

Given $\bar{h}$ in this set, choose a representative $h \in T$, and write $h^* z = g$ for $z \in \tilde{Z}_+$.

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Equivalently the sum is over

$$A = \{h \in T \mid (hz)^n = p(g) \text{ for some } z \in Z\}/Z.$$

On the other hand, we have written the sum over

$$B = \{h \in T \mid h^n = p(g)\}.$$

There is an n to 1 surjective map from B to A given by $h \to \bar{h}$. Finally, if $h^n = p(g)$ then $h^{*-1}g = s(h)^{-n}u(h)g = \tau(g,h)$, and since this is an element of μ_n, $\tilde{\omega}(\tau(g,h)) = \tau(g,h)$. We have incorporated this term, together with the constant b of [**6**, § 24] (divided by n because of the difference between A and B) into the transfer factor $\Delta(h,g)$.

Flicker, Kazhdan and Patterson conjecture that for π an irreducible unitary representation $t_*(\pi)$ is either 0 or $\pm$ the character of a genuine irreducible unitary representation of $\widetilde{GL}(n,\mathbb{F})$. We refine this conjecture into two hypotheses for later use.

Hypothesis I. *Let π be an irreducible representation of $GL(n,\mathbb{F})$ such that $\chi_\pi(\zeta I) = 1$ for all $\zeta \in \mu_n$. We say 'Hypothesis I holds for π' if $t_*(\pi)$ is 0 or $\pm$ the character of an irreducible representation of $\widetilde{GL}(n,\mathbb{F})$. If this holds, we define the virtual representation $t_*(\pi)$ by $t_*(\Theta_\pi) = \Theta_{t_*(\pi)}$. Furthermore, if $t_*(\pi) \neq 0$ define $\epsilon(\pi) = \pm 1$ so that $\epsilon(\pi)t_*(\pi)$ is a representation. We say 'Hypothesis I holds' if it holds for all π.*

Hypothesis II. *Every genuine irreducible unitary representation of $\widetilde{GL}(n,\mathbb{F})$ is isomorphic to $\epsilon(\pi)t_*(\pi)$ for some irreducible unitary representation π satisfying Hypothesis I.*

Hypotheses I and II hold for $n = 2$ (see [**5**]). Hypothesis I is true if π is a discrete series representation, and t_* is a bijection between a subset of the discrete series of $GL(n,\mathbb{F})$ and the genuine discrete series of $\widetilde{GL}(n,\mathbb{F})$ (see [**6**, § 26]). Hence Hypothesis II holds in the context of discrete series representations. For π a discrete series representation $\epsilon(\pi) = 1$. If $t_*(\pi)$ is supercuspidal, then π is supercuspidal, but not conversely.

Hypothesis I holds if π is tempered (see [**6**]), with the caveat that this statement depends on [**6**, Proposition 26.2], and in some cases there is a technical obstruction to this result holding as stated (the construction of an irreducible representation of $\widetilde{M}$ is not valid in all cases). In any event if π is tempered and satisfies Hypothesis I, then $t_*(\pi)$ is tempered and $\epsilon(\pi) = 1$. Subject to the preceding caveat Hypothesis II holds for tempered representations and t_* is a bijection between a subset of the irreducible tempered representations of $GL(n,\mathbb{F})$ and the genuine irreducible tempered representations of $\widetilde{GL}(n,\mathbb{F})$ (see [**6**, Theorem 27.3]).

Assuming Hypothesis II holds for tempered representations, then the Grothendieck group of genuine representations of $\widetilde{GL}(n,\mathbb{F})$ is spanned by the $t_*(\pi)$ for π satisfying Hypothesis I. Furthermore, the non-zero $t_*(\pi)$ as π runs over all standard modules for $GL(n,\mathbb{F})$ is a basis of the Grothendieck group of genuine representations of $\widetilde{GL}(n,\mathbb{F})$.

We are particularly interested in non-tempered representations π satisfying Hypothesis I. For example Hypothesis I holds for any character α satisfying $\alpha(\zeta) = 1$ for all $\zeta \in \mu_n$. In this case $t_*(\alpha)$ is a singular unitary quotient of a minimal principal series with a one-dimensional space of Whittaker functionals (see [**5**] and [**7**, Corollary I.3.6]).

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Hypothesis I should hold for all characters α. For example for $n = 2$, $-t_*(\alpha)$ is the supercuspidal constituent of the oscillator representation if $\alpha(-1) = -1$ (see [**5**]).

The central characters of π and $t_*(\pi)$ are related by

$$\chi_{t_*(\pi)}(x^n I, 1) = \chi_\pi(x). \tag{4.4}$$

We also have for any $\alpha \in \widehat{\mathbb{F}^*}$

$$t_*(\pi\alpha^n) = t_*(\pi)\alpha. \tag{4.5}$$

These follow immediately from (4.3).

5. Parameters for $\widetilde{SL}(n, \mathbb{F})$

We put lifting from $GL(n, \mathbb{F})$ to $\widetilde{GL}(n, \mathbb{F})$ together with restriction from $\widetilde{GL}(n, \mathbb{F})$ to $\widetilde{SL}(n, \mathbb{F})$ to obtain a character formula relating $GL(n, \mathbb{F})$ and $\widetilde{SL}(n, \mathbb{F})$.

We first consider $\widetilde{GL}(n, \mathbb{F})_+$. Suppose for the moment that Hypothesis II is true. We parametrize the genuine irreducible unitary representations of $\widetilde{GL}(n, \mathbb{F})_+$ as follows.

Fix a genuine irreducible unitary representation Π of $\widetilde{GL}(n, \mathbb{F})$. By Proposition 3.1 a constituent of the restriction of Π to $\widetilde{GL}(n, \mathbb{F})_+$ is determined by a character λ of $\tilde{Z}$ satisfying $\lambda|_{\tilde{Z}_+} = \chi_\Pi$, i.e.

$$\lambda(x^n, 1) = \chi_\Pi(x^n, 1), \quad x \in \mathbb{F}^*.$$

By Hypothesis II there exists an irreducible representation π of $GL(n, \mathbb{F})$, with $\chi_\pi(\mu_n) = 1$, such that $t_*(\pi) = \pm\Pi$. By (4.4)

$$\chi_\Pi(x^n, 1) = \chi_\pi(x),$$

so we have

$$\lambda(x^n, 1) = \chi_\pi(x). \tag{5.1}$$

Fix a genuine character μ of $\tilde{Z}$ satisfying (2.3). Then the set of characters λ of $\tilde{Z}$ satisfying (5.1) is (cf. (2.5))

$$\{\chi_\nu \mid \nu^n = \chi_\pi\}.$$

Note that by Lemma 1.1 and (1.5) the set of such ν is parametrized by $\widehat{\mathbb{F}^*/\mathbb{F}^{*n}}$, and by Proposition 3.1 this parametrizes the constituents of $\Pi|_{\widetilde{GL}(n,\mathbb{F})_+}$.

This motivates the following definition.

Definition 5.1. Let X be the set of pairs (π, ν) where the following hold.

(1) π is an irreducible representation of $GL(n, \mathbb{F})$, with central character χ_π satisfying $\chi_\pi(\zeta I) = 1$ for all $\zeta \in \mu_n$.

(2) ν is a character of $\mathbb{F}^*$ satisfying $\nu^n = \chi_\pi$.

Let $(\pi, \nu) \in X$, and assume Hypothesis I holds for π.

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(3) Let $L_+(\pi,\nu)$ be the constituent of $t_*(\pi)$ restricted to $\widetilde{GL}(n,\mathbb{F})_+$ with central character χ_ν (cf. (2.5)).

(4) Let $L(\pi,\nu)$ be the restriction of $L_+(\pi,\nu)$ to $\widetilde{SL}(n,\mathbb{F})$.

Remark 5.2. L and L_+ depend on the choice of μ satisfying (2.3).

By definition $\epsilon(\pi)L(\pi,\nu)$ is the character of a representation. Assuming Hypothesis II every genuine irreducible unitary representation of $\widetilde{GL}(n,\mathbb{F})_+$ is isomorphic to $\epsilon(\pi)L(\pi,\nu)$ for some $(\pi,\nu)\in X$.

If $(\pi,\nu)\in X$, then by (1.6)

$$\chi_{\pi\nu^{-1}} = \chi_\pi \nu^{-n} = 1, \tag{5.2}$$

i.e. $\pi\nu^{-1}$ factors to a representation of $PGL(n,\mathbb{F})$. If π is a representation of $GL(n,\mathbb{F})$ with trivial central character, let $\bar{\pi}$ be the corresponding representation of $PGL(n,\mathbb{F})$.

Definition 5.3. For $(\pi,\nu)\in X$, let $M(\pi,\nu)$ be the irreducible representation $\overline{\pi\nu^{-1}}$ of $PGL(n,\mathbb{F})$.

Thus X is the graph of a correspondence between irreducible genuine representations of $\widetilde{GL}(n,\mathbb{F})_+$ or $\widetilde{SL}(n,\mathbb{F})$ and $PGL(n,\mathbb{F})$. That is for π an irreducible representation of $\widetilde{GL}(n,\mathbb{F})_+$ or $\widetilde{SL}(n,\mathbb{F})$ and π' an irreducible representation of $PGL(n,\mathbb{F})$ we say π corresponds to π' if there exists $x=(\pi,\nu)\in X$, with π satisfying Hypothesis I, such that $L_+(x)=\pi$ or $L(x)=\pi$, and $M(x)=\pi'$. Assuming Hypothesis II every genuine irreducible unitary representation of $\widetilde{SL}(n,\mathbb{F})$ is in the image of the correspondence.

Lemma 5.4.

(1) *If $(\pi,\nu)\in X$, then $(\pi\alpha,\nu\alpha)\in X$ for all $\alpha\in\widehat{\mathbb{F}^*}$. Thus $x=(\pi,\nu)\to\alpha x=(\pi\alpha,\nu\alpha)$ defines an action of $\widehat{\mathbb{F}^*}$ on X.*

For all $\alpha\in\widehat{\mathbb{F}^*}$ and $x\in X$, we have the following.

(2) $M(\alpha x)=x$.

(3) $L_+(\alpha^n x)=L_+(x)\alpha$.

(4) $L(\alpha^n x)=L(x)$.

Proof. (1) and (2) are immediate. By (4.5) $t_*(\alpha^n\pi)=t_*(\pi)\alpha$, and by (1.6) $L_+(\alpha^n x)$ and $L_+(x)\alpha$ have the same central character; (3) follows and (4) is an immediate consequence of (3). □

Remark 5.5. If $\beta\in\widehat{\mathbb{F}^*}$ is non-trivial on μ_n, then $\beta\notin\widehat{\mathbb{F}^{*n}}$, and there is no elementary relationship between $L_+(\beta x)$ and $L_+(x)$.

Remark 5.6. The action of $\widehat{\mathbb{F}^*}/\widehat{\mathbb{F}^{*n}}\approx\hat{\mu}_n$ on genuine representations of $\widetilde{SL}(n,\mathbb{F})$ given by $\alpha: L(x)\to L(\alpha x)$ generalizes the 'Waldspurger involution' for $\widetilde{SL}(2,\mathbb{F})$ (see [**19**]). We intend to return to this point in another paper.

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We compute the set of representations of $\widetilde{SL}(n,\mathbb{F})$ corresponding to a given irreducible representation of $PGL(n,\mathbb{F})$.

Fix $x=(\pi,\nu)\in X$. If $M(x')=M(x)$, then $x'=\alpha x$ for some α. By Lemma 5.4 (4) if $\alpha\in(\widehat{\mathbb{F}^*})^n\approx\widehat{\mathbb{F}^{*n}}$, then $L(\alpha x)=L(x)$. Therefore, the irreducible representations of $\widetilde{SL}(n,\mathbb{F})$ corresponding to $M(x)$ are the $L(\alpha x)$ for $\alpha\in\widehat{\mathbb{F}^*}/\widehat{\mathbb{F}^{*n}}$ (not to be confused with $\widehat{\mathbb{F}^*/\mathbb{F}^{*n}}$), which by (1.4) is isomorphic to $\hat{\mu}_n$.

Definition 5.7. Let π be an irreducible representation of $GL(n,\mathbb{F})$ with central character trivial on μ_n. Suppose Hypothesis I holds for $\pi\alpha$ for all $\alpha\in\widehat{\mathbb{F}^*}$.

(1) For $(\pi,\nu)\in X$ let

$$L_{\mathrm{st}}(\pi,\nu)=\sum_{\alpha}L(\pi\alpha,\nu\alpha),$$

where the sum runs over a set of representatives of $\widehat{\mathbb{F}^*}/\widehat{\mathbb{F}^{*n}}\approx\hat{\mu}_n$.

(2) Let π be an irreducible representation of $PGL(n,\mathbb{F})$, and let π' denote π pulled back to $GL(n,\mathbb{F})$. Assume $\pi'\alpha$ satisfies Hypothesis I for all $\alpha\in\widehat{\mathbb{F}^*}$. Define $L_{\mathrm{st}}(\pi)=L_{\mathrm{st}}(\pi',1)$.

Remark 5.8. $L_{\mathrm{st}}(\pi,\nu)=L_{\mathrm{st}}(\pi\alpha,\nu\alpha)$ for all α, and in particular

$$L_{\mathrm{st}}(\pi,\nu)=L_{\mathrm{st}}(\pi\nu^{-1},1)=L_{\mathrm{st}}(\pi\nu^{-1}).$$

As discussed in § 1, $L_{\mathrm{st}}(\pi)$ and $\Pi(\pi,\nu)=\{L(\pi\alpha,\nu\alpha)\mid\alpha\in\hat{\mu}_n\}$ are our candidates for a 'stable' virtual character and packet of $\widetilde{SL}(n,\mathbb{F})$. Note that the non-zero representations $L(\pi\alpha,\nu\alpha)$ in $\Pi(\pi,\nu)$ are distinct, and in fact have distinct central characters on $\widetilde{SL}(n,\mathbb{F})$. One could define a stable virtual character of $\widetilde{SL}(n,\mathbb{F})$ to be in the span of the $L_{\mathrm{st}}(\pi)$. It is not clear how to characterize the stable virtual characters intrinsically.

Not all $L(\pi\alpha,\nu\alpha)$ are necessarily non-zero. For example suppose π is the principal series representation defined by the character $\lambda(\mathrm{diag}(h_1,\dots,h_n))=\prod\lambda_i(h_i)$. This has central character trivial on μ_n if $\prod_i\lambda_i(\zeta)=1$ for all $\zeta\in\mu_n$. On the other hand, $t_*(\pi)=0$ unless $\lambda_i(\zeta)=1$ for all $i,\zeta\in\mu_n$. Assume this holds. Then $L(\pi,\alpha)$ is a principal series of $\widetilde{GL}(n,\mathbb{F})$, and $L(\pi\alpha,\nu\alpha)=0$ for all $\alpha\neq1$, so $L(\pi,\nu)=L_{\mathrm{st}}(\pi,\nu)$.

If π is a discrete series representation, then π satisfies Hypothesis I; if $\chi_\pi(\mu_n)=1$, then $t_*(\pi\alpha)\neq0$ for all α, and $L(\pi\alpha,\nu\alpha)\neq0$ for all α. Therefore, in this case, $|\Pi(\pi,\alpha)|=n$.

In the case $n=2$, $L(1,1)$ is non-tempered, and is isomorphic to ω_{e}, the even half of the oscillator representation $\omega=\omega_{\mathrm{e}}\oplus\omega_{\mathrm{o}}$ of $\widetilde{SL}(n,\mathbb{F})$ (ω depends on an additive character ψ, which is determined by μ). If $\alpha(-1)=-1$, then $L(\alpha,\alpha)=-\omega_{\mathrm{o}}$ is supercuspidal, and $L_{\mathrm{st}}(1)=\omega_{\mathrm{e}}-\omega_{\mathrm{o}}$ (see [**1**, **5**, **15**]).

6. Orbit correspondence

For $g\in GL(n,\mathbb{F})$ write $\bar{g}$ for the image of g in $PGL(n,\mathbb{F})$.

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Definition 6.1. For $h \in GL(n,\mathbb{F})$ let

$$\phi(h) = \det(h^{-1})h^n \in SL(n,\mathbb{F}).$$

Then $\phi(zg) = \phi(g)$ for all $z \in Z$, so ϕ factors to a map from $PGL(n,\mathbb{F})$ to $SL(n,\mathbb{F})$.

Thus $GL(n,\mathbb{F})$ is the graph of a correspondence between $PGL(n,\mathbb{F})$ and $SL(n,\mathbb{F})$ via the maps the maps $g \to \bar{g} \in PGL(n,\mathbb{F})$ and $g \to \phi(g) \in SL(n,\mathbb{F})$. The following lemma is immediate.

Lemma 6.2.

(1) *For all $h \in PGL(n,\mathbb{F})$, $g \in GL(n,\mathbb{F})$, $\phi(\bar{g}h\bar{g}^{-1}) = g\phi(h)g^{-1}$.*

(2) *If h is a regular semisimple element, then $\phi(h)$ is relevant (cf. Lemma 2.2).*

We also need the *weak* orbit correspondence. Suppose $h \in GL(n,\mathbb{F})$, $g \in SL(n,\mathbb{F})$ satisfy

$$h^n = zg, \quad z \in Z.$$

Multiplying both sides by $\det(h^{-1})$ shows this is equivalent to

$$\phi(h) = \det(h^{-1})zg, \quad z \in Z.$$

Since $\phi(h)$ and g have determinant one this gives

$$\det(h^{-1})z = \phi(h)g^{-1} = \zeta I, \quad \zeta \in \mu_n. \tag{6.1}$$

Definition 6.3. We say $h \in PGL(n,\mathbb{F}), g \in SL(n,\mathbb{F})$ weakly correspond, written

$$h \xleftrightarrow{\text{weak}} g,$$

if for any (equivalently all) $h' \in GL(n,\mathbb{F})$ with $\bar{h}' = h$,

$$h'^n = zg, \quad z \in Z.$$

Equivalently,

$$g = \zeta\phi(h), \quad \zeta \in \mu_n.$$

If $h \xleftrightarrow{\text{weak}} g$, define $\zeta(h,g) \in \mu_n$ by

$$g = \zeta(h,g)\phi(h). \tag{6.2}$$

We give an alternative description of the orbit correspondences in terms of roots and weights. This is not needed for what follows. Given a Cartan subgroup T of $GL(n,\mathbb{F})$, we identify the root and weight lattices of the corresponding Cartan subgroups of $PGL(n,\mathbb{F})$ and $SL(n,\mathbb{F})$.

Lemma 6.4. *Fix a Cartan subgroup T of $GL(n,\mathbb{F})$, with corresponding subgroups $T_{PGL(n)}$ and $T_{SL(n)}$. Suppose $h \in T_{PGL(n)}$ and $g \in T_{SL(n)}$.*

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(1) $h \overset{weak}{\longleftrightarrow} g$ *if and only if* $\alpha(h^n) = \alpha(g)$ *for all roots* α.

(2) $\phi(h) = g$ *if and only if* $(n\lambda)(h) = \lambda(g)$ *for all weights* λ.

Proof. Part (1) is immediate. For (2) we need to show for $h \in GL(n, \mathbb{F})$, $\zeta \in \mu_n$,

$$(n\lambda)(h) = \lambda(\zeta \det(h^{-1})h^n) \quad \text{for all weights } \lambda \Leftrightarrow \zeta = 1.$$

The subtlety is that $\lambda(h)$ is not defined for arbitrary elements of $GL(n, \mathbb{F})$. If $h \in SL(n, \mathbb{F})$, $Z = GL(n, \mathbb{F})_+$, then $\lambda(h)$ is defined and this is immediate. It is enough to work over the algebraic closure $\bar{\mathbb{F}}$, in which case $GL(n, \bar{\mathbb{F}})_+ = GL(n, \bar{\mathbb{F}})$, proving the result. □

Remark 6.5. If g is in the split torus, then $|\{h \mid \phi(h) = g\}| = n^{n-2}$ or 0. In general the cardinality of the inverse image of a $g \in SL(n, \mathbb{F})$ depends on the Cartan subgroup containing g.

7. Transfer factors

We continue with the notation of § 6. Fix a character μ of $\tilde{Z}$ satisfying (2.3). We define transfer factors $\Delta_\mu(h, g)$ (Definition 7.3). These satisfy one of the standard requirements of transfer factors: $|\Delta_\mu(h, g)| = |\Delta(h)/\Delta(g)|$ (see (7.2)), up to a constant which is 1 for almost every residual characteristic.

Definition 7.1. Suppose $h \in GL(n, \mathbb{F})$, $g \in \widetilde{SL}(n, \mathbb{F})$ satisfy

$$h^n = p(zg), \quad z \in \tilde{Z} \tag{7.1}$$

(cf. Definition 4.1).

Let

$$\Delta_\mu(h, g) = \frac{n^2}{|\mathbb{F}^*/\mathbb{F}^{*n}|}\mu(z)^{-1}\Delta(h, zg).$$

This is independent of the choice of z satisfying (7.1).

Lemma 7.2. *For all* $\lambda \in \mathbb{F}^*$

$$\Delta_\mu(\lambda h, g) = \Delta_\mu(h, g).$$

Proof. Choose $z \in \tilde{Z}$ satisfying $h^n = p(zg)$, and $w \in \tilde{Z}$ satisfying $p(w) = \lambda^n I$. Then $(\lambda h)^n = p(wzg)$. We need to show $\Delta_\mu(h, g) = \Delta_\mu(\lambda h, g)$, i.e.

$$\mu(z)^{-1}zgs(h)^{-n}u(h) = \mu(wz)^{-1}wzgs(\lambda h)^{-n}u(\lambda h).$$

After cancellations this is equivalent to

$$s(\lambda h)^n u(\lambda h) = \mu(w)^{-1}ws(h)^n u(h).$$

By [**6**, § 4] $s(\lambda h)^n u(\lambda h) = s(h)^n u(h)s_0(\lambda^n)$, where s_0 is the distinguished section, i.e. $s_0(g) = (g, 1)$. Inserting this we are reduced to showing $s_0(\lambda^n) = \mu(w)^{-1}w$, which is precisely the fact that $\mu|_{\tilde{Z}_+} = \iota$. □

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Definition 7.3. Suppose $h \in PGL(n,\mathbb{F})$, $g \in \widetilde{SL}(n,\mathbb{F})$ satisfy

$$h \overset{\text{weak}}{\longleftrightarrow} p(g).$$

Choose $h' \in GL(n,\mathbb{F})$ satisfying $\bar{h}' = h$. Let

$$\Delta_\mu(h,g) = \Delta_\mu(h',g).$$

By the lemma this is independent of the choice of h'.

Given h, g as in Definition 7.3, choose $h' \in GL(n,\mathbb{F})$ satisfying $\bar{h}' = h$, and choose $z \in \tilde{Z}$ with $h'^n = p(zg)$. Recall τ is given by Definition 4.1, and $|\mathbb{F}^*/\mathbb{F}^{*n}| = n^2/|n|_\mathbb{F}$ (see [**7**, Lemma 0.3.2]). This gives

$$\begin{aligned}\Delta_\mu(h,g) &= \frac{n^2}{|\mathbb{F}^*/\mathbb{F}^{*n}|}\mu(z)^{-1}\Delta(h',zg)\\ &= |n|_\mathbb{F}^{1-n/2}\mu(z)^{-1}\tau(h',zg)\frac{\Delta(h)}{\Delta(g)}\\ &= |n|_\mathbb{F}^{1-n/2}\mu(z)^{-1}zgs(h')^{-n}u(h')\frac{\Delta(h)}{\Delta(g)}.\end{aligned}$$

This is independent of the choices.

Remark 7.4. If $n = 2$ or the residual characteristic of $\mathbb{F}$ does not divide n, then by (2.4)

$$\left(\frac{\Delta_\mu(h,g)}{\Delta(h)/\Delta(g)}\right)^N = 1, \tag{7.2}$$

with $N = n$ (n odd) or $N = 2n$ (n even).

Remark 7.5. If μ' is another character satisfying (2.3), then $\mu'(z) = \mu(z)(y,x)_n$ for some y, where $p(z) = xI$, and

$$\frac{\Delta_{\mu'}}{\Delta_\mu}(h,g) = \frac{\mu}{\mu'}(h) = (\det(h), y)_n$$

($\det(h)$ is a well-defined element of $\mathbb{F}^*/\mathbb{F}^{*n}$).

Although we will not need it we state the invariance property of Δ_μ. Suppose $h \overset{\text{weak}}{\longleftrightarrow} g$. For $y \in \widetilde{GL}(n,\mathbb{F})$ let $y_0 = \overline{p(y)} \in PGL(n,\mathbb{F})$.

Lemma 7.6. *We have*

$$\Delta_\mu(y_0 h y_0^{-1}, ygy^{-1}) = \Delta_\mu(h,g)(\det(h)\zeta(h,g), \det(y))_n.$$

Proof. A straightforward computation which is left to the reader. □

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8. Stable character formula

We state the formula relating the character of an irreducible representation π of $PGL(n,\mathbb{F})$ to the character of the virtual genuine representation $L_{\mathrm{st}}(\pi)$ of $\widetilde{SL}(n,\mathbb{F})$.

Fix μ as in (2.3), define L_{st} as in Definition 5.7, ϕ as in § 6 and Δ_μ as in § 7.

Theorem 8.1 (main theorem). *Let π be an irreducible representation of $PGL(n,\mathbb{F})$, for which $L_{\mathrm{st}}(\pi)$ is defined (Definition 5.7). Then for g a regular semisimple element of $\widetilde{SL}(n,\mathbb{F})$,*

$$\Theta_{L_{\mathrm{st}}(\pi)}(g)=\sum_{\substack{h\in PGL(n,\mathbb{F})\\ \phi(h)=p(g)}}\Delta_\mu(h,g)\Theta_\pi(h). \tag{8.1}$$

Recall the hypothesis on π is that $t_*(\pi\alpha)$ is defined for all $\alpha\in\widehat{\mathbb{F}^*}$ (we have pulled π back to $GL(n,\mathbb{F})$).

Remark 8.2. By Lemma 7.6 the right-hand side of (8.1) is *a priori* $\widetilde{SL}(n,\mathbb{F})$ conjugation invariant. We do not need this, and it is a consequence of the theorem. Note that $\Theta_{L_{\mathrm{st}}(\pi)}$ is not necessarily invariant by conjugation by $\widetilde{GL}(n,\mathbb{F})$, since Δ_μ is only $\widetilde{SL}(n,\mathbb{F})$ conjugation invariant (Lemma 7.6).

Proof. We first give a formula for $\Theta_{L(\pi,\nu)}(g)$ for arbitrary $(\pi,\nu)\in X$ (with π satisfying Hypothesis I).

By Theorem 3.3,

$$\Theta_{L(\pi,\nu)}(g)=\sum_{z\in\tilde{Z}/\tilde{Z}_+}\chi_\nu(z)^{-1}\Theta_{t_*(\pi)}(zg)$$

(sum over any set of coset representatives). Inserting (4.3) gives

$$\Theta_{L(\pi,\nu)}(g)=\frac{1}{|\mathbb{F}^*/\mathbb{F}^{*n}|}\sum_{z\in\tilde{Z}/\tilde{Z}_+}\sum_{h^n=p(zg)}\chi_\nu(z)^{-1}\Delta(h,zg)\Theta_\pi(h). \tag{8.2}$$

Write the summand as follows:

$$\begin{aligned}\chi_\nu(z)^{-1}\Delta(h,zg)\Theta_\pi(h)&=\mu(z)^{-1}\nu(z)^{-1}\Delta(h,zg)\Theta_\pi(h) &&\text{(by (2.6))}\\ &=\frac{|\mathbb{F}^*/\mathbb{F}^{*n}|}{n^2}\nu(z)^{-1}\Delta_\mu(h,g)\Theta_\pi(h) &&\text{(Definition 7.1)}\\ &=\frac{|\mathbb{F}^*/\mathbb{F}^{*n}|}{n^2}\nu(h)\nu(z)^{-1}\Delta_\mu(h,g)\Theta_{\pi\nu^{-1}}(h).\end{aligned}$$

By (5.2) and Lemma 7.2, $\Delta_\mu(h,g)$ and $\Theta_{\pi\nu^{-1}}(h)$ only depend on the image $\bar{h}\in PGL(n,\mathbb{F})$ of h. By (6.1) and (6.2), $\nu(h)\nu(z)^{-1}=\nu(\phi(h)^{-1}g)=\nu(\zeta(\bar{h},g))$. This gives

$$\frac{|\mathbb{F}^*/\mathbb{F}^{*n}|}{n^2}\nu(\zeta(\bar{h},g))\Delta_\mu(\bar{h},g)\theta_{\pi\nu^{-1}}(\bar{h}).$$

Inserting this in (8.2) and changing the order of summation gives the following intermediate result.

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Proposition 8.3. *We have*

$$\begin{aligned}\Theta_{L(\pi,\nu)}(g) &= \frac{1}{n}\sum_{\substack{h\in PGL(n,\mathbb{F})\\ h\xleftrightarrow{\text{weak}} g}} \nu(\zeta(h,g))\Delta_\mu(h,g)\theta_{\pi\nu^{-1}}(h)\\ &= \frac{1}{n}\sum_{\zeta\in\mu_n}\nu(\zeta)\sum_{\substack{h\in PGL(n,\mathbb{F})\\ \phi(h)=\zeta g}}\Delta_\mu(h,g)\theta_{\pi\nu^{-1}}(h).\end{aligned}$$

Replace (π,ν) with $(\pi\alpha,\nu\alpha)$. On the right-hand side only the term $\nu(\zeta(h,g))$ is affected. Summing over α gives

$$\Theta_{L_{\mathrm{st}}(\pi,\nu)}(g) = \frac{1}{n}\sum_{\zeta\in\mu_n}\sum_{\alpha\in\hat{\mu}_n}\nu(\zeta)\alpha(\zeta)\sum_{\substack{h\in PGL(n,\mathbb{F})\\ \phi(h)=\zeta g}}\Delta_\mu(h,g)\theta_{\pi\nu^{-1}}(h).$$

By orthogonality of characters for μ_n this equals

$$\sum_{\substack{h\in PGL(n,\mathbb{F})\\ \phi(h)=g}}\Delta_\mu(h,g)\theta_{\pi\nu^{-1}}(h).$$

This completes the proof. □

9. Inversion

We continue in the setting of the preceding section. Suppose $\pi\alpha$ satisfies Hypothesis I for all α.

Definition 9.1. For $\zeta\in\mu_n$ let

$$L_\zeta(\pi,\nu) = \sum_{\alpha\in\hat{\mu}_n}\alpha(\zeta)L(\pi\alpha,\nu\alpha).$$

This is a virtual character in which we allow rational coefficients, and $L_1(\pi,\nu) = L_{\mathrm{st}}(\pi,\nu)$.

By Fourier inversion on μ_n we have

$$L(\pi,\nu) = \frac{1}{n}\sum_{\zeta\in\mu_n}L_\zeta(\pi,\nu). \tag{9.1}$$

Recall the central character of $L(\pi\alpha,\nu\alpha)$ is $\chi_{\alpha\nu}$, i.e.

$$\chi_{L(\pi\alpha,\nu\alpha)}(z_\zeta) = \chi_\nu(z_\zeta)\alpha(\zeta),$$

where $p(z_\zeta)=\zeta I$. That is,

$$\alpha(\zeta)\Theta_{L(\pi\alpha,\nu\alpha)}(g) = \chi_\nu^{-1}(z_\zeta)\Theta_{L(\pi\alpha,\nu\alpha)}(z_\zeta g).$$

Inserting this into the definition gives the following result.

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Lemma 9.2. *For all* $\zeta \in \mu_n$,

$$\Theta_{L_\zeta(\pi,\nu)}(g) = \chi_\nu^{-1}(z_\zeta)\Theta_{L_{\mathrm{st}}(\pi,\nu)}(z_\zeta g)$$

for any choice of z_ζ *satisfying* $p(z_\zeta) = \zeta I$.

Inserting this in (9.1) gives the following result.

Theorem 9.3 (inversion). *Suppose* $(\pi,\nu) \in X$, *and* $\pi\alpha$ *satisfies Hypothesis I for all* $\alpha \in \widehat{\mathbb{F}^*}$. *Then*

$$\begin{aligned} \Theta_{L(\pi,\nu)}(g) &= \frac{1}{n}\sum_{\zeta\in\mu_n} L_\zeta(\pi,\nu)(g) \\ &= \frac{1}{n}\sum_{\zeta\in\mu_n} \chi_\nu^{-1}(z_\zeta)\Theta_{L_{\mathrm{st}}(\pi,\nu)}(z_\zeta g). \end{aligned} \tag{9.2}$$

By Theorem 8.1 each term on the right-hand side of (9.2) may be expressed in terms the character $\Theta_{\pi\nu^{-1}}$ of $PGL(n,\mathbb{F})$. The resulting formula is Proposition 8.3.

We record the analogue of Theorem 8.1 for $L_\zeta(\pi,\nu)$,

$$\Theta_{L_\zeta(\pi,\nu)}(g) = \nu(\zeta)^{-1} \sum_{\substack{h \overset{\text{weak}}{\longleftrightarrow} g \\ \zeta(h,g)=\zeta^{-1}}} \Delta_\mu(h,g)\Theta_{\pi\nu^{-1}}(h).$$

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