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THE NON-EMPTINESS OF JOINT SPECTRAL SUBSETS OF EUCLIDEAN n-SPACE

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Abstract

A. McIntosh and A. Pryde introduced and gave some applications of a notion of "spectral set", $\gamma(\mathbf{T})$, associated with each finite, commuting family of continuous linear operators $\mathbf{T}$ in a Banach space. Unlike most concepts of joint spectrum, the set $\gamma(\mathbf{T})$ is part of real Euclidean space. It is shown that $\gamma(\mathbf{T})$ is always non-empty whenver there are at least two operators in $\mathbf{T}$.

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Let X be a complex Banach space and $\mathbf{T} = (T_1, \dots, T_n)$ be a commuting n-tuple of continuous linear operators in X; the space of all such operators in X is denoted by $L(X)$. A joint spectral set $\gamma(\mathbf{T}) \subset \mathbf{R}^n$ is defined by

$$(1)\quad \gamma(\mathbf{T}) = \left\{ (u_1, \dots, u_n) \in \mathbf{R}^n : \sum_{j=1}^{n} (T_j - u_j I)^2 \text{ is not invertible in } L(X) \right\},$$

where I is the identity operator in X and $\mathbf{R}$ denotes the real numbers [1]. This notion has proved to be useful in determining functional calculi for certain n-tuples $\mathbf{T}$ with applications to finding estimates for the solution of linear systems of operator equations [1, 2, 3]. For applications to other notions of joint spectra we refer to [4].

It is known that $\gamma(\mathbf{T})$ is always a compact subset of $\mathbf{R}^n$ [3, Theorem 4.1]. The question arises of whether or not $\gamma(\mathbf{T})$ is empty? If $n = 1$, then it is easy

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to check that $\gamma(\mathbf{T}) = \sigma(\mathbf{T}) \cap \mathbf{R}$ and so $\gamma(\mathbf{T})$ may be empty in this case. For $n \geq 2$, it is known that $\gamma(\mathbf{T}) \neq \varnothing$ if each operator T_j, $1 \leq j \leq n$, has real spectrum. Indeed, in this case $\gamma(\mathbf{T})$ coincides with the joint Taylor spectrum of $\mathbf{T}$ [4, Theorem 1] and so is certainly non-empty. By a judicious use of Clifford analysis and monogenic functions, McIntosh and Pryde have shown that $\gamma(\mathbf{T}) \neq \varnothing$ for arbitrary commuting n-tuples $\mathbf{T}$ whenever n is an even integer [3, Section 3]. If the underlying Banach space X is finite dimensional, then it is shown in [5] that $\gamma(\mathbf{T}) \neq \varnothing$ for arbitrary commuting n-tuples $\mathbf{T}$ such that $n \geq 2$. In this note we show that $\gamma(\mathbf{T})$ is always non-empty whenever $n \geq 2$; there are no restrictions on $\mathbf{T}$ or on the Banach space X. Let us record this statement formally.

THEOREM 1. *Let X be a Banach space, $n \geq 2$ be an integer and $\mathbf{T} = (T_1, \ldots, T_n)$ be a commuting n-tuple of elements from $L(X)$. Then $\gamma(\mathbf{T})$ is non-empty.*

The proof is based on some elementary Banach algebra theory combined with an analogue of the computation given in the proof of [5, Theorem 1].

Let $\mathfrak{A}(\mathbf{T}) = \{\mathbf{T}\}^{cc}$ denote the bicommutant of $\{T_j : 1 \leq j \leq n\}$ in $L(X)$. Then $\mathfrak{A}(\mathbf{T})$ is a closed, abelian Banach subalgebra of $L(X)$ containing the identity operator I. In addition, $\mathfrak{A}(\mathbf{T})$ is inverse closed in $L(X)$. That is, if $S \in \mathfrak{A}(\mathbf{T})$ is invertible in $L(X)$, then $S^{-1} \in \mathfrak{A}(\mathbf{T})$. This is a consequence of the identity $\mathfrak{A}(\mathbf{T})^{cc} = \mathfrak{A}(\mathbf{T})$ and the fact that $S^{-1} \in \{S\}^{cc}$ whenever $S \in L(X)$ is invertible. Of course, if $S \in \mathfrak{A}(\mathbf{T})$, then $\{S\}^{cc} \subseteq \mathfrak{A}(\mathbf{T})^{cc}$. It follows that

$$\sigma_{\mathfrak{A}(\mathbf{T})}(S) = \sigma_{L(X)}(S), \qquad S \in \mathfrak{A}(\mathbf{T}), \tag{2}$$

where $\sigma_{\mathfrak{A}(\mathbf{T})}$ and $\sigma_{L(X)}$ denote the spectrum relative to the Banach algebra $\mathfrak{A}(\mathbf{T})$ and $L(X)$, respectively. It is clear from (2) and the definition of $\gamma(\mathbf{T})$ that

$$\gamma(\mathbf{T}) = \left\{ (u_1, \ldots, u_n) \in \mathbf{R}^n : 0 \in \sigma_{\mathfrak{A}(\mathbf{T})} \left(\sum_{j=1}^{n} (T_j - u_j I)^2 \right) \right\}.$$

Furthermore, since $\sum_{j=1}^{n}(T_j - u_j I)^2$ actually belongs to $\mathfrak{A}(\mathbf{T})$, for every $\mathbf{u} = (u_1, \ldots, u_n)$ in $\mathbf{R}^n$, it suffices to show that

$$\gamma(\mathbf{b}) = \left\{ \mathbf{u} \in \mathbf{R}^n ; 0 \in \sigma_{\mathfrak{B}} \left(\sum_{j=1}^{n} (b_j - u_j e)^2 \right) \right\} \tag{3}$$

is non-empty, whenever $\mathfrak{B}$ is a commutative Banach algebra (with unit e), $n \geq 2$ is an integer and $\mathbf{b} = (b_1, \ldots, b_n) \in \mathfrak{B}^n$.

To establish this we proceed as follows. Let $\mathfrak{M}$ be the maximal ideal space of $\mathfrak{B}$. It follows from standard Banach algebra theory that $\mathfrak{B}$ can be identified with a subalgebra $\hat{\mathfrak{B}}$ of the space $C(\mathfrak{M})$ of continuous functions on $\mathfrak{M}$. The Gelfand transform $\hat{}: \mathfrak{B} \to \hat{\mathfrak{B}}$ is a homomorphism such that $\hat{e} = 1$ (the constant function 1 on $\mathfrak{M}$) and

$$\sigma_{\mathfrak{B}}(b) = \hat{b}(\mathfrak{M}) = \{\hat{b}(m); m \in \mathfrak{M}\}, \qquad b \in \mathfrak{B}. \tag{4}$$

The homomorphism property of the Gelfand transform implies that

$$\left[\sum_{j=1}^{n}(b_j - u_j e)^2\right]^{\widehat{}} = \sum_{j=1}^{n}(\hat{b}_j - u_j 1)^2, \qquad \mathbf{u} \in \mathbf{R}^n,$$

for every $\mathbf{b} \in \mathfrak{B}^n$. This identity, together with (4), implies that

$$0 \in \sigma_{\mathfrak{B}}\left(\sum_{j=1}^{n}(b_j - u_j e)^2\right) \quad \text{if and only if} \quad \sum_{j=1}^{n}(\hat{b}_j(m) - u_j)^2 = 0,$$

for some $m \in \mathfrak{M}$. It follows immediately from (3) that

$$\gamma(\mathbf{b}) = \bigcup_{m \in \mathfrak{M}} \left\{\mathbf{u} \in \mathbf{R}^n : \sum_{j=1}^{n}(\hat{b}_j(m) - u_j)^2 = 0\right\}. \tag{5}$$

So, to show $\gamma(\mathbf{b})$ is non-empty it suffices to show that there exists $m \in \mathfrak{M}$ for which the set

$$Z(\hat{\mathbf{b}}, m) = \left\{\mathbf{u} \in \mathbf{R}^n : \sum_{j=1}^{n}(\hat{b}_j(m) - u_j)^2 = 0\right\} \tag{6}$$

is non-empty. Actually, we will show that $Z(\hat{\mathbf{b}}, m) \neq \emptyset$ for every $m \in \mathfrak{M}$. So, fix $m \in \mathfrak{M}$. Write $\hat{b}_j(m) = a_j(m) + ic_j(m)$, $1 \leq j \leq n$, with $a_j(m)$ and $c_j(m)$ being real numbers. Then $\mathbf{u} \in \mathbf{R}^n$ satisfies $\sum_{j=1}^{n}(\hat{b}_j(m) - u_j)^2 = 0$ if and only if

$$\sum_{j=1}^{n}(u_j - a_j(m))^2 = \sum_{j=1}^{n} c_j(m)^2 \tag{7.1}$$

and simultaneously

$$\sum_{j=1}^{n}(u_j - a_j(m))c_j(m) = 0. \tag{7.2}$$

Considering $\mathbf{u} \in \mathbf{R}^n$ as a variable, (7.1) is the equation of a sphere in $\mathbf{R}^n$ centred at $\mathbf{a}(m) = (a_1(m), \ldots, a_n(m))$ and with radius

$$\|\mathbf{c}(m)\|_2 = \left(\sum_{j=1}^{n} c_j(m)^2\right)^{1/2}$$

and (7.2) is a hyperplane in $\mathbf{R}^n$ with normal $\mathbf{c}(m)$ and passing through $\mathbf{a}(m)$. So, if $n \geq 2$, then there certainly are simultaneous solutions of (7.1) and (7.2) and hence, $Z(\hat{\mathbf{b}}, m) \neq \varnothing$ (for every $m \in \mathfrak{M}$). These calculations should be compared with those in [5, page 246]. This completes the proof of Theorem 1.

It is worth pointing out that the formula (5) can also be used to show that $\gamma(\mathbf{b})$ is a compact subset of $\mathbf{R}^n$. Indeed, to see that $\gamma(\mathbf{b})$ is bounded, fix an element $m \in \mathfrak{M}$. Then $\hat{b}_j(m) \in \sigma_{\mathfrak{B}}(b_j)$ and so $|\hat{b}_j(m)| \leq r(b_j)$, $1 \leq j \leq n$, where

$$r(b) = \sup\{|\lambda| : \lambda \in \sigma_{\mathfrak{B}}(b)\} \leq ||b||$$

denotes the spectral radius of any $b \in \mathfrak{B}$. Accordingly, if $\mathbf{u}$ is an element of $Z(\hat{\mathbf{b}}, m)$ (see (6)), then in the notation of (7.1) and (7.2) we have

$$||\mathbf{u}||_2 \leq ||\mathbf{a}(m)||_2 + ||\mathbf{c}(m)||_2 \leq \left(2\sum_{j=1}^{n} |\hat{b}_j(m)|^2\right)^{1/2} \leq \left(2\sum_{j=1}^{n} r(b_j)^2\right)^{1/2}.$$

It follows that $Z(\hat{\mathbf{b}}, m)$ is contained in the ball in $\mathbf{R}^n$ centred at zero with radius

$$r(\mathbf{b}) = \left(2\sum_{j=1}^{n} r(b_j)^2\right)^{1/2}.$$

Since this is valid for every $m \in \mathfrak{M}$, the set $\gamma(\mathbf{b})$ is also contained in this ball; see (5). This should be compared with [3, Theorem 4.1(b)] where it is shown that $\gamma(\mathbf{T})$ is contained in a ball centred at zero with radius $n^{1/2}||\mathbf{T}||$. Here $||\mathbf{T}||$ is a norm satisfying [3, page 423]

$$\max\{||T_j|| : 1 \leq j \leq n\} \leq ||\mathbf{T}|| \leq \sum_{j=1}^{n} ||T_j||$$

which can be associated with $\mathbf{T}$ by identifying $\mathbf{T}$ with the operator $T = \sum_{j=1}^{n} T_j e_j$ acting in the Banach module $X_{(n)}$ defined over the (real) Clifford algebra $\mathbf{R}_{(n)}$ as in [3, Section 3]. In general, $||T||$ is difficult to compute and so in practice the most useful statement would be that $\gamma(\mathbf{T})$ is contained in the ball centred at zero with radius $n^{1/2}\sum_{j=1}^{n} ||T_j||$. Noting that

$$r(\mathbf{T}) = \left(2\sum_{j=1}^{n} r(T_j)^2\right)^{1/2} \leq \left(2\sum_{j=1}^{n} ||T_j||^2)\right)^{1/2} \leq 2^{1/2}\sum_{j=1}^{n} ||T_j||$$

we can improve this statement; it suffices to use a ball of radius $r(\mathbf{T})$. To see that $\gamma(\mathbf{b})$ is a closed set, let $\{\mathbf{u}^{(k)}\} \subseteq \gamma(\mathbf{b})$ be a sequence which is convergent to $\mathbf{u} \in \mathbf{R}^n$. By (5) there exist elements $m_k \in \mathfrak{M}$ such that

$$\text{(8)} \qquad \sum_{j=1}^{n} (\hat{b}_j(m_k) - u_j^{(k)})^2 = 0, \qquad k = 1, 2, \ldots .$$

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The compactness of $\mathfrak{M}$ guarantees the existence of a point $m \in \mathfrak{M}$ and a subnet $\{m_\alpha\}$ of $\{m_k\}$ such that $m_\alpha \to m$ in $\mathfrak{M}$. This induces a subnet $\{\mathbf{u}^{(\alpha)}\}$ of $\{\mathbf{u}^{(k)}\}$ and hence, $\mathbf{u}^{(\alpha)} \to \mathbf{u}$ in $\mathbf{R}^n$. In particular, for each $j = 1, \dots, n$, we have $\lim_\alpha u_j^{(\alpha)} = u_j$ and, by continuity of $\hat{b}_j$, also $\lim_\alpha \hat{b}_j(m_\alpha) = \hat{b}_j(m)$. It follows from (8) that

$$\sum_{j=1}^{n} (\hat{b}_j(m) - u_j)^2 = \lim_\alpha \sum_{j=1}^{n} (\hat{b}_j(m_\alpha) - u_j^{(\alpha)})^2 = 0$$

and hence $\mathbf{u} \in \gamma(\mathbf{b})$.

It is worth summarizing and specializing our Banach algebra results to the original setting of operators on a Banach space. Recall that $\{\mathbf{T}\}^{cc}$ denotes the bicommutant of $\{T_j : 1 \le j \le n\}$. Let $\mathfrak{M}(\mathbf{T})$ denote the maximal ideal space of $\{\mathbf{T}\}^{cc}$.

THEOREM 2. *Let X be a Banach space, $n \ge 2$ be an integer and $\mathbf{T} = (T_1, \dots, T_n)$ be a commuting n-tuple of elements from $L(X)$. Then*

$$\gamma(\mathbf{T}) = \left\{ \mathbf{u} \in \mathbf{R}^n : 0 \le \sigma_{\{\mathbf{T}\}^{cc}} \left(\sum_{j=1}^{n} (T_j - u_j I)^2 \right) \right\}$$

is a non-empty, compact subset of $\mathbf{R}^n$ which is contained in the ball centred at zero with radius $r(\mathbf{T}) = (2\sum_{j=1}^{n} r(T_j)^2)^{1/2}$. Furthermore,

$$\gamma(\mathbf{T}) = \bigcup_{m \in \mathfrak{M}(\mathbf{T})} \left\{ \mathbf{u} \in \mathbf{R}^n : \sum_{j=1}^{n} (\hat{T}_j(m) - u_j)^2 = 0 \right\}. \tag{9}$$

We conclude with some remarks about the cardinality of $\gamma(\mathbf{T})$.

(I) If $\sigma(T_j) \subseteq \mathbf{R}$, for every $j = 1, 2, \dots, n$, then we have, in the notation of Theorem 2 (see (2) and (4) with $\mathfrak{B} = \{\mathbf{T}\}^{cc}$), that

$$\hat{T}_j(m) \in \sigma_{\{\mathbf{T}\}^{cc}}(T_j) = \sigma_{L(X)}(T_j) \subseteq \mathbf{R}, \qquad 1 \le j \le n,$$

for every $m \in \mathfrak{M}(\mathbf{T})$. Accordingly, the only solution in $\mathbf{R}^n$ of the equation $\sum_{j=1}^{n} (\hat{T}_j(m) - u_j)^2 = 0$ is $\mathbf{u} = \mathbf{T}(m)$. It follows from (9) that

$$\gamma(\mathbf{T}) = \{(\hat{T}_1(m), \dots, \hat{T}_n(m)); m \in \mathfrak{M}(\mathbf{T})\}$$

whenever all operators T_j, $1 \le j \le n$, have real spectrum; see also [4, Theorem 1(iii)]. Since $\gamma(S) = \sigma(S) \cap \mathbf{R}$, for every $S \in L(X)$, we have, in particular, that

$$\gamma(\mathbf{T}) \subseteq \gamma(T_1) \times \cdots \times \gamma(T_n); \tag{10}$$

see also [3, Corollary 7.4].

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(II) If $n \geq 3$ and, for some $j_0 \in \{1, \dots, n\}$ the set $\sigma(T_{j_0})$ contains a point from $\mathbb{C}\backslash\mathbf{R}$, then $\gamma(\mathbf{T})$ is an uncountable set. Indeed, in this case there exists $m_0 \in \mathfrak{M}(\mathbf{T})$ such that $\hat{T}_{j_0}(m_0) \in \mathbb{C}\backslash\mathbf{R}$ and hence, if we determine the set

$$Z(\hat{\mathbf{T}}, m_0) = \left\{ \mathbf{u} \in \mathbf{R}^n : \sum_{j=1}^{n} (\hat{T}_j(m_0) - u_j)^2 = 0 \right\}$$

by solving the corresponding equations (7.1) and (7.2), then it is clear that the sphere and the hyperplane in $\mathbf{R}^n$ so specified are not degenerate (that is, they are $(n-1)$-dimensional). It is then clear from (9) that $\gamma(\mathbf{T})$ is uncountably infinite. Surprisingly, perhaps, even if X is a finite dimensional space, the set $\gamma(\mathbf{T})$ is "very large" as soon as at least one of the operators $\{T_j\}$ does not have real spectrum $(n \geq 3)$.

(III) The situation with two commuting operators is quite different. First we note that, unlike for $n \geq 3$, it can happen that $\gamma(\mathbf{T})$ is finite even if at least one of the operators T_1 or T_2, has complex points in its spectrum. Indeed, for each $m \in \mathfrak{M}(\mathbf{T})$, it is clear that

$$\{\mathbf{u} \in \mathbf{R}^2 : (\hat{T}_1(m) - u_1)^2 + (\hat{T}_2(m) - u_2)^2 = 0\}$$

consists of at most 2 elements. Accordingly, if $\sigma(T_j)$ is a finite set, with k_j elements, say, then it follows from (9) and the fact that $\hat{T}_j(m) \in \sigma(T_j)$, for every $m \in \mathfrak{M}(\mathbf{T})$, that $\gamma(\mathbf{T})$ is also a finite set with at most $2^{k_1 k_2}$ elements. We note that with $T_1 = T_2 = I$ the number of elements in $\gamma(T) = \{(1,1)\}$ is less than the maximum number possible, namely 2 $(k_1 = k_2 = 1)$. If we take $T_1 = I$ and $T_2 = iI$, then $\gamma(\mathbf{T})$ has 2 elements and the maximum is obtained. So, in finite dimensional spaces it is the case that $\gamma(\mathbf{T})$ is always a finite set (when $n = 2$). For infinite dimensional spaces this need not be so. For example, if $X = l^2$, $T_1 = I$ and T_2 is the operator with diagonal matrix $\{i/n : n = 1, 2, \dots\}$, then $\gamma(\mathbf{T})$ equals

$$\{(1,0)\} \cup \{(1 \pm n^{-1}, 0) : n = 1, 2, \dots\}.$$

Accordingly, $\gamma(\mathbf{T})$ is infinite and countable. It can also happen that $\gamma(\mathbf{T})$ is infinite and uncountable. Just take $X = l^2$, $T_1 = I$ and T_2 the operator with diagonal matrix $\{ir_n ; n = 1, 2, \dots\}$ where $\{r_n\}$ is dense in $[0, 1]$, say.

(IV) The inclusion (10) is not valid for operators $\{T_j\}$ without real spectra. Indeed, take $T_1 = T_2 = I$ and let T_3 be any operator such that $\sigma(T_3) \cap \mathbf{R}$ is finite and $\sigma(T_3) \cap (\mathbb{C}\backslash\mathbf{R})$ is non-empty. Then $\gamma(\mathbf{T})$ is infinite by Remark (II), but

$$\gamma(T_1) \times \gamma(T_2) \times \gamma(T_3) = \{1\} \times \{1\} \times (\sigma(T_3) \cap \mathbf{R})$$

is a finite set.

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