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The fifth moment of Hecke L-functions in the weight aspect

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Abstract

We prove an upper bound for the fifth moment of Hecke L-functions associated to holomorphic Hecke cusp forms of full level and weight k in a dyadic interval $K \le k \le 2K$, as $K \to \infty$. The bound is sharp on Selberg's eigenvalue conjecture.

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1. *Introduction*

Moments of L-functions, especially at the central point, are extensively studied. They yield valuable data about the distribution of L-functions in families, and can be used for example to infer information about the size and non-vanishing of the central values. For example, let $H_k(q)$ denote the set of (normalised) holomorphic newforms of weight $k \ge 2$ and level q. The fourth moment of the associated Hecke L-functions can be estimated without too much difficulty as follows:

$$\sum_{f \in H_k(q)} L\left(\tfrac{1}{2}, f\right)^4 \ll_k q^{1+\epsilon}.$$

This is an upper bound in the level aspect (k is fixed while q tends to infinity), and it is sharp (on average it is as strong as the Lindelöf hypothesis $L(1/2, f) \ll_k q^{\epsilon}$). Dropping all but one term in the sum yields the upper bound $L(1/2, f) \ll_k q^{\frac{1}{4}+\epsilon}$. In the level aspect this is the so-called convexity bound of the L-function and it is considered to be trivial (for it can also be derived from just the functional equation of the L-function). Thus in the pursuit of a nontrivial bound for $L(1/2, f)$, it becomes very desirable to try to estimate any moment higher than the fourth. This is a difficult problem which was solved just recently, as we describe below. However a subconvexity bound for this L-function was already proved by Duke, Friedlander and Iwaniec [**8**] in a landmark paper using their amplification method applied to the fourth moment.

In a recent paper, Kiral and Young [**16**] established for the first time an upper bound for the fifth moment of the L-functions associated to $H_k(q)$, for certain small weights k and prime values $q \to \infty$. The upper bound depends on the Ramanujan Conjecture at the finite places, and when assuming the truth of this conjecture, the bound is sharp. In [**2**], Blomer and the author established a certain reciprocity-type formula for the twisted fourth moment of Hecke L-functions in the level aspect (see also [3] for the case of nontrivial nebentypus), which gives as a corollary the same bound for the fifth moment as [16] but with more general conditions and also allowing for Maass L-functions. As far as the subconvexity problem is concerned, the subconvexity bound coming from the fifth moment of [16] is superior to that of [8], but the subconvexity bound coming from the amplification method in [2] is the current best.

It is natural to try to establish a similar fifth moment estimate in the weight aspect, and this is the goal of the present paper. We fix $q = 1$ and write $H_k = H_k(1)$. This set has $k/12 + O(1)$ elements and forms a basis of the space of cusp forms of level 1 and weight k. Let $\lambda_f(n)$ denote the (real) eigenvalue corresponding to $f \in H_k$ of the nth Hecke operator (which satisfies Deligne's bound $\lambda_f(n) \ll n^\epsilon$). The L-function associated to f is defined for $\Re(s) > 1$ by

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}.$$

The central point is $s = 1/2$ and by [17] the central value $L(1/2, f)$ is known to be non-negative. Our main theorem is

THEOREM 1·1. *Let*

$$\mathcal{F} = \bigcup_{\substack{K \le k \le 2K \\ k \equiv 0 \bmod 2}} H_k,$$

be a set of $O(K^2)$ *elements. For any* $\epsilon > 0$*, we have*

$$\sum_{f \in \mathcal{F}} L\left(\tfrac{1}{2}, f\right)^5 \ll K^{2+2\theta+\epsilon} \tag{1·1}$$

as $K \to \infty$*, where* $\theta = 7/64$ *is the current best bound towards the Selberg eigenvalue conjecture* [15, appendix 2].

Again, a sharp estimate for the fourth moment would yield only the convexity bound (in the weight aspect it would be $L(1/2, f) \ll k^{1/2+\epsilon}$), so our fifth moment represents the breaking of a barrier. It should also be possibe to prove a similar result for Hecke Maass L-functions in the spectral aspect.

The "log of conductor" to "log of family size" ratio in (1·1) is 5/2, the same as in the level aspect fifth moment considered in [16] and [2]. Thus our result should be considered an analogue of the level aspect estimate. Assuming the Selberg eigenvalue conjecture (which is part of the Ramanujan Conjecture at the infinite place), our bound is sharp. This seems to be the first sharp bound for any moment higher than the fourth in the archimedean (weight or spectral) aspect. Jutila [14] proved a strong upper bound for the twelfth moment of Hecke Maass L-functions in the spectral aspect, but that is not sharp.

There has been a great deal of interest in such moment estimates in the archimedean aspect. Other authors [10, 13, 19] have proven sharp bounds for the third and fourth moments over smaller families (see also [18]). For example, in [19] Peng proved a sharp bound for the third moment over H_k, which yields the Weyl-quality bound $L(1/2, f) \ll k^{\frac{1}{3}+\epsilon}$. Since such a strong subconvexity bound already exists, we do not pursue estimates for a twisted fourth moment and amplification, although our methods would permit it (as we indicate in the sketch below). The aim is not to obtain individual bounds, although our main theorem already implies a weaker subconvexity bound.

Our ideas have a similar flavour to those of [2, 16], but besides uncovering how such ideas work in the archimedean setting (for example it is interesting how we need the Ramanujan Conjecture at the infinite place instead of the finite places), our method has some important differences (for example, we apply "reciprocity" twice, while the other papers apply it once). The difference in method leads to certain advantages. Compared to [16], our proof is simpler and shorter, and as already noted above, our method could also be used to prove a bound for the twisted fourth moment, while this was not the case in [16].

Throughout we will use the convention that ϵ denotes an arbitrarily small positive constant, but not necessarily the same one from one occurrence to the next.

2. *Rough sketch*

The purpose of this sketch is to explain the main ideas, ignoring all technicalities. We will consider only the generic ranges of all sums.

Using approximate function equations, we can write the fifth moment as

$$\begin{aligned}\frac{1}{K^2}\sum_{f\in\mathcal{F}} L\left(\tfrac{1}{2}, f\right)^5 &\approx \frac{1}{K^2}\sum_{f\in\mathcal{F}}\sum_{n_1\asymp K}\frac{\lambda_f(n_1)}{\sqrt{n_1}}\sum_{n_2,n_{3,4},n_5\asymp K}\frac{\lambda_f(n_2n_3n_4n_5)}{\sqrt{n_2n_3n_4n_5}}\\ &\approx \frac{1}{K^{\frac{7}{2}}}\sum_{K\le k\le K}\frac{1}{K}\sum_{f\in H_k}\lambda_f(n_1)\lambda_f(n_2n_3n_4n_5).\end{aligned}$$

We need an upper bound of $O(K^{2\theta+\epsilon})$. We will in fact find that this kind of grouping with n_1 on one side and n_2, n_3, n_4, n_5 on the other leads to cleaner calculations. Applying Petersson's trace formula, the off-diagonal part of this is

$$\frac{1}{K^{\frac{7}{2}}}\sum_{K\le k\le K}\sum_{n_1,n_2,n_3,n_4,n_5\asymp K}\sum_{c\ge 1}2\pi i^k\frac{S(n_1, n_2n_3n_4n_5, c)}{c}J_{k-1}\left(4\pi\frac{\sqrt{n_1n_2n_3n_4n_5}}{c}\right).$$

Summing over k first, we will get that this is

$$\begin{aligned}&\frac{1}{K^{\frac{7}{2}}}\sum_{n_1,n_2,n_3,n_4,n_5\asymp K}\sum_{c\asymp K^{\frac{1}{2}}}\frac{S(n_1, n_2n_3n_4n_5, c)}{c}e\left(\frac{2\sqrt{n_1n_2n_3n_4n_5}}{c}\right)\\ &\approx \frac{1}{K^4}\sum_{n_1,n_2,n_3,n_4,n_5\asymp K}\sum_{c\asymp K^{\frac{1}{2}}}S(n_1, n_2n_3n_4n_5, c)e\left(\frac{2\sqrt{n_1n_2n_3n_4n_5}}{c}\right),\end{aligned}$$

where as usual $e(z)$ denotes $e^{2\pi iz}$. Splitting the sum over n_1 into residue classes mod c and applying Poisson summation (denote the dual variable by m_1) we get

$$\begin{aligned}&\frac{1}{K^{\frac{7}{2}}}\sum_{\substack{n_2,n_3,n_4,n_5\asymp K\\ c\asymp K^{\frac{1}{2}}}}\sum_{-\infty<m_1<\infty}\sum_{a \bmod c}S(a, n_2n_3n_4n_5, c)e\left(\frac{am_1}{c}\right)\\ &\times\int_{x\asymp 1}e\left(\frac{2\sqrt{xKn_2n_3n_4n_5}}{c}\right)e\left(\frac{-xKm_1}{c}\right)dx.\end{aligned}$$

The complete sum over residue classes evaluates to $ce(-n_2n_3n_4n_5\overline{m_1}/c)$. Up to an admissible error, the integral evaluates to a quantity of size about $1/K$ times the exponential

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$e(n_2n_3n_4n_5/m_1c)$, by the stationary phase method (a stationary point exists only when $m_1 \asymp K^{\frac{3}{2}}$). We get

$$\frac{1}{K^4} \sum_{\substack{n_2,n_3,n_4,n_5 \asymp K \\ c \asymp K^{\frac{1}{2}} \\ m_1 \asymp K^{\frac{3}{2}}}} e\left(\frac{-n_2n_3n_4n_5\overline{m_1}}{c}\right) e\left(\frac{n_2n_3n_4n_5}{m_1c}\right).$$

Note that the first exponential has modulus c, and the second exponential has phase of size $n_2n_3n_4n_5/m_1c$, so if the second exponential is part of the weight function then the total "conductor" here is of size $cn_2n_3n_4n_5/m_1c \asymp K^{5/2}$. This conductor can be greatly reduced to $m_1 \asymp K^{\frac{3}{2}}$ if we apply reciprocity, to obtain the expression

$$\frac{1}{K^4} \sum_{\substack{n_2,n_3,n_4,n_5 \asymp K \\ c \asymp K^{\frac{1}{2}} \\ m_1 \asymp K^{\frac{3}{2}}}} e\left(\frac{n_2n_3n_4n_5\overline{c}}{m_1}\right). \tag{2·1}$$

Next we apply Poisson summation (mod m_1) to the n_2 and n_3 sums (in the actual proof, we will apply Voronoi summation once instead of Poisson summation twice). Note that if we were following [**16**] step by step, we would have applied Poisson summation to n_2, n_3 and n_4, but this is not how we proceed. We get

$$\frac{1}{K^{\frac{7}{2}}} \sum_{\substack{n_4,n_5 \asymp K \\ c,m_2,m_3 \asymp K^{\frac{1}{2}} \\ m_1 \asymp K^{\frac{3}{2}}}} e\left(\frac{m_2m_3c\overline{n_4n_5}}{m_1}\right). \tag{2·2}$$

This sum displays only the generic ranges of m_2 and m_3 (the dual variables). The zero frequencies $m_2 = 0$ or $m_3 = 0$, which are omitted, are in fact quite troublesome. For example, return to (2·1) and consider the terms with $m_1 | n_3n_4n_5$ (these terms correspond to $m_2 = 0$). The contribution of such terms is

$$\frac{1}{K^4} \sum_{\substack{n_2,n_3,n_4,n_5 \asymp K \\ c \asymp K^{\frac{1}{2}},\, m_1 \asymp K^{\frac{3}{2}} \\ m_1 | n_3n_4n_5}} 1 \asymp K^{\frac{1}{2}}, \tag{2·3}$$

while we need to prove a bound of $K^{2\theta+\epsilon}$. It seems that we cannot do better because there are no harmonics present to produce further cancellation. Of course, it is not possible (by the Lindelöf hypothesis) for the fifth moment to be so large, so a careful evaluation of the fifth moment must show that these "fake main terms" should cancel out somehow. But there is a shortcut. The weight functions from the approximate functional equations have been suppressed in (2·3). If we take them into account, there is a way to design them carefully so that (2·3) is not so large. This idea was used in [**1**] and [**16**], and [**16**, section 2] contains a nice heuristic about how the idea works.

Back to (2·2), we can apply reciprocity again to get

$$\frac{1}{K^{\frac{7}{2}}} \sum_{\substack{n_4,n_5 \asymp K \\ c,m_2,m_3 \asymp K^{\frac{1}{2}} \\ m_1 \asymp K^{\frac{3}{2}}}} e\left(\frac{m_2m_3c\overline{m_1}}{n_4n_5}\right) e\left(\frac{-m_2m_3c}{m_1n_4n_5}\right) \approx \frac{1}{K^{\frac{7}{2}}} \sum_{\substack{n_4,n_5 \asymp K \\ c,m_2,m_3 \asymp K^{\frac{1}{2}} \\ m_1 \asymp K^{\frac{3}{2}}}} e\left(\frac{m_2m_3c\overline{m_1}}{n_4n_5}\right).$$

This second application of reciprocity actually increases the conductor, but we gain structural advantage as we are able to reduce the largest variable length by Poisson summation. Indeed, applying Poisson summation (mod $n_4 n_5$) to the m_1 sum (denote the dual variable by l_1), we get

$$\frac{1}{K^4} \sum_{\substack{n_4, n_5 \asymp K \\ c, m_2, m_3, l_1 \asymp K^{\frac{1}{2}}}} S(m_2 m_3 c, l_1, n_4 n_5).$$

Now we can sum over n_4 using Kuznetsov's formula. The sum of Kloosterman sums is in the Linnik range as $n_4 n_5 \geq \sqrt{m_2 m_3 c l_1}$. This leads to

$$\frac{1}{K^2} \sum_{n_5 \asymp K} \left(\sum_{c, m_2, m_3, l_1 \asymp K^{\frac{1}{2}}} \sum_{t_j \asymp 1} \frac{\lambda_j(m_2 m_3 c) \lambda_j(l_1)}{\sqrt{m_2 m_3 c l_1}} + \cdots \right), \tag{2·4}$$

where the sum is over an orthonormal basis of Maass cusp forms $\{u_j\}$ of level n_5 and (essentially bounded) Laplacian eigenvalue $1/4 + t_j^2$, and the ellipsis denotes the contribution of the Eisenstein series and holomorphic forms. Actually we lose $O(K^{2\theta+\epsilon})$ here due to the possibility of exceptional eigenvalues, but for the purposes of this sketch we ignore this issue.

The inner sum of (2·4), given within the parentheses, looks like the fourth moment of $L(1/2, u_j)$ in the level aspect, provided that we can decompose $\lambda_j(m_2 m_3 c)$ by multiplicativity. For this, we need to work with a basis comprising of lifts of newforms; such a basis is given in [**5**] or [**2**]. Then the expected bound for the fourth moment, which can be proved using the spectral large sieve, gives

$$\frac{1}{K^2} \sum_{n_5 \asymp K} (n_5 K^\epsilon) \ll K^\epsilon$$

as desired. We never need any cancellation from the n_5-sum, which is why a twisted fourth moment bound would probably be possible in place of the main theorem.

3. *Background*

3·1. *Approximate functional equations*

For $f \in H_k$ we have the functional equation [**11**, theorem 14·7],

$$\Lambda(s, f) := (2\pi)^{-s} \Gamma(s + \tfrac{k-1}{2}) L(s, f) = i^k \Lambda(1 - s, f). \tag{3·1}$$

Let $\tau(m)$ denote the number of divisors of m. We will use the following standard approximate functional equations. For any $f \in H_k$, we have

$$L\left(\tfrac{1}{2}, f\right)^2 = 2 \sum_{m \geq 1} \frac{\lambda_f(m) \tau(m)}{\sqrt{m}} V_k(m), \tag{3·2}$$

where

$$V_k(x) = \frac{1}{2\pi i} \int_{(A)} (2\pi x)^{-s} \mathcal{G}(s) \frac{\Gamma(s + \frac{k}{2})^2}{\Gamma(\frac{k}{2})^2} \zeta(1 + 2s) \frac{ds}{s}$$

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for any $A > 0$ and

$$\mathcal{G}(s) = 4e^{s^2}\left(\tfrac{1}{4} - s^2\right). \tag{3·3}$$

This follows from the functional equation (3·1) and [11, theorem 5·3]. As explained in that theorem, we may insert in the integrand above any even function which is bounded in a fixed horizontal strip about $\Re(s) = 0$, and has value 1 at $s = 0$. Our function $\mathcal{G}(s)$ satisfies these properties and is chosen to decay exponentially in the vertical direction (this is convenient for convergence) and to vanish at $s = 1/2$ (this will be needed later to deal with the "fake main terms").

For $k \equiv 0 \bmod 4$, the root number in the functional equation is 1, and we have

$$L\left(\tfrac{1}{2}, f\right) = 2\sum_{n\geq 1} \frac{\lambda_f(n)}{\sqrt{n}} W_k(n), \tag{3·4}$$

where

$$W_k(x) = \frac{1}{2\pi i}\int_{(A)} (2\pi x)^{-s} e^{s^2} \frac{\Gamma\left(s + \frac{k}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} \frac{ds}{s}.$$

We have

$$V_k^{(j)}(x),\ W_k^{(j)}(x) \ll x^{-j}(1+x)^{-A} \tag{3·5}$$

for any $A > 0$ and integer $j \geq 0$. Using this for $j = 0$, large A and Stirling's estimates for the gamma function, the sums (3·2) and (3·4) may be restricted to $m \ll k^{2+\epsilon}$ and $n \ll k^{1+\epsilon}$ respectively, up to an error of $O(k^{-100})$. Taking $j = 0$ and $A = \epsilon$ shows that $|V_k(x)|$, $|W_k(x)| \ll k^{\epsilon}$.

3·2. *Summation formulae*

We will need the Voronoi summation formula and the Poisson summation formula.

LEMMA 3·1. *Voronoi summation. Given a compactly supported smooth function Φ, and coprime integers h and ℓ, we have*

$$\sum_{m\geq 1} \frac{\tau(m)}{m} e\left(\frac{m\overline{h}}{\ell}\right)\Phi\left(\frac{m}{M}\right) = \frac{1}{\ell}\int_{-\infty}^{\infty}\left(\log\frac{x}{\ell^2} + 2\gamma\right)\Phi\left(\frac{x}{M}\right)\frac{dx}{x} + \sum_{\pm}\frac{1}{\ell}\sum_{r\geq 1}\tau(r)e\left(\frac{\pm r h}{\ell}\right)\check{\Phi}_{\pm}\left(\frac{Mr}{\ell^2}\right), \tag{3·6}$$

where

$$\check{\Phi}_{\pm}(x) = \frac{1}{2\pi i}\int_{(A)} H_1^{\pm}(s)\tilde{\Phi}(-s)x^{-s}\,ds,$$

$\tilde{\Phi}$ is the Mellin transform of Φ,

$$H_1^{\pm}(s) = 2(2\pi)^{-2s}\Gamma(s)^2\cos^{(1\mp 1)/2}(\pi s),$$

and $A > 0$.

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Proof. See [1, section 2·3]. We can take any $A>0$ because $\tilde{\Phi}(-s) \ll (1+|s|)^{-B}$ for any $B \geq 0$ by integration by parts.

LEMMA 3·2. *Poisson summation. Given a compactly supported smooth function Φ with bounded derivatives, and an arithmetic function $S_q(n)$ with period q, we have*

$$\begin{aligned}\sum_{-\infty<n<\infty} \Phi\left(\frac{n}{N}\right) S_q(n) &= \frac{N}{q} \sum_{-\infty<l<\infty} \hat{\Phi}\left(\frac{lN}{q}\right) \sum_{a \bmod q} S_q(a) e\left(\frac{al}{q}\right) \\ &= \frac{N}{q}\hat{\Phi}(0) \sum_{a \bmod q} S_q(a) + \frac{N}{q} \sum_{-\infty<l<\infty} \sum_{a \bmod q} S_q(a) e\left(\frac{al}{q}\right) \\ &\quad \times \frac{1}{2\pi i} \int_{(A)} \int_{-\infty}^{\infty} \left(\frac{-2\pi x l N}{q}\right)^{-s} \Phi(x) H_2(s) dx ds, \end{aligned} \tag{3·7}$$

where $\hat{\Phi}$ denotes the Fourier transform of Φ,

$$H_2(s) = \Gamma(s) \exp\left(\frac{-i\pi s}{2}\right)$$

and $A>0$.

Proof. For the second line of (3·7), separate the n sum into sums over residue classes a modulo q and apply the usual Poisson summation formula to each sum. For the third line we keep aside the contribution of $l=0$, and for $l \neq 0$ we first compute the Mellin transform for $\Re s > 0$:

$$\int_0^{\infty} \hat{\Phi}(y) y^{s-1} dy = \int_0^{\infty} \int_{-\infty}^{\infty} \Phi(x) e(-yx) y^{s-1} dx dy = \int_{-\infty}^{\infty} \Phi(x)(2\pi x)^{-s} H_2(s) dx. \tag{3·8}$$

This can be seen in two steps. The first is truncating the y-integral and swapping the order of integration by the bound

$$\hat{\Phi}(y) = \int_{-\infty}^{\infty} \Phi(x) e(-yx) dx \ll (1+|y|)^{-B} \tag{3·9}$$

for any $B \geq 0$, which follows from the compact support of Φ and integration by parts. The second step is making the substitution $y \leftrightarrow y/2\pi x$ and using the Mellin transform

$$\int_0^{\infty} e^{-iy} y^{s-1} dy = H_2(s)$$

which holds for $0 < \Re(s) < 1$. This analytically continues to $\Re(s) > 0$, and by the Mellin inversion theorem (which we can apply by the absolute convergence of $\int_0^{\infty} \hat{\Phi}(y) y^{s-1} dy$ using (3·9)), we have

$$\hat{\Phi}\left(\frac{lN}{q}\right) = \frac{1}{2\pi i} \int_{(A)} \left(\frac{lN}{q}\right)^{-s} \int_{-\infty}^{\infty} \Phi(x)(-2\pi x)^{-s} H_2(s) dx ds$$

for any $A>0$.

3·3. *An average of the J-Bessel function*

The following result can be found in [12, corollary 8·2].

LEMMA 3·3. *Let $x > 0$ and let h be a smooth function compactly supported on the positive reals and possessing bounded derivatives. We have*

$$\frac{1}{K}\sum_{k\equiv 0 \bmod 2} 2i^k h\left(\frac{k-1}{K}\right) J_{k-1}(x) = -\frac{1}{\sqrt{x}}\Im\left(e^{-2\pi i/8}e^{ix}\hbar\left(\frac{K^2}{2x}\right)\right) + O\left(\frac{x}{K^5}\int_{-\infty}^{\infty} v^4|\hat{h}(v)|dv\right), \qquad (3\cdot10)$$

where for real v,

$$\hbar(v) := \int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}} e^{iuv} du$$

and $\hat{h}$ denotes the Fourier transform of h. The implied constant is absolute.

By integrating by parts several times we get that $\hbar(v) \ll |v|^{-B}$ for any $B \geq 0$. Thus the main term of (3·10) is not dominant if $x < K^{2-\epsilon}$.

Note that we will be able to use this lemma for any fixed h as well as for a function like

$$h(u) = \mathfrak{H}(u)W_{uK}(x) = \mathfrak{H}(u)\frac{1}{2\pi i}\int_{(A)} (2\pi x)^{-s} e^{s^2} \frac{\Gamma\left(s+\frac{uK}{2}\right)}{\Gamma\left(\frac{uK}{2}\right)}\frac{ds}{s}$$

for $x > 0$, where $\mathfrak{H}$ is a fixed smooth function compactly supported on $u \in (1, 2)$. This is because we may restrict the integral to $|s| < K^\epsilon$ by the rapid decay of e^{s^2} and then use Stirling's approximation

$$\frac{\Gamma\left(s+\frac{uK}{2}\right)}{\Gamma\left(\frac{uK}{2}\right)} = \left(\frac{uK}{2}\right)^s\left(1+\sum_{n=1}^{N}\frac{P_n(s)}{(uK)^n} + O\left(K^{-N}\right)\right)$$

for any $N \geq 1$ and some polynomials P_n, to see that h and its derivatives are bounded.

For future use, define for any complex number s the more general function

$$\hbar_s(v) := \int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}} u^{s/2} e^{iuv} du.$$

Integrating by parts, we get

$$\hbar_s^{(j)}(v) \ll_{\Re(s)} (1+|s|)^B |v|^{-B} \qquad (3\cdot11)$$

for any $B \geq 0$. Thus the Mellin transform

$$\tilde{\hbar}_s(w) = \int_0^\infty \hbar_s(v) v^{w-1} dv$$

is holomorphic in the half plane $\Re(w) > 0$, and we have by integrating by parts j times:

$$\tilde{\hbar}_s(w) \ll_{\Re(s)} (1+|s|)^{j+\Re(w)+1}(1+|w|)^{-j}.$$

4. *Hecke relations*

Define

$$\sum_{f\in H_k}^{P} \gamma_f := \sum_{f\in H_k} \left(\frac{2\pi^2}{(k-1)L(1,\operatorname{sym}^2 f)}\right)\gamma_f$$

for any complex numbers γ_f depending on f. The average $\sum^P$ arises in the Petersson trace formula [**11**, proposition 14·5]:

$$\sum_{f\in H_k}^{P} \lambda_f(n)\lambda_f(m) = \delta_{m,n} + 2\pi i^k \sum_{c=1}^{\infty} \frac{S(n,m,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

where the value of $\delta_{m,n}$ is 1 if $m=n$ and 0 otherwise, $S(n,m,c)$ is the Kloosterman sum, and $J_{k-1}(x)$ is the J-Bessel function.

The following lemma explains how we will group together variables in the fifth moment.

LEMMA 4·1. *To prove the main theorem, it suffices to prove that for any smooth functions* h, U_1, U_2, U_3 *compactly supported on* $(1/2, 5/2)$ *with bounded derivatives, and any*

$$\alpha, \beta, \beta_1, \beta_2 \geq 1, \quad 1 \leq N_1, N_2, N_3 < K^{1+\epsilon},$$

with

$$N_3 \geq N_2, \quad N_1 N_2 < \frac{K^{2+\epsilon}}{\alpha}, \quad \beta \geq \alpha, \tag{4·1}$$

we have

$$\frac{1}{K}\sum_{k\equiv 0 \bmod 2} h\left(\frac{k-1}{K}\right) \sum_{f\in H_k}^{P} S_f \ll \sqrt{\alpha} K^{2\theta+\epsilon}, \tag{4·2}$$

where

$$S_f := \sum_{n_1,n_2,n_3,m\geq 1} \frac{\lambda_f(n_1 n_2 m\alpha)\lambda_f(n_3)\tau(m)}{\sqrt{n_1 n_2 n_3 m}}$$
$$\times W_k(n_1\beta_1) W_k(n_2\beta_2) W_k(n_3) V_k(m\beta) U_1\left(\frac{n_1}{N_1}\right) U_2\left(\frac{n_2}{N_2}\right) U_3\left(\frac{n_3}{N_3}\right).$$

Proof. To prove the main theorem, it suffices to prove that

$$\frac{1}{K}\sum_{k\equiv 0 \bmod 2} h\left(\frac{k-1}{K}\right) \sum_{f\in H_k}^{P} L\left(\tfrac{1}{2}, f\right)^5 \ll K^{2\theta+\epsilon},$$

because we have $L(1/2, f) \geq 0$ by [**17**] and $k^{-\epsilon} \ll L(1,\operatorname{sym}^2 f) \ll k^{\epsilon}$ by [**9**, appendix].

We claim that

$$L\left(\tfrac{1}{2}, f\right)^5 = 8\left(\sum_{n\geq 1} \frac{\lambda_f(n)}{\sqrt{n}} W_k(n)\right)^3 L\left(\tfrac{1}{2}, f\right)^2. \tag{4·3}$$

This holds by (3·4) when $k \equiv 0 \bmod 4$. But when $k \equiv 2 \bmod 4$, it also holds because then $L(1/2, f) = 0$ by the functional equation (3·1), so both sides of (4·3) vanish. Now we can

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insert the approximate functional equation for $L(\frac{1}{2}, f)^2$ given in (3·2) to get that

$$L\left(\tfrac{1}{2}, f\right)^5 = 16\left(\sum_{n\geq 1} \frac{\lambda_f(n)}{\sqrt{n}} W_k(n)\right)^3 \left(\sum_{m\geq 1} \frac{\lambda_f(m)\tau(m)}{\sqrt{m}} V_k(m)\right).$$

Expanding the cube and working in dyadic intervals, to establish the main theorem it suffices to prove that

$$\frac{1}{K}\sum_{k\equiv 0 \bmod 2} h\left(\frac{k-1}{K}\right) \sum_{f\in H_k}^{P} S_1 \ll K^{2\theta+\epsilon},$$

where

$$S_1 := \prod_{i=1}^{3}\left(\sum_{n_i\geq 1} \frac{\lambda_f(n_i)}{\sqrt{n_i}} W_i\left(\frac{n_i}{N_i}\right)\right)\left(\sum_{m\geq 1} \frac{\lambda_f(m)\tau(m)}{\sqrt{m}} V_k(m)\right)$$

for

$$W_i(n_i) := W_k(n_i) U_i\left(\frac{n_i}{N_i}\right)$$

and $1 \leq N_1, N_2, N_3 < K^{1+\epsilon}$. By symmetry, we can suppose that $N_3 \geq N_2$. By Hecke multiplicativity, we have

$$\lambda_f(m)\lambda_f(n_1) = \sum_{d|(m,n_1)} \lambda_f\left(\frac{mn_1}{d^2}\right),$$

so replacing m by md and n_1 by $n_1 d$, we get

$$S_1 = \sum_{n_1,n_2,n_3,m,d\geq 1} \frac{\lambda_f(mn_1)\lambda_f(n_2)\lambda_f(n_3)\tau(md)}{d\sqrt{n_1 n_2 n_3 m}} W_1\left(\frac{n_1 d}{N_1}\right) W_2\left(\frac{n_2}{N_2}\right) W_3\left(\frac{n_3}{N_3}\right) V_k(md).$$

Now we combine

$$\lambda_f(mn_1)\lambda_f(n_2) = \sum_{b|(mn_1,n_2)} \lambda_f\left(\frac{mn_1n_2}{b^2}\right) = \sum_{\substack{n_2=b_1 b\\ b|mn_1}} \lambda_f\left(\frac{mn_1 b_1}{b}\right).$$

Ordering by the gcd of n_1 and b, we have the disjoint union

$$\{n_1, m : b|n_1 m\} = \bigsqcup_{\substack{b=b_2b'\\ (b,n_1)=b_2}} \{n_1, m : b|n_1 m\} = \bigsqcup_{b=b_2b'} \left\{n_1, m : b_2|n_1, b'|m, \left(\frac{n_1}{b_2}, b'\right) = 1\right\}, \tag{4·4}$$

and $(n_1/b_2, b') = 1$ can be detected using the Mobius function:

$$\sum_{\substack{b'=b_3b_4\\ b_3|\frac{n_1}{b_2}}} \mu(b_3) = \begin{cases} 1 & \text{if } (\frac{n_1}{b_2}, b') = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{4·5}$$

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Thus replacing b by $b_2b_3b_4$, n_2 by $b_1b_2b_3b_4$, n_1 by $n_1b_2b_3$, and m by mb_3b_4, we get

$$S_1 = \sum_{\substack{n_1,b_1,n_3,m\geq 1\\ b_2,b_3,b_4,d\geq 1}} \frac{\lambda_f(mn_1b_1b_3)\lambda_f(n_3)\tau(mb_3b_4d)\mu(b_3)}{db_2b_3^{\frac{3}{2}}b_4\sqrt{n_1b_1n_3m}} \times W_1\left(\frac{n_1b_2b_3d}{N_1}\right) W_2\left(\frac{b_1b_2b_3b_4}{N_2}\right) W_3\left(\frac{n_3}{N_3}\right) V_k(mb_3b_4d).$$

Splitting the divisor function

$$\tau(mb_3b_4d) = \sum_{r|(m,b_3b_4d)} \mu(r)\tau\left(\frac{m}{r}\right)\tau\left(\frac{b_3b_4d}{r}\right),$$

replacing m by mr, and renaming b_1 to n_2, we have

$$S_1 = \sum_{\substack{n_1,n_2,n_3,m\geq 1\\ b_2,b_3,b_4,d\geq 1\\ r|b_3b_4d}} \frac{\lambda_f(mn_1n_2b_3r)\lambda_f(n_3)\tau(m)\tau(\frac{b_3b_4d}{r})\mu(b_3)\mu(r)}{db_2b_3^{\frac{3}{2}}b_4\sqrt{n_1n_2n_3mr}} \times W_1\left(\frac{n_1b_2b_3d}{N_1}\right) W_2\left(\frac{n_2b_2b_3b_4}{N_2}\right) W_3\left(\frac{n_3}{N_3}\right) V_k(mb_3b_4dr).$$

We plan to find cancellation in the sum over n_1, n_2, n_3, m and to sum trivially over the remaining parameters b_2, b_3, b_4, r, d. Thus it suffices to prove that

$$\sum_{\substack{b_2,b_3,b_4,d\geq 1\\ r|b_3b_4d}} \frac{1}{db_2b_3^{\frac{3}{2}}b_4\sqrt{r}}\left|\frac{1}{K}\sum_{k\equiv 0 \bmod 2} h\left(\frac{k-1}{K}\right)\sum_{f\in H_k}^{P} S_2\right| \ll K^{2\theta+\epsilon}, \tag{4·6}$$

where

$$S_2 := \sum_{n_1,n_2,n_3,m\geq 1} \frac{\lambda_f(mn_1n_2b_3r)\lambda_f(n_3)\tau(m)}{\sqrt{n_1n_2n_3m}} W_1\left(\frac{n_1b_2b_3d}{N_1}\right) W_2\left(\frac{n_2b_2b_3b_4}{N_2}\right) \times W_3\left(\frac{n_3}{N_3}\right) V_k(mb_3b_4dr).$$

For (4·6) it suffices to show that

$$\frac{1}{K}\sum_{k\equiv 0 \bmod 2} h\left(\frac{k-1}{K}\right)\sum_{f\in H_k}^{P} S_2 \ll K^{2\theta+\epsilon}\sqrt{b_3r}.$$

This is given by (4·2), once in S_f we replace N_1 by N_1/b_2b_3d and N_2 by $N_2/b_2b_3b_4$, and take $\alpha = b_3r$, $\beta = b_3b_4dr$, $\beta_1 = b_2b_3d$, $\beta_2 = b_2b_3b_4$. Note that these substitutions lead to a smaller value of N_2, so that $N_3 \geq N_2$ still holds. Since $\beta_1\beta_2 \geq b_3(b_3b_4r) \geq \alpha$, we have $N_1N_2 < K^{2+\epsilon}/\alpha$. Also note that $\beta \geq \alpha$.

5. *Application of the trace formula*

Applying the Petersson trace formula to Lemma 4·1, we need to prove that

$$D + OD \ll \sqrt{\alpha}K^{2\theta+\epsilon},$$

where the diagonal

$$D := \sum_{\substack{n_1,n_2,n_3,m\geq 1\\ n_3=n_1n_2m\alpha}} \frac{\lambda_f(n_1n_2m\alpha)\lambda_f(n_3)\tau(m)}{\sqrt{n_1n_2n_3m}} U_1\left(\frac{n_1}{N_1}\right) U_2\left(\frac{n_2}{N_2}\right) \times U_3\left(\frac{n_3}{N_3}\right) W_k(n_1\beta_1)W_k(n_2\beta_2)W_k(n_3)V_k(m\beta)$$

trivially satisfies the required bound, and the off-diagonal is

$$OD := \sum_{n_1,n_2,n_3,m,c\geq 1} \frac{S(n_1n_2m\alpha, n_3, c)\tau(m)}{c\sqrt{n_1n_2n_3m}} U_1\left(\frac{n_1}{N_1}\right) U_2\left(\frac{n_2}{N_2}\right) U_3\left(\frac{n_3}{N_3}\right) \times \frac{1}{K}\sum_{k\equiv 0 \bmod 2} h\left(\frac{k-1}{K}\right) 2\pi i^k J_{k-1}\left(\frac{4\pi\sqrt{n_1n_2n_3m\alpha}}{c}\right) \times W_k(n_1\beta_1)W_k(n_2\beta_2)W_k(n_3)V_k(m\beta).$$

At this point, we cannot absorb the W_k functions into the arbitrary weight functions U_i because W_k depends on k and we still need to average over k, which is what we do next. Applying Lemma 3·3, and the discussion following it, the contribution of its error term is bounded by

$$\frac{1}{K^{5-\epsilon}} \sum_{\substack{n_1n_2<K^{2+\epsilon}/\alpha\\ n_2<K^{1+\epsilon}\\ m<K^{2+\epsilon}}} \sum_{c\geq 1} \frac{|S(n_1n_2m\alpha, n_3, c)|}{c\sqrt{n_1n_2n_3m}} \frac{\sqrt{n_1n_2n_3m\alpha}}{c} \ll K^{\epsilon},$$

on using Weil's bound for the Kloosterman sum. Thus we need only consider the main term of Lemma 3·3, and it suffices to prove

$$OD_1 := \sum_{n_1,n_2,n_3,m,c\geq 1} \frac{S(n_1n_2m\alpha, n_3, c)\tau(m)}{\sqrt{c}(n_1n_2n_3m)^{\frac{3}{4}}} e\left(\frac{2\pi\sqrt{n_1n_2n_3m\alpha}}{c}\right) \times \Psi_K\left(n_1\beta_1, n_2\beta_2, n_3, m\beta, \frac{K^2c}{8\pi\sqrt{n_1n_2n_3m\alpha}}\right)\prod_{j=1}^{3} U_j\left(\frac{n_j}{N_j}\right) \ll \alpha^{\frac{3}{4}}K^{2\theta+\epsilon},$$

where

$$\Psi_K(x_1,x_2,x_3,x_4,v) := \frac{1}{(2\pi i)^4}\int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}} \int_{(A_4)} \frac{\zeta(1+2s_4)\mathcal{G}(s_4)}{(2\pi x_4)^{s_4}} \frac{\Gamma^2\left(\frac{\sqrt{u}}{2}K+s_4+\frac{1}{2}\right)}{\Gamma^2\left(\frac{\sqrt{u}}{2}K+\frac{1}{2}\right)} \times \prod_{j=1}^{3}\int_{(A_j)} \frac{e^{s_j^2}}{(2\pi x_j)^{s_j}} \frac{\Gamma\left(\frac{\sqrt{u}}{2}K+s_j+\frac{1}{2}\right)}{\Gamma\left(\frac{\sqrt{u}}{2}K+\frac{1}{2}\right)} \frac{ds_j}{s_j}\frac{ds_4}{s_4} e^{iuv}du.$$

By the rapid decay of the s_1, s_2, s_3, s_4 integrands in vertical lines, we may effectively truncate the integrals to $|\Im s_1|, |\Im s_2|, |\Im s_3|, |\Im s_4| < K^{\epsilon}$. For $|s| < K^{\epsilon}$, by Stirling's approximation we have

$$\frac{\Gamma\left(\frac{\sqrt{u}}{2}K+s+\frac{1}{2}\right)}{\Gamma\left(\frac{\sqrt{u}}{2}K+\frac{1}{2}\right)} = \left(\frac{\sqrt{u}}{2}K\right)^s \left(1+\frac{P(s)}{\sqrt{u}K}+O\left(\frac{1}{K^{2-\epsilon}}\right)\right)$$

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for some polynomial P. Thus

$$\Psi_K(x_1, x_2, x_3, x_4, v) = \Psi\Big(\frac{x_1}{K}, \frac{x_2}{K}, \frac{x_3}{K}, \frac{x_4}{K^2}, v\Big) + \frac{1}{K}\Psi_0\Big(\frac{x_1}{K}, \frac{x_2}{K}, \frac{x_3}{K}, \frac{x_4}{K^2}, v\Big) + O\big(K^{-2+\epsilon}\big), \tag{5·1}$$

where for $\xi_i > 0$ and real v we define

$$\Psi(\xi_1, \xi_2, \xi_3, \xi_4, v) := \frac{1}{(2\pi i)^4}\int_{(A_4)}\int_{(A_3)}\int_{(A_2)}\int_{(A_1)} \frac{e^{s_1^2+s_2^2+s_3^2}\zeta(1+2s_4)\mathcal{G}(s_4)}{(4\pi\xi_1)^{s_1}(4\pi\xi_2)^{s_2}(4\pi\xi_3)^{s_3}(8\pi\xi_4)^{s_4}} \times \hbar_{s_1+s_2+s_3+2s_4}(v)\frac{ds_1}{s_1}\frac{ds_2}{s_2}\frac{ds_3}{s_3}\frac{ds_4}{s_4} \tag{5·2}$$

and W_0 has the same definition except for the presence of an extra factor $P(s_1, s_2, s_3, s_4)/\sqrt{u}$ in the integrand for some polynomial P. It suffices to treat only the contribution of Ψ, as the treatment of the secondary term Ψ_0 will be similar. Thus we need to prove

$$OD_2 := \sum_{n_1,n_2,n_3,m,c\geq 1} \frac{S(n_1n_2m\alpha, n_3, c)\tau(m)}{\sqrt{c}(n_1n_2n_3m)^{\frac{3}{4}}} e\Big(\frac{2\sqrt{n_1n_2n_3m\alpha}}{c}\Big) \times \Psi\Big(\frac{n_1\beta_1}{K}, \frac{n_2\beta_2}{K}, \frac{n_3}{K}, \frac{m\beta}{K^2}, \frac{K^2c}{8\pi\sqrt{n_1n_2n_3m\alpha}}\Big)\prod_{j=1}^{3} U_j\Big(\frac{n_j}{N_j}\Big) \ll \alpha^{\frac{3}{4}}K^{2\theta+\epsilon}. \tag{5·3}$$

By (3·11) we may assume (up to negligible error) that

$$c \ll \frac{\sqrt{n_1n_2n_3m\alpha}}{K^{2-\epsilon}}. \tag{5·4}$$

By (3·5) and (3·11), we have that

$$\frac{\partial^{j_1}}{\partial\xi_1^{j_1}}\frac{\partial^{j_2}}{\partial\xi_2^{j_2}}\frac{\partial^{j_3}}{\partial\xi_3^{j_3}}\frac{\partial^{j_4}}{\partial\xi_4^{j_4}}\frac{\partial^{j}}{\partial v^{j}}\Psi(\xi_1, \xi_2, \xi_3, \xi_4, v) \ll K^{\epsilon}\xi_1^{-j_1-A_1}\xi_2^{-j_2-A_2}\xi_3^{-j_3-A_3}\xi_4^{-j_4-A_4}v^{-j-B} \tag{5·5}$$

for $\xi_1, \xi_2, \xi_3, \xi_4, v > 0$, any integers $j_i, B \geq 0$ and any real $A_i > 0$.

6. *Poisson summation and reciprocity*

In (5·3), we sum over n_3 in residue classes mod c and apply Poisson summation (Lemma 3·2), getting

$$OD_2 = \sum_{\substack{n_1,n_2,m,c\geq 1\\ -\infty<\ell<\infty}} \frac{N_3}{c}\sum_{a \bmod c} S(a, n_1n_2m\alpha, c)e\Big(\frac{a\ell}{c}\Big)\frac{\tau(m)}{\sqrt{c}(n_1n_2N_3m)^{\frac{3}{4}}}U_1\Big(\frac{n_1}{N_2}\Big)U_2\Big(\frac{n_2}{N_2}\Big) \times \int_{-\infty}^{\infty} e\Big(\frac{-\ell N_3 x}{c}\Big)e\Big(\frac{2\sqrt{xn_1n_2N_3m\alpha}}{c}\Big) \times \Psi\Big(\frac{n_1\beta_1}{K}, \frac{n_2\beta_2}{K}, \frac{xN_3}{K}, \frac{m\beta}{K^2}, \frac{K^2c}{8\pi\sqrt{xn_1n_2N_3m\alpha}}\Big)\frac{U_3(x)}{x^{\frac{3}{4}}}dx. \tag{6·1}$$

Call the integral above I. We will evaluate it using stationary phase approximation.

LEMMA 6·1. *We have $I \ll K^{-1000}$ unless $|\ell| \asymp \sqrt{n_1 n_2 m\alpha}/\sqrt{N_3}$, in which case*

$$I = \sqrt{\frac{2\ell c}{n_1 n_2 m\alpha}}\, e\Big(\frac{n_1 n_2 m\alpha}{\ell c} - \frac{\pi}{8}\Big)\, \Psi\Big(\frac{n_1\beta_1}{N_1}, \frac{n_2\beta_2}{N_2}, \frac{n_1 n_2 m\alpha}{\ell^2 K}, \frac{m\beta}{K^2}, \frac{K^2\ell c}{8\pi n_1 n_2 m\alpha}\Big)$$
$$\times U_3\Big(\frac{n_1 n_2 m\alpha}{\ell^2 N_3}\Big)\Big(\frac{n_1 n_2 m\alpha}{\ell^2 N_3}\Big)^{\frac{1}{4}} + O\Big(\frac{1}{K^{3-\epsilon}}\Big).$$

Proof. Let $\Omega(y)$ be a smooth function with bounded derivatives which is equal to 1 for $y \in (1/2, 3/2)$ and 0 for $y \in (-\infty, 1/4) \cup (2, \infty)$. We write

$$I = I_1 + I_2,$$

where I_1 has the same definition as I except that its integrand has an extra factor $1 - \Omega(x\ell^2 N_3/n_1 n_2 m\alpha)$, and I_2 has the same definition as I except that its integrand has an extra factor $\Omega(x\ell^2 N_3/n_1 n_2 m\alpha)$. We first show that $I_1 \ll K^{-1000}$. The phase of the exponential factor in the integrand is

$$\phi(x) = \frac{2\sqrt{x n_1 n_2 N_3 m\alpha}}{c} - \frac{\ell N_3 x}{c},$$

which has first derivative

$$\phi'(x) = \frac{\sqrt{n_1 n_2 N_3 m\alpha}}{\sqrt{x} c}\left(1 - \frac{\ell\sqrt{x N_3}}{\sqrt{n_1 n_2 m\alpha}}\right) \gg K^{2-\epsilon}$$

by (5·4) and the support of Ω, and higher derivatives

$$\phi^{(j)}(x) \ll \frac{\sqrt{n_1 n_2 N_3 m\alpha}}{c} \ll K^{2-\epsilon}$$

for $j \geq 2$. Thus integrating multiple times and using (5·5), we get $I_1 \ll K^{-1000}$.

We now turn to I_2. Here the support of Ω makes the restriction $|\ell| \asymp \sqrt{n_1 n_2 m\alpha}/\sqrt{N_3}$. Substituting $y = x\ell^2 N_3/n_1 n_2 m\alpha$ and defining $U_0(x) = x^{\frac{1}{4}} U_3(x)$, we get

$$I_2 = \int_{-\infty}^{\infty} e\Big(\frac{n_1 n_2 m\alpha}{\ell c}(2\sqrt{y} - y)\Big)\, \Psi\Big(\frac{n_1\beta_1}{K}, \frac{n_2\beta_2}{K}, \frac{y n_1 n_2 m\alpha}{\ell^2 K}, \frac{m\beta}{K^2}, \frac{K^2\ell c}{8\pi\sqrt{y} n_1 n_2 m\alpha}\Big)$$
$$\times U_0\left(\frac{n_1 n_2 m\alpha}{\ell^2 N_3} y\right)\frac{\Omega(y)}{y} dy. \tag{6·2}$$

The stationary point occurs at $y = 1$. We apply [4, proposition 8·2], with

$$h(y) = \frac{2\pi n_1 n_2 m\alpha}{\ell c}(2\sqrt{y} - y),$$
$$w(y) = \frac{\Omega(y)}{y}\Psi\Big(\frac{n_1\beta_1}{K}, \frac{n_2\beta_2}{K}, \frac{y n_1 n_2 m\alpha}{\ell^2 K}, \frac{m\beta}{K^2}, \frac{K^2\ell c}{8\pi\sqrt{y} n_1 n_2 m\alpha}\Big) U_0\Big(\frac{n_1 n_2 m\alpha}{\ell^2 N_3} y\Big),$$
$$X = K^{\epsilon},\ V = V_1 = Q = 1,\ Y = \frac{n_1 n_2 m\alpha}{|\ell| c} \asymp \frac{\sqrt{n_1 n_2 N_3 m\alpha}}{c}.$$

By (5·4), we have that $Y \gg K^{2-\epsilon}$. Thus the conditions [4, line (8·7)] are satisfied for $\delta = 1/5$ say, and we get (we have a factor of $e(-1/8)$ instead of $e(1/8)$ because the second derivative of h is negative)

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$$I_2 = \sqrt{\frac{2\ell c}{n_1 n_2 m\alpha}}\, e\left(\frac{n_1 n_2 m\alpha}{\ell c} - \frac{\pi}{8}\right) \Psi\left(\frac{n_1\beta_1}{K}, \frac{n_2\beta_2}{K}, \frac{n_1 n_2 m\alpha}{\ell^2 K}, \frac{m\beta}{K^2}, \frac{K^2 \ell c}{8\pi n_1 n_2 m\alpha}\right) \times U_0\left(\frac{n_1 n_2 m\alpha}{\ell^2 N_3}\right) + \text{error},$$

where

$$\text{error} = O\left(\frac{1}{\sqrt{Y}} \sum_{1\le n\le 10^6} p_n(1) + K^{-100}\right)$$

and

$$p_n(1) = \frac{1}{n!}\frac{|G^{(2n)}(1)|}{Y^n}, \quad G(t) = w(t)e(H(t)), \quad H(t) = h(t) - h(1) - \tfrac{1}{2}h''(1)(t-1)^2.$$

Note that $H(1) = H'(1) = H''(1) = 0$, and so $G^{(2n)}(1) \ll Y^{\lfloor \frac{2n}{3} \rfloor} \ll Y^{n-1}$. Thus

$$\text{error} = O(Y^{-3/2}) = O\left(\frac{1}{K^{3-\epsilon}}\right).$$

Now we are ready to return to (6·1). We evaluate the a-sum there as

$$\sum_{a \bmod c} S(a, n_1 n_2 m\alpha, c) e\left(\frac{a\ell}{c}\right) = \begin{cases} c e\left(\frac{-n_1 n_2 m\alpha\bar{\ell}}{c}\right) & \text{if } (\ell, c) = 1 \\ 0 & \text{otherwise,}\end{cases}$$

and then apply Lemma 6·1 for the integral. The error term of this lemma contributes, using (4·1) and (5·4), at most

$$\frac{1}{K^{3-\epsilon}} \sum_{\substack{n_1\asymp N_1, n_2\asymp N_2 \\ m<K^{2+\epsilon}/\beta \\ c<\sqrt{N_1N_2N_3 m\alpha}/K^{2-\epsilon} \\ |\ell|\asymp\sqrt{n_1n_2m\alpha/N_3}}} \frac{N_3^{\frac{1}{4}}}{c^{\frac{1}{2}}(n_1n_2m)^{\frac{3}{4}}} \ll K^{\epsilon}.$$

Thus we only need to consider the contribution of the main term. It suffices to prove (we only treat the terms with $\ell > 0$)

$$OD_3 := \sum_{n_1,n_2,m,c,\ell\ge1} e\left(\frac{-n_1n_2m\alpha\bar{\ell}}{c}\right) e\left(\frac{n_1n_2m\alpha}{\ell c}\right)\frac{\tau(m)}{n_1n_2m} \times \Psi\left(\frac{n_1\beta_1}{K}, \frac{n_2\beta_2}{K}, \frac{n_1 n_2 m\alpha}{\ell^2 K}, \frac{m\beta}{K^2}, \frac{K^2 \ell c}{8\pi n_1 n_2 m\alpha}\right) U_1\left(\frac{n_1}{N_1}\right) U_2\left(\frac{n_2}{N_2}\right) \times U_3\left(\frac{n_1n_2m\alpha}{\ell^2 N_3}\right) \ll \alpha K^{2\theta+\epsilon}, \tag{6·3}$$

where it is understood that the sum is restricted to $(\ell, c) = 1$. By the reciprocity relation for exponentials, we have

$$OD_3 = \sum_{n_1,n_2,m,c,\ell\ge1} e\left(\frac{n_1n_2m\alpha\bar{c}}{\ell}\right)\frac{\tau(m)}{n_1n_2m} \Psi\left(\frac{n_1\beta_1}{K}, \frac{n_2\beta_2}{K}, \frac{n_1 n_2 m\alpha}{\ell^2 K}, \frac{m\beta}{K^2}, \frac{K^2 \ell c}{8\pi n_1 n_2 m\alpha}\right) \times U_1\left(\frac{n_1}{N_1}\right) U_2\left(\frac{n_2}{N_2}\right) U_3\left(\frac{n_1n_2m\alpha}{\ell^2 N_3}\right).$$

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7. *Voronoi summation and fake main terms*

The next aim is to perform Voronoi summation on m but we cannot do so immediately because in the exponential $e(n_1n_2m\alpha\overline{c}/\ell)$, the integers $n_1n_2\alpha$ and ℓ may not be coprime. We first prepare by eliminating any common factors. Re-ordering the sum OD_3 by $b_1 = (n_1, \ell)$, and replacing n_1 by b_1n_1 and ℓ by $b_1\ell$, we have

$$OD_3 = \sum_{\substack{n_1,n_2,m,c,\ell\geq 1\\ b_1\geq 1\\ (n_1,\ell)=1}} e\left(\frac{n_1n_2m\alpha\overline{c}}{\ell}\right)\frac{\tau(m)}{b_1n_1n_2m}\Psi\left(\frac{b_1n_1\beta_1}{K}, \frac{n_2\beta_2}{K}, \frac{n_1n_2m\alpha}{b_1\ell^2K}, \frac{m\beta}{K^2}, \frac{K^2\ell c}{8\pi n_1n_2m\alpha}\right)$$
$$\times U_1\left(\frac{b_1n_1}{N_1}\right)U_2\left(\frac{b_2n_2}{N_2}\right)U_3\left(\frac{n_1n_2m\alpha}{b_1\ell^2N_3}\right).$$

Next we re-order the sum by $b_2 = (n_2, \ell)$, and replace n_2 by b_2n_2 and ℓ by $b_2\ell$, then re-order the result by $b_3 = (\alpha, \ell)$, and replace ℓ by $b_3\ell$ and α by $b_3\alpha$. In this way, the conditions (4·1) become

$$N_3 \geq N_2, \quad N_1N_2 < \frac{K^{2+\epsilon}}{b_3\alpha}, \quad \beta \geq b_3\alpha, \tag{7·1}$$

and

$$OD_3 = \sum_{\substack{n_1,n_2,m,c,\ell\geq 1\\ b_1,b_2\geq 1, b_3|\alpha\\ (n_1n_2\alpha,\ell)=1}} e\left(\frac{n_1n_2m\alpha\overline{c}}{\ell}\right)\frac{\tau(m)}{b_1b_2n_1n_2m}$$
$$\times \Psi\left(\frac{b_1n_1\beta_1}{K}, \frac{b_2n_2\beta_2}{K}, \frac{n_1n_2m\alpha}{b_1b_2b_3\ell^2K}, \frac{m\beta}{K^2}, \frac{K^2\ell c}{8\pi n_1n_2m\alpha}\right)$$
$$\times U_1\left(\frac{b_1n_1}{N_1}\right)U_2\left(\frac{n_2b_2}{N_2}\right)U_3\left(\frac{n_1n_2m\alpha}{b_1b_2b_3\ell^2N_3}\right), \tag{7·2}$$

for which the required bound (6·3) becomes

$$OD_3 \ll b_3\alpha K^{2\theta+\epsilon}.$$

Working in dyadic intervals of m by taking a partition of unity, it suffices to show that

$$OD_3 = \sum_j \sum_{\substack{n_1,n_2,m,c,\ell\geq 1\\ b_1,b_2\geq 1, b_3|\alpha\\ (n_1n_2\alpha,\ell)=1}} e\left(\frac{n_1n_2m\alpha\overline{c}}{\ell}\right)\frac{\tau(m)}{b_1b_2n_1n_2m}U_1\left(\frac{b_1n_1}{N_1}\right)U_2\left(\frac{n_2b_2}{N_2}\right)U_3\left(\frac{n_1n_2m\alpha}{b_1b_2b_3\ell^2N_3}\right)$$
$$\times U_{4,j}\left(\frac{m}{M_j}\right)\Psi\left(\frac{b_1n_1\beta_1}{K}, \frac{b_2n_2\beta_2}{K}, \frac{n_1n_2m\alpha}{b_1b_2b_3\ell^2K}, \frac{m\beta}{K^2}, \frac{K^2\ell c}{8\pi n_1n_2m\alpha}\right) \ll b_3\alpha K^{\theta+\epsilon}$$

for some smooth functions $U_{4,j}$ compactly supported on $(1/2, 5/2)$, for $j \ll \log K$, and $M_j \asymp 2^j$. We now apply the Voronoi summation formula (Lemma 3·1) to the m sum, getting

$$OD_3 := FM + OD_4, \tag{7·3}$$

where the "fake main term" is

$$FM := \int_{-\infty}^{\infty} \left(\log \frac{x}{\ell^2} + 2\gamma\right) \sum_j U_{4,j}\left(\frac{x}{M_j}\right) \sum_{\substack{n_1,n_2,c,\ell\geq 1\\ b_1,b_2\geq 1, b_3|\alpha\\ (n_1n_2\alpha,\ell)=1}} \frac{1}{\ell b_1 b_2 n_1 n_2}$$
$$\times \Psi\left(\frac{b_1n_1\beta_1}{K}, \frac{b_2n_2\beta_2}{K}, \frac{n_1n_2x\alpha}{b_1b_2b_3\ell^2K}, \frac{x\beta}{K^2}, \frac{K^2\ell c}{8\pi n_1n_2x\alpha}\right) U_1\left(\frac{b_1n_1}{N_1}\right)$$
$$\times U_2\left(\frac{n_2b_2}{N_2}\right) U_3\left(\frac{n_1n_2x\alpha}{b_1b_2b_3\ell^2N_3}\right)\frac{dx}{x},$$

and OD_4 is given in the next section. In the sum FM, we may re-patch the partition of unity and reverse the steps which led to (7·2), getting that

$$FM = \int_{-\infty}^{\infty} \left(\log \frac{x}{\ell^2} + 2\gamma\right) \sum_{n_1,n_2,c,\ell\geq 1} \frac{1}{\ell n_1 n_2}$$
$$\times \Psi\left(\frac{n_1\beta_1}{K}, \frac{n_2\beta_2}{K}, \frac{n_1n_2x\alpha}{\ell^2K}, \frac{x\beta}{K^2}, \frac{K^2\ell c}{8\pi n_1n_2x\alpha}\right) U_1\left(\frac{n_1}{N_1}\right) U_2\left(\frac{n_2}{N_2}\right) U_3\left(\frac{n_1n_2x\alpha}{\ell^2N_3}\right)\frac{dx}{x}.$$

The trivial bound for M is $O(K^{1/2+\epsilon})$, from the length of the c-sum given by (5·4). It seems like we cannot do better because there are no exponentials or other harmonics present which may produce further cancellation (hence the name "fake main term"). However we can exploit our judicious choice of weight function in the approximate functional equation, as follows.

Making the substitution $y = xn_1n_2\alpha/\ell^2N_3$, we have

$$FM = \int_{-\infty}^{\infty} \sum_{n_1,n_2,c,\ell\geq 1} \left(\log \frac{yN_3}{n_1n_2\alpha} + 2\gamma\right) \frac{1}{\ell n_1 n_2}$$
$$\times \Psi\left(\frac{n_1\beta_1}{K}, \frac{n_2\beta_2}{K}, \frac{yN_3}{K}, \frac{y\ell^2N_3\beta}{K^2n_1n_2\alpha}, \frac{K^2c}{8\pi\ell N_3 y}\right) U_1\left(\frac{n_1}{N_1}\right) U_2\left(\frac{n_2}{N_2}\right) U_3(y)\frac{dy}{y}.$$

It suffices to show that

$$FM' := \sum_{c,\ell\geq 1} \frac{1}{\ell}\Psi\left(\frac{n_1\beta_1}{K}, \frac{n_2\beta_2}{K}, \frac{yN_3}{K}, \frac{y\ell^2N_3\beta}{K^2n_1n_2\alpha}, \frac{K^2c}{8\pi\ell N_3 y}\right) \ll K^{\epsilon}$$

for any $n_1 \asymp N_1$, $n_2 \asymp N_2$, $y \asymp 1$. Using (5·2) and Mellin inversion, we have

$$FM' = \frac{1}{(2\pi i)^5}\int_{(1+\epsilon)}\int_{\left(\frac{1}{2}+\epsilon\right)}\int_{(\epsilon)}\int_{(\epsilon)}\int_{(\epsilon)} \frac{e^{s_1^2+s_2^2+s_3^2}\zeta(1+2s_4)\mathcal{G}(s_4)}{(4\pi)^{s_1+s_2+s_3}(8\pi)^{s_4}}\zeta(1+2s_4-w)\zeta(w)$$
$$\times \left(\frac{K}{n_1\beta_1}\right)^{s_1}\left(\frac{K}{n_2\beta_2}\right)^{s_2}\left(\frac{K}{yN_3}\right)^{s_3}\left(\frac{K^2n_1n_2\alpha}{yN_3\beta}\right)^{s_4}\left(\frac{8\pi N_3 y}{K^2}\right)^{w}$$
$$\times \tilde{h}_{s_1+2s_2+2s_3}(w)\frac{ds_1}{s_1}\frac{ds_2}{s_2}\frac{ds_3}{s_3}\frac{ds_4}{s_4}dw.$$

Here $\zeta(1+2s_4-w)$ comes from the ℓ-sum and $\zeta(w)$ comes from the c-sum. We must initially keep the lines of integration at $\Re(w) = 1+\epsilon$ and $\Re(s_4) = 1/2+\epsilon$ in order to stay in the region of absolute convergence. The goal is to move all the lines of integration to (ϵ), and this would prove the claim. We first move the w-integral to $\Re(w) = \epsilon$. This crosses a simple pole at $w = 1$, with residue

$$FM'' := \frac{1}{(2\pi i)^4} \int_{(\frac{1}{2}+\epsilon)} \int_{(\epsilon)} \int_{(\epsilon)} \int_{(\epsilon)} \frac{e^{s_1^2+s_2^2+s_3^2}\zeta(1+2s_4)\mathcal{G}(s_4)}{(4\pi)^{s_1+s_2+s_3}(8\pi)^{s_4}} \zeta(2s_4)$$
$$\times \left(\frac{K}{n_1\beta_1}\right)^{s_1} \left(\frac{K}{n_2\beta_2}\right)^{s_2} \left(\frac{K}{yN_3}\right)^{s_3} \left(\frac{K^2 n_1 n_2 \alpha}{yN_3\beta}\right)^{s_4} \left(\frac{8\pi N_3 y}{K^2}\right)$$
$$\times \tilde{h}_{s_1+2s_2+2s_3}(1) \frac{ds_1}{s_1}\frac{ds_2}{s_2}\frac{ds_3}{s_3}\frac{ds_4}{s_4}.$$

On the shifted integral at $\Re(w) = \epsilon$, which is not displayed, we may move the s_4 integral to $\Re(s_4) = \epsilon$ and then estimate (this does not cross any pole of $\zeta(1 + 2s_4 - w)$ so this straight forward). Thus the shifted integral is $O(K^\epsilon)$ and we are left to estimate FM''. In the integral FM'', we move the line of integration to $\Re(s_4) = \epsilon$. This does not cross any poles because the simple pole of $\zeta(2s_4)$ at $s_4 = 1/2$ is cancelled out by the zero at $s_4 = 1/2$ of $\mathcal{G}(s_4)$. See the definition (3·3). Thus FM'' is $O(K^{-1+\epsilon})$.

8. *Second application of reciprocity*

We now return to (7·3) and give the definition of OD_4 corresponding to the sum on the right-hand side of (3·6). We have (r is the dual variable)

$$OD_4 := \sum_j \sum_{\pm} \frac{1}{2\pi i} \int_{(A)} \int_0^\infty \sum_{\substack{n_1,n_2,r,c,\ell \geq 1 \\ b_1,b_2 \geq 1, b_3 | \alpha \\ (c,\ell)=1}} e\left(\frac{\pm rc\overline{n_1 n_2 \alpha}}{\ell}\right) \frac{\tau(r)}{\ell b_1 b_2 n_1 n_2} \left(\frac{M_j r x}{\ell^2}\right)^{-w}$$
$$\times H_1^{\pm}(w) U_1\left(\frac{b_1 n_1}{N_1}\right) U_2\left(\frac{n_2 b_2}{N_2}\right) U_3\left(\frac{n_1 n_2 x M_j \alpha}{b_1 b_2 b_3 \ell^2 N_3}\right)$$
$$\times U_{4,j}(x) \Psi\left(\frac{b_1 n_1 \beta_1}{K}, \frac{b_2 n_2 \beta_2}{K}, \frac{n_1 n_2 x M_j \alpha}{b_1 b_2 b_3 \ell^2 K}, \frac{x M_j \beta}{K^2}, \frac{K^2 \ell c}{8\pi n_1 n_2 x M_j \alpha}\right) \frac{dx}{x} dw,$$

where it is understood that the sum is restricted to $(n_1 n_2 \alpha, \ell) = 1$ and we need $OD_4 \ll b_3 \alpha K^{2\theta+\epsilon}$.

We first simplify the notation a bit (we did not do this earlier because we needed the exact form of the weight functions in order to deal with fake main terms). First, we observe that since there are $O(K^\epsilon)$ dyadic intervals, it is enough to consider any one smooth function $U_{4,j} = U_4$ and $M_j = M$. From the fourth component of Ψ and the assumption $\beta \geq b_3\alpha$ from (4·1), we can assume

$$M < \frac{K^{2+\epsilon}}{b_3 \alpha}.$$

We can also consider the sum in dyadic intervals $r \asymp R$ by inserting a smooth bump function $U_5(r/R)$, where U_5 is supported on $(1/2, 5/2)$. We can assume that

$$R < \frac{K^\epsilon \ell^2}{M}$$

because the contribution of $r \geq K^\epsilon \ell^2/M$ is $O(K^{-100})$ say. This can be seen by moving the w-integral in OD_4 far to the right (taking A large). By repeatedly integrating by parts the

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x-integral, we may restrict the w-integral to $|\Im w| < K^\epsilon$ (the real part is already fixed at A). Doing so, we may absorb r^{-w} and $(\ell^2)^{-w}$ into U_5 and U_3 respectively. Similarly we may expand the function Ψ using (5·2), truncate the integrals there to $|\Im s_1|, |\Im s_2|, |\Im s_3|, |\Im s_4| < K^\epsilon$ (with $\Re(s_i)$ fixed of course) and absorb part of this function into the bump functions U_1, U_2, U_3, U_4. Thus it suffices to prove (we do not seek cancellation in the sum over b_1, b_2, b_3)

$$\sum_{\substack{n_1,n_2,r,c,\ell\geq 1\\(c,\ell)=1}} e\left(\frac{\pm rc\overline{n_1n_2\alpha}}{\ell}\right)\frac{\tau(r)}{\ell n_1n_2}U_1\left(\frac{b_1n_1}{N_1}\right)U_2\left(\frac{n_2b_2}{N_2}\right)U_3\left(\frac{n_1n_2M\alpha}{b_1b_2b_3\ell^2N_3}\right)$$
$$\times U_5\left(\frac{r}{R}\right)\hbar_s\left(\frac{K^2\ell c}{8\pi xn_1n_2M\alpha}\right)\ll b_3\alpha K^{\theta+\epsilon}$$

for any $b_1, b_2, b_3 \geq 1$, $x \asymp 1$, $|s| < K^\epsilon$ and any compactly supported functions U_j with jth derivative bounded by $(K^\epsilon)^j$. We simplify the notation a bit more. We suppress the factor $8\pi x$ in $\hbar_s$, rename b_1b_2 to a, b_3 to b, $M\alpha$ to M, N_1/b_1 to N_1 and N_2/b_2 to N_2. Thus it suffices to prove

$$\sum_{\substack{n_1,n_2,r,c,\ell\geq 1\\(c,\ell)=1}} e\left(\frac{\pm rc\overline{n_1n_2\alpha}}{\ell}\right)\frac{\tau(r)}{\ell n_1n_2}U_1\left(\frac{n_1}{N_1}\right)U_2\left(\frac{n_2}{N_2}\right)U_3\left(\frac{n_1n_2M}{ab\ell^2N_3}\right)$$
$$\times U_5\left(\frac{r}{R}\right)\hbar_s\left(\frac{K^2\ell c}{n_1n_2M}\right)\ll b\alpha K^{\theta+\epsilon}, \tag{8·1}$$

for any integers a, b, α and

$$N_1, N_2, N_3 < K^{1+\epsilon},\quad N_1N_2 < \frac{K^{2+\epsilon}}{\alpha ab},\quad M < \frac{K^{2+\epsilon}}{b},\quad R < \frac{K^\epsilon\ell^2\alpha}{M} \asymp \frac{K^\epsilon N_1N_2\alpha}{abN_3},\quad N_3 \geq N_2. \tag{8·2}$$

The approximation $K^\epsilon\ell^2\alpha/M \asymp K^\epsilon N_1N_2\alpha/abN_3$ follows from the support of U_3. This updates (7·1).

Now using reciprocity for exponentials, we have

$$e\left(\frac{\pm rc\overline{n_1n_2\alpha}}{\ell}\right)=e\left(\frac{\mp rc\overline{\ell}}{n_1n_2\alpha}\right)e\left(\frac{\pm rc}{\ell n_1n_2\alpha}\right)=e\left(\frac{\mp rc\overline{\ell}}{n_1n_2\alpha}\right)\left(1+O\left(\frac{\pm rc}{\ell n_1n_2\alpha}\right)\right).$$

The contribution to (8·1) of this error term is less than

$$\sum_{\substack{n_1\asymp N_1\\ n_2\asymp N_2\\ \ell\asymp\frac{\sqrt{N_1N_2M}}{\sqrt{abN_3}}}}\ \sum_{r<\frac{K^\epsilon N_1N_2\alpha}{N_3ab}}\ \sum_{c<\frac{N_1N_2M}{\ell K^{2-\epsilon}}}\frac{K^\epsilon}{\ell n_1n_2}\cdot\frac{rc}{\ell n_1n_2\alpha}\ll\frac{N_1^{\frac{3}{2}}N_2^{\frac{3}{2}}M^{\frac{1}{2}}\alpha}{K^{4-\epsilon}N_3^{\frac{1}{2}}}\ll\frac{1}{K^{\frac{1}{2}-\epsilon}},$$

by (8·2). So in (8·1) we can replace the exponential with $e(\mp rc\overline{\ell}/n_1n_2\alpha)$ and detect the condition $(\ell, c) = 1$ using the Möbius function:

$$\sum_{l|(\ell,c)}\mu(l)=\begin{cases}1 & \text{if } (\ell, c) = 1\\ 0 & \text{otherwise.}\end{cases} \tag{8·3}$$

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Thus replacing ℓ by ℓl and c by cl, it suffices to prove

$$\sum_{\substack{n_1,n_2,r,c,\ell\geq 1\\ l\geq 1}} e\left(\frac{\mp rc\overline{\ell}}{n_1n_2\alpha}\right)\frac{\mu(l)\tau(r)}{l\ell n_1n_2}U_1\left(\frac{n_1}{N_1}\right)U_2\left(\frac{n_2}{N_2}\right)U_3\left(\frac{n_1n_2M}{abl^2\ell^2N_3}\right)$$
$$\times U_5\left(\frac{r}{R}\right)\hbar_s\left(\frac{K^2l^2\ell c}{n_1n_2M}\right)\ll b\alpha K^{\theta+\epsilon}.$$

We do not seek cancellation over the l-sum, so it suffices to prove

$$OD_5 := \sum_{n_1,n_2,r,c,\ell\geq 1} e\left(\frac{\mp rc\overline{\ell}}{n_1n_2\alpha}\right)\frac{\tau(r)}{\ell n_1n_2}U_1\left(\frac{n_1}{N_1}\right)U_2\left(\frac{n_2}{N_2}\right)U_3\left(\frac{n_1n_2M}{abl^2\ell^2N_3}\right)$$
$$\times U_5\left(\frac{r}{R}\right)\hbar_s\left(\frac{K^2l^2\ell c}{n_1n_2M}\right)\ll b\alpha K^{\theta+\epsilon}$$

for any integer $l\geq 1$ and assuming (8·2). Also keep in mind that it is understood that the sum is restricted to $(\ell, n_1n_2\alpha)=1$.

9. *Second Poisson summation*

Now we split the ℓ-sum in OD_5 into (primitive) residue classes mod $n_1n_2\alpha$ and apply Poisson summation (Lemma 3·2). Note that ℓ is supported in compact interval of size $\sqrt{n_1n_2M}/l\sqrt{abN_3}$. The result is that (the dual variable is d)

$$OD_5 = \sum_{n_1,n_2,r,c\geq 1}\frac{S(\mp rc, d, n_1n_2\alpha)}{n_1n_2\alpha}\frac{\tau(r)}{n_1n_2}U_1\left(\frac{n_1}{N_1}\right)U_2\left(\frac{n_2}{N_2}\right)U_5\left(\frac{r}{R}\right)$$
$$\times\left(\int_{-\infty}^{\infty}U_3\left(\frac{1}{y}\right)\hbar_s\left(\frac{yK^2cl}{\sqrt{abn_1n_2N_3M}}\right)\frac{dy}{y}\right.$$
$$+\sum_{d\neq 0}\frac{1}{2\pi i}\int_{(A)}\int_0^{\infty}H_2(z)\left(\frac{-2\pi yd\sqrt{M}}{l\alpha\sqrt{abn_1n_2N_3}}\right)^{-z}$$
$$\left.\times U_3\left(\frac{1}{y}\right)\hbar_s\left(\frac{yK^2cl}{\sqrt{abn_1n_2N_3M}}\right)\frac{dy}{y}dz\right). \tag{9·1}$$

We first consider the contribution of the second line of (9·1). This is the zero frequency contribution, and it is bounded by

$$\sum_{\substack{n_1\asymp N_1\\ n_2\asymp N_2}}\sum_{r<\frac{K^\epsilon N_1N_2\alpha}{N_3ab}}\sum_{c<\frac{\sqrt{abn_1n_2N_3M}}{lK^{2-\epsilon}}}\frac{K^\epsilon|S(\mp rc,0,n_1n_2\alpha)|}{n_1^2n_2^2\alpha}\ll\frac{\sqrt{N_1N_2M}}{K^{2-\epsilon}\sqrt{N_3}}\ll\frac{1}{K^{\frac{1}{2}-\epsilon}},$$

on using $N_3\geq N_2$ and that the Ramanujan sum is $O(K^\epsilon)$ on average.

Now we consider the contribution of the third line of (9·1), arising from the sum over $d\neq 0$. We consider this sum in dyadic intervals $d\asymp D$ (for simplicity, we restrict to only positive values of d) and $c\asymp C$ by inserting smooth bump functions $U_6(d/D)$ and $U_7(c/C)$ say. We can assume that

$$D<\frac{K^\epsilon l\alpha\sqrt{abN_1N_2N_3}}{\sqrt{M}} \tag{9·2}$$

because the contribution of $d > l\alpha\sqrt{abN_1N_2N_3}/\sqrt{M}$ is $O(K^{-100})$ by moving z-integral in (9·1) far to the right. Restricting to $|\Im z| < K^\epsilon$ and $\Re(z)$ fixed, which we may do up to negligible error by repeatedly by parts with respect to y, we may absorb d^{-z}, n_1^z, n_2^z into the existing weight functions. We can also assume that

$$C < \frac{\sqrt{abN_1N_2N_3M}}{lK^{2-\epsilon}} \tag{9·3}$$

and absorb the function $\hbar_s$ into U_7, by using Mellin inversion and separating variables as above. Thus it suffices to prove

$$\sum_{n_1,n_2,r,c,d\ge 1} \frac{S(\mp rc, d, n_1n_2\alpha)}{n_1n_2\alpha}\frac{\tau(r)}{n_1n_2}U_1\left(\frac{n_1}{N_1}\right)U_2\left(\frac{n_2}{N_2}\right)U_5\left(\frac{r}{R}\right)U_6\left(\frac{d}{D}\right)U_7\left(\frac{c}{C}\right) \ll b\alpha K^{2\theta+\epsilon}.$$

Finally, we need this to be in a form to which we can apply Kuznetsov's formula. To this end we define

$$X := \frac{N_1N_2\alpha}{\sqrt{RDC}},$$

and replace $U_2(n_2/N_2)$ with a different bump function

$$Y_1\left(\frac{4\pi X\sqrt{rcd}}{n_1n_2\alpha}\right)$$

with properties given below. We can also replace $\tau(r)/n_1n_2$ with $\tau(r)/N_1N_2$. Thus it suffices to prove (we do not seek cancellation in the n_1 sum)

$$OD_6 := \frac{1}{N_1N_2}\sum_{n_1\asymp N_1}\left|\sum_{n_2,r,c,d\ge 1}\frac{S(\pm rc, d, n_1n_2\alpha)}{n_1n_2\alpha}Y_1\left(\frac{4\pi X\sqrt{rcd}}{n_1n_2\alpha}\right)Y_2\left(\frac{d}{D}\right)Y_3\left(\frac{r}{R}\right)Y_4\left(\frac{c}{C}\right)\right|$$
$$\ll b\alpha K^{2\theta+\epsilon},$$

where Y_i are smooth functions compactly supported on $(1/2, 5/2)$ with $\|Y_j^{(j)}\|_\infty \ll (K^\epsilon)^j$ and we assume (8·2), (9·2) and (9·3).

10. *Kuznetsov's formula*

The aim now is to prove the required bound for OD_6 using Kuznetsov's formula and the spectral large sieve. We consider only the case of positive sign; the negative sign case is similar. By [**11**, theorem 16·5], we have that

$$\sum_{n_2\ge 1}\frac{S(rc, d, n_1n_2\alpha)}{n_1n_2\alpha}Y_1\left(\frac{4\pi X\sqrt{rcd}}{n_1n_2\alpha}\right) = \text{Maass} + \text{Eis} + \text{Hol}, \tag{10·1}$$

where these are the contributions of the Maass cusp forms, Eisenstein series, and holomorphic forms as given in the referenced theorem. For the Maass forms we have

$$\text{Maass} = \sum_{j\ge 1}\mathcal{M}_{Y_1}(t_j)\frac{\rho_j(rc)\overline{\rho}_j(d)}{\cosh(\pi t_j)},$$

where the sum is over an orthonormal basis of Maass cusp forms of level $n_1\alpha$ with Fourier coefficients $\rho_j(n)$ and Laplacian eigenvalue $1/4+t_j^2$, and

$$\mathcal{M}_{Y_1}(t)=\frac{\pi i}{2\sinh(\pi t)}\int_0^\infty (J_{2it}(x)-J_{-2it}(x))Y_1(xX)\frac{dx}{x}.$$

By [6, lemma 3·6], for example, we have that $\mathcal{M}_{Y_1}(t)\ll K^{-100}$ if $|t|\geq K^\epsilon$, so we can restrict the sum Maass to $|t_j|<K^\epsilon$, in which range $\mathcal{M}_{Y_1}(t_j)\ll X^{2\theta+\epsilon}$, by the same Lemma. We have $X\ll K^{1-\epsilon}$ by (10·2), so it suffices to prove that

$$\sum_{n_1\asymp N_1}\frac{1}{N_1N_2}\sum_{|t_j|<K^\epsilon}\left|\sum_{r,d,c\geq 1}\frac{\rho_j(rc)\overline{\rho}_j(d)}{\cosh(\pi t_j)}\tau(r)Y_2\left(\frac{d}{D}\right)Y_3\left(\frac{r}{R}\right)Y_4\left(\frac{c}{C}\right)\right|\ll b\alpha K^\epsilon.$$

Now we would like to decompose $\rho_j(rc)$, so that Cauchy–Schwarz and the spectral large sieve may be applied. To do this we need to work with newforms, whose Fourier coefficients are multiplicative. We consult [2, section 3] to see how to choose a basis consisting of lifts of newforms. By [2, equation (3·10)], and the $\cosh(\pi t_j)^{1/2}$ normalisation from the first display of [2, section 3·2], it suffices to prove that

$$\sum_{n_1\asymp N_1}\frac{1}{N_1N_2}\sum_{|t_j|<K^\epsilon}\frac{K^\epsilon(uv)^{\frac12}}{N_1\alpha}\left|\sum_{\substack{r,d,c\geq 1\\ u|rc\\ v|d}}\lambda_j\left(\frac{rc}{u}\right)\lambda_j\left(\frac{d}{v}\right)\tau(r)Y_2\left(\frac{d}{D}\right)Y_3\left(\frac{r}{R}\right)Y_4\left(\frac{c}{C}\right)\right|\ll b\alpha K^\epsilon$$

for any integers u, $v\geq 1$ and $N_0|n_1\alpha$, where $\lambda_j(n)$ are the Hecke eigenvalues corresponding to newforms of level N_0. We now replace d by dv and, proceeding exactly like in steps (4·4) to (4·5), we can write $u=u_1u_2u_3$ and replace r by ru_1u_2 and c by cu_2u_3 to see that it suffices to prove

$$\sum_{n_1\asymp N_1}\frac{1}{N_1N_2}\sum_{|t_j|<K^\epsilon}\frac{K^\epsilon(u_1u_2u_3v)^{\frac12}}{N_1\alpha}$$
$$\times\left|\sum_{r,d,c\geq 1}\lambda_j(rcu_2)\lambda_j(d)\mu(u_2)\tau(ru_1u_2)Y_2\left(\frac{dv}{D}\right)Y_3\left(\frac{ru_1u_2}{R}\right)Y_4\left(\frac{cu_2u_3}{C}\right)\right|\ll b\alpha K^\epsilon.$$

To simplify notation, we may replace r by ru_2. Thus it suffices to prove

$$\sum_{n_1\asymp N_1}\frac{1}{N_1N_2}\sum_{|t_j|<K^\epsilon}\frac{K^\epsilon(u_1u_2u_3v)^{\frac12}}{N_1\alpha}\left|\sum_{\substack{r\asymp R/u_1\\ c\asymp C/u_2u_3\\ d\asymp D/v}}\lambda_j(rc)\lambda_j(d)\gamma_1(r)\gamma_2(d)\gamma_3(c)\right|\ll b\alpha K^\epsilon$$

for any $\gamma_1(r)$, $\gamma_2(c)$, $\gamma_3(d)\ll K^\epsilon$. By Hecke multiplicativity, we have

$$\lambda_j(rc)\lambda_j(d)=\sum_{\substack{s|(r,c)\\ (s,N_0)=1}}\mu(s)\lambda_j\left(\frac{r}{s}\right)\lambda_j\left(\frac{c}{s}\right)\lambda_j(d)=\sum_{\substack{s|(r,c)\\ w|(c/s,d)\\ (sw,N_0)=1}}\mu(s)\lambda_j\left(\frac{r}{s}\right)\lambda_j\left(\frac{cd}{sw^2}\right)$$

and so, after replacing r by rs, c by csw, and d by dw, it suffices to prove

$$OD_7:=\sum_{\substack{n_1\asymp N_1\\ s,w\leq K^{10}}}\frac{1}{N_1N_2}\sum_{|t_j|<K^\epsilon}\frac{K^\epsilon(u_1u_2u_3v)^{\frac12}}{N_1\alpha}\left|\sum_{\substack{r\asymp R/u_1s\\ cd\asymp CD/u_2u_3vsw^2}}\lambda_j(r)\lambda_j(cd)\gamma_1(r)\gamma_2(cd)\right|$$
$$\ll b\alpha K^\epsilon,$$

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for any $\gamma_1(r), \gamma_2(cd) \ll K^\epsilon$. By the Cauchy–Schwarz inequality and the spectral large sieve [**7**, theorem 2], we have that OD_7 is bounded by

$$\sum_{\substack{n_1 \asymp N_1 \\ s,w \le K^{10}}} \frac{1}{N_1 N_2} \frac{K^\epsilon (u_1 u_2 u_3 v)^{\frac{1}{2}}}{N_1 \alpha} \left(\sum_{|t_j|<K^\epsilon} \left| \sum_{r \asymp R/u_1 s} \lambda_j(r) \gamma_1(r) \right|^2 \right)^{\frac{1}{2}}$$

$$\times \left(\sum_{|t_j|<K^\epsilon} \left| \sum_{cd \asymp CD/u_2 u_3 v s w^2} \lambda_j(cd) \gamma_2(cd) \right|^2 \right)^{\frac{1}{2}}$$

$$\ll \sum_{\substack{n_1 \asymp N_1 \\ s,w \le K^{10}}} \frac{1}{N_1 N_2} \frac{K^\epsilon (u_1 u_2 u_3 v)^{\frac{1}{2}}}{N_1 \alpha} \left(\left(N_1 \alpha + \frac{R}{u_1 s} \right) \frac{R}{u_1 s} \right)^{\frac{1}{2}}$$

$$\times \left(\left(N_1 \alpha + \frac{CD}{u_2 u_3 v s w^2} \right) \frac{CD}{u_2 u_3 v s w^2} \right)^{\frac{1}{2}}.$$

Thus it suffices to prove

$$\frac{K^\epsilon}{N_1 N_2 \alpha} \big((N_1 \alpha + R) R \big)^{\frac{1}{2}} \big((N_1 \alpha + CD) CD \big)^{\frac{1}{2}} \ll b \alpha K^\epsilon.$$

By (8·2), (9·2) and (9·3), we have

$$(RCD)^{\frac{1}{2}} \ll \frac{\alpha N_1 N_2}{K^{1-\epsilon}}, \tag{10·2}$$

so it suffices to prove

$$\frac{1}{K} (N_1 \alpha + R)^{\frac{1}{2}} (N_1 \alpha + CD)^{\frac{1}{2}} \ll b \alpha K^\epsilon. \tag{10·3}$$

We have

$$\frac{N_1 \alpha}{K} \ll K^\epsilon \alpha,$$

$$\frac{(N_1 \alpha CD)^{\frac{1}{2}}}{K} \ll \frac{N_1 \alpha (ab N_2 N_3)^{\frac{1}{2}}}{K^2} \ll \frac{(N_1 N_3 \alpha)^{\frac{1}{2}} (ab \alpha N_1 N_2)^{\frac{1}{2}}}{K^2} \ll K^\epsilon \alpha^{\frac{1}{2}},$$

$$\frac{(N_1 \alpha R)^{\frac{1}{2}}}{K} \ll \frac{N_1 \alpha N_2^{\frac{1}{2}}}{K N_3^{\frac{1}{2}}} \ll K^\epsilon \alpha,$$

where in the last inequality we use crucially that $N_3 \ge N_2$. This establishes (10·3).

It remains to consider Eis and Hol in (10·1). These are similarly treated using the large sieve, once we use the multiplicative Fourier coefficients provided explicitly in [**2**, section 3].

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