Student Problems

Students up to the age of 19 are invited to send solutions to either of the following problems to Stan Dolan, 126A Harpenden Road, St. Albans, Herts AL3 6BZ.

Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive solutions for either problem. It is, therefore, not necessary to submit solutions to both. Solutions should arrive by September 20th 2011. Please give your School year, the name and address of your School or College, and the name of a teacher through whom the award may be made. The names of all successful solvers will be published in the November 2011 edition.

Problem 2011.3

ABCD is a quadrilateral with *ABC* an equilateral triangle and $AD = BD + CD$. Prove that $AB^2 = AD^2 - BD \times CD$.

Problem 2011.4

What happens to the value of the product $\prod_{i=m}^{2m-1}\left(1 + \frac{1}{2i}\right)$ as the positive integer m becomes large?

Solutions to 2011.1 and 2011.2

Problem 2011.1 was solved by Chris Hampson (Farnborough Sixth Form College), Ian Robson (Nottingham High School) and Alan Wills (Bishop Wordsworth's Grammar School). A variant of Problem 2011.1 with p a prime greater than or equal to 5 was solved by Robert Spencer (Westerford High School, Cape Town). A partial solution to problem 2011.2 was given by Chulue Zhong (King Edward's School, Birmingham). Both problems were solved by Kira Düsterwald (Springfield Convent Senior School, Cape Town), Xuedi Liu (Portchester Community School), William Salkeld (Bishop Wordsworth's Grammar School) and Sean Wentzel (Westerford High School, Cape Town).

Problem 2011.1

Find the greatest common divisor of all numbers of the form $p^4 - 1$, for p a prime greater than 5.

Solution

The following solution uses a combination of ideas and is closely based on the solutions of Kira, Robert and Xuedi.

Suppose p is a prime greater than 5 and note that

$$p^4 - 1 = (p^2 - 1)(p^2 + 1) = (p - 1)(p + 1)(p^2 + 1).$$

Each of $p - 1$, $p + 1$ and $p^2 + 1$ is even. Furthermore, $p - 1$ and $p + 1$ are consecutive even numbers and so one is divisible by 4. Therefore $p^4 - 1 \equiv 0 \pmod{16}$. By Fermat's Little Theorem, $p^4 - 1 \equiv 0 \pmod 5$ and $p^2 - 1 \equiv 0 \pmod 3$. Hence all numbers of the form $p^4 - 1$ are divisible by $16 \times 5 \times 3 = 240$.

The greatest common divisor of all such numbers is 240 since the greatest common divisor of $7^4 - 1 = 2400$ and $11^4 - 1 = 14640$ is 240.

The use of Fermat's Little Theorem in the above proof is not necessary since the congruences modulo 3 and 5 can easily be proved directly and Ian and William did this especially neatly. However, two entrants erroneously supposed that divisibility by 10 and by 8 implied divisibility by 80.

Problem 2011.2

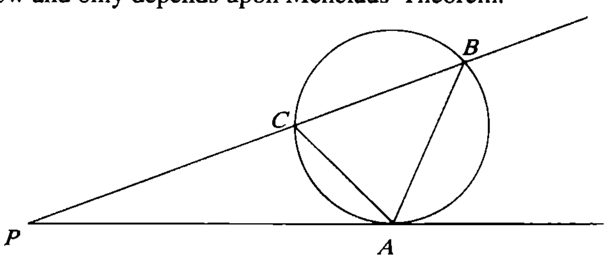
Let the triangle ABC be inscribed in a circle and let the tangents at A, B and C intersect the lines BC, CA and AB at P, Q and R, respectively. Prove that P, Q and R are collinear.

Solution

I was very impressed by the variety and ingenuity of the methods that students used to solve this problem. Kira gave an elegant proof obtained by applying Desargues' Theorem to triangle ABC and the triangle formed by the tangents. However, the shortest proof was one of the two proofs submitted by Sean:

Consider triangle ABC to be a hexagon $AABBCC$. The result is then a special case of Pascal's Theorem.

The relatively elementary approach adopted by William and Xuedi is given below and only depends upon Menelaus' Theorem.



$\angle PAC = \angle PBA$ (Alternate segments) and so ΔPAC is similar to ΔPBA. Then

$$\frac{PA}{PC} = \frac{PB}{PA} = \frac{AB}{CA} \Rightarrow \frac{PB}{PC} = \left(\frac{AB}{CA}\right)^2,$$

with similar results for Q and R. Therefore

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$$\frac{AR}{RB} \times \frac{BP}{PC} \times \frac{CQ}{QA} = \left(\frac{CA}{BC} \times \frac{AB}{CA} \times \frac{BC}{AB}\right)^2 = 1.$$

Then P, Q and R are collinear, by Menelaus.

The first prize of £25 is awarded to Xuedi Liu and the second prize of £20 is awarded to Kira Düsterwald. William Salkeld, who was a very close third, also deserves a special mention.

STAN DOLAN

Reviewers wanted!

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Gerry Leversha