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AN EQUIVARIANT THEORY FOR THE BIVARIANT CUNTZ SEMIGROUP

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Abstract We provide an equivariant extension of the bivariant Cuntz semigroup introduced in previous work for the case of compact group actions over C*-algebras. Its functoriality properties are explored, and some well-known classification results are retrieved. Connections with crossed products are investigated, and a concrete presentation of equivariant Cuntz homology is provided. The theory that is here developed can be used to define the equivariant Cuntz semigroup. We show that the object thus obtained coincides with the one recently proposed by Gardella ['Regularity properties and Rokhlin dimension for compact group actions', *Houston J. Math.* **43**(3) (2017), 861–889], and we complement their work by providing an open projection picture of it.

Keywords: C*-algebras; group actions; Cuntz semigroup; crossed product; Cuntz homology

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1. Introduction

During the last couple of decades, the Cuntz semigroup, an invariant for C*-algebras originally proposed by Cuntz [**6**] in the late 1970s, has gained a prominent role in the classification programme of C*-algebras initiated by George Elliott. Important contributions came from the results of Rørdam [**13**] and Toms [**14**], which brought to light the fact that there are non-isomorphic C*-algebras with the same Elliott invariant and that can be told apart by other invariants, like the real rank and their Cuntz semigroup data.

The investigation into the theory of completely positive maps with the order zero property (hereafter, c.p. order zero maps for short) undertaken in [**16**], has led to the discovery of deep connections between this special class of maps between C*-algebras and the Cuntz semigroup. In particular, they have shown that every such map induces a semigroup homomorphism between the associated Cuntz semigroups. This has opened the doors for a bivariant extension of the theory of the Cuntz semigroup, which has been established in [**4**]. The main idea is to provide a new framework, in spirit similar to Kasparov's KK-theory, but that is reminiscent of the idea of Cuntz comparison of positive structures. The classification result for the class of unital and stably finite C*-algebras, obtained by the author in his PhD work, and contributed to [**4**], can be regarded as an analogue of the

Kirchberg–Phillips classification of purely infinite C*-algebras by KK-theory. Thus, the bivariant Cuntz semigroup complements the classification of purely infinite C*-algebras in the sense that it gives a suitable Cuntz-analogue of the KK-theoretic Kirchberg–Phillips classification result for stably finite C*-algebras.

The main subject of the present paper is the introduction of an equivariant extension of the bivariant Cuntz semigroup as a novel tool for the problem of classification of C*-dynamical systems. The theory proposed in this paper is specifically designed for *compact* group actions on C*-algebras. With such a tool at our disposal, the definition of an equivariant extension of the ordinary Cuntz semigroup appears as a natural consequence, since it has been shown in [**4**] that the ordinary Cuntz semigroup can be recovered from the bivariant theory there defined. A notion of equivariant Cuntz semigroup, however, has recently appeared in the work of Gardella and Santiago [**9**], where it is also used to provide a classification result of group actions on certain C*-algebras. Indeed, we shall see that the equivariant theory developed in this paper coincides with the new definitions that have appeared in [**9**], and that the classification result just mentioned can also be *restated* within the framework of the present work.

1.1. Outline

The present paper is organized as follows. In § 2, we introduce the equivariant theory of the bivariant Cuntz semigroup, and we show that most of the properties of the ordinary bivariant Cuntz semigroup of [**4**] carry over to the equivariant setting. Here, we restrict our attention to *separable* C*-dynamical systems, i.e. those for which the underlying C*-algebra is separable and the group is second countable. We also investigate the relations with crossed products to strengthen the analogy with equivariant KK-theory. In particular, we show that a Green–Julg-type theorem holds for the equivariant theory of the bivariant Cuntz semigroup developed in this paper (Theorem 2.27). This section concludes with an equivariant extension of the *Cuntz homology* for compact Hausdorff spaces introduced in [**4**].

In § 3, we introduce an open projection picture for the *equivariant Cuntz semigroup*. This new object emerges as the special case $\mathsf{Cu}^G(\mathbb{C}, A)$, in complete analogy with the way we can recover the ordinary Cuntz semigroup from the bivariant theory defined in [**4**], and coincides with that introduced in [**9**].

In § 4, we show how to use the equivariant bivariant Cuntz semigroup for the problem of classification of actions. In particular, we show how to use the theory developed in this paper to retrieve the classification result of Handelman and Rossmann [**10**] on locally representable actions by compact groups on approximately finite-dimensional (AF) algebras (Corollary 4.11), and the more recent result of Gardella and Santiago of locally representable finite Abelian group actions on inductive limits of one-dimensional non-commutative CW complexes (Corollary 4.8).

1.2. Notation

In what follows, we shall make use of capital letters $A, B, C, \ldots$ to denote C*-algebras, and the notation A_+ to denote the cone of the positive elements of A. The multiplier algebra of A is denoted by $\mathcal{M}(A)$, while the letter K is used to denote the C*-algebra of compact operators on an infinite-dimensional separable Hilbert space. For $k \in \mathbb{N}$, we

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shall denote by M_k the complex $k \times k$ matrix algebra, and by $M_k(A)$ the A-valued $k \times k$ matrix algebra. With this notation we then have $M_k = M_k(\mathbb{C})$.

The Greek letters $\pi, \omega, \dots$ are used to denote $*$-homomorphisms between C*-algebras, while ϕ, ψ are reserved for c.p. order zero maps, whose definition is recalled in this section for the reader's convenience.

In describing equivariant theories, we shall reserve the capital letter G for denoting a topological group, usually assumed to be compact and second countable, unless otherwise specified.

2. Definitions and main properties

In this section, we introduce an equivariant extension of the bivariant Cuntz semigroup developed in [**4**]. As already mentioned, in this paper we restrict our attention to *second countable*, *compact* groups.

Definition 2.1. Let A and B be C*-algebras. A completely positive map $\phi : A \to B$ has the order zero property if $\phi(a)\phi(b) = 0$ whenever $ab = 0$, with $a, b \in A_+$.

The structure of completely positive maps with the order zero property has been established in [**16**], where the authors have built on previous work by Wolff [**17**] on orthogonality preserving maps. A c.p.c. order zero map is a c.p. map with the order zero property that is also contractive, i.e. with norm bounded by 1.

A c.p.c. order zero map ϕ between two G-algebras (A, G, α) and (B, G, β) is said to be *equivariant* if it is an intertwiner for the actions α and β, that is

$$\phi \circ \alpha_g = \beta_g \circ \phi, \quad \forall g \in G.$$

Unless otherwise stated, we shall assume that a c.p.c. order zero map $\phi : A \to B$ between the G-algebras A and B is always equivariant. The Cuntz comparison of equivariant c.p.c. order zero maps then takes the following form.

Definition 2.2. Let A and B be G-algebras, and let $\phi, \psi : A \to B$ be c.p.c. order zero maps. We say that ϕ is equivariantly Cuntz-subequivalent to ψ (in symbols $\phi \precsim_G \psi$) if there exists a G-invariant sequence $\{b_n\}_{n\in\mathbb{N}} \subset B^G$ such that

$$\lim_{n\to\infty} \|b_n \psi(a) b_n^* - \phi(a)\| = 0$$

for any $a \in A$.

We observe that, in the separable case, the above definition has a standard *localization.* Two c.p.c. order zero maps $\phi, \psi : A \to B$ satisfy $\phi \precsim_G \psi$ if and only if, for every finite subset $F \Subset A$ and $\epsilon > 0$ there is $b \in B^G$ such that $\|b\psi(a)b^* - \phi(a)\| < \epsilon$ for any $a \in F$.

Let $\sim_G$ denote the relation arising from the antisymmetrization of the relation $\precsim_G$ just defined, that is $\phi \sim_G \psi$ if $\phi \precsim_G \psi$ and $\psi \precsim_G \phi$. Reflexivity follows trivially from the fact that any G-algebra, with G compact, admits a G-invariant approximate unit.

Lemma 2.3. *Let A and B be G-algebras. For any equivariant c.p.c. order zero map $\phi : A \to B$, finite subset $F \Subset A$ and $\epsilon > 0$, there exists $x \in C^*(\phi(A))^G$ such that $\|x\phi(a)x^* - \phi(a)\| < \epsilon$.*

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Proof. Fix a finite subset $F \Subset A$ and $\epsilon > 0$. By the existence of a G-invariant approximate unit for A, there is $e \in A^G$ such that $\|eae^* - a\| < \epsilon$ for any $a \in F$. Let h_ϕ and π_ϕ be the positive element and the $*$-homomorphism coming from Theorem 2.5 applied to ϕ. As $h_\phi^{2/n} h_\phi$ converges to h_ϕ in norm, there is $m \in \mathbb{N}$ such that $\|h_\phi^{1/m}\phi(eae^*)h_\phi^{1/m} - \phi(eae^*)\| < \epsilon$ for every $a \in F$. With $x := h_\phi^{1/m}\pi_\phi(e) = \phi^{1/m}(e) \in B^G$, we have the estimate

$$\begin{aligned}\|x\phi(a)x^* - \phi(a)\| &= \|h_\phi^{1/m}\phi(eae^*)h_\phi^{1/m} - \phi(a)\| \\ &\leq \|h_\phi^{1/m}\phi(eae^*)h_\phi^{1/m} - \phi(eae^*)\| + \|\phi(eae^*) - \phi(a)\| \\ &< 2\epsilon\end{aligned}$$

for any $a \in F$. □

If (A, G, α) is a G-algebra, we shall always assume that the tensor product $A \otimes K_G$ is equipped with the diagonal action $\alpha \otimes (\lambda_G \otimes \mathrm{id}_K)$, where λ_G is the left-regular representation of G on $L^2(G)$. As an equivariant generalization of the bivariant Cuntz semigroup Cu of [4] we then give the following definition.

Definition 2.4. Let A and B be G-algebras. The equivariant bivariant Cuntz semigroup $\mathsf{Cu}^G(A, B)$ of A and B is the set of equivalence classes

$$\mathsf{Cu}^G(A, B) := \{\phi : A \otimes K_G \to B \otimes K_G \mid \phi \text{ equivariant c.p.c. order zero map}\}/\sim_G .$$

The above semigroup can be equipped with a positive order structure by requiring $[\phi] \leq_G [\psi]$ whenever $\phi \precsim_G \psi$. Hence, $(\mathsf{Cu}^G(A, B), \leq_G)$ becomes a positively ordered Abelian monoid.

As discussed in [16], given any two C*-algebras A and B, there is a one-to-one correspondence between c.p.c. order zero maps from A to B and $*$-homomorphisms from the cone over A, i.e. $C_0((0, 1]) \otimes A$, to B. This result generalizes to the equivariant setting by equipping the cone over A with the diagonal action, as shown by [8, Corollary 2.10].

We now give an equivariant extension of the structure theorem for c.p.c. order zero maps of [16].

Theorem 2.5. *Let (A, G, α) and (B, G, β) be G-algebras and let $\phi : A \to B$ be an equivariant c.p.c. order zero map. Set $C_\phi := C^*(\phi(A))$ and introduce a strictly continuous action of G on $\mathcal{M}(C_\phi)$ by restricting the bidual maps β_g^{**}, $g \in G$, onto it. Then there exist a G-invariant positive element $h_\phi \in \mathcal{M}(C_\phi)_+^G \cap C_\phi'$, with $\|h_\phi\| = \|\phi\|$, and a non-degenerate $*$-homomorphism $\pi_\phi : A \to \mathcal{M}(C_\phi) \cap \{h_\phi\}'$ such that $\beta_g^{**} \circ \pi_\phi = \pi_\phi \circ \alpha_g$ for any $g \in G$, and*

$$\phi(a) = h_\phi \pi_\phi(a), \quad \forall a \in A.$$

Proof. By the structure theorem of [16], there are a positive element $h_\phi \in \mathcal{M}(C_\phi)_+ \cap C_\phi'$ with $\|h_\phi\| = \|\phi\|$ and a non-degenerate $*$-homomorphism $\pi_\phi : A \to \mathcal{M}(C_\phi) \cap \{h_\phi\}'$ such that $\phi(a) = h_\phi \pi_\phi(a)$ for any $a \in A$. If $\{e_n\}_{n \in \mathbb{N}} \subset A$ is an approximate unit, the equivariance of ϕ implies $h_\phi\pi_\phi(\alpha_g(e_n)) = \beta_g(h_\phi\pi_\phi(e_n))$ for any $n \in \mathbb{N}$, whence

$$\begin{aligned} 0 &= \text{SOT}\lim_{n\to\infty}[\phi(\alpha_g(e_n)) - \beta_g(\phi(e_n))] \\ &= h_\phi - \beta_g^{**}(h_\phi), \quad \forall g \in G, \end{aligned}$$

which shows that h_ϕ is G-invariant in $\mathcal{M}(C_\phi)$, with the strictly continuous action given by the restriction of β^{**} to this multiplier algebra. Since $h_\phi^{1/n}$ is also G-invariant, equivariance also implies $h_\phi^{1/n}[\pi_\phi(\alpha_g(a)) - \beta_g^{**}(\pi_\phi(a))] = 0$ for any $n \in \mathbb{N}$ and $a \in A$, whence

$$\begin{aligned} 0 &= \text{SOT}\lim_{n\to\infty} h_\phi^{1/n}[\pi_\phi(\alpha_g(a)) - \beta_g^{**}(\pi_\phi(a))] \\ &= \pi_\phi(\alpha_g(a)) - \beta_g^{**}(\pi_\phi(a)), \quad \forall a \in A \end{aligned}$$

i.e. $\pi_\phi \circ \alpha_g = \beta_g^{**} \circ \pi_\phi$, for any $g \in G$. □

The proof of the above result does not make use of the compactness of G and therefore it applies to the non-compact case as well. Furthermore, separability is also not important, as the same argument would work for the non-separable case as well, with the due modifications.

The fact that such a result holds for the equivariant case allows us to give equivariant generalizations of some of the results in [**4**].

Proposition 2.6. *Let A, B and C be G-algebras, and let $\phi, \psi : A \to B$, $\eta, \theta : B \to C$ be equivariant c.p.c. order zero maps such that $\phi \precsim_G \psi$ and $\eta \precsim_G \theta$. Then $\eta \circ \phi \precsim_G \theta \circ \phi$ and $\eta \circ \phi \precsim_G \eta \circ \psi$.*

Proof. The implication $\eta \precsim_G \theta \Rightarrow \eta \circ \phi \precsim_G \theta \circ \phi$ is trivial. For the other implication, let h_η and π_η be the positive element and support $*$-homomorphism coming from Theorem 2.5 applied to η. For a finite subset $F \Subset A$ and $\epsilon > 0$, find $b \in B^G$ such that $\|b\psi(a)b^* - \phi(a)\| < \epsilon$ for any $a \in F$. Since $h_\eta^{2/n}h_\eta$ converges to h_η in norm, there exists $n \in \mathbb{N}$ such that

$$\|h_\eta^{1/n}\eta(b\psi(a)b^*)h_\eta^{1/n} - \eta(b\psi(a)b^*)\| < \epsilon,$$

for any $a \in F$. Therefore, with the element $d := h_\eta^{1/n}\pi_\eta(b) = \eta^{1/n}(b) \in C^G$, we have

$$\begin{aligned} \|d(\eta\circ\psi)(a)d^* - (\eta\circ\phi)(a)\| &\le \|h_\eta^{1/n}\eta(b\psi(a)b^*)h_\eta^{1/n} - \eta(b\psi(a)b^*)\| \\ &\quad + \|\eta(b\psi(a)b^*) - (\eta\circ\phi)(a)\| \\ &< \epsilon + \|\eta\|\,\|b\psi(a)b^* - \phi(a)\| \\ &< 2\epsilon. \end{aligned}$$

Hence $\eta \circ \phi \precsim_G \eta \circ \psi$. □

Observe that the $C_0((0,1])_+$-functional calculus of [**16**] extends to the equivariant case. Indeed, $f(\phi) := f(h_\phi)\pi_\phi$ is an equivariant c.p. map for every equivariant c.p.c. order zero map ϕ and $f \in C_0((0,1])_+$.

Proposition 2.7. *Let A and B be C^*-algebras, and let $\phi : A \to B$ be an equivariant c.p.c. order zero map. For any pair of positive continuous functions $f, g \in C_0((0,1])_+$ such that $\operatorname{supp} f \subseteq \operatorname{supp} g$ we have that $f(\phi) \precsim_G g(\phi)$.*

Proof. Fix a finite subset F of A and an $\epsilon > 0$. For a given pair of positive continuous functions $f, g \in C_0((0,1])_+$ such that $\operatorname{supp} f \subseteq \operatorname{supp} g$, find $k \in C_0((0,1])_+$ with the property that $\|gk - f\| < \epsilon/M$, where $M := \max_{a\in F} \|a\|$, e.g. like in the proof of [2, Proposition 2.5]. Find $e \in B^G$ such that

$$\|eg(\phi)(a)e^* - (gk)(\phi)(a)\| < \epsilon$$

for any $a \in F$. We then have the estimate

$$\begin{aligned}\|eg(\phi)(a)e^* - f(\phi)(a)\| &\leq \|eg(\phi)(a)e^* - (gk)(\phi)(a)\| + \|(gk)(\phi)(a) - f(\phi)(a)\| \\ &< \epsilon + \frac{\epsilon\,\|a\|}{M} \\ &\leq 2\epsilon,\end{aligned}$$

for any $a \in F$. □

2.1. Stability

We recall that for every $*$-isomorphism $\gamma : K \otimes K \to K$ and minimal projection $e \in K$ there exists an isometry $v \in B(\ell^2(\mathbb{N}))$ such that $\operatorname{Ad}_v \circ\gamma \circ (\operatorname{id}_K \otimes e) = \operatorname{id}_K$. Observe that, inside the algebra K_G there is a minimal G-invariant projection e_G. This is given by $e \otimes e_0$, where e is any minimal projection of K and e_0 is the projection in $K(L^2(G))$ onto the constant functions of $L^2(G)$. Furthermore, the flip $a \otimes e_G \mapsto e_G \otimes a$ is implemented by a G-invariant unitary.

Proposition 2.8. *The G-algebras $K_G \otimes K_G$ and K_G are equivariantly isomorphic. Furthermore, there exists a G-invariant isometry $v \in B(L^2(G) \otimes \ell^2(\mathbb{N}))^G$ with the property that $\operatorname{Ad}_v \circ\gamma_G \circ (\operatorname{id}_{K_G} \otimes e_G) = \operatorname{id}_{K_G}$.*

Proof. To ease the notation in this proof, let $K_G^\lambda = (K \otimes K(L^2(G)), \operatorname{id}_K \otimes\lambda_G)$ and $K_G = (K \otimes K(L^2(G)), \operatorname{id}_K \otimes \operatorname{id}_{K(L^2(G))})$. By Fell's absorption principle, the G-algebras $K_G^\lambda \otimes K_G^\lambda$ and $K_G^\lambda \otimes K_G$ are equivariantly isomorphic by a map ϕ satisfying $\phi \circ (\operatorname{id}_{K_G} \otimes e_G) = \operatorname{id}_{K_G} \otimes e_G$. By stability, there is an equivariant isomorphism γ between $K_G^\lambda \otimes K_G$ and K_G^λ with the property that $\operatorname{Ad}_v \circ\gamma \circ (\operatorname{id}_{K^G} \otimes e_G) = \operatorname{id}_{K_G}$, for some G-invariant isometry $v \in \mathcal{M}(K_G)$. An equivariant isomorphism between $K_G^\lambda \otimes K_G^\lambda$ and K_G^λ is then given by $\gamma \circ \sigma$, and

$$\operatorname{Ad}_v \circ\gamma \circ \phi \circ (\operatorname{id}_{K_G} \otimes e_G) = \operatorname{Ad}_v \circ\gamma \circ (\operatorname{id}_{K_G} \otimes e_G) = \operatorname{id}_{K_G},$$

as it was to be shown. □

Lemma 2.9. *Let A, B, C and D be G-algebras, and let $\phi, \psi : A \to B$, $\eta : C \to D$ be equivariant c.p.c. order zero maps such that $\phi \precsim_G \psi$. Then $\eta \otimes \phi \precsim_G \eta \otimes \psi$ for any tensor norms on $A \otimes C$ and $B \otimes D$.*

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Proof. If $\{b_n\}_{n\in\mathbb{N}} \subset B^G$ is the sequence that witnesses $\phi \precsim_G \psi$, then $\{d_n \otimes b_n\}_{n\in\mathbb{N}}$, where $\{d_n\}_{n\in\mathbb{N}} \subset D^G$ is a G-invariant approximate unit for D, witnesses the sought equivariant Cuntz subequivalence between $\eta \otimes \phi$ and $\eta \otimes \psi$. □

Corollary 2.10. *Let A and B be G-algebras and let $\phi, \psi : A \to B$ be equivariant c.p.c. order zero maps. Then $\phi \precsim_G \psi$ in B if and only if $\phi \otimes \mathrm{id}_{K_G} \precsim_G \psi \otimes \mathrm{id}_{K_G}$ in $B \otimes K_G$.*

Proof. The implication $\phi \precsim_G \psi \Rightarrow \phi \otimes \mathrm{id}_{K_G} \precsim_G \psi \otimes \mathrm{id}_{K_G}$ follows from the previous lemma. For the other implication observe that B embeds into $B \otimes K_G$ by means of the injective map $b \overset{\iota}{\mapsto} b \otimes e_G$. If $\{b_n\}_{n\in\mathbb{N}} \subset (B \otimes K_G)^G$ is the sequence that witnesses the relation $\phi \otimes \mathrm{id}_{K_G} \precsim_G \psi \otimes \mathrm{id}_{K_G}$ then, with $x_n := (1_{\tilde{B}} \otimes e_G) b_n (1_{\tilde{B}} \otimes e_G) \in B \otimes \{e_G\}$, where $1_{\tilde{B}}$ is the unit of the minimal unitization $\tilde{B}$ of B, we have

$$\|x_n^*(\psi(a) \otimes e_G)x_n - \phi(a) \otimes e_G\| \to 0, \quad \forall a \in \ A$$

which can be pulled back to B through ι to give

$$\left\|\iota^{-1}(x_n)^*\psi(a)\iota^{-1}(x_n) - \phi(a)\right\| \to 0, \quad \forall a \in A.$$

Since $x_n \in (B \otimes \{e_G\})^G = B^G \otimes \{e_G\}$, it follows that $\iota^{-1}(x_n) \in B^G$, for any $n \in \mathbb{N}$, whence $\phi \precsim_G \psi$. □

Thanks to the above proposition and the map γ_G of Proposition 2.8, the stability* of Cu^G holds in the rather general form of the following result.

Theorem 2.11. *For any pair of G-algebras A and B, the partially ordered Abelian monoids $\mathsf{Cu}^G(A, B)$ and $\mathsf{Cu}^G(A \otimes K_G, B \otimes K_G)$ are order-isomorphic.*

Proof. Mutual inverses are given by the pair of semigroup homomorphisms

$$[\phi] \mapsto [\phi \otimes \mathrm{id}_{K_G}], \quad [\phi] \in \mathsf{Cu}^G(A, B)$$

and

$$[\Phi] \mapsto [(\mathrm{id}_B \otimes \gamma_G) \circ \Phi \circ (\mathrm{id}_{A\otimes K_G} \otimes e_G)],$$

where e_G is the minimal G-invariant projection in K_G mentioned at the beginning of this section. Indeed, by making use of Proposition 2.6 and Lemma 2.9, we have

$$\begin{aligned}
(\mathrm{id}_B \otimes \gamma_G) \circ (\phi \otimes \mathrm{id}_{K_G}) \circ (\mathrm{id}_{A\otimes K_G} \otimes e_G) &= (\mathrm{id}_B \otimes \gamma_G) \circ (\phi \otimes e_G) \\
&= (\mathrm{id}_B \otimes \gamma_G) \circ (\mathrm{id}_B \otimes \mathrm{id}_{K_G} \otimes e_G) \circ \phi \\
&\sim_G (\mathrm{id}_B \otimes \mathrm{id}_{K_G}) \circ \phi \\
&= \phi
\end{aligned}$$

* We are here assuming that the action on the tensor factors K_G is the inner one given by $\mathrm{id}_K \otimes \lambda_G$.

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and

$$\begin{aligned}((\mathrm{id}_B \otimes \gamma_G) \circ \Phi \circ (\mathrm{id}_{A\otimes K_G} \otimes e_G)) \otimes \mathrm{id}_{K_G} \\ &= (\mathrm{id}_B \otimes \gamma_G \otimes \mathrm{id}_{K_G}) \circ (\Phi \otimes \mathrm{id}_{K_G}) \circ (\mathrm{id}_{A\otimes K_G} \otimes e_G \otimes \mathrm{id}_{K_G}) \\ &\quad \sim_G (\mathrm{id}_B \otimes \gamma_G \otimes \mathrm{id}_{K_G}) \circ (\Phi \otimes \mathrm{id}_{K_G}) \circ (\mathrm{id}_{A\otimes K_G} \otimes \mathrm{id}_{K_G} \otimes e_G) \\ &= (\mathrm{id}_B \otimes \gamma_G \otimes \mathrm{id}_{K_G}) \circ (\Phi \otimes e_G) \\ &= (\mathrm{id}_B \otimes \gamma_G \otimes \mathrm{id}_{K_G}) \circ (\mathrm{id}_B \otimes \mathrm{id}_{K_G} \otimes \mathrm{id}_{K_G} \otimes e_G) \circ \Phi \\ &\quad \sim_G (\mathrm{id}_B \otimes \gamma_G \otimes \mathrm{id}_{K_G}) \circ (\mathrm{id}_B \otimes \mathrm{id}_{K_G} \otimes e_G \otimes \mathrm{id}_{K_G}) \circ \Phi \\ &\quad \sim_G (\mathrm{id}_B \otimes \mathrm{id}_{K_G} \otimes \mathrm{id}_{K_G}) \circ \Phi \\ &= \Phi,\end{aligned}$$

which become equalities at the level of equivariant Cuntz classes. □

Corollary 2.12. *Let A and B be G-algebras. For every $\Phi \in \mathsf{Cu}^G(A, B)$ there exists an equivariant c.p.c. order zero map $\phi : A \to B \otimes K_G$ such that $\Phi = [(\mathrm{id}_B \otimes \gamma_G) \circ (\phi \otimes \mathrm{id}_{K_G})]$.*

Proof. Define a semigroup $\mathsf{cu}^G(A, B)$ of Cuntz-equivalence classes of equivariant c.p.c. order zero maps from A to $B \otimes K_G$, equipped with the same binary operation of $\mathsf{Cu}^G(A, B)$ and repeat the argument of the previous proof to show that they are isomorphic. This amounts to replacing $\mathrm{id}_{A\otimes K_G}$ with id_A. □

The following example shows that Definition 2.4 gives an equivariant extension of the bivariant Cuntz semigroup defined in [4].

Example 2.13. Let G be the trivial group $\{e\}$. Then $K_G = \mathbb{C} \otimes K \cong K$ with the trivial action and therefore $\mathsf{Cu}^G(A, B) \cong \mathsf{Cu}(A, B)$.

The example that follows shows that Definition 2.4 can be regarded as an bivariant extension of the equivariant Cuntz semigroup defined in [9].

Example 2.14. Let (A, G, α) and (B, G, β) be G-algebras. Theorem 2.11 implies that, for every class $\Phi \in \mathsf{Cu}^G(A, B)$, there exists a representative of the form* $\phi \otimes \mathrm{id}_{K_G}$, where $\phi : A \to B \otimes K_G$ is an equivariant c.p.c. order zero map. When $A = \mathbb{C}$ with the trivial action of G then

$$\phi(z) = zh_\phi, \quad \forall z \in \mathbb{C},$$

where h_ϕ is a G-invariant positive element in $B \otimes K_G$ by Theorem 2.5. Hence, $\mathsf{Cu}^G(\mathbb{C}, B)$ can be identified with Cuntz-equivalence classes of G-invariant positive elements from $B \otimes K_G$, i.e.

$$\mathsf{Cu}^G(\mathbb{C}, B) \cong \mathsf{Cu}^G(B).$$

More generally, if G acts trivially on A, then any equivariant c.p.c. order zero map $\phi : A \to B \otimes K_G$ maps into the fixed point algebra $(A \otimes K_G)^G \cong (A \rtimes G) \otimes K$, whence

* Here we are tacitly dropping the map $\mathrm{id}_B \otimes \gamma_G$ of Corollary 2.12 to ease the notation.

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the natural isomorphism

$$\mathsf{Cu}^G(A,B) \cong \mathsf{Cu}(A, B \rtimes G).$$

2.2. Functoriality

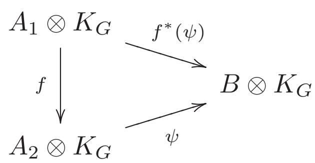
We now investigate the functoriality of the equivariant bivariant Cuntz semigroup $\mathsf{Cu}^G(\,\cdot\,,\,\cdot\,)$.

Theorem 2.15. *Let A and B be G-algebras. $\mathsf{Cu}^G(\,\cdot\,,B)$ (respectively $\mathsf{Cu}^G(A,\,\cdot\,)$) is a contravariant (respectively covariant) functor from the category of G-algebras to that of ordered Abelian monoids.*

Proof. Let A_1, A_2 be any G-algebras, and consider a homomorphism $f : A_1 \to A_2$ and an equivariant c.p.c. order zero map $\psi : A_2 \otimes K_G \to B \otimes K_G$. By defining $f^*(\psi)$ as

$$f^*(\psi) := \psi \circ (f \otimes \mathrm{id}_{K_G}),$$

the following diagram



commutes. Hence $f^*(\psi)$ is an equivariant c.p.c. order zero map and therefore f^* defines a pull-back which can be projected onto equivalence classes by setting

$$\mathsf{Cu}^G(f,B)([\psi]) = [f^*(\psi)], \quad \forall[\psi] \in \mathsf{Cu}^G(A_2, B).$$

It is easy to check that this yields a well-defined map.

Similarly for the second part, let B_1 and B_2 be any G-algebras, $g : B_1 \to B_2$ an equivariant $*$-homomorphism and $\psi : A \otimes K_G \to B_1 \otimes K_G$ a c.p.c. order zero map, and define $g_*(\psi)$ as

$$g_*(\psi) := (g \otimes \mathrm{id}_{K_G}) \circ \psi.$$

Such map is clearly equivariant c.p.c. order zero and defines a push-forward between c.p.c. order zero maps that gives rise to the well-defined semigroup homomorphism

$$\mathsf{Cu}^G(A,g)([\psi]) = [g_*(\psi)], \quad \forall[\psi] \in \mathsf{Cu}^G(A, B),$$

since $g(A^G) \subset B^G$. □

Remark 2.16. The above theorem still holds if we consider equivariant c.p.c. order zero maps instead of equivariant $*$-homomorphisms as arrows between G-algebras.

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2.3. Additivity

Let A_1, A_2 and B be G-algebras. We shall say that two equivariant c.p.c. order zero maps $\phi : A_1 \to B$ and $\psi : A_2 \to B$ are orthogonal, and we shall indicate this by $\phi \perp \psi$, if $\phi(A_1)\psi(A_2) = \{0\}$. This implies, in particular, that the positive elements $h_\phi, h_\psi \in B^{**}$ coming from Theorem 2.5 applied to ϕ and ψ respectively are orthogonal, i.e. $h_\phi h_\psi = 0$ in B^{**}.

Proposition 2.17. *Let A_1, A_2 and B be G-algebras and let $\phi_1 : A_1 \to B$ and $\phi_2 : A_2 \to B$ be c.p.c. order zero maps such that $\phi_1 \perp \phi_2$. Then $\phi_1(A_1) \cap \phi_2(A_2) = \{0\}$.*

Proof. Assume that there are $a_1 \in A_1, a_2 \in A_2$ such that $b = \phi_1(a_1) = \phi_2(a_2)$. Then

$$\|b\|^2 = \|b^* b\| = \|\phi_1(a_1)^* \phi_2(a_2)\| = \|\phi_1(a_1^*)\phi_2(a_2)\| = 0$$

by orthogonality of ϕ_1 and ϕ_2. Hence, $b = 0$. □

Let A_1 and A_2 be G-algebras. We observe that, given two equivariant c.p.c. order zero maps $\phi_1 : A_1 \to B$ and $\phi_2 : A_2 \to B$, their direct sum $\phi_1 \oplus \phi_2$ is easily seen to be an equivariant c.p.c. order zero map from $A_1 \oplus A_2$ to $M_2(B)$, where the action on $A_1 \oplus A_2$ is $\alpha_1 \oplus \alpha_2$ and that on $M_2(B) \cong M_2 \otimes B$ is $\mathrm{id}_2 \otimes \beta$. For the converse of this property we provide the following results.

Lemma 2.18. *Let A_1, A_2, B be G-algebras. A map $\phi : A_1 \oplus A_2 \to B$ is an equivariant c.p.c. order zero if and only if there are equivariant c.p.c. order zero maps $\phi_1 : A_1 \to B$ and $\phi_2 : A_2 \to B$ such that*

(i) *$\phi_1(a_1) + \phi_2(a_2) = \phi(a_1, a_2)$, for any $a_1 \in A_1$ and $a_2 \in A_2$;*

(ii) *$\phi_1 \perp \phi_2$.*

Lemma 2.19. *Let A_1, A_2 and B be G-algebras, and let $\phi_1 : A_1 \to B$ and $\phi_2 : A_2 \to B$ be equivariant c.p.c. order zero maps such that $\phi_1 \perp \phi_2$. Then*

$$\begin{bmatrix} \phi_1 + \phi_2 & 0 \\ 0 & 0 \end{bmatrix} \sim_G \begin{bmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{bmatrix}$$

in $M_2(B)$, where $\phi_1 + \phi_2$ is the equivariant c.p.c. order zero map from $A_1 \oplus A_2$ to B given by $(\phi_1 + \phi_2)(a_1, a_2) := \phi_1(a_1) + \phi_2(a_2)$.

Proof. By the previous lemma, the matrix on the left is a well-defined equivariant c.p.c. order zero map from $A_1 \oplus A_2$ to $M_2(B)$. Fix a G-invariant approximate unit $\{e_n\}_{n\in\mathbb{N}}$ for A, and introduce the G-invariant sequences of $M_2(B)$

$$x_n := \begin{bmatrix} \phi_1^{1/2}(e_n) & 0 \\ \phi_2^{1/2}(e_n) & 0 \end{bmatrix}, \quad y_n := \begin{bmatrix} \phi_1^{1/4}(e_n) & \phi_2^{1/4}(e_n) \\ 0 & 0 \end{bmatrix}.$$

We can easily see from Theorem 2.5 and the definition of functional calculus for equivariant c.p.c. order zero maps that

$$\lim_{n\to\infty} \left\| x_n \begin{bmatrix} (\phi_1 + \phi_2)(a) & 0 \\ 0 & 0 \end{bmatrix} x_n^* - \begin{bmatrix} \phi_1^2(a) & 0 \\ 0 & \phi_2^2(a) \end{bmatrix} \right\| = 0$$

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and

$$\lim_{n\to\infty}\left\|y_n\begin{bmatrix}\phi_1^{1/2}(a) & 0\\ 0 & \phi_2^{1/2}(a)\end{bmatrix}y_n^* - \begin{bmatrix}(\phi_1+\phi_2)(a) & 0\\ 0 & 0\end{bmatrix}\right\| = 0$$

for any $a \in A_1 \oplus A_2$, whence

$$\begin{bmatrix}\phi_1^2 & 0\\ 0 & \phi_2^2\end{bmatrix} \precsim_G \begin{bmatrix}\phi_1+\phi_2 & 0\\ 0 & 0\end{bmatrix} \quad\text{and}\quad \begin{bmatrix}\phi_1+\phi_2 & 0\\ 0 & 0\end{bmatrix} \precsim_G \begin{bmatrix}\phi_1^{1/2} & 0\\ 0 & \phi_2^{1/2}\end{bmatrix}.$$

By Proposition 2.7 we then have

$$\begin{bmatrix}\phi_1^2 & 0\\ 0 & \phi_2^2\end{bmatrix} \sim_G \begin{bmatrix}\phi_1^{1/2} & 0\\ 0 & \phi_2^{1/2}\end{bmatrix} \sim_G \begin{bmatrix}\phi_1 & 0\\ 0 & \phi_2\end{bmatrix},$$

which concludes the proof. □

Proposition 2.20. *For any triple of G-algebras A_1, A_2 and B, the partially ordered semigroup isomorphism*

$$\mathsf{Cu}^G(A_1 \oplus A_2, B) \cong \mathsf{Cu}^G(A_1, B) \oplus \mathsf{Cu}^G(A_2, B)$$

holds.

Proof. Let $\sigma : \mathsf{Cu}^G(A_1, B) \oplus \mathsf{Cu}^G(A_2, B) \to \mathsf{Cu}^G(A_1 \oplus A_2, B)$ be the map given by

$$\sigma([\phi_1] \oplus [\phi_2]) = [\phi_1 \oplus \phi_2].$$

By the above two lemmas it is clear that this map is surjective. To prove injectivity and the order-isomorphism we show that $\phi_1 \oplus \phi_2 \precsim_G \psi_1 \oplus \psi_2$ implies $\phi_k \precsim_G \psi_k$, $k = 1, 2$. By assumption, there exists a sequence $\{b_n\}_{n\in\mathbb{N}} \subset (B \otimes K_G)^G$ such that

$$b_n^*(\psi_1(a_1) \oplus \psi_2(a_2))b_n \to \phi_1(a_1) \oplus \phi_2(a_2)$$

in norm for every $a_1 \in A_1, a_2 \in A_2$. As $M_2(B \otimes K_G) \cong B \otimes K_G$ equivariantly, there are $b_{n,ij} \in (B \otimes K_G)^G$, $i, j = 1, 2$ such that the sequence b_n has the structure

$$b_n = \sum_{i,j=1}^{2} b_{n,ij} \otimes e_{ij},$$

with $\{e_{ij}\}_{i,j=1,2}$ denoting the standard basis of matrix units for M_2. Thus, for $a_2 = 0$, we find that

$$\lim_{n\to\infty}\left\|b_{n,11}^*\psi_1(a_1)b_{n,11} - \phi_1(a_1)\right\| = 0$$

for any $a_1 \in A_1$, i.e. $\phi_1 \precsim_G \psi_1$. A similar argument with $a_1 = 0$ leads to the conclusion that $\phi_2 \precsim_G \psi_2$ as well. To check that σ preserves the semigroup operations it suffices to

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show that

$$(\phi_1 \mathbin{\hat{\oplus}} \phi_2) \oplus (\psi_1 \mathbin{\hat{\oplus}} \psi_2) \sim_G (\phi_1 \oplus \psi_1) \mathbin{\hat{\oplus}} (\phi_2 \oplus \psi_2).$$

A direct computation reveals that such equivalence is witnessed by the sequence $\{b_n\}_{n\in\mathbb{N}} \subset M_4((B \otimes K_G)^G)$ given by

$$b_n := u_n \otimes (e_{11} + e_{44} + e_{23} + e_{32}),$$

where $\{u_n\}_{n\in\mathbb{N}} \subset (B \otimes K_G)^G$ is an approximate unit for $B \otimes K_G$. □

As in the case of the bivariant Cuntz semigroup and of KK-theory, $\mathsf{Cu}^G(\ \cdot\ ,\ \cdot\ )$ is countably additive in the first argument.

Lemma 2.21. *Let A and B be G-algebras, $\phi : A \to B$ a countable sum of pair-wise orthogonal equivariant c.p.c. order zero maps, that is*

$$\lim_{n\to\infty} \left\| \phi(a) - \sum_{k=1}^{n} \phi_k(a) \right\| = 0, \quad \forall a \in A,$$

where each ϕ_k is an equivariant c.p.c. order zero map and $\phi_k \perp \phi_i$ for any $i \neq k$ in $\mathbb{N}$, and $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ a sequence that sums up to 1. Then

$$\phi \sim_G \sum_{k=1}^{\infty} \lambda_k \phi_k.$$

Proof. Fix a finite subset F of A and $\epsilon > 0$. Find an $n \in \mathbb{N}$ such that

$$\left\| \sum_{k=n+1}^{\infty} \phi_k(a) < \epsilon \right\|$$

for any $a \in F$. Define the C*-subalgebras $B_k := C^*(\phi_k(A)) \subset B$ for any $k \in \mathbb{N}$. By Lemma 2.3, find $e_k \in B_k^G$ such that $\|e_k \phi_k(a) e_k^* - \phi_k(a)\| < \epsilon/n$ for any $a \in F$ and $k = 1, \ldots, n$. Observe that the orthogonality of the maps ϕ_k implies that $e_i \perp e_k$ for any $i \neq k$. With the element $x \in B^G$ defined as $x := \sum_{k=1}^{n} e_k/\sqrt{\lambda_k}$, we have the estimate

$$\begin{aligned}\left\| x\left(\sum_{k=1}^{\infty} \lambda_k \phi_k(a)\right)x^* - \phi(a)\right\| &\leq \sum_{k=1}^{n} \|e_k \phi_k(a) e_k^* - \phi_k(a)\| + \left\| \sum_{k=n+1}^{\infty} \phi_k(a)\right\| \\ &< \sum_{k=1}^{n} \frac{\epsilon}{n} + \epsilon \\ &\leq 2\epsilon\end{aligned}$$

for any $a \in F$. Hence, $\phi \precsim_G \sum_{k=1}^{\infty} \lambda_k \phi_k$. For the converse subequivalence, find, if necessary, a new n for which

$$\left\| \sum_{k=n+1}^{\infty} \lambda_k \phi_k(a) \right\| < \epsilon$$

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for any $a \in F$, and new elements $e_k \in B_k^G$, $k = 1, \ldots, n$ such that $\|e_k\phi_k(a)e_k^* - \phi_k(a)\| < \epsilon/n$. With the element $y \in B^G$ defined as $y := \sum_{k=1}^n \sqrt{\lambda_k} e_k$, we have the estimate

$$\left\| y\phi(a)y^* - \sum_{k=1}^{\infty} a_k\phi_k(a) \right\| \leq \sum_{k=1}^{n} \|e_k\phi_k(a)e_k^* - \phi_k(a)\| + \left\| \sum_{k=n+1}^{\infty} a_k\phi_k(a) \right\|$$
$$< \sum_{k=1}^{n} \frac{\epsilon}{n} + \epsilon$$
$$\leq 2\epsilon,$$

for any $a \in F$, which implies that $\sum_{k=1}^{\infty} \lambda_k\phi_k \precsim \phi$. □

Lemma 2.22. *Let $\{A_n\}_{n\in\mathbb{N}} \cup \{B\}$ be a countable family of G-algebras. Then any equivariant c.p.c. order zero map $\phi : A := \bigoplus_{k=1}^{\infty} A_k \to B$ satisfies*

$$\phi \otimes e \sim \psi := \bigoplus_{k=1}^{\infty} \frac{\phi|_{A_k}}{2^k}$$

in $B \otimes K$, where $e \in K$ is a minimal projection and $\phi|_{A_k}$ is defined as

$$\phi|_{A_k}(a_k) := \phi(0, \ldots, 0, a_k, 0, \ldots), \quad \forall k \in \mathbb{N}, a_k \in A_k.$$

Proof. Assume, without loss of generality, that $e = e_{11}$. Fix a G-invariant approximate unit $\{e_n\}_{n\in\mathbb{N}}$ of A, set

$$\xi_{k,n} := \frac{\phi^{1/2}|_{A_k}(e_n)}{2^{k/2}}, \quad \eta_{k,n} := \frac{\phi^{1/4}|_{A_k}(e_n)}{2^{k/4}}$$

for any $k, n \in \mathbb{N}$, and define the G-invariant sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \in (B \otimes K)^G$ by

$$x_n := \sum_{k=1}^{\infty} \xi_{k,n} \otimes e_{k1} = \begin{bmatrix} \xi_{1,n} & 0 & \cdots \\ \xi_{2,n} & 0 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}, \quad y_n := \sum_{k=1}^{\infty} \eta_{k,n} \otimes e_{1k} = \begin{bmatrix} \eta_{1,n} & \eta_{2,n} & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

By Theorem 2.5 we reach the conclusion that

$$\lim_{n\to\infty} \left\| x_n(\phi \otimes e)(a)x_n^* - \psi^2(a) \right\| = 0$$

and

$$\lim_{n\to\infty} \left\| y_n\psi^{1/2}(a)y_n^* - \left(\sum_{k=1}^{\infty} \frac{\phi|_{A_k}}{2^k} \otimes e \right)(a) \right\| = 0$$

for any $a \in A$. Since $\psi \sim_G \psi^2 \sim_G \psi^{1/2}$ by Proposition 2.7 and $\phi \sim_G \sum_{k=1}^{\infty} \phi|_{A_k}/2^k$ by the previous lemma, the result now follows. □

Proposition 2.23. *Let $\{A_n\}_{n\in\mathbb{N}} \cup \{B\}$ be a countable family of G-algebras. Then the semigroups $\prod_{n\in\mathbb{N}} \mathsf{Cu}^G(A_n, B)$ and $\mathsf{Cu}^G\left(\bigoplus_{n\in\mathbb{N}} A_n, B\right)$ are order-isomorphic.*

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Proof. Let $\sigma : \prod_{n\in\mathbb{N}} \mathsf{Cu}^G(A_n, B) \to \mathsf{Cu}^G\left(\bigoplus_{n\in\mathbb{N}} A_n, B\right)$ be the semigroup homomorphism defined by

$$\sigma\left([\phi_1], [\phi_2], \ldots\right) := \left[(\mathrm{id}_B \otimes \gamma) \circ \bigoplus_{n\in\mathbb{N}} \frac{1}{2^n}\phi_n\right],$$

where $\gamma : K_G \otimes K \to K_G$ is any equivariant $*$-isomorphism. For any minimal projection $e \in K$, $\gamma \circ (\mathrm{id}_{K_G} \otimes e)$ is conjugate to id_{K_G} by a G-invariant isometry $w \in (B(L^2(G)), \lambda_G)$ and therefore* $\gamma \circ (\mathrm{id}_G \otimes e) \sim_G \mathrm{id}_{K_G}$. The inverse of σ is provided by the semigroup homomorphism $\rho : \mathsf{Cu}^G\left(\bigoplus_{n\in\mathbb{N}} A_n, B\right) \to \prod_{n\in\mathbb{N}} \mathsf{Cu}^G(A_n, B)$ given by

$$\rho([\phi]) := ([\phi|_{A_1}], [\phi|_{A_2}], \ldots),$$

where $\phi|_{A_k}(a_k) := \phi(a_k \otimes e_{kk})$ for any $k \in \mathbb{N}$ and $a_k \in A_k$. Indeed, by the previous lemma

$$(\mathrm{id}_B \otimes \gamma) \circ \bigoplus_{n\in\mathbb{N}} \frac{\phi|_{A_n}}{2^n} \sim_G (\mathrm{id}_B \otimes \gamma) \circ (\phi \otimes e) \sim_G \phi$$

and

$$\gamma_G \circ \left(\frac{\phi_k}{2^k} \otimes e_{kk}\right) \sim_G \gamma_G \circ (\phi_k \otimes e) \sim_G \phi_k, \quad \forall k \in \mathbb{N}$$

since every minimal projection e_{kk} is Cuntz-equivalent to e, and $\lambda\phi_k \sim_G \phi_k$ for any $\lambda \in (0, 1)$. □

Proposition 2.24. *For any triple of G-algebras A, B_1 and B_2, the partially ordered semigroup isomorphism*

$$\mathsf{Cu}^G(A, B_1 \oplus B_2) \cong \mathsf{Cu}^G(A, B_1) \oplus \mathsf{Cu}^G(A, B_2)$$

holds.

2.4. Relation with crossed products

In KK-theory there is a group homomorphism between the equivariant KK-group and the KK-group of the crossed product, [**3**, § 2.6]. We now provide an analogue of this result within the framework of the equivariant extension of the bivariant Cuntz semigroup. First, we record some intermediate results.

Proposition 2.25. *Let A and B be G-algebras. Every equivariant c.p.c. order zero map $\phi : A \to B$ induces a c.p.c. order zero map $\phi_\rtimes : A \rtimes G \to B \rtimes G$ between the crossed products.*

For a proof of the above result we refer the reader to [**7**, Proposition 2.3].

Proposition 2.26. *Let A and B be G-algebras and let $\phi, \psi : A \to B$ be equivariant c.p.c. order zero maps such that $\phi \precsim_G \psi$. Then $\phi_\rtimes \precsim \psi_\rtimes$.*

* Observe that the element w in the multiplier algebra of K_G yields a sequence in K_G by multiplication with a G-invariant approximate unit from K_G, which ultimately witnesses Cuntz-subequivalence.

Proof. Let $\{f_n\}_{n\in\mathbb{N}} \subset L^1(G)$ be an approximate unit for $L^1(G)$. If $\{b_n\}_{n\in\mathbb{N}} \subset B^G$ is the sequence witnessing the subequivalence $\phi \precsim_G \psi$, then a direct computation shows that the sequence $\{d_n\}_{n\in\mathbb{N}} \subset L^1(G, B)$ given by* $d_n := b_n \otimes f_n$ satisfies

$$\lim_{n\to\infty} \|d_n \psi_{\rtimes}(a \otimes f) {d_n}^* - \phi_{\rtimes}(a \otimes f)\| = 0, \quad \forall a \otimes f \in L^1(G, A),$$

whence $\phi_{\rtimes} \precsim \psi_{\rtimes}$. □

This last result shows that the assignment $\phi \mapsto \phi_{\rtimes}$ becomes well defined when considered at the level of classes. Furthermore, we can easily check that $(\mathrm{id}_A)_{\rtimes} = \mathrm{id}_{A\rtimes G}$ for any G-algebra (A, G, α). Therefore, we reach the following conclusion.

Theorem 2.27. *Let A and B be G-algebras. There is a natural semigroup homomorphism*

$$j^G : \mathsf{Cu}^G(A, B) \to \mathsf{Cu}(A \rtimes G, B \rtimes G)$$

which is functorial in A and B and compatible with the composition product.

Proof. The sought map j^G is defined as $j^G([\phi]) := [\phi_{\rtimes}]$, which is well defined as a consequence of the above proposition. □

2.5. Equivariant Cuntz homology

A notion of *Cuntz homology* for compact Hausdorff spaces has been introduced in [**4**]. Its definition follows the way K-homology is obtained from KK-theory, namely by fixing the second argument to be the algebra of complex numbers $\mathbb{C}$. More generally, we can see that $\mathsf{Cu}(A, \mathbb{C})$ encodes information relative to the finite-dimensional representation theory of the C*-algebra A in the first argument. However, this topic will be touched upon in detail elsewhere (see [**15**]).

We now proceed to define an equivariant version of Cuntz homology analogously to the non-equivariant case of [**4**], and provide a concrete realization for compact group actions on compact Hausdorff spaces.

Definition 2.28. A *topological dynamical system* is a triple (X, G, α) consisting of a topological space X, a topological group and a continuous G-action α of G on X.

When the specification of the action is not necessary, we shall refer to the topological space X to denote the topological dynamical system (X, G, α). A topological dynamical system (X, G, α) is compact if its underlying topological space X and the group G are compact.

Definition 2.29. Let (X, G, α) be a compact topological dynamical system. The equivariant Cuntz homology of (X, G, α) is the partially ordered Abelian monoid

$$\mathsf{Cu}^G(X) := \mathsf{Cu}^G(C(X), \mathbb{C}).$$

* For $f \in L^1(G)$ and $b \in B$, the tensor product $b \otimes f$, sometimes also denoted simply by bf, can be identified with the B-valued function of class L^1 on G with respect to the Haar measure given by $g \mapsto f(g)b$ almost everywhere.

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We refer the reader to [4, § 5.3] for the terminology related to Cuntz homology, in particular to the notion of the *spectrum* of a c.p.c. order zero map, and its decomposition into the essential and isolated parts.

Theorem 2.30. *For any compact topological dynamical system* (X, G, α) *there is a natural monoid isomorphism*

$$\mathsf{Cu}^G(X) \cong \mathsf{Cu}(X/G).$$

Proof. Since $K(L^2(G)) \otimes K \cong K$, every equivariant representation $\pi : C(X) \to K_G$ is easily seen to decompose, up to equivariant unitary equivalence, into one of the form

$$\pi(f) = \sum_{k=1}^{\infty} M_f^{x_k} \otimes e_{kk}, \quad \forall f \in C(X),$$

where $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $M_f^{x_k} \in L^\infty(G) \subset B(L^2(G))$ is the multiplication operator associated with the function

$$g \mapsto f(gx_k), \quad \forall g \in G, k \in \mathbb{N}.$$

Hence, for every x_k, its full orbit Gx_k appears in this decomposition, and the multiplicity of each of the points in Gx_k is evidently constant. Therefore, the multiplicity functions can be assumed to be defined on the orbit space X/G, whence the presentation of Cuntz homology given in [4] applies. □

Corollary 2.31. *Let* G *be a compact group and let* (X, G, α) *and* (Y, G, β) *be topological dynamical systems. The equivariant Cuntz homologies* $\mathsf{Cu}^G(X)$ *and* $\mathsf{Cu}^G(Y)$ *are isomorphic as partially ordered Abelian monoid if and only if the orbit spaces* X/G *and* Y/G *are homeomorphic.*

Proof. By results in [4] for the non-equivariant Cuntz homology theory, we have $\mathsf{Cu}^G(X) \cong \mathsf{Cu}(X/G) \cong \mathsf{Cu}(Y/G) \cong \mathsf{Cu}^G(Y)$. □

3. The equivariant Cuntz semigroup

Analogously to the ordinary Cuntz semigroup $\mathsf{Cu}(A)$ of a C*-algebra A, which can be obtained from the bivariant Cuntz semigroup as $\mathsf{Cu}(A) \cong \mathsf{Cu}(\mathbb{C}, A)$, in Example 2.14 we have defined the *equivariant* Cuntz semigroup of the G-algebra (A, G, α) as

$$\mathsf{Cu}^G(A) := \mathsf{Cu}^G(\mathbb{C}, A).$$

We have also shown that this object has a natural identification with the set of Cuntz-equivalence classes of G-invariant positive elements from $A \otimes K_G$, and therefore it turns out to coincide with the equivariant Cuntz semigroup defined in [9].

In this section we propose an open projection picture for the equivariant theory of the Cuntz semigroup. To do so, we shall generalize many of the results of [11] to the equivariant setting first. Then the sought open projection picture will follow naturally. For concreteness, we now give the explicit definition of the equivariant Cuntz semigroup

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that we will employ throughout this section. We also recall that we are still under the assumption that every group we consider is compact and second countable.

Definition 3.1 (equivariant Cuntz semigroup). Let (A, G, α) be a G-algebra. Its equivariant Cuntz semigroup is the set of classes

$$\mathsf{Cu}^G(A) := (A \otimes K_G)_+^G / \sim_G,$$

where Cuntz comparison is now witnessed by G-invariant sequences; that is, if B is a G-algebra and $a, b \in B_+^G$, then

$$a \precsim_G b \quad \text{if} \quad \exists \{x_n\}_{n\in\mathbb{N}} \subset B^G \mid \|x_n b x_n^* - a\| \to 0,$$

where B^G denotes the fixed point algebra of B with respect to the action of G. The binary operation is still derived from the direct sum of positive elements, that is

$$[a] + [b] := [a \oplus b],$$

for any $[a], [b] \in \mathsf{Cu}^G(A)$.

The approach of [**9**] is different, closer in spirit to the original definition of equivariant K-theory (cf. [**9**, Definition 2.4]). Finite-dimensional representations of G are replaced by separable ones, i.e. those representations μ of G over a separable Hilbert space H_μ, and Cuntz classes of G-invariant positive elements from the C*-algebras $K(H_\mu \otimes A)$ are now considered. Cuntz comparison is then implemented by G-invariant elements from $K(H_\mu \otimes A, H_\nu \otimes A)$, where ν is any other separable representation of G (cf. [**9**, Definition 2.6]).

As with the ordinary Murray–von Neumann and Cuntz semigroups, there are similar connections between the equivariant versions of these objects. Let (A, G, α) be a G-algebra and $p \in (A \otimes K_G)^G$ a projection. The map that sends the class of p in $V^G(A)$ to the class of p in $\mathsf{Cu}^G(A)$ is a well-defined semigroup homomorphism, as a consequence of the following result, which generalizes [**2**, Lemma 2.18] to the equivariant setting. Here, $\preceq_G$ denotes the equivariant Murray–von Neumann subequivalence relation between projections.

Lemma 3.2. *Let (A, G, α) be a G-algebra and let $p, q \in A^G$ be G-invariant projections. Then $p \preceq_G q$ if and only if $p \precsim_G q$.*

Proof. Thanks to [**9**, Proposition 2.5], the same proof of [**2**, Lemma 2.18] applies almost verbatim by taking all the elements to be G-invariant. □

The above result does not imply that Murray–von Neumann equivalence is equivalent to Cuntz equivalence on projections. However, as in the non-equivariant theory, there are special cases where the equivariant Murray–von Neumann semigroup embeds into the equivariant Cuntz semigroup. A stably finite G-algebra (A, G, α) is a G-algebra whose underlying C*-algebra A is stably finite. The following result is an equivariant generalization of [**2**, Lemma 2.20].

Lemma 3.3. *Let (A, G, α) be a stably finite G-algebra. Then the natural map $V^G(A) \to \mathsf{Cu}^G(A)$ is injective.*

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Proof. Since the algebras are stably finite and G is compact, the fixed point algebras and the crossed products are also stably finite. Hence, the same proof of [2, Lemma 2.20] applies almost verbatim by taking all the elements to be G-invariant. □

The *completed* representation semiring $\mathsf{Cu}(G)$, or simply the representation semiring, as defined in [9, Definition 3.1], is the semiring arising by considering separable representations G rather than just the finite-dimensional ones. We choose to include the word *complete* here because $\mathsf{Cu}(G)$ can be regarded as a sup-completion of the semiring $V^G(\mathbb{C})$. However, we sometimes refrain from specifying this explicitly. As in the case of K-theory, where $R(G) \cong K_0^G(\mathbb{C})$, it turns out that there is an ordered semigroup isomorphism between $\mathsf{Cu}(G)$ and $\mathsf{Cu}^G(\mathbb{C})$ [9, Theorem 3.4], which is then an object in the category Cu.

Let (A, G, α) be a G-algebra. Definition 3.10 and Theorem 3.11 of [9] show that the equivariant Cuntz semigroup $\mathsf{Cu}^G(A)$ has a natural $\mathsf{Cu}(G)$-semimodule structure and, as such, $\mathsf{Cu}^G(A)$ belongs to a subcategory of Cu, denoted Cu^G [9, Definition 3.7]. As we are not particularly interested in this category, we refer the reader to the already cited work of Gardella and Santiago for more details. Here we limit ourselves to observing that, thanks to [9, Theorem 3.11], by equipping every equivariant Cuntz semigroup with this $\mathsf{Cu}(G)$-semimodule structure, Cu^G becomes a functor from the category of G-algebras to the category Cu^G.

3.1. The open projection picture

A module picture for the equivariant Cuntz semigroup is introduced in § 4 of [9]. We now introduce an open projection picture for the equivariant Cuntz semigroup as defined in this section.

In [1], Akemann has given a generalization of the notion of open subsets to non-commutative C*-algebras by naturally replacing sets with projections.

Definition 3.4. Let A be any C*-algebra. A projection $p \in A^{**}$ is *open* if it is the strong limit of an increasing net of positive elements $\{a_i\}_{i\in I} \subseteq A_+$.

Equivalently [1], a projection $p \in A^{**}$ is open if it belongs to the strong closure of the hereditary subalgebra $A_p \subseteq A$, where

$$A_p := pA^{**}p \cap A = pAp \cap A.$$

Observe that, for any positive element $a \in A_+$ that has $p \in A^{**}$ as support projection, we have $A_p = A_a$, where A_a is the hereditary C*-subalgebra of A generated by a, that is, $A_a := \overline{aAa}$.

Throughout, the set of all the open projections of A in A^{**} will be denoted $P_o(A^{**})$.

A projection $p \in A^{**}$ is said to be *closed* if its complement $1 - p \in A^{**}$ is an open projection. The supremum of an arbitrary set $P \subset P_o(A^{**})$ of open projections in A^{**} is still an open projection and, likewise, the infimum of an arbitrary family of closed projections is still a closed projection, by the results in [1]. Therefore, the closure of an

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open projection $p \in A^{**}$ can be defined as

$$\overline{p} := \inf\{q^*q = q \in A^{**} \mid 1 - q \in P_o(A^{**}) \ \wedge \ p \leq q\}.$$

Let B be a C*-subalgebra of A. A closed projection $p \in A^{**}$ is said to be *compact* in B if there exists a positive contraction $a \in B_+$ such that $pa = p$.

For a positive contraction a of a C*-algebra A, its *support projection* p_a is the open projection in A^{**} given by

$$p_a := \text{SOT} \lim_{n\to\infty} a^{1/n}.$$

Definition 3.5. Let A be a G-algebra. A G-invariant open projection is an open projection in $(A^G)^{**}$.

The above definition entails that every G-invariant open projection is the strong limit of an increasing sequence of positive elements from the fixed point algebra.

Lemma 3.6. *If (E, ρ) is an equivariant Hilbert A-module of the form $\overline{aA}$ for some $a \in A_+$, then there exists $\bar{a} \in A^G$ such that $E \cong_G \overline{\bar{a}A}$.*

Proof. Clearly $a \in E$. Since the map $g \mapsto \rho_g(a)$ is continuous, for every $\epsilon > 0$ there exists a neighbourhood N of the identity e of the group G such that $\|\rho_g(a) - a\| < \epsilon$, for any $g \in N$. Hence,

$$\begin{aligned}\int_G \rho_g(a) \mathrm{d}\mu(g) &\geq \int_N \rho_g(a)\, \mathrm{d}\mu(g) \\ &\geq \int_N (a - \epsilon)_+ \, \mathrm{d}\mu(g) \\ &= \mu(N)(a - \epsilon)_+,\end{aligned}$$

with $\mu(N) > 0$ by the regularity of the Haar measure μ on G. By setting

$$\bar{a} := \int_G \rho_g(a)\, \mathrm{d}\mu(g)$$

we have $\bar{a} \in A_+$ and $p_{\bar{a}} \geq p_{(a-\epsilon)_+}$ for any $\epsilon > 0$, so that $E \cong \overline{\bar{a}A}$, and $\rho_g(\bar{a}) = \bar{a}$ for any $g \in G$. For the inner product, we have

$$\begin{aligned}(\rho_g(\bar{a}b), \rho_g(\bar{a}c)) &= \rho_g(\bar{a}b)^* \rho_g(\bar{a}c) \\ &= \alpha_g(b)^* \bar{a}^2 \alpha_g(c) \\ &= \alpha_g(b^* \bar{a}^2 c), \quad \forall g \in G\end{aligned}$$

and by taking approximate units for b and c, we then find $\bar{a}^2 = \alpha_g(\bar{a}^2)$ for any $g \in G$, whence $\bar{a} \in A^G$. □

Let $a \in A^G$ be a G-invariant positive element and, like in the non-equivariant case, use E_a to denote the equivariant Hilbert A-module generated by $(\overline{aA}, \rho)$, where the strongly

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continuous action ρ is given by $\rho_g(ab) := a\alpha_g(b)$ for any $g \in G$. We give the following equivariant version of Blackadar equivalence.

Definition 3.7. Let A be a G-algebra. Two positive elements $a, b \in A^G$ are said to be *equivariantly Blackadar equivalent*, in symbols $a \sim_{G,s} b$, if there exists $x \in A^G$ such that $A_a = A_{x^*x}$ and $A_b = A_{xx^*}$.

We give the following equivariant version of Peligrad–Zsidó (PZ) equivalence.

Definition 3.8. Let A be a G-algebra. Two G-invariant open projections $p, q \in (A^G)^{**}$ are said to be *equivariantly PZ equivalent*, in symbols $p \sim_{G,\mathrm{PZ}} q$, if there exists a partial isometry $v \in (A^G)^{**}$ such that

$$p = v^*v, \quad q = vv^*,$$

and

$$v(A^G)_p \subset A^G, \quad v^*(A^G)_q \subset A^G.$$

A direct application of the Kaplansky density theorem and the dominated convergence theorem shows that we might as well use the notation A_p^G to denote either $(A^G)_p$ or $(A_p)^G$, since both these hereditary subalgebras coincide.

Proposition 3.9. *Let A be a G-algebra and let p be a G-invariant open projection. Then $(A^G)_p = (A_p)^G$.*

The result that follows can be regarded as an equivariant extension of Proposition 4.3 of [**11**].

Proposition 3.10. *Let A be a G-algebra and let a and b be G-invariant positive elements of A. The following are equivalent:*

(i) $a \sim_{G,s} b$

(ii) *E_a and E_b are equivariantly isomorphic*

(iii) *there exists $x \in A^G$ such that $E_a = E_{x^*x}$ and $E_b = E_{xx^*}$*

(iv) $p_a \sim_{G,PZ} p_b$.

Proof. (i) $\Rightarrow$ (iv). As a direct consequence of [**12**, Theorem 1.4], we have $p_{x^*x} \sim_{G,PZ} p_{xx^*}$, since this is true for $p_{x^*x} \sim_{PZ} p_{xx^*}$ in A^G. Furthermore, $A_a = A_b$, with $a, b \in A^G$, implies that $p_a = p_b$, with p_a and p_b in $(A^G)^{**}$.

(iv) $\Rightarrow$ (i). By the arguments of [**11**, Proposition 4.3], we can see that, if v denotes the partial isometry that witnesses the PZ equivalence of p_a and p_b, then $vav^* \in A^G$ has the same support projection of b, i.e. p_b, in A^G.

(ii) $\Rightarrow$ (iii). Let u be the map that implements the equivariant isomorphism and set $x := ua$. Then $E_{xx^*} = \overline{xA} = \overline{uaA} = E_b$ and

$$\sigma_g(x) = (\sigma_g \circ u \circ \rho_g^{-1} \circ \rho_g)(a) = u\rho_g(a) = ua = x, \quad \forall g \in G,$$

therefore $x \in A^G$. Furthermore, $x^*x = a^2$ since u is isometric, so that $E_a = E_{x^*x}$.

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(iii) $\Rightarrow$ (ii). Let $x = v|x|$ be the polar decomposition of x, with $v \in (A^G)^{**}$. Then

$$\begin{aligned} v\rho_g(|x|b) &= v|x|\alpha_g(b) \\ &= |x^*|v\alpha_g(b) \\ &= \sigma_g(|x^*|v)\alpha_g(xb) \\ &= \sigma_g(|x^*|vb) \\ &= \sigma_g(v|x|b) \end{aligned}$$

for any $b \in A$, whence $v \in B(E_{x^*x}, E_{xx^*})^G$ is the sought equivariant isomorphism.

(i) $\Leftrightarrow$ (iii). This is a restatement of the definitions involved and based on the one-to-one correspondence between hereditary subalgebras and right ideals. □

The following is an equivariant version of the compact containment relation for open projections.

Definition 3.11. Let A be a G-algebra, and let $p, q \in (A^G)^{**}$ be G-invariant open projections. We say that q is compactly contained in p (in symbols $q \Subset_G p$) if there exists $e \in A_p^G$ such that $\bar{q}e = \bar{q}$, where $\bar{q}$ denotes the closure of q.

With both Definitions 3.8 and 3.11, we can define the Cuntz comparison of two G-invariant open projections in the usual way of [**5**] and [**11**].

Definition 3.12. Let (A, G, α) be a G-algebra and let p, q be G-invariant open projections from $(A^G)^{**}$. We shall say that p is equivariantly Cuntz-subequivalent to q ($p \precsim_G q$ in symbols) if

$$\forall p' \Subset_G p \quad \exists q' \Subset_G q \quad | \quad p' \sim_{G,\mathrm{PZ}} q'.$$

Hence, two G-invariant open projections p and q are said to be Cuntz-equivalent if both $p \precsim_G q$ and $q \precsim_G p$ hold.

The proposition below can be regarded as an equivariant extension of part of the results established in [**11**, Proposition 4.10].

Proposition 3.13. *Let A be a G-algebra and let a, b be G-invariant positive elements. Then $E_a \Subset_G E_b$ if and only if $p_a \Subset_G p_b$.*

Proof. Identify $K(E_b)$ with A_b and observe that the rank-1 operator $\theta_{bd,bc}$ is sent to the element bdc^*b for any $c, d \in A$. Hence, the action Ad_{ρ_g} on $K(E_b)$ coincides with the action of α_g on A_b. Therefore, if $e \in K(E_b)^G$ is such that $e|_{E_a} = \mathrm{id}_{E_a}$, then $e \in A_b^G$ satisfies $\overline{p_a}e = \overline{p_a}$. □

Theorem 3.14. *Let G be a second countable compact group. Then $\mathsf{Cu}^G(A) \cong P_o(((A \otimes K_G)^G)^{**})/\sim_G$.*

Proof. By Proposition 3.10, equivariant isomorphism of modules coincides with equivariant PZ equivalence of the corresponding G-invariant open projections, while by Proposition 3.13 compact containment of equivariant modules corresponds to compact

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containment of G-invariant open projections. Hence, it is enough to show that there exists a bijection between $E_a^{\Subset_G} := \{X \mid X \Subset_G E_a\}$ and $p_a^{\Subset_G} := \{p \mid p \Subset_G p_a\}$ for any positive element $a \in (A \otimes K_G)^G$. To this end, suppose that $X \Subset_G E_a$. Since $A \otimes K_G$ is a stable C*-algebra, there exists $a' \in (A \otimes K_G)_+$ such that $X = \overline{a'(A \otimes K_G)}$, and by Lemma 3.6, we can assume that a' is G-invariant. By Proposition 3.13, $E_{a'} \Subset_G E_a$ is equivalent to $p_{a'} \Subset_G p_a$, so that we can associate the G-invariant projection $p_{a'}$ to the equivariant module X. To see that this correspondence is well-defined and independent from the choice of a', observe that, if $a'' \in A \otimes K_G$ is another G-invariant positive element such that $X = \overline{a''(A \otimes K_G)}$, then the hereditary subalgebra generated by a'' is the same as that generated by a', and therefore $p_{a''} = p_{a'}$. Conversely, for every $p \Subset_G p_a$, there exists $a' \in (A \otimes K_G)^G$ such that $p = p_{a'}$, and by Proposition 3.13, again this implies that $E_{a'} \Subset_G E_a$. Any other choice of a positive element that gives the same open projection leads to the same hereditary subalgebra and hence to the same module, whence it follows that the correspondence $p \mapsto E_{a'}$ is well-defined and independent from the choice of a'. It is now easy to verify that this correspondence is the inverse of the one above, and therefore it provides a bijection between $p_a^{\Subset_G}$ and $E_a^{\Subset_G}$. □

4. Classification of actions

In this section, we show how to use the equivariant extension of the bivariant Cuntz semigroup, defined in this paper, to establish classification results for actions by compact groups on C*-algebras. In particular, we show how to rephrase the classical result of Handelman and Rossmann [**10**], and the more recent one of Gardella and Santiago [**9**] in the language of the new theory proposed in this paper.

Definition 4.1. Let A and B be G-algebras. An element $\Phi \in \mathsf{Cu}^G(A, B)$ is said to be *strictly invertible* if there exist equivariant c.p.c. order zero maps $\phi : A \to B$ and $\psi : B \to A$ such that

(i) $[\phi \otimes \mathrm{id}_{K_G}] = \Phi$;

(ii) $\psi \circ \phi \sim_G \mathrm{id}_A$ and $\phi \circ \psi \sim_G \mathrm{id}_B$.

As in the case of the non-equivariant theory of the bivariant Cuntz semigroup of [**4**], we can regard strictly invertible elements as invertible elements of $\mathsf{Cu}^G(A, B)$, with the obvious meaning of invertibility, that come from the *scale* of $\mathsf{Cu}^G(A, B)$, where the latter is defined as follows.

Definition 4.2. Let A and B be G-algebras. The scale of $\mathsf{Cu}^G(A, B)$ is the set of classes

$$\Sigma(\mathsf{Cu}^G(A, B)) := \{[\phi \otimes \mathrm{id}_{K_G}] \in \mathsf{Cu}^G(A, B) \mid \phi : A \to B \text{ equiv. c.p.c. order zero}\}.$$

An isomorphism criterion for crossed products follows from the following result about the map j_G of § 2.4.

Proposition 4.3. *Let A and B be G-algebras. If $\Phi \in \mathsf{Cu}^G(A, B)$ is a strictly invertible element then so is $j_G(\Phi) \in \mathsf{Cu}(A \rtimes G, B \rtimes G)$, in the sense of [**4**].*

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Proof. Observe that $(\phi \otimes \mathrm{id}_{K_G})_{\rtimes}$ can be identified with $\phi_{\rtimes} \otimes \mathrm{id}_K$ for any equivariant c.p.c. order zero map $\phi : A \to B$. Since Φ is strictly invertible in $\mathsf{Cu}^G(A, B)$, there are equivariant c.p.c. order zero maps $\phi : A \to B$ and $\psi : B \to A$ such that $[\phi \otimes \mathrm{id}_{K_G}] = \Phi$ and $\psi \circ \psi \sim_G \mathrm{id}_A$, $\phi \circ \psi \sim_G \mathrm{id}_B$. From Theorem 2.27 it follows that

$$j_G(\Phi) = [(\phi \otimes \mathrm{id}_{K_G})_{\rtimes}]$$

whereas from Proposition 2.26, we have $\psi_{\rtimes} \circ \phi_{\rtimes} \sim \mathrm{id}_{A \rtimes G}$ and $\phi_{\rtimes} \circ \psi_{\rtimes} \sim \mathrm{id}_{B \rtimes G}$. Hence, $j_G(\Phi) \in \mathsf{Cu}(A \rtimes G, B \rtimes G)$ is a strictly invertible element. □

For any pair of G-algebras A and B, it is easy to see that there is a well-defined map σ_G from the scale of $\mathsf{Cu}^G(A, B)$ to the scale of $\mathsf{Cu}(A^G, B^G)$, which is given by

$$\sigma_G([\phi \otimes \mathrm{id}_{K_G}]) := [\phi|_{A^G} \otimes \mathrm{id}_K].$$

In particular, it follows that any strictly invertible element of $\mathsf{Cu}^G(A, B)$ yields a strictly invertible element of $\mathsf{Cu}(A^G, B^G)$. Hence

Theorem 4.4. *Let A and B be unital and stably finite G-algebras. If there is a strictly invertible element in $\mathsf{Cu}^G(A, B)$ then the fixed point algebras A^G and B^G are isomorphic. Furthermore, if G finite, then the crossed products $A \rtimes G$ and $B \rtimes G$ are isomorphic.*

Proof. By the above considerations, if $\Phi \in \mathsf{Cu}^G(A, B)$ is strictly invertible, then so is $\sigma_G(\Phi) \in \mathsf{Cu}(A^G, B^G)$. Since A and B are unital, A^G and B^G are unital and stably finite, and the classification theorem of [4] applies. For the second part, if $\Phi \in \mathsf{Cu}^G(A, B)$ is a strictly invertible element, then $j_G(\Phi)$ is strictly invertible in $\mathsf{Cu}(A \rtimes G, B \rtimes G)$. Furthermore, $A \rtimes G$ and $B \rtimes G$ are unital and stably finite, and therefore the classification theorem of [4] applies. □

As in the standard theory of the bivariant Cuntz semigroup, we have the following result for the equivariant setting.

Proposition 4.5. *Let A, B be unital and stably finite G-algebras. If $\phi : A \to B$ and $\psi : B \to A$ are two equivariant c.p.c. order zero maps such that $\psi \circ \phi \sim_G \mathrm{id}_A$ and $\phi \circ \psi \sim_G \mathrm{id}_B$ then there are equivariant unital $*$-homomorphisms $\pi_\phi : A \to B$ and $\pi_\psi : B \to A$ such that*

(i) $[\pi_\phi] = [\phi]$ *and* $[\pi_\psi] = [\psi]$;

(ii) $\pi_\psi \circ \pi_\phi \sim_G \mathrm{id}_A$ *and* $\pi_\phi \circ \pi_\psi \sim_G \mathrm{id}_B$.

Proof. By Theorem 2.5 we can find G-invariant positive elements h_ϕ, h_ψ and equivariant $*$-homomorphisms π_ϕ, π_ψ such that $\phi = h_\phi \pi_\phi$ and $\psi = h_\psi \pi_\psi$. Evaluating on the unit of A and B respectively we get

$$h_\psi^{1/2} \pi_\psi(h_\phi) h_\psi^{1/2} \sim_G 1_A \quad \text{and} \quad h_\phi^{1/2} \pi_\phi(h_\psi) h_\phi^{1/2} \sim_G 1_B,$$

where by $\sim_G$ we mean that the sequences that witness the Cuntz equivalences are taken from the fixed point algebras. Hence, there exists $\{x_n\}_{n \in \mathbb{N}} \subset A^G$ such that

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$x_n h_\psi^{1/2} \pi_\psi(h_\phi) h_\psi^{1/2} x_n^*$ converges to 1_A, and therefore $x_n h_\psi^{1/2} \pi_\psi(h_\phi) h_\psi^{1/2} x_n^*$ is eventually invertible. From this we conclude that, for large enough values of n, there exist $c_n \in A$ such that

$$x_n h_\psi^{1/2} \pi_\psi(h_\phi) h_\psi^{1/2} x_n^* c_n = 1_A,$$

which shows that x_n is eventually right invertible. Since A is stably finite, it follows that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is eventually *invertible*, and therefore

$$h_\psi^{1/2} \pi_\psi(h_\phi) h_\psi^{1/2} x_n^* c x_n = 1_A,$$

which shows that h_ψ is also right invertible, hence invertible. Similarly, we also deduce the invertibility of h_ϕ, and so π_ϕ and π_ψ satisfy (i) and (ii). Now set $p = \pi_\phi(1_A)$ and $q = \pi_\psi(1_B)$. Since $\pi_\psi(p) \sim_G 1_A$ and $\pi_\phi(q) \sim_G 1_B$, stable finiteness of A and B implies $\pi_\phi(q) = 1_B$ and $\pi_\psi(p) = 1_A$. Now $1_A - \pi_\psi(q)$ is a positive element in A^G, but

$$\pi_\phi(1_A - \pi_\psi(q)) = p - 1_B \leq 0,$$

which is possible only if $p = 1_B$. Similarly, we find that $q = 1_A$, and therefore π_ϕ and π_ψ are unital. Finally, observe that the invertibility of h_ϕ and h_ψ implies that the ranges of π_ϕ and π_ψ are B and A respectively. □

The *completed* representation semiring $\mathsf{Cu}(G)$, or simply the representation semiring, as defined in [**9**, Definition 3.1], is the semiring arising by considering separable representations G rather than just the finite-dimensional ones.

Theorem 4.6. *Let A and B be unital and stably finite G-algebras. Every strictly invertible element $\Phi \in \mathsf{Cu}^G(A, B)$ induces a $\mathsf{Cu}(G)$-semimodule isomorphism $\rho : \mathsf{Cu}^G(A) \to \mathsf{Cu}^G(B)$ such that $\rho([1_A]) = [1_B]$ and $\rho([1_A \otimes e_G]) = [1_B \otimes e_G]$.*

Proof. Thanks to Proposition 4.5, if $\Phi \in \mathsf{Cu}^G(A, B)$ is a strictly invertible element, there are equivariant c.p.c. order zero maps $\phi : A \to B$ and $\psi : B \to A$ such that $\psi \circ \phi \sim_G \mathrm{id}_A$ and $\phi \circ \psi \sim_G \mathrm{id}_B$, which can then be replaced by their support $*$-homomorphisms π_ϕ and π_ψ respectively. Then $\rho := \mathsf{Cu}^G(\pi_\phi)$ is a $\mathsf{Cu}(G)$-semimodule isomorphism that clearly satisfies $\rho([1_A]) = [1_B]$ and $\rho([1_A \otimes e_G]) = [1_B \otimes e_G]$. □

Definition 4.7. Let (A, G, α) be a G-algebra. The action α on A is said to be *representable* if there exists a strongly continuous group homomorphism $u : G \to U(\mathcal{M}(A))$ such that $\alpha_g = \mathrm{Ad}(u_g)$ for any $g \in G$. The action α is said to be *locally representable* if there exists an increasing sequence $\{A_n\}_{n\in\mathbb{N}}$ of α-invariant C*-subalgebras of A such that $\bigcup_{n\in\mathbb{N}} A_n$ is dense in A and $\alpha|_{A_n}$ is representable for every $n \in \mathbb{N}$.

For the following corollary, we borrow the definition of the class of algebras $\mathbf{R}$ and that of locally representable actions from [**9**] (see the discussion that precedes [**9**, Theorem 8.4]).

Corollary 4.8. *Let G be a finite Abelian group and let (A, G, α) and (B, G, β) be unital G-algebras in the class $\mathbf{R}$ with locally representable actions α and β along given inductive sequences for A and B respectively, that lie in the class $\mathbf{R}$. Then (A, G, α) and*

(B, G, β) are equivariantly isomorphic if and only if there is a strictly invertible element in $\mathsf{Cu}^G(A, B)$.

Proof. It follows directly from the above theorem, together with the classification results of [**9**]. □

Locally representable actions for the larger class of compact groups have been considered by Handelman and Rossmann. Their definition of local representability is restricted to AF algebras, and it is assumed that an action α over an AF algebra A is locally representable if it is representable along a given inductive sequence of *finite-dimensional* C*-algebras whose limit is A. We shall say that a G-algebra (A, G, α) is AF if the underlying C*-algebra A is. Their main classification result [**10**, Theorem III.1], for the purposes of this paper, can be stated in the following way.

Definition 4.9. The equivariant Murray–von Neumann semigroup $V^G(A)$ of a unital G-algebra (A, G, α) is the set of isomorphism classes of finitely generated projective (A, G, α) modules equipped with the operation $+$ derived from the direct sum of modules.

Theorem 4.10 (Handelman–Rossmann). *Let G be a compact group and let (A, G, α) and (B, G, β) be unital AF G-algebras, with α and β locally representable actions along given inductive sequences for A and B respectively. Then A and B are equivariantly isomorphic if and only if there exists a $V^G(\mathbb{C})$-semimodule isomorphism $\rho : V^G(A) \to V^G(B)$ such that $\rho([1_A]) = [1_B]$.*

This classification result can be restated within the theory of the equivariant bivariant Cuntz semigroup as a corollary to Theorem 4.6, as it is now shown.

Corollary 4.11. *Let G be a compact group and let (A, G, α) and (B, G, β) be unital AF G-algebras, with α and β locally representable actions along given inductive sequences for A and B respectively. Then A and B are equivariantly isomorphic if and only if there exists a strictly invertible element in $\mathsf{Cu}^G(A, B)$.*

Proof. Recall that, by Lemma 3.3, $V^G(A)$ injects in $\mathsf{Cu}^G(A)$ for any stably finite C*-algebra A. By Theorem 4.6, every strictly invertible element is represented by an equivariant $*$-homomorphism, which maps G-invariant projections to G-invariant projections, and therefore induces a $V^G(\mathbb{C})$-semimodule homomorphism between $V^G(A)$ and $V^G(B)$ that satisfies all the hypotheses of Theorem 4.10. □

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