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HILBERTIANITY OF FIELDS OF POWER SERIES

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Abstract Let R be a domain contained in a rank-1 valuation ring of its quotient field. Let $R[\![X]\!]$ be the ring of formal power series over R, and let F be the quotient field of $R[\![X]\!]$. We prove that F is Hilbertian. This resolves and generalizes an open problem of Jarden, and allows to generalize previous Galois-theoretic results over fields of power series.

Keywords: Hilbertian fields; formal power series; field arithmetic; generalized Krull domains

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Introduction

Field arithmetic studies the connection between the arithmetic properties of a field and its Galois theoretic properties. A central conjecture in field arithmetic, which widely generalizes the inverse Galois problem, was coined by Dèbes and Deschamps.

Conjecture A (Dèbes and Deschamps [2, § 2.1.2]). *If F is a Hilbertian field, then every finite split embedding problem over F is solvable.*

Conjecture A is proven in [**17**] in the case where F is ample (called 'large' in [**17**]). In particular, this holds if F is complete with respect to a discrete valuation. Following this came a series of works studying Galois theory over complete valued domains whose quotient fields are not complete. The archetype of such a domain is $A[\![X]\!]$, where A is some domain which is not a field. The first Galois theoretic result over such fields is due to Lefcourt [**10**], who showed that if A is integrally closed and Noetherian, then the inverse Galois problem has a positive solution over $F = \mathrm{Quot}(A[\![X]\!])$. We call the field F the *field of formal power series over* A (note that F is usually smaller than the field $K((X)) = \mathrm{Quot}(K[\![X]\!])$ of formal power series over $K = \mathrm{Quot}(A)$). The next result is due to Harbater and Stevenson [**7**], who showed that Conjecture A holds over F, in the case where A is a complete discrete valuation ring (moreover, they showed that each such problem has $|F|$-many distinct solutions). Then the author [**14**] showed that Conjecture A holds in a more general situation.

Theorem B. *Let A be a Krull domain (e.g. an integrally closed Noetherian domain). Then every finite split embedding problem over* $\mathrm{Quot}(A[\![X]\!])$ *is solvable.*

In [**18**], Pop showed that the quotient field of a Henselian domain is ample. Using his result, one can give a short proof of Theorem B: given a split embedding problem over F, extend it to $F(t)$, solve it there using the result of [**17**], then specialize the solution into a solution over F. The second part of the proof, specialization, is possible since F is Hilbertian by a theorem of Weissauer [**21**]. That theorem asserts that the quotient field of a domain R of dimension exceeding 1 is Hilbertian, provided that R is a generalized Krull domain. We recall the definition.

Definition C. A domain R is called a *generalized Krull domain* if there exists a non-empty family $\mathcal{F}$ of non-trivial rank-1 valuations on $K = \mathrm{Quot}(R)$, satisfying the following properties.

(a) Denoting the valuation ring of v by R_v for each $v \in \mathcal{F}$, we have $\bigcap_{v\in\mathcal{F}} R_v = R$.

(b) For each $a \in K^{\times}$, $v(a) = 0$ for all but finitely many $v \in \mathcal{F}$.

(c) For each $v \in \mathcal{F}$, R_v is the localization of R with respect to the centre $\mathfrak{p}(v) = \{a \in R \mid v(a) > 0\}$ of v on R.

If every $v \in \mathcal{F}$ is discrete, then R is called a *Krull domain* [**22**, § VI.13]. Note that by assuming that $\mathcal{F}$ is non-empty, we do not consider fields as Krull domains.

It is well known [**11**, Theorem 12.4] that if A is a Krull domain (in particular, if A is integrally closed and Noetherian) then so is $R = A[\![X]\!]$. Applying Weissauer's theorem to R proves the Hilbertianity of $F = \mathrm{Quot}(R)$ in all of the above-mentioned works.

The first part in the above-mentioned proof of Theorem B, the ampleness of $\mathrm{Quot}(A[\![X]\!])$, holds for an arbitrary A. However, the second part of the proof is limited by the conditions of Weissauer's theorem. One may ask whether Theorem B holds for an arbitrary domain A (which is not a field). This leads to the question: is $\mathrm{Quot}(A[\![X]\!])$ Hilbertian? A special case of the question was posed as an open problem by Jarden in [**3**, § 15]. That problem consisted of two parts. The first part is, assuming A is a generalized Krull domain, whether so is $A[\![X]\!]$. The second part of the problem was whether $\mathrm{Quot}(A[\![X]\!])$ is Hilbertian. A positive answer to the first part of Jarden's problem would have implied a positive answer to the second part as well, by Weissauer's theorem. However, the work [**15**] showed that the answer for the first part of Jarden's problem is negative. In fact, the situation for generalized Krull domains is essentially opposite to the situation for Krull domains. That is, if A is a generalized Krull domain which is not a Krull domain, then $A[\![X]\!]$ is never a generalized Krull domain [**15**, Theorem 2.5].

Due to the negative answer to the first part of Jarden's problem, one cannot apply Weissauer's theorem to $A[\![X]\!]$ and deduce that $\mathrm{Quot}(A[\![X]\!])$ is Hilbertian. Despite this, we prove that the answer to the second part of the problem is positive: such a field is Hilbertian. In fact, we prove a more general result.

Main Theorem. *Let A be a domain, contained in a rank-1 valuation ring of* $\mathrm{Quot}(A)$*. Then* $\mathrm{Quot}(A[\![X]\!])$ *is Hilbertian.*

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By definition, any generalized Krull domain is contained in a rank-1 valuation ring, implying the positive answer to Jarden's problem [**3**, Problem 15.5.9(b)].

The idea of the proof of the Main Theorem is to embed $A[[X]]$ into the ring $A_v\{X\}$ of convergent (with respect to the valuation v given in the hypothesis of the theorem) power series over the valuation ring A_v of v, and extend this to an embedding $\varphi\colon \operatorname{Quot}(A[[X]]) \to \operatorname{Quot}(A_v\{X\})$. We then observe that $A_v\{X\}$ is a generalized Krull domain (which is essentially a consequence of the Weierstrass division theorem), and consider the inverse image S of $A_v\{X\}$ under φ. We show that S is an over-ring of $A[[X]]$ of dimension exceeding 1, and show that it is a generalized Krull domain. Then by applying Weissauer's theorem to S instead of $A[[X]]$, we deduce that $\operatorname{Quot}(S) = \operatorname{Quot}(A[[X]])$ is Hilbertian. A difficulty arises in establishing the fact that S is indeed a generalized Krull domain—the ring S is the intersection of two generalized Krull domains, however, in general such an intersection need not be a generalized Krull domain. In order to overcome this problem, we develop in § 1 a theory concerning the characterization of generalized Krull domains which generalizes the theory for Krull domains. Using our characterization we are able to prove that S is a Krull domain.

As a Galois-theoretic consequence of the Main Theorem, we deduce a generalization of Theorem B.

Corollary D. *Let A be a domain, contained in a rank-1 valuation ring of $\operatorname{Quot}(A)$. Then every finite split embedding problem over $\operatorname{Quot}(A[[X]])$ is solvable.*

1. Characterizing generalized Krull domains

In [**9**] Krull showed that a valuation ring R is completely integrally closed if and only if $\operatorname{rank}(R) \leqslant 1$, which led him to conjecture that a completely integrally closed domain is an intersection of valuations rings of rank less than or equal to 1. This conjecture was disproven by Nakayama [**13**]. Generalized Krull domains were defined by Ribenboim in [**19**] and were shown in [**12**] to have a natural role in commutative algebra, as a class of rings for which Krull's conjecture holds. Their properties were then studied in several works (see, for example, [**1**, **12**, **15**, **16**, **20**]).

The aim of this section is to establish an equivalent characterization of generalized Krull domains which is easier to establish. It is well known [**11**, Theorem 12.3] that in order to prove that a domain R is a Krull domain, it suffices to find a family $\mathcal{F}$ of discrete valuations satisfying conditions (a) and (b) of Definition C (the idea of the proof of [**11**, Theorem 12.3] is to show that there exists a subfamily of valuations satisfying all three conditions). This allows one to give an alternative definition of a Krull domain, which is easier to establish. This is useful, for example, to show that the intersection of two Krull domains is a Krull domain (a property which is not immediate from the definition, because of the difficulty in establishing condition (c)). Unfortunately, a similar property for generalized Krull domains does not hold. In [**4**, § 4], an example is given of a domain R and a family of rank-1 valuations on $\operatorname{Quot}(R)$ satisfying conditions (a) and (b) of Definition C, while R is not a generalized Krull domain. Moreover, analysing that example, one sees it also provides an example of two generalized Krull domains

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whose intersection is not a generalized Krull domain. However, in this section we show that one can at least give an equivalent definition of a generalized Krull domain where condition (c) is weakened. Namely, it suffices that each $v \in \mathcal{F}$ satisfies $v(R_v) = v(R)$ *or* $\inf(v(\mathfrak{p}(v))) > 0$.

Throughout this section, whenever R is a domain and v a valuation of $\mathrm{Quot}(R)$, we denote by R_v the valuation ring of v in $\mathrm{Quot}(R)$, and by $\mathfrak{p}(v) = \{a \in R \mid v(a) > 0\}$ the centre of v on R. We recall some common convenient terminology (following [**5**], for example). A family $\mathcal{F}$ of valuations on a field K is said to be of *finite type* if for each $x \in K^\times$ there exist only finitely many $v \in \mathcal{F}$ with $v(x) \neq 0$. We say that the ring $R = \bigcap_{v \in \mathcal{F}} R_v$ is *defined* by $\mathcal{F}$. If R is a domain and v a valuation of $\mathrm{Quot}(R)$ satisfying $R_v = R_{\mathfrak{p}(v)}$, we say that v is *essential* (for R). Thus a generalized Krull domain is a domain R defined by a non-empty family of finite type $\mathcal{F}$ of valuations on $\mathrm{Quot}(R)$, where each $v \in \mathcal{F}$ is essential and of rank 1. Lemma 1.3 and Remark 1.4 below show that such a family is unique (up to equivalence of valuations) and consists of all rank-1 valuations on K which are essential for R. Thus we shall refer to $\mathcal{F}$ as the *essential family* of R.

A rank-1 valuation v of a field K is said to be *well-centred* on a subring R, if $v(R_v) = v(R)$ (where $v(R_v)$ is the non-negative part of the value group of v), and we say that a valuation v of K is *positive* on R if $\inf(v(\mathfrak{p}(v))) > 0$. In particular, any discrete rank-1 valuation of K is positive on any subring of K (note that by definition, a trivial valuation of K is positive on each subring of K).

The rest of this section proves the equivalence detailed above (Proposition 1.6 below). The proof is somewhat complicated technically speaking, and the reader may at first prefer to read only the assertion of Proposition 1.6 and then skip to the next section, which proves the Main Theorem.

We begin with some general lemmas.

Lemma 1.1. *Let R be an integral domain, let v be a rank-1 valuation of $\mathrm{Quot}(R)$, and $\mathfrak{p} = \mathfrak{p}(v)$. If v is not positive on R, then $v(\mathfrak{p})$ is dense in $(0, \infty)$.*

Proof. Let $0 < a < b \in \mathbb{R}$. We show that there exists $x \in \mathfrak{p}$ with $v(x) \in (a, b)$. Choose some $1 < \delta < b/a$. Then $b - a\delta > 0$, so for a sufficiently large $n \in \mathbb{N}$ we have $b - a\delta > 1/\delta^n$. Since $\inf(v(\mathfrak{p})) = 0$, there exists $y \in \mathfrak{p}$ with $0 < v(y) < 1/\delta^n$. We have

$$\left(0, \frac{1}{\delta^n}\right) = \bigcup_{m=n}^{\infty} \left[\frac{1}{\delta^{m+1}}, \frac{1}{\delta^m}\right),$$

hence there exists an integer $n \leqslant m$ such that $1/\delta^{m+1} \leqslant v(y) < 1/\delta^m$. Since $b - a\delta > 1/\delta^n \geqslant 1/\delta^m$, we have $\delta^m b - \delta^{m+1} a > 1$, hence there exists an integer k with $\delta^{m+1} a < k < \delta^m b$. Then $a < k/\delta^{m+1} \leqslant kv(y) < k/\delta^m < b$. Thus $x = y^k$ satisfies $a < v(x) < b$. □

Lemma 1.2. *Let R be a local domain of dimension exceeding 1, defined by a family $\mathcal{F}$ of rank-1 valuations. Then $\mathcal{F}$ is infinite.*

Proof. Suppose $\mathcal{F} = \{v_1, \dots, v_k\}$ is finite, where $v_1, \dots, v_k$ are distinct (i.e. inequivalent) valuations. That is, the corresponding valuation rings $R_1, \dots, R_k$ are distinct. For each

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$1 \leqslant i \leqslant k$ let $\mathfrak{p}_i = \mathfrak{p}(v_i)$. Then by [**11**, Theorem 12.2], each $\mathfrak{p}_i$ is a maximal ideal, and $R_i = R_{\mathfrak{p}_i}$. Since R is local, we deduce that $\mathfrak{p}_i = \mathfrak{p}_j$ and hence $R_i = R_j$ for each $1 \leqslant i, j \leqslant k$. Thus $k = 1$, and $R = \bigcap_{v\in\mathcal{F}} R_v = R_1$ is a rank-1 valuation ring, hence it is a maximal subring of K. But since R is of dimension exceeding 1, it is strictly contained in its localization by some non-maximal prime ideal, a contradiction. □

Lemma 1.3. *If R is a generalized Krull domain, then there is a unique family (up to equivalence) $\mathcal{F}$ satisfying the conditions of Definition C. The corresponding family of valuation rings is*

$$\{R_v \mid v \in \mathcal{F}\} = \{R_{\mathfrak{p}} \mid \mathfrak{p} \text{ is a minimal non-zero prime ideal of } R\}.$$

Proof. Suppose R is a generalized Krull domain, and let $\mathcal{F}$ be a family of valuations on $K = \mathrm{Quot}(R)$, satisfying the conditions of Definition C. Suppose R_v is a valuation ring for some $v \in \mathcal{F}$. Since R_v is a rank-1 valuation ring, R_v is a maximal subring of K. Since $R_v = R_{\mathfrak{p}(v)}$, $\mathfrak{p}(v)$ is a minimal non-zero prime ideal. Thus each of the valuation rings R_v is obtained by localizing R by some minimal prime ideal. Conversely, we show that for each minimal non-zero prime ideal $\mathfrak{p}$, there exists $v \in \mathcal{F}$ such that $R_v = R_{\mathfrak{p}}$. Indeed, suppose there exists no such v. Choose $0 \neq a \in \mathfrak{p}$. Let $v_1, \dots, v_k$ be all valuations in $\mathcal{F}$ which are positive at a. By our assumption, $R_{\mathfrak{p}(v_i)} \neq R_{\mathfrak{p}}$ for each $1 \leqslant i \leqslant k$, hence $\mathfrak{p}(v_i) \neq \mathfrak{p}$ for each $1 \leqslant i \leqslant k$. Since these are minimal non-zero primes, this implies that $\mathfrak{p}(v_i) \not\subseteq \mathfrak{p}$, so we may choose $a_i \in \mathfrak{p}(v_i) \smallsetminus \mathfrak{p}$, for each $1 \leqslant i \leqslant k$. Choose sufficiently large integers $e_1, \dots, e_k$, such that $v_i(a_i^{e_i}) > v_i(a)$ for each $1 \leqslant i \leqslant k$, and define $b = a_1^{e_1} \cdot \dots \cdot a_k^{e_k} \in R$. Then $v_i(b/a) > 0$ for each $1 \leqslant i \leqslant k$. For each $v \in \mathcal{F} \smallsetminus \{v_1, \dots, v_k\}$ we have $v(b/a) = v(b) \geqslant 0$. Thus $v(b/a) \geqslant 0$ for each $v \in \mathcal{F}$, hence $b/a \in R$. Since $a \in \mathfrak{p}$, $b = (b/a) \cdot a \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, there exists $1 \leqslant i \leqslant k$ such that $a_i \in \mathfrak{p}$, a contradiction. □

Remark 1.4. It follows from Lemma 1.3, that if R is a generalized Krull domain, then the unique family given in the lemma consists of *all* rank-1 valuations on $K = \mathrm{Quot}(R)$ that are essential for R. Indeed, if v is a rank-1 valuation on K which is essential for R, then $R_v = R_{\mathfrak{p}(v)}$ is a rank-1 valuation ring, hence a maximal subring of K, hence $\mathfrak{p}(v)$ is a minimal non-zero ideal of R.

The following technical lemma is the key ingredient in the proof of Proposition 1.6 below.

Lemma 1.5. *Let R be a domain, $v_1, \dots, v_r, v'$ rank-1 valuations on $\mathrm{Quot}(R)$ whose valuation rings contain R, and with $\mathfrak{p}(v') \subseteq \mathfrak{p}(v_1)$. Let $a \in R$ such that $v'(a) = 0$. For each $1 \leqslant i \leqslant r$ put $\mathfrak{q}_i = \{c \in R \mid v_i(c) \geqslant v_i(a)\}$ and suppose that $\mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \dots \cap \mathfrak{q}_r = aR \neq \mathfrak{q}_2 \cap \dots \cap \mathfrak{q}_r$. Suppose that v_1 is well-centred or positive on R. Then there exist $d \in \mathfrak{q}_2 \cap \dots \cap \mathfrak{q}_r \smallsetminus aR$ and a real number $\delta > 0$, such that $\mathfrak{p}(v') \subseteq \{c \in R \mid v_1(c) + v_1(d) \geqslant \delta\}$ and $aR = \{c \in R \mid v_1(c) \geqslant \delta\} \cap \mathfrak{q}_2 \cap \dots \cap \mathfrak{q}_r$.*

Proof. Define $\lambda = v_1(a) \in \mathbb{R}$, $\mathfrak{p}_1 = \mathfrak{p}(v_1)$, $\mathfrak{p}' = \mathfrak{p}(v')$, and for each $\alpha > 0$ put $\mathfrak{a}_\alpha = \{c \in R \mid v_1(c) \geqslant \alpha\}$. Then $\mathfrak{q}_1 = \mathfrak{a}_\lambda$. Define $\lambda_1 = \inf(v_1(\mathfrak{p}_1))$. Then $\lambda_1 \leqslant \lambda$, and we put $\mu = \inf\{\alpha \in [\lambda_1, \lambda] \mid aR = \mathfrak{a}_\alpha \cap \mathfrak{q}_2 \cap \dots \cap \mathfrak{q}_r\}$. Clearly, $\lambda_1 \leqslant \mu$. We claim that

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$\mu > 0$. Indeed, suppose $\mu = 0$ and let $b \in \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r$. If $v_1(b) < \lambda$, choose $c \in \mathfrak{p}_1$ with $0 < v_1(c) < \lambda - v_1(b)$ (since $\mu = 0$ we have $\lambda_1 = 0$, hence such a choice is possible). Choose a real number $0 < \alpha < v_1(c)$ (so that $c \in \mathfrak{a}_\alpha$). Then $bc \in \mathfrak{a}_\alpha \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r = aR$. Thus $v_1(a) \leqslant v_1(bc) < \lambda$, a contradiction. Hence $v_1(b) \geqslant \lambda$, so $b \in \mathfrak{a}_\lambda \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r = aR$. Thus $\mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r = aR$, a contradiction. This proves that $\mu > 0$.

We now distinguish between two cases. First suppose that $\lambda_1 = 0$. Then by our assumptions we must have $v_1(R_{v_1}) = v_1(R)$. Choose some $0 < \varepsilon < \mu$. If $\lambda > \mu$ choose a real number $\mu < \delta < \lambda$, and if $\lambda = \mu$ put $\delta = \mu$. Either way we have $\mathfrak{a}_\delta \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r = aR$. Moreover, by the definition of μ, aR is strictly contained in $\mathfrak{a}_\varepsilon \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r$. Let $d \in (\mathfrak{a}_\varepsilon \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r) \setminus aR$. Then $\varepsilon \leqslant v_1(d) \leqslant \mu$, and if $\lambda = \mu$ then $v_1(d) < \mu$. Thus either way $\varepsilon' = \delta - v_1(d)$ is positive. Put $\mathfrak{a}' = \{c \in R \mid v_1(c) \geqslant \varepsilon'\}$, and we show that $\mathfrak{p}' \subseteq \mathfrak{a}'$. Indeed, suppose there exists $c_1 \in \mathfrak{p}'$ with $v_1(c_1) < \varepsilon'$. Since $\delta \leqslant \lambda$ we have $\lambda - v_1(d) - v_1(c_1) > 0$. If $\mu = \delta = \lambda$ then $v_1(a/c_1 d) = \delta - v_1(d) - v_1(c_1) > 0$, and since $v_1(R) = v_1(R_{v_1})$ we may choose $c_2 \in R$ with $v_1(c_2) = \delta - v_1(d) - v_1(c_1)$. If $\mu < \delta < \lambda$ apply Lemma 1.1 to choose $c_2 \in R$ with $\delta - v_1(d) - v_1(c_1) < v_1(c_2) < \lambda - v_1(d) - v_1(c_1)$. Then either way, $c = c_1 c_2 \in \mathfrak{p}'$ satisfies $\varepsilon' \leqslant v_1(c) \leqslant \lambda - v_1(d)$. Thus $cd \in \mathfrak{a}_\delta \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r$, hence $cd/a \in R$. Moreover, $v'(cd/a) = v'(cd) > 0$, thus $cd/a \in \mathfrak{p}' \subseteq \mathfrak{p}_1$, hence $v_1(cd/a) > 0$. But $v_1(cd/a) = v_1(c) + v_1(d) - \lambda \leqslant 0$, a contradiction. Thus $\mathfrak{p}' \subseteq \mathfrak{a}'$.

Next, suppose that $\lambda_1 > 0$. Choose some $\max\{\mu - 2\lambda_1, 0\} < \varepsilon < \mu$. If $\lambda > \mu$ put $\delta = \min\{2\lambda_1 + \varepsilon, \lambda\}$ (so $\mu < \delta$), and if $\lambda = \mu$ put $\delta = \mu$. Either way we have $\delta - \varepsilon \leqslant 2\lambda_1$, and $\mathfrak{a}_\delta \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r = aR$. Moreover, aR is strictly contained in $\mathfrak{a}_\varepsilon \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r$. Let $d \in (\mathfrak{a}_\varepsilon \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r) \setminus aR$. Then $\varepsilon \leqslant v_1(d) \leqslant \mu$, and if $\lambda = \mu$ then $v_1(d) < \mu$. Thus $\varepsilon' = \delta - v_1(d)$ is positive. Put $\mathfrak{a}' = \{c \in R \mid v_1(c) \geqslant \varepsilon'\}$. We show that $\mathfrak{p}' \subseteq \mathfrak{a}'$. Indeed, suppose there exists $c_1 \in \mathfrak{p}'$ with $v_1(c_1) < \varepsilon'$. Since $\delta \leqslant \lambda$ we have $\lambda - v_1(d) - v_1(c_1) > 0$. Since $\lambda_1 = \inf(v_1(\mathfrak{p}_1))$ we may choose $c_2 \in R$ with $\lambda_1 \leqslant v_1(c_2) < \lambda_1 + \lambda - v_1(d) - v_1(c_1)$. Then

$$\varepsilon' - v_1(c_1) \leqslant \varepsilon' - \lambda_1 = \delta - v_1(d) - \lambda_1 \leqslant \delta - \varepsilon - \lambda_1 \leqslant 2\lambda_1 + \varepsilon - \varepsilon - \lambda_1 = \lambda_1 \leqslant v_1(c_2).$$

Hence $c = c_1 c_2 \in \mathfrak{p}'$ satisfies $\varepsilon' \leqslant v_1(c) < \lambda + \lambda_1 - v_1(d)$. Thus $cd \in \mathfrak{a}_\delta \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r$, hence $cd/a \in R$. Moreover, $v'(cd/a) = v'(cd) > 0$, thus $cd/a \in \mathfrak{p}' \subseteq \mathfrak{p}_1$, hence $v_1(cd/a) \geqslant \lambda_1$. But $v_1(cd/a) = v_1(c) + v_1(d) - \lambda < \lambda + \lambda_1 - \lambda = \lambda_1$, a contradiction. Thus $\mathfrak{p}' \subseteq \mathfrak{a}'$. □

We now prove our result concerning generalized Krull domains. The proof strategy is similar to the proof of [**11**, Theorem 12.4], where an analogues claim (using a slightly different terminology) is proven for Krull domains. We note that for discrete valuations, Lemma 1.5 is trivial and has no counterpart in [**11**, § 12].

Proposition 1.6. *Let $\mathcal{F}$ be a non-empty family of finite type of rank-1 valuations on a field K, and let R be the ring defined by $\mathcal{F}$. Suppose that each $v \in \mathcal{F}$ is positive on R or well-centred on R. Then R is a generalized Krull domain, and its essential family is contained in $\mathcal{F}$.*

Proof. Define $\mathcal{F}' = \{v \in \mathcal{F} \mid R_v = R_{\mathfrak{p}(v)}\}$. Then $\mathcal{F}'$ is of finite type, and each $v \in \mathcal{F}$ is essential for R. It remains to show that $\mathcal{F}'$ defines R. Let $b/a \in \bigcap_{v \in \mathcal{F}'} R_v$, with

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$0 \neq a, b \in R$. Let $v_1, \dots, v_t$ be all the valuations in $\mathcal{F}$ which are positive at a, and for each $1 \leqslant i \leqslant t$ define $\mathfrak{p}_i = \mathfrak{p}(v_i)$, $\mathfrak{q}_i = \{c \in R \mid v_i(c) \geqslant v_i(a)\}$. Then $aR = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$, by condition (a). Discard finitely many of the $\mathfrak{q}_i$ and reorder them, to assume that $\{\mathfrak{q}_1, \dots, \mathfrak{q}_r\}$ is a minimal subset of $\{\mathfrak{q}_1, \dots, \mathfrak{q}_t\}$ satisfying $aR = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$. We will show that $v_1, \dots, v_r \in \mathcal{F}'$, and hence $v_i(x) \geqslant 0$ for each $1 \leqslant i \leqslant r$. This then implies that $b \in \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$, so $b/a \in R$.

To show that for each $1 \leqslant i \leqslant r$ we have $v_i \in \mathcal{F}'$, we must show that $R_{\mathfrak{p}_i}$ is the valuation ring R_{v_i}. That is, $R_{\mathfrak{p}_i}$ is a maximal subring of $\mathrm{Quot}(R)$, or equivalently, $\mathfrak{p}_i$ is a minimal non-zero prime ideal. Suppose that, say, $\mathfrak{p}_1$ is a non-minimal prime. Define $R_1 = R_{\mathfrak{p}_1}$ and $\mathcal{F}_1 = \{w \in \mathcal{F} \mid R_1 \subseteq R_w\}$. In particular, $v_1 \in \mathcal{F}_1$.

Note that $\bigcap_{w \in \mathcal{F}_1} R_w = R_1$. Indeed, suppose $y \in \bigcap_{w \in \mathcal{F}_1} R_w$, and let $w'_1, \dots, w'_m$ be all the valuations in $\mathcal{F}$ that are negative at y. Then for each $1 \leqslant j \leqslant m$ we have $w'_j \notin \mathcal{F}_1$, hence $(R \smallsetminus \mathfrak{p}_1) \cap \mathfrak{p}(w'_j) \neq \emptyset$. Thus by taking a sufficiently large power of a non-zero element in $(R \smallsetminus \mathfrak{p}_1) \cap \mathfrak{p}(w'_j)$, we may choose $a_j \in R$ such that $w'_j(a_j) > -w'_j(y)$ and $v_1(a_j) = 0$. Then $y \cdot a_1 \cdot \cdots \cdot a_m \in \bigcap_{v \in \mathcal{F}} R_v = R$, hence $y \in R_{\mathfrak{p}_1} = R_1$.

Next, since $\mathfrak{p}_1$ is non-minimal, R_1 is a local ring of dimension exceeding 1. By Lemma 1.2, $\mathcal{F}_1$ is infinite. Choose $v' \in \mathcal{F}_1$ such that $v'(a) = 0$, and let $\mathfrak{p}' = \mathfrak{p}(v')$. Then $a \notin \mathfrak{p}'$ and since $R_1 \subseteq R_{v'}$ we have $\mathfrak{p}' \subseteq \mathfrak{p}_1$. By our assumptions, aR is strictly contained in $\mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r$. Also, $v_1 \in \mathcal{F}$ is well-centred or positive on R. Thus all the conditions of Lemma 1.5 hold, so we may choose $d \in \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r \smallsetminus aR$ and $\delta > 0$ such that $aR = \mathfrak{a}_\delta \cap \mathfrak{q}_2 \cdots \cap \mathfrak{q}_r$ and $\mathfrak{p}' \subseteq \mathfrak{a}'$, where $\mathfrak{a}_\delta = \{c \in R \mid v_1(a) \geqslant \delta\}$ and $\mathfrak{a}' = \{c \in R \mid v_1(c) \geqslant \delta - v_1(d)\}$.

Choose $0 \neq c \in \mathfrak{p}' \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r$. In particular $c \in \mathfrak{a}'$. We claim that for each $n \geqslant 0$, $(d/a)^n c \in \mathfrak{p}' \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r$. Suppose by induction that we have proven this for some $n \geqslant 0$. In particular, $(d/a)^n c \in \mathfrak{a}'$, hence $v_1(d^{n+1}c/a^n) \geqslant \delta - v_1(d) + v_1(d) = \delta$. Hence $d^{n+1}c/a^n \in \mathfrak{a}_\delta \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r = aR$. Thus $(d/a)^{n+1}c \in R$. Since $c, d \in \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r$, we have $v_j((d/a)^{n+1}c) \geqslant v_j(a)$ for each $2 \leqslant j \leqslant r$, hence $(d/a)^{n+1}c \in \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r$. Finally,

$$v'\left(\left(\frac{d}{a}\right)^{n+1} c\right) = v'\left(\frac{d}{a}\right) + v'\left(\left(\frac{d}{a}\right)^{n} c\right) = v'(d) - 0 + v'\left(\left(\frac{d}{a}\right)^{n} c\right) \geqslant v'\left(\left(\frac{d}{a}\right)^{n} c\right) > 0,$$

hence $(d/a)^{n+1}c \in \mathfrak{p}'$. This concludes the induction.

In particular, $(d/a)^n c \in R$ for each $n \geqslant 0$. For each $v \in \mathcal{F}$, R_v is a rank-1 valuation ring, hence it is a completely integrally closed domain (see [**11**, Exercise 9.5]), hence so is the intersection R. Thus $d/a \in R$, a contradiction. □

2. Power series and Hilbertianity

We are almost ready to prove the Main Theorem. We first recall some properties of rings of convergent power series.

Remark 2.1 (convergent power series). Let R be an integral domain, and let $R[\![X]\!]$ be the ring of formal power series over R. Let v be a rank-1 valuation on $\mathrm{Quot}(R)$, and define

$$R\{X\} = \left\{ \sum_{i=0}^{\infty} f_i X^i \in R[\![X]\!] \;\middle|\; v(f_i) \to \infty \right\}.$$

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Then clearly $R\{X\}$ is a subring of $R[\![X]\!]$, called the ring of convergent power series over R. The ring R_v is naturally contained in $R_v\{X\}$, and v extends to a rank-1 valuation v^* on $\mathrm{Quot}(R\{X\})$, given on $R\{X\}$ by

$$v^*\left(\sum_{i=0}^{\infty} f_i X^i\right) = \min_{i\geqslant 0} v(f_i).$$

If R is complete with respect to v, then $R\{X\}$ is complete with respect to v^* [**6**, Lemma 1.3(ii)]. If R is also a field, then by [**6**, Theorem 1.10], $R\{X\}$ is a unique factorization domain, hence it is a Krull domain [**22**, § VI.13].

A proof of the following lemma appears in the proof of [**8**, Theorem 2.3.3], in the case where O is complete.

Lemma 2.2. *Let O be a rank-1 valuation ring, and v the corresponding valuation on $K = \mathrm{Quot}(O)$. Extend v to a rank-1 valuation v^* on $\mathrm{Quot}(K\{X\})$ as in Remark 2.1. Then $O\{X\}$ is a generalized Krull domain with quotient field $\mathrm{Quot}(K\{X\}) = \mathrm{Quot}(O\{X\})$. Moreover, if $\mathcal{F}$ is the essential family (Remark 1.4) of $O\{X\}$, then $v^* \in \mathcal{F}$, and all valuations in $\mathcal{F} \setminus \{v^*\}$ are discrete.*

Proof. Let $\hat{K}$ be the completion of $K = \mathrm{Quot}(O)$ with respect to v. The ring $K[\![X]\!]$ is a discrete valuation ring, and in particular a Krull domain. By Remark 2.1, so is $\hat{K}\{X\}$. Hence the intersection $K\{X\} = K[\![X]\!] \cap \hat{K}\{X\}$ (taken inside $\hat{K}[\![X]\!]$) is also a Krull domain, and let $\mathcal{F}_0$ be its essential family (Remark 1.4). By the uniqueness of the essential family, each $w \in \mathcal{F}_0$ is discrete.

Put $F = \mathrm{Quot}(K\{X\})$ and $D = O\{X\}$. Each element of $K\{X\}$ can be written in the form $af(X)$, with $a \in K$ and $f(X) \in K\{X\}$ with $v^*(f(X)) = 0$ (by dividing by a coefficient of minimal value). Thus $K\{X\} = K \cdot D$, hence $F = \mathrm{Quot}(K\{X\}) = \mathrm{Quot}(D)$. Let O' be the valuation ring of v^* in F. Then $O' \cap K\{X\} = D$, by the definition of v^*. Let $\mathfrak{p}$ be the maximal ideal of O. Then

$$\mathfrak{p}^* = \left\{ \sum_{i=0}^{\infty} f_i X^i \in D \;\middle|\; f_i \in \mathfrak{p} \text{ for all } i \geqslant 0 \right\}$$

is the centre of v^* on D. Put $\mathcal{F}' = \mathcal{F}_0 \cup \{v^*\}$. For each $w \in \mathcal{F}'$, let D_w be the valuation ring of w in F. Then

$$\bigcap_{w\in\mathcal{F}'} D_w = \left(\bigcap_{w\in\mathcal{F}_0} D_w\right) \cap O' = K\{X\} \cap O' = D.$$

Since $\mathcal{F}_0$ is of finite type, so is $\mathcal{F}'$. Each $w \in \mathcal{F}_0$ is discrete, hence positive on D. Note that $D_{\mathfrak{p}^*} = O'$. Indeed, if $f(X)/g(X) \in O'$, where $0 \neq g(X)$, $f(X) \in D$ and $v^*(f(X)) \geqslant v^*(g(X))$, then (by dividing both $f(X)$ and $g(X)$ by a coefficient of minimal value) we have $f(X)/g(X) = a(f_1(X)/g_1(X))$ with $a \in O$, $f_1(X), g_1(X) \in D \setminus \mathfrak{p}^*$. Thus $f(X)/g(X) = (a \cdot f_1(X))/g_1(X) \in D_{\mathfrak{p}^*}$. In particular, v^* is well-centred on D. We have shown that each $w \in \mathcal{F}'$ is positive or well-centred on D, hence by Proposition 1.6, D is a generalized Krull domain, with an essential family $\mathcal{F}$ contained in $\mathcal{F}'$. Since $D_{\mathfrak{p}^*} = O'$, we have $v^* \in \mathcal{F}$, by Lemma 1.3. Thus $\mathcal{F} \setminus \{v^*\} \subseteq \mathcal{F}' \setminus \{v^*\} = \mathcal{F}_0$, hence each valuation in $\mathcal{F} \setminus \{v^*\}$ is discrete. □

Remark 2.3. In the proof of Lemma 2.2, one can show that in fact $\mathcal{F} = \mathcal{F}'$, but we shall not need this.

Theorem 2.4. *Let R be a domain contained in a rank-1 valuation ring R_v of $K = \mathrm{Quot}(R)$. Then $\mathrm{Quot}(R[\![X]\!])$ is Hilbertian.*

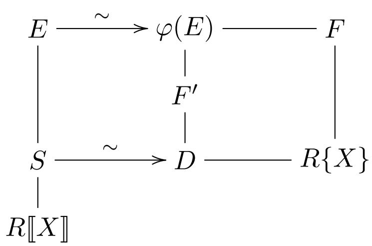
Proof. Let v be the valuation that corresponds to R_v. If $v(a) = 0$ for each $0 \neq a \in R$, then $R_v = K$, a contradiction. Thus we may choose $0 \neq a \in R$ with $v(a) > 0$. We have a monomorphism $\varphi\colon R[\![X]\!] \to R_v\{X\}$ given by

$$\varphi\biggl(\sum_{i=0}^{\infty} f_i X^i\biggr) = \sum_{i=0}^{\infty} f_i a^i X^i.$$

Put $E = \mathrm{Quot}(R[\![X]\!])$, $F = \mathrm{Quot}(R_v\{X\})$, and extend φ (in the unique possible way) to an embedding $\varphi\colon E \to F$. By Lemma 2.2, $R_v\{X\}$ is a generalized Krull domain, where its essential family $\mathcal{F}$ consists of v^* and discrete valuations. Put $D = R_v\{X\} \cap \varphi(E)$. Then $F' = \mathrm{Quot}(D)$ is a subfield of F, and $\mathcal{F}$ induces a family of finite type $\mathcal{F}'$ of rank-1 valuations on F', by restriction (throwing away valuations that are trivial on F'). For each $w \in \mathcal{F}'$, denote the valuation ring of w in F' by D_w. Then $D \subseteq \bigcap_{w \in \mathcal{F}} D_w \subseteq R_v\{X\} \cap F' \subseteq R_v\{X\} \cap \varphi(E) = D$, hence D is defined by $\mathcal{F}'$. Each $v \in \mathcal{F}' \setminus \{v^*|_{F'}\}$ is discrete, hence positive on D.

Note that φ is a K-embedding, hence $R_v \subseteq \varphi(E)$. Also, $R_v \subseteq R_v\{X\}$, hence $R_v \subseteq D$. This implies that v^* is not trivial on F', and that $v^*(R_v) \subseteq v^*(D) \subseteq v^*(R_v\{X\})$. On the other hand, by the definition of v^* we have $v(R_v) = v^*(R_v) = v^*(R_v\{X\})$, hence $v^*(D) = v^*(R_v\{X\}) = v(R_v)$. Let O' be the valuation ring of v^* in F'. Since v^* belongs to the essential family of $R_v\{X\}$, its valuation ring in F is $R_v\{X\}_{\mathfrak{p}^*}$, where $\mathfrak{p}^*$ is the centre of v^* on $R_v\{X\}$. Thus O' is contained in $R_v\{X\}_{\mathfrak{p}^*}$. Hence $v^*(O') \subseteq v^*(R_v\{X\}) = v^*(D)$, and since $D \subseteq O'$ we have $v^*(O') = v^*(D)$. That is, v^* is well-centred on D. Thus each valuation in $\mathcal{F}'$ is positive or well-centred on D. Since $\mathcal{F}'$ is not empty (for example since v^* is not trivial on F'), Proposition 1.6 implies that D is a generalized Krull domain.

Note that $S = \varphi^{-1}(R_v\{X\})$ maps isomorphically onto D via φ, hence S is a generalized Krull domain:



We now claim that $\dim S > 1$. Indeed, we have an epimorphism $\psi\colon R_v\{X\} \to R_v$ given by $\psi(\sum_{i=0}^{\infty} f_i X^i) = f_0$. Composing $\varphi|_S$ with ψ, we get a homomorphism $\psi \circ \varphi\colon S \to R_v$.

Note that the restriction of $\psi \circ \varphi$ to R_v is the identity map, so in particular $\psi \circ \varphi$ is an epimorphism, but not an isomorphism (since $(\psi \circ \varphi)(X) = 0$). Since R_v is a domain but not a field, $\mathrm{Ker}(\psi \circ \varphi)$ is a (non-zero) prime ideal in S, but not a maximal ideal. Hence $\dim S > 1$.

Thus S is a generalized Krull domain of dimension exceeding 1, hence by Weissauer's theorem [**3**, Theorem 15.4.6], $\mathrm{Quot}(S)$ is Hilbertian. But $R[\![X]\!] \subseteq S \subseteq E$, hence $\mathrm{Quot}(R[\![X]\!]) = \mathrm{Quot}(S) = E$ is Hilbertian. □

The immediate Galois-theoretic consequence of Theorem 2.4 is the following corollary (Corollary D of the introduction).

Corollary 2.5. *Let R be a domain, contained in a rank-1 valuation ring of* $\mathrm{Quot}(R)$. *Then every finite split embedding problem over $F = \mathrm{Quot}(R[\![X]\!])$ is solvable.*

Proof. Since $R[\![X]\!]$ is complete with respect to $\langle X \rangle$, F is ample by [**18**] and Hilbertian by Theorem 2.4. Thus the result follows by the main theorem of [**17**]. □

A special case of Theorem 2.4 is where R is a generalized Krull domain. Thus Theorem 2.4 provides a positive answer to [**3**, Problem 15.5.9(b)]. Together with [**15**, Theorem 2.4], this gives a complete answer to [**3**, Problem 15.5.9].

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