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Blow-up of solutions of a quasilinear parabolic equation

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We consider non-negative solutions of the Cauchy problem for quasilinear parabolic equations $u_t = \Delta u^m + f(u)$, where $m > 1$ and $f(\xi)$ is a positive function in $\xi > 0$ satisfying $f(0) = 0$ and a blow-up condition

$$\int_1^\infty \frac{1}{f(\xi)}\,\mathrm{d}\xi < \infty.$$

We show that if $\xi^{m+2/N}/(-\log \xi)^\beta = O(f(\xi))$ as $\xi \downarrow 0$ for some $0 < \beta < 2/(mN + 2)$, one of the following holds: (i) all non-trivial solutions blow up in finite time; (ii) every non-trivial solution with an initial datum u_0 having compact support exists globally in time and grows up to ∞ as $t \to \infty$: $\lim_{t\to\infty} \inf_{|x|<R} u(x, t) = \infty$ for any $R > 0$. Moreover, we give a condition on f such that (i) holds, and show the existence of f such that (ii) holds.

1. Introduction

In this paper, we discuss the Cauchy problem of quasilinear parabolic equations

$$u_t = \Delta u^m + f(u), \quad (x, t) \in \mathbb{R}^N \times (0, T), \tag{1.1}$$

$$u(x, 0) = u_0(x), \qquad x \in \mathbb{R}^N, \tag{1.2}$$

where $u_t = \partial u/\partial t$, Δ is the N-dimensional Laplacian and $f(\xi)$ with $\xi \geqslant 0$ and $u_0(x)$ with $x \in \mathbb{R}^N$ are non-negative functions. We consider only non-negative solutions.

Throughout this paper, we assume the following:

(A1) $u_0 \in \mathrm{BC}(\mathbb{R}^N)$ (bounded continuous functions) and $u_0(x) \geqslant 0$ in $\mathbb{R}^N$.

(A2) $f \in C^1[0, \infty) \cap C^\infty(0, \infty)$ and $f(\xi) > 0$ in $\xi > 0$.

Under assumptions (A1) and (A2), a unique non-negative weak solution of (1.1), (1.2) exists locally in time [1,2,4,6,21,29]. The definition of a weak solution is given in § 2. We know from the uniqueness of solutions and the existence theorem that if the solution does not exist globally in time, it blows up in finite time, that is, for some $T \in (0, \infty)$,

$$\lim_{t \uparrow T} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} = \infty.$$

Moreover, we assume the following two conditions:

(A3) $\displaystyle\int_\xi^\infty \frac{1}{f(\eta)} \, \mathrm{d}\eta < \infty \quad \text{for } \xi > 0.$

(A4) $f(0) = 0$.

Conditions (A3) and (A4) restrict the behaviours of $f(\xi)$ near $\xi = \infty$ and $\xi = 0$, respectively. Condition (A3) is a necessary condition for a solution to blow up in finite time. If (A3) fails, all solutions exist globally in time. Under condition (A3), the solution u of (1.1), (1.2) blows up in finite time if $u_0(x)$ tends to $\|u_0\|_{L^\infty(\mathbb{R}^N)}$ in some direction as $|x| \to \infty$ (see [31, theorem 1.5] and [13–15] when $m = 1$).

Condition (A4) is a necessary condition for a solution to exist globally in time, provided (A3) holds. If (A3) holds but (A4) does not, all solutions blow up in finite time. Under condition (A4), $u \equiv 0$ is a global solution of (1.1), (1.2) in time.

However, for a general initial datum $u_0 \not\equiv 0$ decaying as $|x| \to \infty$, we do not know whether the solution blows up in finite time or not. We are interested in the conditions on f for which a solution blows up in finite time.

For this problem, when $m = 1$, we have the familiar result proposed by Kaplan (in 1963) [17]: if $f''(\xi) \geqslant 0$ in $\xi \geqslant 0$, there exists an initial datum with compact support such that the solution of (1.1), (1.2) with $m = 1$ blows up in finite time. Kaplan showed this result for the Dirichlet problem in a bounded domain, and it holds for the Cauchy problem, as confirmed by the comparison theorem.

When at least one solution with the initial datum having compact support blows up in finite time, we call the phenomenon *C_0-blow-up* and say that (1.1) causes C_0-blow-up. For (1.1), we are more interested in the conditions on f for which C_0-blow-up occurs.

For the special equation

$$u_t = \Delta u^m + u^p, \quad (x, t) \in \mathbb{R}^N \times (0, T), \tag{1.3}$$

with $m \geqslant 1$ and $p > 1$, the blow-up problem has been studied by many authors (see [5, 24] for a review), following the pioneering work of Fujita in 1966. He showed [9] that the number $p_1^* = 1 + 2/N$ (called the Fujita exponent) divides exponents p into the following two cases when $m = 1$:

(i) if $1 < p < p_1^*$, all non-trivial solutions of (1.2), (1.3) with $m = 1$ blow up infinite time;

(ii) if $p > p_1^*$, there exists a global solution of (1.2), (1.3) with $m = 1$ when the initial datum u_0 is sufficiently small.

For the general equation (1.1), this result leads to the following question: under what condition on f do all non-trivial solutions blow up in finite time? When all

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non-trivial solutions blow up in finite time, we call the phenomenon *all-blow-up* and say that the equation causes all-blow-up.

Fujita's result was covered by Hayakawa [16] and Kobayashi *et al*. [19] (see also [32]) for the case $m = 1$ and $p = p_1^*$, and was extended to the case $m > 1$ by Galaktionov and co-workers [11, 12] (see also [18, 27]). We can summarize their results as follows.

(a) Let $1 < p \leqslant m + 2/N$. Then, all non-trivial solutions $u(x, t)$ of (1.2), (1.3) blow up in finite time, i.e. all-blow-up occurs.

(b) Let $p > m + 2/N$. Then, there exists a global solution of (1.2), (1.3) when the initial datum u_0 is sufficiently small, i.e. all-blow-up does not occur.

Case (a), in which all-blow-up occurs, is called the blow-up case. Case (b), in which all-blow-up does not occur, is called the global existence case. The cut-off number

$$p_m^* = m + \frac{2}{N}$$

is called the critical exponent (or the Fujita exponent). We can say that $f(u) = u^p$ ($1 < p \leqslant p_m^*$) belongs to the blow-up case and $f(u) = u^p$ ($p > p_m^*$) belongs to the global existence case. We can also say that the function $f(u) = u^{p_m^*}$ is a cut-off function between the blow-up case and the global existence case.

Thus, as is said above, for the general equation (1.1), we are interested in the conditions on f for which all non-trivial solutions blow up in finite time and the conditions on f for which at least one solution with the initial datum having compact support blows up in finite time. Thus, we consider

(i) what condition on f leads to C_0-blow-up and

(ii) what condition on f leads to all-blow-up.

These problems have already been studied for the semilinear case $m = 1$ by Kobayashi *et al*. [19]. In particular, problem (ii) for $m = 1$ has been completely solved by them. Roughly speaking, they showed the following results, by assuming (A1)–(A4).

(I) If $\xi^{1+2/N}/(-\log\xi) = O(f(\xi))$ as $\xi \downarrow 0$, every non-trivial solution of (1.1), (1.2) with $m = 1$ blows up in finite time or at $t = \infty$ (in other words, all global solutions are unbounded in $\mathbb{R}^N \times [0, \infty)$), i.e. $\lim_{t\uparrow T} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} = \infty$ for some $T \in (0, \infty]$. In particular, in the case $T = \infty$, the solution grows up to ∞ as $t \to \infty$, i.e. $\lim_{t\uparrow\infty} \inf_{|x|<R} u(x, t) = \infty$ for any $R > 0$. Moreover, if $f(\xi)$ is non-decreasing in $\xi \geqslant \xi_0$ for some $\xi_0 > 0$, all non-trivial solutions blow up in finite time, i.e. all-blow-up occurs.

(II) If $f(\xi) = O(\xi^{1+2/N}/(-\log\xi)^\beta)$ as $\xi \downarrow 0$ for some $\beta > 1$, there exists a global solution of (1.1), (1.2) with $m = 1$ converging to 0 as $t \to \infty$, when the initial datum u_0 is sufficiently small.

We now give some notation. For two functions $f(\xi)$ and $g(\xi)$ in $[0, \infty)$, we say that $f(\xi) = O(g(\xi))$ as $\xi \downarrow 0$ if $\limsup_{\xi\downarrow 0} |f(\xi)/g(\xi)| < \infty$. Similarly, we define

$f(\xi) = O(g(\xi))$ as $\xi \to \infty$. Also, for two functions $f(x)$ and $g(x)$ in $\mathbb{R}^N$, we define $f(x) = O(g(x))$ as $|x| \to \infty$.

We note that Kobayashi *et al.* also obtained more general results in which some assumptions are given in the integral form [19, theorems 3.5 and 4.1].

Their results are interesting for several reasons. First, they showed that for the general equation (1.1) with $m = 1$, the cut-off function (which divides the functions $f(u)$ into the blow-up case and the global-existence case) is not $f(u) = u^{p_1^*}$.

Second, from their results, we can easily see the mechanism of all-blow-up, i.e. all-blow-up is determined only by the behaviour of $f(\xi)$ near $\xi = 0$. More precisely, they showed that every non-trivial solution grows up by the behaviour of $f(\xi)$ near $\xi = 0$, and then we can see that if the function $f(\xi)$ causes C_0-blow-up, the non-trivial solution blows up in finite time, i.e. all-blow-up occurs (otherwise, the non-trivial solution exists globally in time and grows up to ∞ as $t \to \infty$).

Third, they posed a new problem of under what condition on f every non-trivial solution blows up in finite time or at $t = \infty$ (in other words, all global non-trivial solutions are unbounded in $\mathbb{R}^N \times (0, \infty)$).

Fourth, they relaxed the above mentioned C_0-blow-up condition on f proposed by Kaplan [17]. Specifically, they showed that if $f(\xi)$ satisfies (A3) and is non-decreasing in $\xi \geqslant \xi_0$ for some $\xi_0 > 0$, there exists an initial datum with compact support such that the solution of (1.1), (1.2) with $m = 1$ blows up in finite time.

However, in result (I) of [19], the following problem was not solved: when

$$\frac{\xi^{1+2/N}}{-\log \xi} = O(f(\xi)) \quad \text{as } \xi \downarrow 0$$

and for any $\xi_0 > 0$, $f(\xi)$ is not non-decreasing in $\xi \geqslant \xi_0$, does any global non-trivial solution (i.e. infinite-time blow-up solution) exist?

For $m = 1$, Fila *et al.* [7, 8] recently obtained interesting results, and they essentially answered the above question. They showed that assumption (A3) is not sufficient for finite-time blow-up if, for any $\xi_0 > 0$, $f(\xi)$ is not non-decreasing in $\xi \geqslant \xi_0$. Specifically, they showed the following result: for $p > 1$, there exists a function f satisfying (A2)–(A4) such that if $u_0(x) = O(|x_i|^{-1/(p-1)})$ as $|x| \to \infty$, $x = (x_1, \dots, x_N)$, for some $i \in (1, \dots, N)$, any solution of (1.1), (1.2) with $m = 1$ never blows up in finite time. We note that in this result the only condition on f near $\xi = 0$ is the condition $f(\xi) = O(\xi^p)$ $(\xi \downarrow 0)$. Combining this with the result of [19], we can easily see that there exists a function $f(\xi)$ satisfying (A2)–(A4) and $\xi^{1+2/N}/(-\log \xi) = O(f(\xi))$ as $\xi \downarrow 0$, such that every non-trivial solution with the initial datum having compact support exists globally in time and grows up to ∞ as $t \to \infty$. We note that such a function f is not non-decreasing in $\xi \geqslant \xi_0$ for any $\xi_0 > 0$.

Our aim in this paper is to extend the results of [7, 8, 19] to the quasilinear case $m > 1$. Specifically, we address the following three questions for the case $m > 1$, assuming (A1)–(A4).

(i) Under what condition on f does every non-trivial solution blow up in finite time or at $t = \infty$? In other words, under what condition on f are all global non-trivial solutions unbounded in $\mathbb{R}^N \times (0, \infty)$?

(ii) Under what condition on f do all non-trivial solutions blow up in finite time? In other words, under what condition on f does all-blow-up occur?

(iii) Under what condition on f does every non-trivial solution with the initial datum having compact support exist globally in time and grow up to ∞ as $t \to \infty$?

Moreover, as in [19], our results include an answer to the following question, which is first posed in our paper.

(iv) Under what condition on f do finite-time blow-up solutions with the initial data having compact support exist? In other words, under what condition on f does C_0-blow-up occur?

The method of the proof in [19], especially in the blow-up case, cannot be applied to the quasilinear case $m > 1$. It depends strongly on the integral expression of a solution by the heat kernel, and it never uses the Jensen inequality and Kaplan's method.

To state our results exactly, we introduce the following class of functions f associated with (1.1), where all functions f in the class lead to C_0-blow-up:

$$\begin{aligned}\mathfrak{M} = \{f \in C^1[0,\infty) \cap C^\infty(0,\infty) \mid\ & f \text{ satisfies condition (A2)} \\ & \text{and there exists an initial datum } u_0 \text{ having a compact support} \\ & \text{such that the solution of (1.1), (1.2) blows up in finite time}\}.\end{aligned}$$

We call this class *the C_0-blow-up class associated with (1.1)*. From the result of [19], we can say that a non-decreasing function f satisfying (A3) belongs to the C_0-blow-up class associated with (1.1) when $m = 1$, and hence, we can also say that such a function f leads to C_0-blow-up. We note that if $f \in \mathfrak{M}$, f must satisfy condition (A3).

Our results are as follows. Theorems 1.1, 1.3 and 1.4 attempt to answer questions (i), (ii), and (iii), respectively. Theorem 1.3 also tries to answer question (iv). Theorem 1.8 is a counterpart of theorem 1.1.

Theorem 1.1. *Let $m > 1$. Assume (A1), (A2) and (A4) hold. Let $u_0 \not\equiv 0$. Suppose that*

$$\text{(A5)}\quad \frac{\xi^{m+2/N}}{(-\log \xi)^\beta} = O(f(\xi)) \quad \text{as } \xi \downarrow 0 \text{ for some } 0 < \beta < \frac{2}{mN+2}.$$

Then, the solution u of (1.1), (1.2) blows up in finite time or at $t = \infty$, i.e. if $T \in (0,\infty]$ is the maximal existence time of u,

$$\lim_{t \uparrow T} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} = \infty.$$

Moreover, in the case $T = \infty$, the solution grows up to ∞ as $t \to \infty$:

$$\lim_{t\uparrow\infty} \inf_{|x|<R} u(x,t) = \infty \quad \text{for any } R > 0. \tag{1.4}$$

Condition (A5) restricts the behaviour of $f(\xi)$ near $\xi = 0$. In the above result, we see that only the behaviour of $f(\xi)$ near $\xi = 0$ leads to blow-up in finite time or at $t = \infty$ for non-trivial solutions. This leads to the following corollary, in which we can see that all-blow-up essentially occurs only by the behaviour of $f(\xi)$ near $\xi = 0$. We note that the blow-up condition (A3) is not assumed in theorem 1.1 but is assumed in the next corollary.

Corollary 1.2. *Let $m > 1$. Assume (A1)–(A5) hold. Then, the following hold.*

(i) *If f belongs to the C_0-blow-up class associated with (1.1), all non-trivial solutions of (1.1), (1.2) blow up in finite time.*

(ii) *If f does not belong to the C_0-blow-up class associated with (1.1), every non-trivial solution of (1.1), (1.2) with the initial datum u_0 having compact support exists globally in time and grows up to ∞ as $t \to \infty$, i.e. (1.4) holds.*

The existence of cases (i) and (ii) in corollary 1.2 is guaranteed by the next two theorems. In addition, in theorem 1.3, we give a condition for f to belong to the C_0-blow-up class associated with (1.1) when $m > 1$ (see condition (A6), below). We note that such a condition on f restricts the behaviour of $f(\xi)$ near $\xi = \infty$.

Theorem 1.3. *Let $m > 1$. Assume (A1)–(A5) and*

$$\text{(A6)} \quad \inf_{\xi \geqslant 1} \frac{\xi f'(\xi)}{f(\xi)} > -\infty \quad \text{and} \quad \lim_{A \to \infty} \inf_{\xi \geqslant 1} \frac{f(A\xi)}{Af(\xi)} = \infty$$

hold.

Then, f belongs to the C_0-blow-up class associated with (1.1), and hence all non-trivial solutions of (1.1), (1.2) blow up in finite time.

Theorem 1.4. *Let $m > 1$. Assume (A1) holds. Let $m < p \leqslant p_m^* = m + 2/N$. Let $g \in C^1[0, \infty) \cap C^\infty(0, \infty)$ satisfy (A4), (A5), and let $g(\xi) = O(\xi^p)$ as $\xi \downarrow 0$. Then, there exists a function f satisfying (A2), (A3), and $f(\xi) = g(\xi)$ in $(0, 1)$ such that if $u_0(x) = O(|x_i|^{-1/(p-m)})$ as $|x| \to \infty$, $x = (x_1, \dots, x_N)$, for some $i \in \{1, \dots, N\}$, the solution u of (1.1), (1.2) exists globally in time, and hence it grows up to ∞ as $t \to \infty$, i.e. (1.4) holds.*

Remark 1.5. A wide class of functions would satisfy condition (A6). For example, if $f \in C^1[0, \infty) \cap C^\infty(0, \infty)$ satisfies $f(\xi) = \xi^p/(\log \xi)^\beta$ in $\xi > L$ for some $p > 1$, $\beta \in \mathbb{R}$, and $L > 1$, f satisfies (A6).

Remark 1.6. As is stated above, when $m = 1$, combining the results of [19] and [7, 8], we can easily see that theorems 1.1, 1.3 and 1.4 and corollary 1.2 also hold, provided conditions (A5) and (A6) are replaced by the conditions that $\xi^{1+2/N}/(-\log \xi) = O(f(\xi))$ as $\xi \downarrow 0$ and $f(\xi)$ is non-decreasing in $\xi \geqslant \xi_0$ for some $\xi_0 > 0$, respectively.

Remark 1.7. In theorem 1.3, we can replace assumption (A6) by the condition that f is estimated from below by a function satisfying (A6).

The next theorem shows the existence of global solutions tending to 0 as $t \to \infty$.

THEOREM 1.8. *Let $m > 1$. Assume (A1), (A2) and*

$$\text{(A7)}\quad f(\xi) = O\biggl(\frac{\xi^{m+2/N}}{(-\log \xi)^{\beta}}\biggr) \quad \text{as } \xi \downarrow 0 \text{ for some } \beta > 1$$

hold. Then, there exists a global solution of (1.1), (1.2) in time, and it tends to 0 *as $t \to \infty$.*

REMARK 1.9. Our results are not complete, since we have no results for the case where $f(\xi)$ behaves like $\xi^{m+2/N}/(-\log \xi)^{\beta}$ as $\xi \downarrow 0$ with $\beta \in [2/(mN+2), 1]$.

In the proof of theorem 1.1, the following lemma, which will be given as lemma 3.2 below, plays a crucial role.

LEMMA A. *Let $m > 1$. Assume (A1), (A2) and (A5) hold. Suppose $u_0 \not\equiv 0$. Let u be a global solution of (1.1), (1.2). Then, there exists a constant $\delta > 0$ independent of u_0 such that, for some $T \in [0, \infty)$ (depending on u_0),*

$$\|u(\cdot, T)\|_{L^{\infty}(\mathbb{R}^N)} \geqslant \delta.$$

This lemma says that if (A5) holds, for some $\delta > 0$ independent of u_0, any global solution with $\|u_0\|_{L^{\infty}(\mathbb{R}^N)} < \delta$ must attain δ in finite time. We note that this lemma does not require assumption (A3) (a blow-up condition). It leads to theorem 1.1 and hence corollary 1.2. We also note that this lemma was proved in [19] for $m = 1$ using the integral expression of a solution by the heat kernel. However, this method cannot be applied to our quasilinear case $m > 1$, as explained above. Our method relies on the test function method used in [22, 23, 25, 26] (see also the references therein), which involves the judicious choice of a test function and the use of the Jensen inequality. This method is useful for showing the non-existence of global solutions to various problems (see, for example, [22, 23, 25]) and can be applied to our problem, albeit not directly. We must develop this method together with the Barenblatt solution (for $m > 1$) (see the proof of lemma 3.2).

To prove theorem 1.3, it suffices to show that the function f in theorem 1.3 belongs to the C_0-blow-up class associated with (1.1). Specifically, we shall show that if f satisfies (A3), (A4) and (A6), f belongs to the C_0-blow-up class (see proposition 4.1). This result is a generalization of the result of [12] (see also [20, chap. IV, § 3]), which treats the special equation (1.3). Therefore, the method of the proof is similar to that of [12].

Theorem 1.4 follows from corollary 1.2 if we show that the result of [7, 8] can be extended to the case $m > 1$ (see proposition 5.1). Theorem 1.8 is proved by the usual method.

The rest of the paper is organized as follows. In § 2, we define a weak solution of (1.1) and give several preliminary lemmas. In § 3, we prove theorem 1.1 and corollary 1.2. In § 4, we prove theorem 1.3 and in § 5 we prove theorem 1.4. In § 6, we prove theorem 1.8.

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2. Definitions and preliminaries

In this section, we define a weak solution of (1.1) and give several preliminary lemmas used in the next section. We begin with the definition. Let G be a domain in $\mathbb{R}^N$ with smooth boundary ∂G.

Definition 2.1. By a (weak) solution of (1.1) in $G \times (0, T)$, we mean a function $u(x, t)$ in $\bar{G} \times [0, T)$ such that

(i) $u(x, t) \geqslant 0$ in $\bar{G} \times [0, T)$ and $\in \mathrm{BC}(\bar{G} \times [0, \tau])$ for each $0 < \tau < T$,

(ii) for any bounded domain $\Omega \subset G$ with a smooth boundary $\partial\Omega$, $0 < \tau < T$ and non-negative $\varphi \in C^{2,1}(\bar{\Omega} \times [0, T))$ which vanishes on the boundary $\partial\Omega$,

$$\int_\Omega u(x, \tau)\varphi(x, \tau)\,\mathrm{d}x - \int_\Omega u(x, 0)\varphi(x, 0)\,\mathrm{d}x = \int_0^\tau \int_\Omega \{u\partial_t\varphi + u^m\Delta\varphi + f(u)\varphi\}\,\mathrm{d}x\,\mathrm{d}t - \int_0^\tau \int_{\partial\Omega} u^m \partial_\nu\varphi\,\mathrm{d}S\,\mathrm{d}t, \quad (2.1)$$

where ν denotes the outer unit normal to the boundary $\partial\Omega$.

A supersolution (subsolution) is similarly defined with the equality of (2.1) replaced by $\geqslant$ ($\leqslant$).

For a supersolution and a subsolution, the usual comparison theorem holds (cf. [31, proposition 2.3]).

We first show the following lemma: let $\{R_n\}$ and $\{t_n\}$ be sequences of positive numbers satisfying $\lim_{n\to\infty} R_n = \infty$.

Lemma 2.2. *Assume (A1) and (A2) hold. Let u be a global solution of (1.1), (1.2). Suppose that for some $\delta > 0$*

$$u(x, t_n) \geqslant \delta \ \textit{in}\ |x| < R_n \quad \textit{for}\ n \geqslant 1. \quad (2.2)$$

Then there exists a sequence of positive numbers $\{t'_n\}$ such that, for any $R > 0$,

$$\lim_{n\to\infty} \inf_{|x|<R} u(x, t'_n) = \infty. \quad (2.3)$$

Proof. Let u be a global solution of (1.1), (1.2) satisfying (2.2) for some $\delta > 0$. Let $u_{0,n} \in C_0^\infty(\mathbb{R}^N)$ $(n \geqslant 1)$ be a non-negative function such that $0 \leqslant u_{0,n}(x) \leqslant \delta$ for $x \in \mathbb{R}^N$ and

$$u_{0,n}(x) = \begin{cases} \delta & \text{in } |x| < \frac{1}{2}R_n, \\ 0 & \text{in } |x| > R_n, \end{cases}$$

and let $u_n(x, t)$ $(n \geqslant 1)$ be a solution of (1.1), (1.2) with the initial datum $u_0(x) = u_{0,n}(x)$. Then, it follows from the comparison theorem that $u(x, t + t_n) \geqslant u_n(x, t)$ in $\mathbb{R}^N \times (0, \infty)$.

On the other hand, since $u_{0,n}(x) \to \delta$ as $n \to \infty$ locally uniformly in $\mathbb{R}^N$, one can easily see that $u_n(x, t) \to v(t)$ as $n \to \infty$ locally uniformly in $\mathbb{R}^N \times [0, T_\delta)$, where

$v(t)$ is a solution of the ordinary equation $v'(t) = f(v(t))$ in $t > 0$ with $v(0) = \delta$, and T_δ ($\leqslant \infty$) is the maximal existence time of $v(t)$, i.e.

$$t = \int_\delta^{v(t)} \frac{1}{f(\eta)}\,\mathrm{d}\eta \quad \text{and} \quad T_\delta = \int_\delta^{\infty} \frac{1}{f(\eta)}\,\mathrm{d}\eta < \infty.$$

Let $\{s_k\}$ be a sequence of positive numbers satisfying $s_k \uparrow T_\delta$ as $k \to \infty$. Then, for any $k \geqslant 1$, there exists $n_k \geqslant 1$ such that

$$u_{n_k}(x, s_k) \geqslant \tfrac{1}{2} v(s_k) \quad \text{in } |x| < R_k,$$

where R_n is as above. Therefore, since $v(t) \to \infty$ as $t \uparrow T_\delta$, we see that

$$\inf_{|x|<R_k} u(x, s_k + t_{n_k}) \geqslant \inf_{|x|<R_k} u_{n_k}(x, s_k) \geqslant \tfrac{1}{2} v(s_k) \to \infty \quad \text{as } k \to \infty.$$

Setting $t'_k = s_k + t_{n_k}$, we get (2.3). The proof is complete. □

The next lemma is a familiar result on the positivity of solutions, which is stated with the (elementary) solution $E_m(x, t; L)$ $(L > 0)$ to the initial-value problem

$$\left.\begin{aligned} v_t &= \Delta v^m, \quad (x,t) \in \mathbb{R}^N \times (0, \infty), \\ v(x, 0) &= L\delta(x), \qquad x \in \mathbb{R}^N, \end{aligned}\right\} \tag{2.4}$$

where $L > 0$ and $\delta(x)$ is Dirac's δ-function. $E_m(x, t; L)$ is expressed concretely by

$$E_m(x, t; L) = L(L^{m-1}t)^{-\ell} G_m(\eta) = L^{2\ell/N} t^{-\ell} G_m(\eta) \tag{2.5}$$

with $\eta = x/(L^{m-1}t)^{\ell/N}$, where

$$\ell = \frac{N}{N(m-1)+2} = \left(m - 1 + \frac{2}{N}\right)^{-1} = (p_m^* - 1)^{-1}$$

and

$$G_m(\eta) = \begin{cases} (4\pi)^{-N/2} \mathrm{e}^{-|\eta|^2/4}, & m = 1, \\ [\tilde{A} - \tilde{B}|\eta|^2]_+^{1/(m-1)}, & m > 1, \end{cases}$$

with $[y]_+ = \max\{0, y\}$, $\tilde{B} = (m-1)\ell/2mN$ and $\tilde{A}$ chosen to satisfy

$$\int_{\mathbb{R}^N} G_m(x)\,\mathrm{d}x = 1.$$

$E_m(x, t; L)$ is the Barenblatt solution if $m > 1$ [30] and the usual heat kernel if $m = 1$.

Lemma 2.3. *Let $m > 1$. Assume (A1) and (A2) hold. Let u be a global solution of (1.1), (1.2). Suppose $u_0(0) > 0$. Then, there exist $L_1 > 0$ and $t_1 > 0$ such that*

$$u(x, t) \geqslant E_m(x, t + t_1; L_1) \quad \text{in } (x, t) \in \mathbb{R}^N \times [0, \infty).$$

Proof. See, for example, [27, lemma 3.4]. □

The following result follows from the above lemma.

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Lemma 2.4. *Assume (A1) and (A2) hold. Let* u *be a global solution of (1.1), (1.2). Let* $u_0 \not\equiv 0$. *Then, for any* $R > 0$, *there exists* $t_1 > 0$ *such that*

$$u(x, t_1) > 0 \quad \text{in } x \in B_R.$$

3. Proof of theorem 1.1 and corollary 1.2

In this section, we prove theorem 1.1 (and hence corollary 1.2). To do this, we need the following proposition, which immediately follows from lemma 3.2, below. This lemma plays a crucial role in the proof of the proposition and thus theorem 1.1. The method of the proof of the lemma, as mentioned in § 1, relies on the test function method used in [22, 23, 25, 26], which involves the judicious choice of a test function and the use of the Jensen inequality. However, we must develop it together with the Barenblatt solution (when $m > 1$).

Proposition 3.1. *Let* $m > 1$. *Assume (A1), (A2) and (A5) hold. Let* $u_0 \not\equiv 0$. *Let* u *be a global solution of (1.1), (1.2). There then exists a constant* $\delta \in (0, \frac{1}{2})$ *independent of* u_0 *such that the following holds: for any* $R > 0$, *there exists* $t_R > 0$ (*depending on* u_0) *such that*

$$u(x, t_R) \geqslant \delta \quad \text{in } x \in B_R.$$

Lemma 3.2. *Let* $m > 1$. *Assume (A1), (A2) and (A5) hold. Suppose* $u_0 \not\equiv 0$. *Let* u *be a global solution of (1.1), (1.2). There then exists a constant* $\delta \in (0, \frac{1}{2})$ *independent of* u_0 *such that, for some* $T \in [0, \infty)$ (*depending on* u_0),

$$\|u(\cdot, T)\|_{L^\infty(\mathbb{R}^N)} \geqslant \delta.$$

Proof. Without loss of generality, we may assume that $u_0(0) > 0$. By (A5), there exists $c_0 > 0$ such that

$$f(\xi) \geqslant c_0 \frac{\xi^{m+2/N}}{(-\log \xi)^\beta} \quad \text{for } \xi \in [0, \tfrac{1}{2}),$$

where $0 < \beta < 2/(mN + 2)$.

It suffices to show this lemma in the case where $f(\xi)$ satisfies

$$f(\xi) = c_0 \frac{\xi^{p_m^*}}{(-\log \xi)^\beta} \quad \text{in } \xi \in (0, \tfrac{1}{2}),$$

where $p_m^* = m + 2/N$.

We prove the lemma by contradiction. Assume that, for any $\delta \in (0, \frac{1}{2})$, there exists a global solution $u(x, t)$ of (1.1), (1.2) such that u never attains any value greater than or equal to δ in $\mathbb{R}^N \times (0, \infty)$. Then

$$u(x, t) < \delta < \tfrac{1}{2} \quad \text{in } \mathbb{R}^N \times (0, \infty). \tag{3.1}$$

Set

$$\tilde{f}(\xi) = c_0 \frac{\xi^{p_m^*/m}}{(-(1/m)\log \xi)^\beta} \quad \text{in } \xi \in (0, 1).$$

Clearly, $f'(\xi), f''(\xi), \tilde{f}'(\xi), \tilde{f}''(\xi) \geqslant 0$ in $\xi > 0$ and $\tilde{f}(\xi^m) = f(\xi)$ in $\xi \in (0, \frac{1}{2})$.

Let $\eta(t) \in C_0^\infty[0,\infty)$ and $\varphi(x) \in C_0^\infty(\mathbb{R}^N)$ be non-negative functions satisfying

$$
\begin{gathered}
0 \leqslant \eta(t) \leqslant 1 \text{ in } [0,\infty), \quad \eta(t) = 1 \text{ in } 0 \leqslant t \leqslant \tfrac{1}{2}, \quad \eta(t) = 0 \text{ in } t \geqslant 1, \\
0 \leqslant \varphi(x) \leqslant 1 \text{ in } \mathbb{R}^N, \quad \varphi(x) = 1 \text{ in } |x| \leqslant \tfrac{1}{2}, \quad \varphi(x) = 0 \text{ in } |x| \geqslant 1,
\end{gathered}
$$

and set, for $R > 1$,

$$\psi_R(x,t) = \varphi_R(x)\eta_R(t),$$

where $\varphi_R(x) = \varphi(x/R)$ and $\eta_R(t) = \eta(t/R^{2+N(m-1)})$. Then

$$
\begin{aligned}
&\eta_R(t) = 1 \text{ in } 0 \leqslant t \leqslant \tfrac{1}{2}R^{2+N(m-1)}, &&\eta_R(t) = 0 \text{ in } t \geqslant R^{2+N(m-1)}, \\
&\varphi_R(x) = 1 \text{ in } |x| \leqslant \tfrac{1}{2}R, &&\varphi_R(x) = 0 \text{ in } |x| \geqslant R,
\end{aligned}
$$

and

$$|\eta_R'(t)| \leqslant \frac{C_1}{R^{2+N(m-1)}} \text{ in } t \geqslant 0 \quad \text{and} \quad |\Delta\varphi_R(x)| \leqslant \frac{C_1}{R^2} \text{ in } \mathbb{R}^N,$$

where

$$C_1 = \sup_{t\geqslant 0}|\eta'(t)| + \sup_{x\in\mathbb{R}^N}|\Delta\varphi|.$$

Let $q > 1$ satisfy

$$\frac{1}{p_m^*/m} + \frac{1}{q} = 1.$$

As in [22, 23, 25], we consider $\psi(x,t) = \psi_R(x,t)^q$ as a test function in the integral identity satisfied by u (see (2.1)). A simple calculation gives

$$
\begin{aligned}
I_R &\equiv \int_0^\infty \int_{\mathbb{R}^N} f(u)\psi_R^q \,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^\infty \int_{\mathbb{R}^N} u \times q\psi_R^{q-1}\{\psi_R\}_t \,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}^N} u\psi_R^q \,\mathrm{d}x\bigg|_{t=0} \\
&\quad - \int_0^\infty \int_{\mathbb{R}^N} u^m\{q(q-1)\psi_R^{q-2}|\nabla\psi_R|^2 + q\psi_R^{q-1}\Delta\psi_R\} \,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant \frac{qC_1}{R^{2+N(m-1)}} \int_0^{R^{2+N(m-1)}} \int_{|x|<R} u\psi_R^{q-1} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{qC_1}{R^2} \int_0^{R^{2+N(m-1)}} \int_{|x|<R} u^m\psi_R^{q-1} \,\mathrm{d}x\,\mathrm{d}t \\
&= qc_1C_1R^N \int_0^{R^{2+N(m-1)}} \int_{|x|<R} \frac{u\psi_R^{q-1}}{k(R)} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + qc_1C_1R^{Nm} \int_0^{R^{2+N(m-1)}} \int_{|x|<R} \frac{u^m\psi_R^{q-1}}{k(R)} \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

where

$$k(R) = c_1R^{2+Nm} \quad \text{and} \quad c_1 = \int_0^1 \int_{|x|<1} \mathrm{d}x\,\mathrm{d}t.$$

Then, setting

$$J_R = \int_0^{R^{2+N(m-1)}} \int_{|x|<R} \frac{u\psi_R^{q-1}}{k(R)}\,\mathrm{d}x\,\mathrm{d}t$$

and

$$K_R = \int_0^{R^{2+N(m-1)}} \int_{|x|<R} \frac{u^m\psi_R^{q-1}}{k(R)}\,\mathrm{d}x\,\mathrm{d}t,$$

we get

$$I_R \leqslant qc_1C_1R^N J_R + qc_1C_1R^{Nm}K_R. \tag{3.2}$$

Here, we note that

$$\int_0^{R^{2+N(m-1)}} \int_{|x|<R} \frac{1}{k(R)}\,\mathrm{d}x\,\mathrm{d}t = 1$$

and by (3.1),

$$J_R < \delta \quad \text{and} \quad K_R < \delta^m (< \delta).$$

We estimate J_R and K_R in the following manner. Set

$$g(\xi) = \xi^{1/p_m^*}\{-\log\xi\}^{\beta/p_m^*} \quad \text{and} \quad \tilde{g}(\xi) = \xi^{m/p_m^*}\{-\log\xi\}^{m\beta/p_m^*} \quad \text{in } \xi \in (0, \mathrm{e}^{-\beta}).$$

Then, $g'(\xi), \tilde{g}'(\xi) \geqslant 0$ in $\xi > 0$. Furthermore, choosing $\delta \in (0, \min\{\frac{1}{2}, \mathrm{e}^{-\beta}\})$ to be sufficiently small, we have

$$f(\xi) < \mathrm{e}^{-\beta} \quad \text{and} \quad \tilde{f}(\xi) < \mathrm{e}^{-\beta} \quad \text{in } \xi \in (0,\delta) \tag{3.3}$$

and

$$\xi \leqslant C_2 \times (g\circ f)(\xi) \quad \text{and} \quad \xi \leqslant C_2 \times (\tilde{g}\circ\tilde{f})(\xi) \quad \text{in } \xi \in (0,\delta), \tag{3.4}$$

where $C_2 > 0$ is a positive constant. Hence, due to the Jensen inequality,

$$J_R \leqslant C_2 g(f(J_R)) \leqslant C_2 g\biggl(\int_0^{R^{2+N(m-1)}} \int_{|x|<R} \frac{f(u\psi_R^{q-1})}{k(R)}\,\mathrm{d}x\,\mathrm{d}t\biggr)$$

and

$$K_R \leqslant C_2 \tilde{g}(\tilde{f}(K_R)) \leqslant C_2 \tilde{g}\biggl(\int_0^{R^{2+N(m-1)}} \int_{|x|<R} \frac{\tilde{f}(u^m\psi_R^{q-1})}{k(R)}\,\mathrm{d}x\,\mathrm{d}t\biggr).$$

We note here that, by the relations $p_m^*(q-1) = mq \geqslant q$ and $p_m^*(q-1)/m = q$,

$$f(u\psi_R^{q-1}) \leqslant f(u)\psi_R^q \quad \text{and} \quad \tilde{f}(u^m\psi_R^{q-1}) \leqslant f(u)\psi_R^q \quad \text{in } \in \mathbb{R}^N \times (0,T).$$

Thus, we have

$$J_R \leqslant C_2 g\biggl(\int_0^{R^{2+N(m-1)}} \int_{|x|<R} \frac{f(u)\psi_R^q}{k(R)}\,\mathrm{d}x\,\mathrm{d}t\biggr) = C_2 g\biggl(\frac{I_R}{k(R)}\biggr)$$

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and

$$K_R \leqslant C_2\tilde{g}\biggl(\int_0^{R^{2+N(m-1)}}\int_{|x|<R}\frac{f(u)\psi_R^q}{k(R)}\,\mathrm{d}x\,\mathrm{d}t\biggr) = C_2\tilde{g}\biggl(\frac{I_R}{k(R)}\biggr).$$

Therefore, by (3.2), we obtain

$$\begin{aligned}I_R &\leqslant qc_1C_1C_2R^N g\biggl(\frac{I_R}{k(R)}\biggr) + qc_1C_1C_2R^{Nm}\tilde{g}\biggl(\frac{I_R}{k(R)}\biggr)\\ &= C_3 I_R^{1/p_m^*}\{-\log I_R + \log c_1 + (2+Nm)\log R\}^{\beta/p_m^*}\\ &\quad + C_4 I_R^{m/p_m^*}\{-\log I_R + \log c_1 + (2+Nm)\log R\}^{m\beta/p_m^*},\end{aligned}$$

where $C_3 = qc_1^{1-1/p_m^*}C_1C_2$ and $C_4 = qc_1^{1-m/p_m^*}C_1C_2$, and hence

$$I_R \leqslant 1 + C(1+\log R)^{m\beta/(p_m^*-m)} = 1 + C(1+\log R)^{m\beta N/2} \quad \text{for all } R > 1, \tag{3.5}$$

where C is a positive constant.

On the other hand, by lemma 2.3 and $u(0) > 0$, we have for some $t_1 > 0$ and $L_1 > 0$,

$$u(x,t) \geqslant E_m(x, t+t_1; L_1) \quad \text{in } (x,t) \in \mathbb{R}^N \times [0,\infty).$$

Hence, by (3.1), we get

$$\begin{aligned}I_R &= \int_0^\infty \int_{\mathbb{R}^N} f(u)\psi_R^q\,\mathrm{d}x\,\mathrm{d}t\\ &\geqslant c_0 \int_0^{R^{2+N(m-1)}/2}\int_{|x|\leqslant R/2}\frac{u^{m+2/N}}{(-\log u)^\beta}\,\mathrm{d}x\,\mathrm{d}t\\ &\geqslant c_0 \int_0^{R^{2+N(m-1)}/2}\int_{|x|\leqslant R/2}\frac{E_m(x,t+t_1;L_1)^{m+2/N}}{(-\log E_m(x,t+t_1;L_1))^\beta}\,\mathrm{d}x\,\mathrm{d}t\\ &\geqslant c_0 \int_0^{R^{N/\ell}/2}\int_{|\eta|\leqslant R/2(L_1^{m-1}(t+t_1))^{\ell/N}}\frac{L_1^{1+2/N}(t+t_1)^{-1}G_m(\eta)^{m+2/N}}{(-\log(L_1^{2\ell/N}(t+t_1)^{-\ell}G_m(\eta)))^\beta}\,\mathrm{d}\eta\,\mathrm{d}t\\ &\geqslant c_0 \int_0^{R^{N/\ell}/2}\int_{|\eta|\leqslant 1/2L_1^{\ell(m-1)/N}}\frac{L_1^{1+2/N}(t+t_1)^{-1}G_m(\eta)^{m+2/N}}{(-\log(L_1^{2\ell/N}(t+t_1)^{-\ell}G_m(\eta)))^\beta}\,\mathrm{d}\eta\,\mathrm{d}t\\ &\geqslant c_0 L_1^{1+2/N}\int_0^{R^{N/\ell}/2}\frac{(t+t_1)^{-1}}{(-\log(L_1^{2\ell/N}(t+t_1)^{-\ell}G_m(\tilde{R})))^\beta}\,\mathrm{d}t\int_{|\eta|\leqslant\tilde{R}}G_m(\eta)^{m+2/N}\,\mathrm{d}\eta,\end{aligned}$$

where $R > 1$ and $\tilde{R} > 0$ are taken to satisfy $R^{N/\ell} > 2t_1$, $\tilde{R} < 1/2L_1^{\ell(m-1)/N}$, and $\tilde{R}^2 < \tilde{A}/\tilde{B}$. Setting

$$c_5 = c_0 L_1^{1+2/N}\int_{|\eta|\leqslant\tilde{R}}G_m(\eta)^{m+2/N}\,\mathrm{d}\eta \quad (>0) \quad\text{and}\quad c_6 = -\log L_1^{2\ell/N}G_m(\tilde{R})$$

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and setting $s = \log(t + t_1)$, we have

$$I_R \geqslant c_5 \int_{\log t_1}^{(N/\ell)\log R - \log 2} \frac{1}{(\ell s + c_6)^\beta}\,\mathrm{d}s = \frac{c_5}{(1-\beta)\ell}\{(N\log R - \ell\log 2 + c_6)^{1-\beta} - c_7\}, \tag{3.6}$$

where $c_7 = (\ell \log t_1 + c_6)^{1-\beta}$.

Combining (3.5) and (3.6), we get, for large $R > 1$,

$$1 + C(1 + \log R)^{m\beta N/2} \geqslant \frac{c_5}{(1-\beta)\ell}\{(N\log R - \ell\log 2 + c_6)^{1-\beta} - c_7\},$$

which implies $m\beta N/2 \geqslant 1-\beta$, i.e. $\beta \geqslant 2/(mN+2)$. This contradicts the assumption that $\beta < 2/(mN+2)$. Thus, we reach a contradiction if $\delta \in (0, \min\{\frac{1}{2}, \mathrm{e}^{-\beta}\})$ is chosen to satisfy (3.3) and (3.4), and hence, for such $\delta > 0$, we see that

$$\|u(\cdot, T)\|_{L^\infty(\mathbb{R}^N)} \geqslant \delta$$

for some $T \in [0, \infty)$. The proof is complete. □

Proof of proposition 3.1. Let u be a global solution of (1.1), (1.2). Assume $u_0 \not\equiv 0$. Then, by lemma 2.4, for any $R > 0$, there exists $t_1 > 0$ such that

$$u(x, t_1) > 0 \quad \text{in } x \in B_{R+2},$$

where $B_R = \{|x| < R\}$.

Hence, without loss of generality, we may assume that

$$u_0(x) > 0 \quad \text{in } x \in B_{R+2}.$$

Set

$$\varepsilon_0 = \inf_{x \in B_{R+1}} u_0(x) \quad (> 0).$$

Let $0 < \varepsilon < \min\{\varepsilon_0, \delta\}$, where $\delta \in (0, \frac{1}{2})$ is as in lemma 3.2. Let $h_\varepsilon(x) = h_\varepsilon(r) \in C_0^\infty(B_1)$ $(r = |x|)$ be a radially symmetric non-negative function in $x \in \mathbb{R}^N$ such that $h_\varepsilon(r)$ is non-increasing in $r \geqslant 0$ and $h_\varepsilon(0) = \varepsilon$, where $B_1 = \{|x| < 1\}$. We extend h_ε in B_1 to $\mathbb{R}^N \setminus B_1$ as $h_\varepsilon = 0$. Let u_ε be a solution of (1.1), (1.2) with the initial datum $u_0(x) = h_\varepsilon(x)$. Then, for each $t > 0$, $u_\varepsilon(x, t) = u_\varepsilon(r, t)$ $(r = |x|)$ is also radially symmetric in $x \in \mathbb{R}^N$ and is non-increasing in $r \geqslant 0$; hence, $u_\varepsilon(x, t)$ in $x \in \mathbb{R}^N$ has maximum value at $x = 0$ for each $t > 0$. It follows from lemma 3.2 that, for some $t_R \in (0, \infty)$,

$$u_\varepsilon(0, t_R) = \delta.$$

Let $x_0 \in B_R$. Since $u_0(x + x_0) \geqslant h_\varepsilon(x)$ in $\mathbb{R}^N$, by the comparison theorem, we have

$$u(x + x_0, t) \geqslant u_\varepsilon(x, t) \quad \text{in } \mathbb{R}^N \times (0, \infty),$$

and thus,

$$u(x_0, t_R) \geqslant u_\varepsilon(0, t_R) = \delta.$$

This is the assertion of proposition 3.1, since $x_0 \in B_R$ can be chosen arbitrarily. The proof is complete. □

For the proof of theorem 1.1, we also need the next lemma.

LEMMA 3.3. *Let $\delta > 0$. Let $u_0 \in C(\mathbb{R}^N)$ be a non-negative function such that, for some $R > 0$,*

$$u_0(x) \geqslant \delta \quad \text{in } B_R.$$

If $R > 0$ is sufficiently large, there exists a non-negative function v_0 $(\not\equiv 0) \in C_0(\mathbb{R}^N)$ with $v_0^m \in C^\infty(\mathbb{R}^N)$ such that

$$v_0(x) \leqslant u_0(x) \quad \text{in } \mathbb{R}^N \tag{3.7}$$

and

$$\Delta v_0^m + f(v_0) \geqslant 0 \quad \text{in } \mathbb{R}^N. \tag{3.8}$$

Proof. It is not difficult to find a function $h(x)$ such that $h(x) = h(r) \in C_0^\infty[0,\infty)$ $(r = |x|)$ is a radially symmetric non-negative function in $x \in \mathbb{R}^N$, $h(r)$ is non-increasing in $r \geqslant 0$ and

$$h(r) = 1 \text{ in } r < \tfrac{1}{2}, \quad h(r) > 0 \text{ in } \tfrac{1}{2} \leqslant r < 1 \quad \text{and} \quad h(r) = 0 \text{ in } r \geqslant 1,$$

$$h''(r) + \frac{N-1}{r}h'(r) \geqslant 0 \quad \text{in } \tfrac{3}{4} \leqslant r < 1.$$

Set

$$h_R(x) = \left\{\delta h\left(\frac{x}{R}\right)\right\}^{1/m}.$$

Then, $h_R^m(x) \in C_0^\infty(\mathbb{R}^N)$, $0 \leqslant h_R(x) \leqslant h_R(0) = \delta^{1/m}$ in $\mathbb{R}^N$, $h_R(x) > 0$ in $|x| < R$ and $h_R(x) = 0$ in $|x| \geqslant R$. Furthermore,

$$\begin{aligned}\Delta h_R^m(x) &= \frac{\delta}{R^2}\Delta h\left(\frac{x}{R}\right) \\ &= \frac{\delta}{R^2}\left\{h''\left(\frac{|x|}{R}\right) + \frac{N-1}{|x/R|}h'\left(\frac{|x|}{R}\right)\right\} \geqslant 0 \quad \text{in } \tfrac{3}{4}R < |x| < R.\end{aligned}$$

If $R > 0$ is sufficiently large,

$$\Delta h_R^m(x) + f(h_R(x)) \geqslant -\frac{\delta}{R^2}\sup_{x\in\mathbb{R}^N}|\Delta h(x)| + \inf_{\delta h(3/4)\leqslant\xi^m\leqslant\delta} f(\xi) \geqslant 0 \quad \text{in } |x| \leqslant \tfrac{3}{4}R.$$

Setting $v_0(x) = h_R(x)$ for large $R > 0$, we have (3.7) and (3.8). The proof is complete. □

Proof of theorem 1.1. Let $u_0 \not\equiv 0$ and let u be a global solution of (1.1), (1.2). By proposition 3.1 and lemma 3.3, we may assume, without loss of generality, that $u_0^m(x) \in C^\infty(\mathbb{R}^N)$ and

$$\Delta u_0^m + f(u_0) \geqslant 0 \quad \text{in } \mathbb{R}^N.$$

Since $u_0(x)$ is a subsolution of (1.1), as in [3] (see also [10]), we see that $u(x,t)$ is non-decreasing with $t > 0$ for each $x \in \mathbb{R}^N$. Combining lemma 2.2 and proposition 3.1, we can find a sequence of positive numbers $\{t_n\}$ such that

$$\inf_{t\geqslant t_n}\inf_{|x|<R} u(x,t_n) \geqslant \inf_{|x|<R} u(x,t_n) \to \infty \quad \text{as } n \to \infty \text{ for each } R > 0.$$

The proof is complete. □

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Proof of corollary 1.2. Corollary 1.2 follows from theorem 1.1. □

4. Proof of theorem 1.3

In this section, we prove theorem 1.3. It suffices to show the next proposition, which gives a condition for f to belong to the C_0-blow-up class associated with (1.1) when $m > 1$.

PROPOSITION 4.1. *Assume $m > 1$. Assume (A2)–(A4) and (A6) hold. Then, f belongs to the C_0-blow-up class associated with (1.1). Specifically, there exists an initial datum u_0 having compact support such that the solution of (1.1), (1.2) blows up in finite time.*

As noted in § 1, this proposition is a generalization of the result of [12] (see also [20, chap. IV, § 3]) and the method of the proof of this proposition is similar to that of [12].

Proof. We construct a blow-up subsolution of (1.1) in the form

$$w(x,t) = Ah(t)\Theta^{1/(m-1)}$$

with

$$\Theta(x,t) = \left[1 - (m-1)\frac{\xi(t)}{a^2}|x|^2\right]_+, \tag{4.1}$$

where $[a]_+ = \max\{a, 0\}$, $h(t)$ is a solution of the ordinary differential equation

$$h' = f(h), \quad t > 0, \qquad h(0) = 1, \tag{4.2}$$

$$\xi(t) = \frac{f(h(t))}{h^m(t)}, \tag{4.3}$$

and $A > 0$ and $a > 0$ will be specified later. We note that w blows up at

$$T = \int_1^\infty \frac{1}{f(\xi)}\,\mathrm{d}\xi < \infty,$$

if (A3) holds.

We calculate

$$J(w) = \Delta w^m + f(w) - w_t$$

in the domain $D = \{(x,t) \in \mathbb{R}^N \times (0,T) \| x|^2 < a^2/(m-1)\xi(t)\}$. Note that $\Theta(x,t) > 0$ in D. Since

$$\Delta w^m = 4mA^m h^m \times \frac{\xi^2}{a^4}|x|^2\Theta^{(-m+2)/(m-1)} - 2mNA^m h^m \times \frac{\xi}{a^2}\Theta^{1/(m-1)}$$

and

$$w_t = Ah'\Theta^{1/(m-1)} - Ah|x|^2\frac{\xi'}{a^2}\Theta^{(-m+2)/(m-1)},$$

by (4.1), we have

$$
\begin{aligned}
J(w) &= \left(\frac{4m}{m-1}\frac{A^m}{a^2}h^m\xi + \frac{Ah}{m-1}\frac{\xi'}{\xi}\right) \times (m-1)\frac{\xi}{a^2}|x|^2\Theta^{(-m+2)/(m-1)} \\
&\quad + \left(-2mN\frac{A^m}{a^2}h^m\xi - Ah'\right)\Theta^{1/(m-1)} + f(Ah\Theta^{1/(m-1)}) \\
&= \Theta^{(-m+2)/(m-1)}\left\{\left(\frac{4m}{m-1}\frac{A^m}{a^2}h^m\xi + \frac{Ah}{m-1}\frac{\xi'}{\xi}\right)\right. \\
&\quad - \left(\frac{4m}{m-1}\frac{A^m}{a^2}h^m\xi + \frac{Ah}{m-1}\frac{\xi'}{\xi} + 2mN\frac{A^m}{a^2}h^m\xi + Ah'\right)\Theta \\
&\quad \left. + f(Ah\Theta^{1/(m-1)})\Theta^{(m-2)/(m-1)}\right\}.
\end{aligned}
$$

Thus, if we use the relations $\xi'/\xi = f'(h) - mf(h)/h$ and $\xi h^m = f(h)$ from (4.2) and (4.3), we get

$$
\begin{aligned}
J(w) &= \frac{1}{m-1}Af(h)\Theta^{(-m+2)/(m-1)} \\
&\quad \times \left\{4m\frac{A^{m-1}}{a^2} - m + (1-\Theta)\frac{hf'(h)}{f(h)}\right. \\
&\quad - \left(2m(2+mN-N)\frac{A^{m-1}}{a^2} - 1\right)\Theta \\
&\quad \left. + (m-1)\Theta^{(m-2)/(m-1)}\frac{f(Ah\Theta^{1/(m-1)})}{Af(h)}\right\} \\
&\geqslant \frac{1}{m-1}Af(h)\Theta^{(-m+2)/(m-1)}I(w),
\end{aligned}
$$

where

$$
\begin{aligned}
I(w) = &\left\{4m\frac{A^{m-1}}{a^2} - m - c_0\right. \\
&\left. - \left(2m(2+mN-N)\frac{A^{m-1}}{a^2} - 1\right)\Theta + (m-1)\Theta\frac{f(A\Theta^{1/(m-1)}h)}{A\Theta^{1/(m-1)}f(h)}\right\}
\end{aligned}
$$

and

$$c_0 = \left|\inf_{\xi\geqslant 1}\frac{\xi f'(\xi)}{f(\xi)}\right| < \infty$$

(see condition (A6)).

Therefore, we shall estimate $I(w)$. Setting, for $A > 0$,

$$a = a(A) = \sqrt{\frac{2m}{m+c_0}A^{m-1}},$$

we have

$$I(w) = m + c_0 - ((m+c_0)(2+mN-N) - 1)\Theta + (m-1)\Theta\frac{f(A\Theta^{1/(m-1)}h)}{A\Theta^{1/(m-1)}f(h)}.$$

Setting

$$\Theta_0 = \frac{m + c_0}{(m + c_0)(2 + mN - N) - 1} \in (0, 1)$$

and choosing $A > 0$ sufficiently large to satisfy

$$(m + c_0)(1 + mN - N) - 1 < (m - 1)\Theta_0 \inf_{\Theta_0 \leqslant \Theta \leqslant 1} \inf_{\xi \geqslant 1} \frac{f(A\Theta^{1/(m-1)}\xi)}{A\Theta^{1/(m-1)} f(\xi)}$$

(see condition (A6)), we get $I(w) \geqslant 0$ and thus $J(w) \geqslant 0$ in D. We can clearly see that $w(x, t)$ is a subsolution of (1.1) in $\mathbb{R}^N \times (0, T)$ [20, chap. IV, § 3].

Now, let u be a solution of (1.1), (1.2). If $u_0(x) \geqslant w(x, 0)$ in $\mathbb{R}^N$, $u \geqslant w$ in $\mathbb{R}^N \times (0, T)$, and therefore u blows up in finite time. The proof is complete. □

Proof of theorem 1.3. Theorem 1.3 follows from corollary 1.2 and proposition 4.1. □

5. Proof of theorem 1.4

In this section, we prove theorem 1.4. It suffices to show the next proposition.

Proposition 5.1. *Let $m > 1$. Assume (A1) holds. Let $g(\xi) = c_0\xi^p$ for some $p > m$ and $c_0 > 0$. Then, there exists a function f satisfying (A2), (A3), $f(\xi) = g(\xi)$ in $(0, 1)$ and $f(\xi) \leqslant g(\xi)$ in $(0, \infty)$ such that if $u_0(x) = O(|x_i|^{-1/(p-m)})$ as $|x| \to \infty$, $x = (x_1, \dots, x_N)$, for some $i \in \{1, \dots, N\}$, the solution u of (1.1), (1.2) exists globally in time.*

We shall prove this proposition only for $c_0 = 1$, when $m > 1$. The method of the proof is similar to that of [7, 8]. We first prove this proposition for $N = 1$. For this purpose, we need several lemmas. The next two lemmas are given by [7, 8].

Lemma 5.2. *Let $p > m$ (> 1). Let $\lambda > (p - m)/6m$. Let β_n be positive numbers such that*

$$\sum_{n=1}^{\infty} \beta_n < \infty,$$

and let $\{a_n\}$ be an increasing sequence such that $a_1 > \max\{1, \beta_1^{-1/2\lambda}\}$ and $a_n > \max\{2a_{n-1}, \beta_n^{-1/2\lambda}\}$ for $n \geqslant 2$. Set

$$b_n = a_n + (1 - \exp(-a_n^{4\lambda}))a_n^{(p/m)-6\lambda} \quad (< 2a_n < a_{n+1}). \tag{5.1}$$

Then, there exists a positive function $h(\eta) \in C^1[0, \infty) \cap C^\infty(0, \infty)$ satisfying

$$\int_1^\infty \frac{1}{h(\eta)}\, \mathrm{d}\eta < \infty,$$

$h(\eta) = \eta^{p/m}$ in $(0, 1)$ and $h(\eta) \leqslant \eta^{p/m}$ in $(0, \infty)$, and there exist functions $v_n \in C^2(-a_n^\lambda, a_n^\lambda)$ $(n \geqslant 1)$ such that

$$v_{n,xx} + h(v_n) = 0 \quad \text{in } -a_n^\lambda < x < a_n^\lambda,$$
$$v_n(0) = b_n, \quad v_{n,x}(0) = 0, \quad v_n(x) \geqslant a_n \quad \text{for } -a_n^\lambda < x < a_n^\lambda.$$

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Proof. See the proofs of [7, lemma 5.2] and [8, lemma 2]. We use $g(s) = s^{p/m}$ in the proof of [7, lemma 5.2]. □

The next lemma is a result about a travelling-wave solution $u(x,t) = w(\xi)$ of

$$\frac{1}{m}u_t = u_{xx} + u^{p/m} \quad \text{in } (x,t) \in \mathbb{R} \times (0,\infty) \tag{5.2}$$

with $\xi = x - t$.

LEMMA 5.3. *Let $p > m$ (> 1). Then, there exist $\xi_0 > 0$ and a solution $w(\xi) \in C^2[-\xi_0,\infty)$ of*

$$-\frac{1}{m}w_\xi = w_{\xi\xi} + w^{p/m} \quad \text{in } [-\xi_0,\infty)$$

such that

$$\left.\begin{aligned} w(-\xi_0) = 0, \quad w(\xi) > 0 \text{ in } \xi > -\xi_0, \\ w(0) < 1, \quad w_\xi(\xi) < 0 \text{ in } \xi \geqslant 0, \\ \lim_{\xi\to\infty} \xi^{m/(p-m)} w(\xi) = (p-m)^{-m/(p-m)}. \end{aligned}\right\} \tag{5.3}$$

Proof. See the proof of [8, lemma 3]. □

Hence, $w(x,t) = w(x-t)$ is a travelling solution of (5.2), and thus, we can see that $u(x,t) = \tilde{w}(x,t) \equiv w^{1/m}(x,t)$ is a supersolution of

$$u_t = \{u^m\}_{xx} + u^p \tag{5.4}$$

in $\{(x,t) \in \mathbb{R}^N \times (0,\infty) \mid x \geqslant t > 0\}$. This is a key tool for proving proposition 5.1 and is stated in the next lemma.

LEMMA 5.4. *Let $\tilde{w}(\xi) = w^{1/m}(\xi)$ and let $\tilde{w}(x,t) = \tilde{w}(x-t)$, where $w(\xi)$ is as in lemma 5.3. Then, $\tilde{w}(x,t)$ is a supersolution of (5.4) in $\{(x,t) \in \mathbb{R} \times (0,\infty) \mid x \geqslant t > 0\}$, i.e.*

$$\tilde{w}_t \geqslant \{\tilde{w}^m\}_{xx} + \tilde{w}^p \quad \text{in } \{(x,t) \in \mathbb{R} \times (0,\infty) \mid x \geqslant t > 0\}. \tag{5.5}$$

Moreover, it satisfies

$$\left.\begin{aligned} \tilde{w}(-\xi_0) = 0, \quad \tilde{w}(\xi) > 0 \text{ in } \xi > -\xi_0, \\ \tilde{w}(0) < 1, \quad \tilde{w}_\xi(\xi) < 0 \text{ in } \xi \geqslant 0, \\ \lim_{\xi\to\infty} \xi^{1/(p-m)} \tilde{w}(\xi) = (p-m)^{-1/(p-m)}. \end{aligned}\right\} \tag{5.6}$$

Proof. We prove only (5.5). Since $0 < w(\xi) < 1$ and $w_\xi(\xi) < 0$ in $\xi \geqslant 0$ by (5.3), we have

$$\begin{aligned} \tilde{w}_t - \{\tilde{w}^m\}_{xx} - \tilde{w}^p &= -\frac{1}{m}w^{-(m-1)/m}w_\xi(\xi) - w_{\xi\xi}(\xi) - w^{p/m}(\xi) \\ &\geqslant -\frac{1}{m}w_\xi(\xi) - w_{\xi\xi}(\xi) - w^{p/m}(\xi) = 0 \quad \text{in } \xi = x - t \geqslant 0. \end{aligned}$$

□

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Proof of proposition 5.1 for $N=1$. See the proof of [8, theorem 2]. Let λ, a_n, b_n, $h(\eta)$ and v_n be as in lemma 5.2. Let $\tilde{w}$ be as in lemma 5.4. Set $u_n = v_n^{1/m}$ and $f(\xi) = h(\xi^m)$. Then, we see that $f(\xi) = \xi^p$ in $(0,1)$, $f(\xi) \leqslant \xi^p$ in $(0,\infty)$ and

$$\int_1^\infty \frac{1}{f(\xi)}\,\mathrm{d}\xi = \frac{1}{m}\int_1^\infty \frac{\eta^{-(m-1)/m}}{h(\eta)}\,\mathrm{d}\eta \leqslant \frac{1}{m}\int_1^\infty \frac{1}{h(\eta)}\,\mathrm{d}\eta < \infty. \tag{5.7}$$

Furthermore,

$$\left.\begin{aligned} \{u_n^m\}_{xx} + f(u_n) = 0 \quad \text{in } -a_n^\lambda < x < a_n^\lambda, \\ u_n(0) = b_n^{1/m}, \quad u_{n,x}(0) = 0, \quad u_n(x) \geqslant a_n^{1/m} \quad \text{for } -a_n^\lambda < x < a_n^\lambda. \end{aligned}\right\} \tag{5.8}$$

In the following manner, we shall construct a supersolution $\psi_n(x,t)$ $(n \geqslant 1)$ of (1.1) with $N=1$ whose existence time goes to ∞ as $n \to \infty$. To do this, we use $\tilde{w}$ in lemma 5.4 to construct a supersolution of (1.1) with $N=1$ outside $[-a_n^\lambda, a_n^\lambda]$.

First, let $k>0$ and let

$$\tilde{w}_k(x,t) = k^{2/(p-m)}\tilde{w}(kx, k^{2(p-1)/(p-m)}t) = k^{2/(p-m)}\tilde{w}(k(x - k^{(p+m-2)/(p-m)}t)).$$

We then see that $\tilde{w}_k$ is a supersolution of (5.4) in

$$\{(x,t) \mid x \geqslant k^{(p+m-2)/(p-m)}t,\ t>0\}$$

and $\tilde{w}_k(k^{(p+m-2)/(p-m)}t, t) = k^{2/(p-m)}\tilde{w}(0)$ for $t \geqslant 0$. We note by (5.6) that, for some $\tilde{\xi}_0 > 0$,

$$\xi^{1/(p-m)}\tilde{w}(\xi) \geqslant \tfrac{1}{2}c_* \quad \text{for } \xi \geqslant \tilde{\xi}_0,$$

where $c_* = (p-m)^{-1/(p-m)}$. Therefore, we have

$$\tilde{w}_k(x,0) = k^{2/(p-m)}\tilde{w}(kx) \geqslant \tfrac{1}{2}c_* k^{1/(p-m)}|x|^{-1/(p-m)} \quad \text{for } x \geqslant \frac{\tilde{\xi}_0}{k}. \tag{5.9}$$

Next, to consider $\tilde{w}_{k_n}$ and u_n in $\{(x,t) \mid k_n^{(p+m-2)/(p-m)}t \leqslant x \leqslant a_n^\lambda,\ 0 \leqslant t \leqslant t_n\}$, we can choose $k_n > 0$ $(n \geqslant 1)$ and $t_n > 0$ $(n \geqslant 1)$ to satisfy, for large n,

$$k_n^{(p+m-2)/(p-m)}t < a_n^\lambda \quad \text{in } 0 \leqslant t \leqslant t_n, \tag{5.10}$$

$$\tilde{w}_{k_n}(k_n^{(p+m-2)/(p-m)}t, t) \geqslant u_n(k_n^{(p+m-2)/(p-m)}t) \quad \text{in } 0 \leqslant t \leqslant t_n, \tag{5.11}$$

$$\tilde{w}_{k_n}(a_n^\lambda, t) < u_n(a_n^\lambda) \quad \text{in } 0 \leqslant t \leqslant t_n, \tag{5.12}$$

and

$$t_n \to \infty \quad \text{as } n \to \infty. \tag{5.13}$$

In fact, choosing

$$k_n = \left(\frac{b_n^{1/m}}{\tilde{w}(0)}\right)^{(p-m)/2}, \tag{5.14}$$

we have

$$\tilde{w}_{k_n}(k_n^{(p+m-2)/(p-m)}t, t) = b_n^{1/m} = u_n(0) \geqslant u_n(k_n^{(p+m-2)/(p-m)}t) \quad \text{for } t \geqslant 0.$$

Since $\tilde{w}(\xi)$ is non-increasing in $\xi > 0$ and converges to 0 as $\xi \to \infty$, there are positive constants $\xi_n > 0$ $(n \geqslant 1)$ such that

$$\tilde{w}(k_n \xi_n) = \frac{\tilde{w}(0)(a_n^{1/m} - 1)}{b_n^{1/m}} \quad (< \tilde{w}(0)). \tag{5.15}$$

We note that

$$k_n \xi_n \to 0 \text{ as } n \to \infty \quad \text{and so} \quad \xi_n \to 0 \text{ as } n \to \infty,$$

since $\lim_{n\to\infty} a_n/b_n = 1$ by (5.1) and $\lambda > (p-m)/6m$.

Set

$$t_n = \frac{a_n^{\lambda} - \xi_n}{k_n^{(p+m-2)/(p-m)}}$$

and choose λ to satisfy $\lambda > (p+m-2)/2m$ $(\geqslant (p-m)/6m)$. Then, we have $t_n > 0$ for large $n \geqslant 1$ and

$$\begin{aligned} t_n &= \frac{\tilde{w}(0)^{(p+m-2)/2}}{b_n^{(p+m-2)/2m}} a_n^{\lambda} - \frac{\xi_n}{k_n^{(p+m-2)/2m}} \\ &= \tilde{w}(0)^{(p+m-2)/2} \left(\frac{a_n}{b_n}\right)^{(p+m-2)/2m} a_n^{\lambda-(p+m-2)/2m} \\ &\quad - \frac{\xi_n}{k_n^{(p+m-2)/(p-m)}} \to \infty \quad \text{as } n \to \infty. \end{aligned}$$

Moreover, due to (5.14) and (5.15),

$$\begin{aligned} \tilde{w}_{k_n}(a_n^{\lambda}, t_n) &= k_n^{2/(p-m)} \tilde{w}(k_n \xi_n) \\ &= \frac{b_n^{1/m}}{\tilde{w}(0)} \frac{\tilde{w}(0)(a_n^{1/m} - 1)}{b_n^{1/m}} \\ &= a_n^{1/m} - 1 \\ &< u_n(a_n^{\lambda}) \quad \text{in } 0 \leqslant t \leqslant t_n. \end{aligned}$$

Therefore, these k_n and t_n satisfy (5.10)–(5.13).

Thus, setting, for $t \in [0, t_n]$,

$$x_n(t) = \inf\{x \mid k_n^{(p+m-2)/(p-m)} t \leqslant x \leqslant a_n^{\lambda},\ \tilde{w}_{k_n}(y,t) \leqslant u_n(y) \text{ for } y \in [x, a_n^{\lambda}]\}$$

and taking, for $t \in [0, t_n]$,

$$\psi_n(x,t) = \begin{cases} \tilde{w}_{k_n}(-x,t) & \text{for } x \leqslant -x_n(t), \\ u_n(x,t) & \text{for } -x_n(t) < x < x_n(t), \\ \tilde{w}_{k_n}(x,t) & \text{for } x_n(t) \leqslant x, \end{cases}$$

we can consider ψ_n a supersolution of (1.1) with $N = 1$. In fact, assume $u_0(x) = O(|x|^{-1/(p-m)})$ as $|x| \to \infty$, and let u be a solution of (5.4), (1.2) in $\mathbb{R} \times (0,T)$. If n is sufficiently large, by (5.8) and (5.9), we have

$$u_0(x) < \psi_n(x, 0) \quad \text{in } \mathbb{R}.$$

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Hence, if we use the comparison theorem and the strong maximum principle, it is not difficult to see that, for large $n \geqslant 1$ [8],

$$u(x,t) \leqslant \psi_n(x,t) \quad \text{in } \mathbb{R} \times [0,t_n].$$

Since $t_n \to \infty$ as $n \to \infty$, we see that u exists globally in time. □

Proof of proposition 5.1 for $N > 1$. For the higher-dimensional case $N > 1$, we can obtain the same result as for $N = 1$ with $\psi_n(x,t)$ replaced by $\psi_n(x_i,t)$. □

Proof of theorem 1.4. The theorem follows from corollary 1.2 and proposition 5.1. □

6. Proof of theorem 1.8

In this section, we show the existence of global solutions of (1.1), (1.2) tending to 0 as $t \to \infty$, by assuming (A7) holds.

Proposition 6.1. *Assume (A1), (A2) and (A7) hold. Then, there exists a global solution of (1.1), (1.2) in time such that it tends to* 0 *as $t \to \infty$.*

Proof. We prove this proposition only when

$$f(\xi) = \frac{\xi^{m+2/N}}{(-\log \xi)^{\beta}} \quad \text{in } 0 < \xi < \tfrac{1}{2} \text{ for some } \beta > 1.$$

The method of the proof is similar to that of [28]. For the proof, we use the elementary solution $E(x,t) = E_m(x,t;1)$ of (2.4) with $L = 1$, which is represented by (2.5) with $L = 1$. We note that, for large $t_1 > 0$,

$$E(x,t+t_1) \leqslant M(t+t_1)^{-\ell} \leqslant Mt_1^{-\ell} \leqslant \tfrac{1}{4} \quad \text{for } (x,t) \in \mathbb{R}^N \times (0,T),$$

where $M = \sup_{\eta \in \mathbb{R}} G_m(\eta)$, and hence

$$\frac{E(x,t+t_1)^{m-1+2/N}}{(-\log E(x,t+t_1))^{\beta}} = \frac{E(x,t+t_1)^{1/\ell}}{(-\log E(x,t+t_1))^{\beta}} \leqslant k(t;t_1) \quad \text{in } \mathbb{R}^N \times (0,T), \tag{6.1}$$

where

$$k(t;t_1) = \frac{M^{1/\ell}(t+t_1)^{-1}}{(-\log M(t+t_1)^{-\ell})^{\beta}} \quad \text{for } t \geqslant 0.$$

Let $\alpha(t)$ be a solution of the ordinary equation $\alpha'(t) = k(t;t_1)\alpha(t)^{(2/N)+1}$ in $t > 0$ with the initial datum $\alpha(0) = \frac{1}{2}$, that is,

$$\alpha(t) = \left\{ 2^{2/N} - \frac{2}{N} \int_0^t k(t;t_1)\,\mathrm{d}t \right\}^{-N/2}.$$

Then, if we choose $t_1 > 0$ sufficiently large to satisfy

$$\int_0^{\infty} k(t;t_1)\,\mathrm{d}t = \frac{M^{1/\ell}}{(\beta-1)\ell}(-\log Mt_1^{-\ell})^{-\beta+1} \leqslant \frac{N}{2}(2^{2/N}-1),$$

$\alpha(t)$ exists in $(0,\infty)$ and

$$\tfrac{1}{2} \leqslant \alpha(t) \leqslant 1 \quad \text{for } t > 0. \tag{6.2}$$

Let $b(t)$ ($\geqslant 0$) be the solution of the ordinary equation

$$b'(t) = \{\alpha(b(t))\}^{m-1}, \qquad b(0) = 0,$$

and set

$$w(x,t) = \alpha(b(t))E(x, b(t) + t_1) \quad (\leqslant \tfrac{1}{4}).$$

Then, by (6.1) and (6.2),

$$\begin{aligned} w_t - \Delta w^m &= k(b(t); t_1)\alpha(b(t))^{m+2/N} E(x, b(t) + t_1) \\ &\geqslant \frac{E(x, b(t) + t_1)^{m+2/N}}{(-\log E(x, b(t) + t_1))^{\beta}} \alpha(b(t))^{m+2/N} \\ &\geqslant \frac{w^{m+2/N}}{(-\log w)^{\beta}} \\ &= f(w) \end{aligned}$$

in $\{(x,t) \in \mathbb{R}^N \times (0,\infty) \mid |x| < \sqrt{\tilde{A}/\tilde{B}}(b(t) + t_1)^{\ell/N}\}$ (when $m > 1$). It is apparent that w is a supersolution of (1.1) in $\mathbb{R}^N \times (0,\infty)$, as in the proof of proposition 4.1. Therefore, if $u_0(x) \leqslant w(x,0)$, there is a global solution u of (1.1), (1.2) and $u(x,t) \leqslant w(x,t)$ in $\mathbb{R}^N \times (0,\infty)$. The proof is complete. □

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