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LOCALLY ANALYTIC CUSPIDAL REPRESENTATIONS FOR GL_2 AND RELATED GROUPS

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Abstract For a finite extension $F/\mathbb{Q}_p$ we construct and study a class of locally analytic representations of $\mathrm{GL}_2(F)$ and related groups such as the quaternion algebra over F. The construction is based on inducing a locally analytic character of a maximal torus. We show that for a generic character the resulting representation is topologically irreducible, and not isomorphic to a locally analytic principal series, when the torus is non-split.

Keywords: locally analytic; cuspidal representation; p-adic group

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Contents

1. Introduction

For F a finite extension of $\mathbb{Q}_p$, the study of locally analytic principal series representations of $\mathrm{SL}_2(F)$ was initiated by Morita [**Mo1**,**Mo2**], and has recently been taken up again by Schneider and Teitelbaum [**S-T1**], for the group GL_2. These representations are attached to locally analytic characters of the diagonal torus in $\mathrm{GL}_2(F)$, and may therefore be viewed as a sort of analytic interpolation of smooth principal series which are attached to smooth characters. This is of course only a heuristic, since applying the locally analytic construction to a smooth character does not give rise to a smooth representation.

In this paper, we introduce a new class of locally analytic representations of $\mathrm{GL}_2(F)$, and certain related groups, which may be viewed as interpolating smooth supercuspidal representations. To define these we consider a finite extension E/F, a locally E-analytic character χ of the diagonal torus in $\mathrm{GL}_2(E)$, and a locally F-analytic subgroup G of $\mathrm{GL}_2(E)$ such that

(i) the E-span of $\mathrm{Lie}(G)$ contains $\mathrm{Lie}(\mathrm{SL}_2(E))$;

(ii) if $P \subset \mathrm{GL}_2(E)$ denotes the subgroup of upper triangular matrices, then G has an open orbit $\mathcal{H}$ on $P \setminus \mathrm{GL}_2(E)$.

Using these data we define a locally F-analytic representation $V_{\chi,\mathcal{H}}$ of G. As the notation suggests, it depends on the choice of an open orbit $\mathcal{H}$. We prove that these representations are topologically irreducible for a generic character χ. Although these representations are not always admissible in the sense of [**S-T3**], they do satisfy a natural condition, analogous to that of admissibility for smooth representations (see § 7.1.7). In fact, it was pointed out to us by Schneider that, for SL_2, a closely related construction already appears in [**Mo1**] (but without a proof of the irreducibility).

The two conditions above put a strong restriction on the extension E/F. For example, if $G \subset \mathrm{GL}_2(F)$, then $[E : F] \leqslant 3$. If $E = F$ and $G = \mathrm{SL}_2(F)$ or $G = \mathrm{GL}_2(F)$, we recover the locally analytic principal series of Morita and Schneider–Teitelbaum.

If E/F has degree 2, then we obtain representations of $\mathrm{GL}_2(F)$ and $\mathrm{SL}_2(F)$, and also of the groups of units $D^\times$ of the quaternion algebra over F. These are the representations which may be thought of as interpolating smooth supercuspidal representations. They are not admissible in the sense of Schneider–Teitelbaum, except when $G = D^\times$. The theory in this case bears a striking resemblance to that of the discrete series representations of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{GL}_2(\mathbb{R})$. For example, the representations of $\mathrm{SL}_2(F)$ thus obtained naturally come in pairs, which may be thought of as 'p-adic L-packets'.

Moreover, there is a connection with finite-dimensional locally analytic representations of Weil groups (these groups have a natural analytic structure), and there is a relation between the representations constructed for $\mathrm{GL}_2(F)$ and $D^\times$. This may be the germ of locally analytic analogues of the Langlands and Jacquet–Langlands correspondences. Unfortunately, there is as yet no reasonable theory of traces which would allow one to intrinsically characterize such correspondences. Thus they are, for now, defined by fiat. We do however check that each of these correspondences is compatible with isomorphisms. That is, their definitions are intrinsic, and depend only on the representation $V_{\chi,\mathcal{H}}$ and not on χ or $\mathcal{H}$.

When E/F has degree 3 we again obtain representations of $\mathrm{GL}_2(F)$ and $\mathrm{SL}_2(F)$, and we show that they are indeed different from the ones obtained in the cases of degree 1 and degree 2.* (In fact there are never any intertwiners between representations coming from different extensions E, or attached to different orbits $\mathcal{H}$.) These representations present an enigma for the theory: they appear not to be in any way related to smooth representations, and they do not seem to fit into the scheme of a locally analytic analogue of the Langlands or Jacquet–Langlands correspondence. One might be tempted to dismiss them as pathological, however from the purely function theoretic point of view, they are as well behaved as the representations constructed in the cases of degree 2 or degree 3.

It is worth remarking that although we refer to the representations we construct as *cuspidal* when $[E:F]>1$, this terminology should be regarded as provisional. We have not given a formal definition of this notion as this seems premature given the early state of development of the theory of locally analytic representations. Nevertheless, there is some justification for this terminology. For example these representation have a trivial Jacquet module (in the sense of § 5.1.7), and they admit no non-trivial intertwiners with the principal series representations (i.e. those arising from the case $E=F$). Moreover, there is an obvious analogy with the smooth theory, where one attaches supercuspidal representations of $\mathrm{GL}_2(F)$ to characters of quadratic extensions of F.

Notation

The ring of integers of a non-archimedean valued field L is denoted by $\mathfrak{o}_L$, and its maximal ideal by $\mathfrak{p}_L$. When $\mathfrak{p}_L$ is a principal ideal, we let ϖ_L denote a generator. By a *p-adic field* we will mean an extension K of $\mathbb{Q}_p$, which is complete with respect to a non-archimedean valuation, which makes K into a $\mathbb{Q}_p$-Banach space. Throughout the paper, K will denote such a field.

2. Preliminaries

2.1. Locally analytic representations

We begin by recalling some notions from the theory of topological vector spaces over non-archimedean fields. A general reference for this theory is [**S**]. At the end of this section we recall the key notion of a locally analytic representation.

* After we completed this paper, T. Finis pointed out that there is an even more general way to construct representations using not necessarily open orbits of G on $\mathbb{P}^1(E)$ and germs of locally E-analytic functions along the orbit. This construction allows the extension E/F to be of arbitrary degree. We did not pursue the investigation of such representations.

The reader who is primarily interested in a description of the representations we are going to study in this paper may skip this section. All topological K-vector spaces are assumed to be Hausdorff.

2.1.1. Let V be a topological K-vector space. V is called *locally convex* if there is a fundamental system of neighbourhoods of 0 consisting of $\mathfrak{o}_K$-modules. An $\mathfrak{o}_K$-submodule of V is called a *lattice* if its K-span is V. V is called *barrelled* if any closed lattice in V is open. A subspace $U \subset V$ is called a BH-space for V, if there exists a Banach space topology on U which is finer than (or equal to) the subspace topology. A subset B of V is called *bounded* if for any open lattice $L \subset V$ there is an $a \in K^\times$ such that $B \subset aL$.

For a locally convex K-vector space V we denote by V'_b the space V' of continuous K-linear maps from V to K, equipped with the strong topology. This is the locally convex topology having the lattices

$$\left\{ l \in V' \,\middle|\, \sup_{v \in B} |l(v)| < \varepsilon \right\}$$

as a fundamental system of neighbourhoods of 0, where B runs over all bounded subsets of V and ε over all positive real numbers. The strong topology is also called the topology of convergence on bounded subsets.

A subset B of a locally convex K-vector space V is called *compactoid* if for any open lattice $A \subset V$ there are finitely many $v_1, \dots, v_n \in V$ such that $B \subset A + \mathfrak{o}_K v_1 + \cdots + \mathfrak{o}_K v_n$. For a bounded $\mathfrak{o}_K$-submodule D of V, the subspace $\langle D \rangle_K$ generated by D can be equipped with the norm $|\cdot|_D$ defined by $|v|_D = \inf\{|\lambda| \mid v \in \lambda D\}$. If $\langle D \rangle_K$ is complete with respect to $|\cdot|_D$, then D is called *completing*.

A continuous linear map $f : W \to V$ between locally convex spaces is called *semi-compact* if there is a compactoid completing $\mathfrak{o}_K$-module $D \subset V$ such that $f^{-1}(D)$ is a neighbourhood of 0 in W. A locally convex Hausdorff space V is called a *semi-compact inductive limit*, if it is the locally convex inductive limit of a sequence $V_1 \hookrightarrow V_2 \hookrightarrow V_3 \hookrightarrow \cdots$ of locally convex K-vector spaces V_i with injective semi-compact maps $V_i \hookrightarrow V_{i+1}$. In this case, the spaces V_i can be taken to be Banach spaces. A semi-compact map between Banach spaces is *compact*, which means that the image of an open ball is a compactoid subset. As is shown in [**GKPS**, Theorem 3.1.7], semi-compact inductive limits are Hausdorff, barrelled, complete and reflexive (in the sense that the canonical map $V \to (V'_b)'_b$ is a topological isomorphism), and the strong dual V'_b of a semi-compact inductive limit V is a Fréchet space. Moreover, a semi-compact inductive limit, as well as its strong dual, is a *Montel space* (cf. [**GKPS**, p. 164]), which means in particular that any bounded subset is compactoid. Examples of semi-compact maps and semi-compact inductive limits will be given in Example 2.1.5.

2.1.2. Fix a subfield $F \subset K$, which is a finite extension of $\mathbb{Q}_p$. Let d be a positive integer. For $x \in F^d$, and $r = (r_1, \dots, r_d) \in |F^\times|^d$ denote by $\mathbb{B}_r(x) \subset \mathbb{A}^d_F$ the rigid-analytic ball of multi-radius r centred at x. We let $B_r(x) = \mathbb{B}_r(x)(F)$ denote the F-valued points of $\mathbb{B}_r(x)$. Let $(V, \|\cdot\|)$ be a Banach space over K. Let $\mathcal{O}(\mathbb{B}_r(x))$ be the space of rigid-analytic functions on $\mathbb{B}_r(x)$. If $\tilde{f} = \sum_{i \geqslant 0} a_i \otimes v_i$ is an element of the completed tensor product

$\mathcal{O}(\mathbb{B}_r(x))\hat{\otimes}_F V$, then evaluation at F-valued points gives a map

$$f : B_r(x) \to V, \quad x \mapsto \sum_{i \geqslant 0} a_i(x) \otimes v_i.$$

The functions f which are obtained in this way are called *V-valued rigid-analytic function* [**S-T1**, § 2]. Concretely this means that f is given by a convergent power series in x with coefficients in V. We remark that $\tilde{f}$ is uniquely determined by f.

2.1.3. One can define the notion of locally analytic functions on any locally F-analytic manifold M. (This is what Bourbaki calls an F-analytic manifold [**Bou**, § 5.1].) We always assume that M is strictly paracompact: this means that any open covering of M can be refined into a covering by disjoint open subsets. Let V be a Hausdorff locally convex K-vector space. A *V-index* $\mathcal{I}$ on M is a family of triples $\{(D_i, \phi_i, V_i)\}_{i \in I}$, where the D_i are pairwise disjoint open subsets of M which cover M, each $\phi_i : D_i \to F^d$ is a chart of the manifold M whose image is an affinoid ball, and $V_i \hookrightarrow V$ is a BH-space for V. For $i \in I$ denote by $\mathcal{F}_{\phi_i}(V_i)$ the K-vector space of functions $f : D_i \to V_i$ such that $f \circ \phi_i^{-1} : \phi_i(D_i) \to V_i$ is a V_i-valued rigid-analytic function on the ball $\phi_i(D_i)$, as defined in the previous section. $\mathcal{F}_{\phi_i}(V_i)$ carries the structure of a Banach space, and we can define the locally convex direct product

$$\mathcal{F}_{\mathcal{I}}(V) := \prod_{i \in I} \mathcal{F}_{\phi_i}(V_i).$$

The set of V-indices is partially ordered by declaring that

$$\mathcal{I} = (D_i, \phi_i, V_i)_{i \in \mathcal{I}} \leqslant \mathcal{J} = (D_j, \phi_j, V_j)_{j \in \mathcal{J}}$$

if $\{D_j\}_{j \in \mathcal{J}}$ refines $\{D_i\}_{i \in \mathcal{I}}$ and whenever $D_j \subset D_i$, then $V_i \subset V_j$ and the composite

$$\phi_j(D_j) \xrightarrow{\phi_j^{-1}} D_j \subset D_i \xrightarrow{\phi_i} \phi_i(D_i) \subset F^d$$

is an F^d-valued rigid-analytic function on $\phi_j(D_j)$. If $\mathcal{I} \leqslant \mathcal{J}$ then there is a natural map $\mathcal{F}_{\mathcal{I}}(V) \to \mathcal{F}_{\mathcal{J}}(V)$ obtained by restricting functions from D_i to D_j, where $D_j \subset D_i$.

The space of V-valued locally analytic functions on M is defined as

$$C^{\mathrm{an}}(M, V) := \varinjlim_{\mathcal{I}} \mathcal{F}_{\mathcal{I}}(V),$$

equipped with the locally convex inductive limit topology. We remark that the definitions of an F-analytic manifold, and of a V-index imply that the above direct limit is filtering.

When we want to emphasize that M is considered as a locally analytic manifold over F (and not over $\mathbb{Q}_p$, for instance), then we write $C_F^{\mathrm{an}}(M, K)$ instead of $C^{\mathrm{an}}(M, K)$.

2.1.4. The topological vector space of *compactly supported locally analytic functions* with values in V is defined as follows. A *compact V-index* $\mathcal{I}$ on M is a family of triples $\{(D_i, \phi_i, V_i)\}_{i \in I}$, where the D_i are pairwise disjoint open subsets of M such that $\bigcup_{i \in I} D_i$ is compact, each $\phi_i : D_i \to F^d$ is a chart of the manifold M whose image is an affinoid ball, and $V_i \hookrightarrow V$ is a BH-space of V. Note that the index set I is finite. For $i \in I$ denote by $\mathcal{F}_{\phi_i}(V_i)$ the K-vector space of functions $f : D_i \to V_i$ such that $f \circ \phi_i^{-1} : \phi_i(D_i) \to V_i$ is a V_i-valued rigid-analytic function on the ball $\phi_i(D_i)$, and define $\mathcal{F}_{\mathcal{I}}(V)$ to be the product of the $\mathcal{F}_{\phi_i}(V_i)$. Extend any function in $\mathcal{F}_{\mathcal{I}}(V)$ by zero to a function on M. As in 2.1.3, the set of compact V-indices is partially ordered, and we can form the inductive limit of the Banach spaces $\mathcal{F}_{\mathcal{I}}(V)$, equipped with the locally convex inductive limit topology:

$$C^{\mathrm{an}}_{F,\mathrm{c}}(M,V) := \varinjlim_{\mathcal{I}} \mathcal{F}_{\mathcal{I}}(V).$$

There is a canonical continuous inclusion $C^{\mathrm{an}}_{F,\mathrm{c}}(M,V) \to C^{\mathrm{an}}_{F}(M,V)$, but the topology on $C^{\mathrm{an}}_{F,\mathrm{c}}(M,V)$ is in general finer than the subspace topology. That is, the inclusion is not a homeomorphism onto its image.

Example 2.1.5. Here we give two examples of semi-compact inductive limits.

(1) Let $M = \mathfrak{o}_F$, and denote by $\mathfrak{p}_F$ the maximal ideal of $\mathfrak{o}_F$. For each $m \geqslant 0$ define the K-index $\mathcal{I}_m$ as $(a + \mathfrak{p}_F^m, \phi_a, K)_{a \in \mathfrak{o}_F/\mathfrak{p}_F^m}$, where $\phi_a : a + \mathfrak{p}_F^m \to F$ is the tautological inclusion. Then the canonical map $\mathcal{F}_{\mathcal{I}_m}(K) \hookrightarrow \mathcal{F}_{\mathcal{I}_{m+1}}(K)$ is compact (cf. the example after Proposition 16.10 in [**S**]). Therefore, $C^{\mathrm{an}}_F(M,K)$ is a semi-compact inductive limit.

(2) More generally, for any locally F-analytic manifold M which can be written as a countable union of compact-open subsets, the space $C^{\mathrm{an}}_{F,\mathrm{c}}(M,K)$ is a semi-compact inductive limit. To see this, take a sequence $\{(D_{m,i}, \phi_{m,i}, K)\}_{i \in I_m}$ of compact K-indices of M, such that $\bigcup_{i \in I_m} D_{m,i} \subset \bigcup_{i \in I_{m+1}} D_{m+1,i}$, and such that any $\phi_{m,j}(D_{m,j})$ is the union of certain $\phi_{m+1,i}(D_{m+1,i})$, as in the first example.

2.1.6. The strong dual $C^{\mathrm{an}}(M,K)'_b$ of $C^{\mathrm{an}}(M,K)$, is called the space of *K-valued distributions* on M and is denoted by $D(M,K)$. The second example shows that for a compact manifold M the space $C^{\mathrm{an}}(M,K)$ is a semi-compact inductive limit, and $D(M,K)$ is then a Fréchet space. We write $D_F(M,K)$ for $D(M,K)$ if we want to make it clear that M is considered as a locally F-analytic manifold.

2.1.7. We recall the definition of a locally analytic representation (cf. [**S-T1**, § 3]). Consider a locally F-analytic group G. Let V be a barrelled locally convex Hausdorff K-vector space, and let $\rho : G \to (\mathrm{End}_K^{\mathrm{cont}}(V))^\times$ be a homomorphism from G into the group of invertible continuous K-linear endomorphisms of V.

Definition 2.1.8. (V,ρ) is called *locally F-analytic* if for each $v \in V$ the orbit map

$$G \to V, \quad g \mapsto \rho(g)(v)$$

is a V-valued locally F-analytic function on G. When there is no risk of confusion, we sometimes omit the field F, and refer to these representations as *locally analytic*.

The prototypical example of a locally F-analytic representation of G is the space $C^{\mathrm{an}}(G,K)$ equipped with a G-action by

$$(g.f)(g') = f(g'g), \quad g, g' \in G, \quad f \in C^{\mathrm{an}}(G,K).$$

2.1.9. Given a locally analytic representation (V,ρ) of the locally F-analytic group G, we have bilinear maps

$$V_b' \times D(G,K) \to V_b', \quad (\lambda,\delta) \mapsto \lambda \cdot \delta = (v \mapsto \delta(g \mapsto \lambda(\rho(g)v))), \tag{2.1.10}$$

and

$$D(G,K) \times V_b' \to V_b', \quad (\delta,\lambda) \mapsto \delta \cdot \lambda = (v \mapsto \delta(g \mapsto \lambda(\rho(g^{-1})v))). \tag{2.1.11}$$

If we apply this to $V = C^{\mathrm{an}}(G,K)$, then this gives two ways to define $\delta_1 \cdot \delta_2$ for $\delta_1,\delta_2 \in D(G,K)$. We will use the first of these. That is, we define $\delta_1 \cdot \delta_2$ by taking $\lambda = \delta_1$ and $\delta = \delta_2$ in (2.1.10) above. This gives $D(G,K)$ the structure of a K-algebra (which agrees with the one defined in [**S-T1**]), with the product of two distributions $\delta_1,\delta_2 \in D(G,K)$ given by

$$(\delta_1\delta_2)(f) = (\delta_1 \cdot \delta_2)(f) = \delta_2(g_1 \mapsto \delta_1(g_2 \mapsto f(g_2g_1))).$$

With this structure (2.1.10) makes V_b' into a right $D(G,K)$-module, and (2.1.11) makes V_b' into a left $D(G,K)$-module. We will usually use the left $D(G,K)$-module structure on V_b'.

2.1.12. If V is a locally F-analytic representation of G, then the Lie algebra $\mathrm{Lie}(G)$ acts on V by

$$\mathfrak{z}.v = \frac{\mathrm{d}}{\mathrm{d}t}(\exp(t\mathfrak{z}).v)|_{t=0}, \quad \mathfrak{z} \in \mathrm{Lie}(G), \quad v \in V.$$

Here $\exp : \mathrm{Lie}(G) \dashrightarrow G$ denotes the exponential map, which is defined in a neighbourhood of 0, and $\mathrm{d}/\mathrm{d}t$ is defined by the usual limit formula. This induces a natural action of the universal enveloping algebra $\mathfrak{U}(\mathrm{Lie}(G))$ of $\mathrm{Lie}(G)$ on V.

Denote by $\mathfrak{z} \mapsto \dot{\mathfrak{z}}$ the unique anti-automorphism of $\mathfrak{U}(\mathrm{Lie}(G))$ which extends the multiplication by -1 on $\mathrm{Lie}(G)$. Then, for any $\mathfrak{z} \in \mathfrak{U}(\mathrm{Lie}(G))$, we define a distribution $\delta_{\mathfrak{z}} \in D_F(G,K)$ by $\delta_{\mathfrak{z}}(f) = (\dot{\mathfrak{z}}.f)(1_G)$. This extends to an embedding of K-algebras $\mathfrak{U}(\mathrm{Lie}(G)) \otimes_F K \hookrightarrow D_F(G,K)$. For $\lambda \in V_b'$ one has $(\delta_{\mathfrak{z}} \cdot \lambda)(v) = \lambda(\mathfrak{z}.v)$.

2.2. The p-adic Fourier transform

Now let F be a finite extension of $\mathbb{Q}_p$ which is contained in K. We will recall essential properties of the p-adic Fourier transform which identifies the space of distributions $D_F(\mathfrak{o}_F,K)$ with the ring of rigid-analytic functions on the *character variety* $\widehat{\mathfrak{o}_F}$ over K, which is a rigid-analytic space. The results are due to Amice [**A**] in the case $F = \mathbb{Q}_p$, and to Schneider and Teitelbaum [**S-T2**] in general.

2.2.1. Denote by $\boldsymbol{B}_1$ the rigid-analytic open unit disc around 1 over $\mathbb{Q}_p$, so that $\boldsymbol{B}_1(K) = 1+\mathfrak{m}_K$, where $\mathfrak{m}_K$ is the maximal ideal of $\mathfrak{o}_K$. Associating to each $z \in 1+\mathfrak{m}_K$ the K-valued character

$$\mathbb{Z}_p \to K^{\times}, \quad a \mapsto z^a,$$

gives a bijection between $\boldsymbol{B}_1(K)$ and the group $\widehat{\mathbb{Z}_p}(K) := \mathrm{Hom}_{\mathrm{loc\,an}}(\mathbb{Z}_p, K^\times)$ of locally analytic K-valued characters of $\mathbb{Z}_p$ (cf. [**Sch**, §§ 32, 47]). Moreover, $\boldsymbol{B}_1(K)$ gets the structure of a $\mathbb{Z}_p$-module, by $a \cdot z = z^a$ for $a \in \mathbb{Z}_p$ and $z \in \boldsymbol{B}_1(K)$. Now consider $\mathfrak{o}_F$ as a locally analytic manifold over $\mathbb{Q}_p$, and denote this manifold by $(\mathfrak{o}_F)_0$. We write

$$\widehat{(\mathfrak{o}_F)_0} := \boldsymbol{B}_1 \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(\mathfrak{o}_F, \mathbb{Z}_p),$$

where the right-hand side denotes the rigid-analytic variety over $\mathbb{Q}_p$ whose points in a rigid-analytic variety X are given by $\boldsymbol{B}_1(X) \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(\mathfrak{o}_F, \mathbb{Z}_p)$. The variety $\widehat{(\mathfrak{o}_F)_0}$ is (non-canonically) isomorphic to $(\boldsymbol{B}_1)^d$, where $d = [F : \mathbb{Q}_p]$. Its K-valued points are in natural bijection with the group $\mathrm{Hom}_{\mathrm{loc\,an}}((\mathfrak{o}_F)_0, K^\times)$ of locally analytic K-valued characters of $\mathfrak{o}_F$, considered as a locally analytic manifold over $\mathbb{Q}_p$. The ring of rigid-analytic functions on $\widehat{(\mathfrak{o}_F)_0}/K$ (the base-change of $\widehat{(\mathfrak{o}_F)_0}$ from $\mathbb{Q}_p$ to K) is in a natural way a Fréchet-algebra over K. This follows from the fact that the ring of rigid-analytic functions on the latter space is the projective limit of the rings of functions on affinoid subdomains inside $\boldsymbol{B}_1 \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(\mathfrak{o}_F, \mathbb{Z}_p)$. The following theorem is a several-variable version of Amice's Theorem 1.3 in [**A**].

Theorem 2.2.2 (Amice). *There is a topological isomorphism of Fréchet algebras*

$$D_{\mathbb{Q}_p}((\mathfrak{o}_F)_0, K) \to \mathcal{O}(\widehat{(\mathfrak{o}_F)_0}/K), \quad \lambda \mapsto \Phi_\lambda,$$

where Φ_λ is given on

$$\widehat{(\mathfrak{o}_F)_0}(\bar{K}^\wedge) = \mathrm{Hom}_{\mathrm{loc\,an}}((\mathfrak{o}_F)_0, (\bar{K}^\wedge)^\times)$$

by $\Phi_\lambda(\chi) = \lambda(\chi)$.

2.2.3. In fact Amice proves her theorem only in the case where K is a subfield of $\mathbb{C}_p$. Presumably this assumption is not essential for the proof, but one can in any case deduce the result for any field K from the case $K = \mathbb{Q}_p$: first we check that there is a (necessarily unique) $\Phi_\lambda \in \mathcal{O}(\widehat{(\mathfrak{o}_F)_0}/K)$ which satisfies $\Phi_\lambda(\chi) = \lambda(\chi)$. In fact, let $U \subset \widehat{(\mathfrak{o}_F)_0}$ be a quasi-compact, admissible open subset. It suffices to construct Φ_λ over U. A straightforward calculation in local coordinates on $\widehat{(\mathfrak{o}_F)_0}$ shows that for any $a \in (\mathfrak{o}_F)_0$ the function $\chi \mapsto \chi(a)$ is a rigid-analytic function on $\widehat{(\mathfrak{o}_F)_0}$, and hence on U, and that the function

$$\Phi : (\mathfrak{o}_F)_0 \to \mathcal{O}(U) : a \mapsto (\chi \mapsto \chi(a))$$

is a locally $\mathbb{Q}_p$-analytic $\mathcal{O}(U)$-valued function on $(\mathfrak{o}_F)_0$. Thus given a distribution $\lambda \in D_{\mathbb{Q}_p}((\mathfrak{o}_F)_0, K)$ we have $\Phi_\lambda = (\Phi, \lambda) \in \mathcal{O}(\widehat{(\mathfrak{o}_F)_0}/K)$. It follows that we have a commutative diagram:

$$\begin{array}{ccc} D_{\mathbb{Q}_p}((\mathfrak{o}_F)_0, K) & \longrightarrow & \mathcal{O}(\widehat{(\mathfrak{o}_F)_0}/K) \\ \uparrow & & \uparrow \\ D_{\mathbb{Q}_p}((\mathfrak{o}_F)_0, \mathbb{Q}_p)\hat{\otimes}_{\mathbb{Q}_p} K & \longrightarrow & \mathcal{O}(\widehat{(\mathfrak{o}_F)_0})\hat{\otimes}_{\mathbb{Q}_p} K \end{array}$$

Now the right vertical map is easily seen to be an isomorphism, since $\mathcal{O}(\widehat{(\mathfrak{o}_F)_0})$ has a simple description in terms of power series. The map on the bottom is an isomorphism by Amice's theorem. This shows that the top map is surjective. Finally, [**S-T2**, Proposition 1.4] shows that the top map is injective (although K is assumed to be a subfield of $\mathbb{C}_p$ in [**S-T2**], the same argument goes through verbatim without this assumption once one replaces $\mathbb{C}_p$ by $\bar{K}$ everywhere).

2.2.4. Now let us consider K-valued characters of $\mathfrak{o}_F$ which are locally analytic when $\mathfrak{o}_F$ is considered as a locally analytic manifold over F. These are the homomorphisms from $\mathfrak{o}_F$ into $K^\times$ which can be developed locally in one-variable power series with coefficients in K. We put

$$\widehat{\mathfrak{o}_F}(K) := \mathrm{Hom}(\mathfrak{o}_F, K^\times) \cap C_F^{\mathrm{an}}(\mathfrak{o}_F, K).$$

Theorem 2.2.5 (see Theorem 2.3 in [S-T2]). *There is a closed rigid-analytic subspace* $\widehat{\mathfrak{o}_F} \subset \widehat{(\mathfrak{o}_F)_0}$ *over* F*, which has the property that for any extension* K *of* F *as above, the set of* K*-valued points of* $\widehat{\mathfrak{o}_F}$ *is equal to* $\widehat{\mathfrak{o}_F}(K) \subset \mathrm{Hom}_{\mathrm{loc\,an}}((\mathfrak{o}_F)_0, K^\times)$*. Moreover, the isomorphism from Theorem 2.2.2 induces an isomorphism of* K*-Fréchet algebras*

$$D_F(\mathfrak{o}_F, K) \to \mathcal{O}(\widehat{\mathfrak{o}_F}/K), \quad \lambda \mapsto \Phi_\lambda.$$

Again, this theorem is proved in [**S-T2**] under the assumption that $K \subset \mathbb{C}_p$. One can reduce the general case to this one as in 2.2.3.

2.2.6. Only in the case $F = \mathbb{Q}_p$ is the rigid-analytic variety $\widehat{\mathfrak{o}_F}$ isomorphic to an open unit disc over F. In general $\widehat{\mathfrak{o}_F}$ is a form of an open unit disc which becomes isomorphic to an open unit disc over an extension which contains all torsion points of a Lubin–Tate formal group over $\mathfrak{o}_F$. This has certain consequences which we are going to use later on.

Let $\mathcal{G}$ be a Lubin–Tate formal group over $\mathfrak{o}_F$ with multiplication by $\mathfrak{o}_F$. It is a one-dimensional formal group of height $[F : \mathbb{Q}_p]$ over $\mathfrak{o}_F$, which is equipped with a homomorphism $\mathfrak{o}_F \to \mathrm{End}_{\mathfrak{o}_F}(\mathcal{G})$. For $a \in \mathfrak{o}_F$ we denote by $[a] : \mathcal{G} \to \mathcal{G}$ the corresponding endomorphism of $\mathcal{G}$. Denote by $\mathcal{G}'$ the p-divisible group dual to $\mathcal{G}$, and let $T' = T(\mathcal{G}')$ be the Tate-module of $\mathcal{G}'$, which is a free $\mathfrak{o}_F$-module of rank one.

Assume for the rest of this section that K contains all p-power torsion points of $\mathcal{G}$. Then T' is isomorphic to the group $\mathrm{Hom}_{\mathrm{f\,gps}/\mathfrak{o}_K}(\mathcal{G}_{\mathfrak{o}_K}, (\hat{\mathbb{G}}_m)_{\mathfrak{o}_K})$ of homomorphisms of formal groups over $\mathfrak{o}_K$, where $\hat{\mathbb{G}}_m$ denotes the formal multiplicative group. One then gets a pairing

$$T' \otimes_{\mathfrak{o}_F} \mathcal{G}(\mathfrak{o}_K) \to \hat{\mathbb{G}}_m(\mathfrak{o}_K) = 1 + \mathfrak{p}_K, \quad t' \otimes z \mapsto \langle t', z \rangle,$$

and this gives rise to an isomorphism of groups (cf. [**S-T2**, Proposition 3.1])

$$T' \otimes_{\mathfrak{o}_F} \mathcal{G}(\mathfrak{o}_K) \to \widehat{\mathfrak{o}_F}(K), \quad t' \otimes z \mapsto \kappa_{t' \otimes z},$$

where $\kappa_{t' \otimes z} : \mathfrak{o}_F \to K^\times$ is defined by $\kappa_{t' \otimes z}(a) = \langle t', [a](z) \rangle$. Fix a generator t_0' of T' as $\mathfrak{o}_F$-module, and denote by $\mathcal{G}^{\mathrm{rig}}$ the rigid-analytic variety associated to the formal scheme $\mathcal{G}$. This rigid space is isomorphic to an open unit disc over F. For $t' \in T'$ define the rigid-analytic function $\Phi_{t'}$ on $\mathcal{G}_K^{\mathrm{rig}}$ by $\Phi_{t'}(z) = \langle t', z \rangle - 1$. Let $\log_{\mathcal{G}} : \mathcal{G}^{\mathrm{rig}} \to \mathbb{G}_a^{\mathrm{rig}}$ be

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a logarithm for $\mathcal{G}$, then there is $\Omega \in \mathfrak{o}_K - \{0\}$ such that $\Phi_{t'_0}(z) = \exp(\Omega \log_{\mathcal{G}}(z)) - 1$. Define a differential operator $\partial : \mathcal{O}(\mathcal{G}^{\text{rig}}) \to \mathcal{O}(\mathcal{G}^{\text{rig}})$ by

$$\partial\Phi(z) = \lim_{w\to 0}(1/w)(\Phi(z +_{\mathcal{G}} w) - \Phi(z)).$$

We have the following results.

Theorem 2.2.7 (see Theorem 3.6 and Lemma 4.6 in [S-T2]). *There is an isomorphism of the rigid-analytic varieties $\mathcal{G}_K^{\text{rig}}$ and $\widehat{\mathfrak{o}_F}$ which is given on K-valued points by $z \mapsto \kappa_{t'_0 \otimes z}$. This induces a pairing*

$$\mathcal{O}(\mathcal{G}_K^{\text{rig}}) \times C_F^{\text{an}}(\mathfrak{o}_F, K) \to K, \quad (\Phi, f) \mapsto \{\Phi, f\},$$

and we have the following formulae:

(1) $\{\Phi_{at'_0}\Phi, f\} = \{\Phi, f(a + \cdot) - f\}$;

(2) $\{\Phi, f(a\cdot)\} = \{\Phi \circ [a], f\}$;

(3) $\{\Phi, f'\} = \{\Omega \log_{\mathcal{G}} \cdot \Phi, f\}$;

(4) $\{\Phi, xf(x)\} = \{\Omega^{-1}\partial\Phi, f\}$.

3. The representations

For the rest of the paper $E \subset K$ will denote a finite extension of F, contained in K.

3.1. General definition and main results

3.1.1. Denote by P an E-rational parabolic subgroup of $\mathrm{GL}_{2/E}$, and use the same letter to denote its group of E-rational points. Consider a locally F-analytic group $G \subset \mathrm{GL}_2(E)$ satisfying the following conditions.

(i) The E-linear span of $\mathrm{Lie}(G)$ inside $\mathrm{Lie}(\mathrm{GL}_2(E))$ contains $\mathrm{Lie}(\mathrm{SL}_2(E))$.

(ii) The action of G on $P\backslash \mathrm{GL}_2(E)$, given by multiplication from the right, has an open orbit.

From now on we fix such a locally F-analytic subgroup, together with an open G-orbit $\mathcal{H} \subset P\backslash \mathrm{GL}_2(E)$. Let $\tilde{\mathcal{H}} \subset \mathrm{GL}_2(E)$ be its preimage (for the canonical projection $\mathrm{GL}_2(E) \to P\backslash \mathrm{GL}_2(E)$). It is an open submanifold of the locally E-analytic manifold $\mathrm{GL}_2(E)$. Let $P \twoheadrightarrow T$ be the reductive quotient of P. A K-valued locally E-analytic character $\chi : T \to K^\times$ (i.e. an element of $C_E^{\text{an}}(T, K) \cap \mathrm{Hom}(T, K^\times)$) defines a character of P, again denoted by χ, by composing χ with the canonical projection from P to T. Denote by $V_{\chi,\mathcal{H}}$ the subspace of $C_E^{\text{an}}(\tilde{\mathcal{H}}, K)$ consisting of functions f satisfying the following conditions.

(i) For all $q \in P$, $g \in \tilde{\mathcal{H}}$: $f(qg) = \chi(q)f(g)$.

(ii) $P\backslash \mathrm{supp}(f) \subset \mathcal{H}$ is compact.

G acts on $V_{\chi,\mathcal{H}}$ by $gf(x) = f(xg)$, $x \in \tilde{\mathcal{H}}$, $g \in G$.

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We will shortly define a topology on these spaces, which gives them the structure of semi-compact inductive limits. The main result is then that these G-representations are topologically irreducible, provided a certain simple numerical condition on the character χ is fulfilled. We continue with a discussion of the topology on these spaces.

3.1.2. Consider a locally E-analytic section $s : \mathcal{H} \to \tilde{\mathcal{H}}$ of the canonical projection map $\tilde{\mathcal{H}} \to P\backslash\tilde{\mathcal{H}} = \mathcal{H}$. Such a section exists for any surjective smooth map of locally analytic manifolds [**Bou**, § 5.9.1]. The induced map $s^* : V_{\chi,\mathcal{H}} \to C^{\mathrm{an}}_{E,\mathrm{c}}(\mathcal{H}, K), f \mapsto f \circ s$, is then easily seen to be an isomorphism of K-vector spaces. In 2.1.4, we defined the structure of a locally convex vector space on $C^{\mathrm{an}}_{E,\mathrm{c}}(\mathcal{H}, K)$. Being an open subset of the compact manifold $P\backslash \mathrm{GL}_2(E)$, $\mathcal{H}$ can be written as an increasing union of a countable number of compact subsets, so by Example 2.1.5 $C^{\mathrm{an}}_{E,\mathrm{c}}(\mathcal{H}, K)$ is a semi-compact inductive limit. We equip $V_{\chi,\mathcal{H}}$ with the topology which makes s^* a homeomorphism.

Proposition 3.1.3. *The topology on $V_{\chi,\mathcal{H}}$ coincides with the topology induced by the inclusion $V_{\chi,\mathcal{H}} \hookrightarrow C^{\mathrm{an}}_E(\mathrm{GL}_2(E), K)$ obtained by extending functions by 0. In particular, it is independent of the section s.*

Proof. Consider first the case $G = \mathrm{GL}_2(E)$. Then $\mathcal{H} = P\backslash \mathrm{GL}_2(E)$ is compact. Choose a section s as above, and define $\tilde{\chi} \in C^{\mathrm{an}}_E(\mathrm{GL}_2(E), K)$ by the formula $\tilde{\chi}(g) = \chi(gs(\mathrm{pr}(g))^{-1})$, where $\mathrm{pr} : \mathrm{GL}_2(E) \to P \backslash \mathrm{GL}_2(E)$ denotes the natural projection. If we use s^* to identify $V_{\chi,P\backslash \mathrm{GL}_2(E)}$ with $C^{\mathrm{an}}_E(P\backslash \mathrm{GL}_2(E), K)$, then the inclusion of the proposition becomes the composite

$$V_{\chi,P\backslash \mathrm{GL}_2(E)} \xrightarrow{\sim} C^{\mathrm{an}}_E(P\backslash \mathrm{GL}_2(E), K) \xrightarrow{\mathrm{pr}^*} C^{\mathrm{an}}_E(\mathrm{GL}_2(E), K) \xrightarrow{\cdot\tilde{\chi}} C^{\mathrm{an}}_E(\mathrm{GL}_2(E), K).$$

It follows that $V_{\chi,P\backslash \mathrm{GL}_2(E)} \hookrightarrow C^{\mathrm{an}}_E(\mathrm{GL}_2(E), K)$ is continuous, and it is a homeomorphism onto its image, because it admits the retraction

$$C^{\mathrm{an}}_E(\mathrm{GL}_2(E), K) \xrightarrow{s^*} C^{\mathrm{an}}_E(P\backslash \mathrm{GL}_2(E), K) \xrightarrow{\sim} V_{\chi,P\backslash \mathrm{GL}_2(E)}.$$

For general G, we begin by checking that the topology on $V_{\chi,\mathcal{H}}$ is independent of s. If s' is another locally analytic section of $\tilde{\mathcal{H}} \to \mathcal{H}$, then for $w \in \mathcal{H}$ write $h(w) = s'(w)s^{-1}(w) \in P$. For $f \in V_{\chi,\mathcal{H}}$ and $w \in \mathcal{H}$ we have

$$(s'^*f)(w) = f(s'(w)) = f(h(w)s(w)) = \chi(h(w))f(s(w)) = \chi(h(w))(s^*f)(w).$$

Since multiplication by $\chi(h(w))$ induces a topological automorphism of $C^{\mathrm{an}}_{\mathrm{c}}(\mathcal{H}, K)$, this proves the topologies induced on $V_{\chi,\mathcal{H}}$ by s and s' coincide.

To show that the natural inclusion $V_{\chi,\mathcal{H}} \hookrightarrow V_{\chi,P\backslash \mathrm{GL}_2(E)}$, obtained by extending functions by 0, is a homeomorphism onto its image, note that the independence of s just shown implies that we may choose s so that it extends to a locally analytic section of $\mathrm{GL}_2(E) \to P\backslash\mathrm{GL}_2(E)$. For such a choice of s it suffices to show that the natural inclusion

$$C^{\mathrm{an}}_{E,\mathrm{c}}(\mathcal{H}, K) \hookrightarrow C^{\mathrm{an}}_E(P\backslash \mathrm{GL}_2(E), K) = C^{\mathrm{an}}_{E,\mathrm{c}}(P\backslash \mathrm{GL}_2(E), K)$$

is a homeomorphism onto its image. In fact it is easily seen from the definitions that for any locally E-analytic manifold M_1, and any open submanifold $M_2 \subset M_1$, the natural map $C^{\mathrm{an}}_{E,\mathrm{c}}(M_2, K) \to C^{\mathrm{an}}_{E,\mathrm{c}}(M_1, K)$ is a topological embedding. $\square$

Proposition 3.1.4. *$V_{\chi,\mathcal{H}}$ is a semi-compact inductive limit, and it is a locally analytic G-representation.*

Proof. The first assertion follows from Example 2.1.5 and the very definition of the topology on $V_{\chi,\mathcal{H}}$. To see that $V_{\chi,\mathcal{H}}$ is a locally analytic representation, note that, since $C_E^{\mathrm{an}}(\mathrm{GL}_2(E), K)$ is a locally E-analytic representation of $\mathrm{GL}_2(E)$, it is *a fortiori* an F-analytic G-representation. By Proposition 3.1.3, $V_{\chi,\mathcal{H}}$ is closed in $C_E^{\mathrm{an}}(\mathrm{GL}_2(E), K)$, and it is easily seen from the definitions that a closed G-stable subspace of a locally F-analytic G-representation is again locally F-analytic. □

3.1.5. Before stating the main result about these representations, we make a definition. The torus T acts by the adjoint representation on the Lie algebra of the unipotent subgroup of P via a character $\alpha : T \to \mathbb{G}_m$. Its differential is a map $\mathrm{d}\alpha : \mathrm{Lie}(T) \to \mathrm{Lie}(\mathbb{G}_{m,E})$. Similarly, for any locally analytic character $\chi : T \to K^\times = \mathbb{G}_m(K)$, we have its differential $\mathrm{d}\chi : \mathrm{Lie}(T) \to \mathrm{Lie}(\mathbb{G}_{m,K}) = \mathrm{Lie}(\mathbb{G}_{m,E}) \otimes_E K$. Define $T^{(1)}$ to be the image of $P \cap \mathrm{SL}_2$ in T. Then there is then an element $c(\chi) \in K$ such that

$$\mathrm{d}\chi|_{\mathrm{Lie}(T^{(1)})} = -\tfrac{1}{2} c(\chi)\, \mathrm{d}\alpha|_{\mathrm{Lie}(T^{(1)})}.$$

Now we can state the following theorem.

Theorem 3.1.6. *If $c(\chi) \notin \boldsymbol{Z}_{\geqslant 0}$, then $V_{\chi,\mathcal{H}}$ is a topologically irreducible representation of G.*

The proof will be given in § 4.

3.1.7. Now suppose that $c(\chi)$ is a non-negative integer. In this case, following Morita [**Mo1**], there is a non-zero closed proper subrepresentation $V_{\chi,\mathcal{H}}^{\mathrm{loc\,alg}}$ of $V_{\chi,\mathcal{H}}$. Let U^- be the unipotent radical of the parabolic subgroup opposite to P, and fix a generator $\mathfrak{u}^-$ of $\mathrm{Lie}(U^-)$ as an E-vector space. We let $V_{\chi,\mathcal{H}}^{\mathrm{loc\,alg}}$ be the space of functions f in $V_{\chi,\mathcal{H}}$ such that for any $g \in \tilde{\mathcal{H}}$ the function $z \mapsto f(\exp(z\mathfrak{u}^-)g)$ (which is defined on a sufficiently small neighbourhood of zero in E, depending on g) is a polynomial in z of degree less than or equal to $c(\chi)$, for sufficiently small z. The coefficients of this polynomial depend on g. This space is G-stable since $\tilde{\mathcal{H}}$ is G-stable. The Lie algebra element $\mathfrak{u}^-$ acts by left translation on functions f on $\tilde{\mathcal{H}}$ via the formula

$$(\mathfrak{u}^- ._\ell f)(g) = \lim_{\lambda \to 0} \frac{1}{\lambda}(f(\exp(\lambda\mathfrak{u}^-)g) - f(g)).$$

In general $\mathfrak{u}^- ._\ell f$ will no longer be equivariant for the action of P by left translation. However, if we define the character χ' of T by $\chi' = \chi \cdot \alpha^{c(\chi)+1}$, then $(\mathfrak{u}^-)^{c(\chi)+1} ._\ell f$ is equivariant with respect to the character χ'.

Proposition 3.1.8. *$V_{\chi,\mathcal{H}}^{\mathrm{loc\,alg}}$ is an infinite-dimensional closed G-stable subspace of $V_{\chi,\mathcal{H}}$. The differential operator $(\mathfrak{u}^-)^{c(\chi)+1}._\ell$ induces a continuous, strict surjection from $V_{\chi,\mathcal{H}}$ to $V_{\chi',\mathcal{H}}$ whose kernel is precisely $V_{\chi,\mathcal{H}}^{\mathrm{loc\,alg}}$.*

The proof will be given in the next section.

Corollary 3.1.9. *$V_{\chi,\mathcal{H}}^{\mathrm{loc\,alg}}$ consists of those functions $f \in V_{\chi,\mathcal{H}}$ such that for any $g \in \tilde{\mathcal{H}}$, and z sufficiently small, the function $z \mapsto f(\exp(z\mathfrak{u}^-)g)$ is a polynomial of some fixed degree, not depending on g.*

Proof. In fact, if $V_{\chi,\mathcal{H}}^n$ denotes the set of f such that for any $g \in \tilde{\mathcal{H}}$ the function $z \mapsto f(\exp(z\mathfrak{u}^-)g)$ is (for sufficiently small z) a polynomial of degree less than or equal to n, then $V_{\chi,\mathcal{H}}^n$ is a proper closed G-stable subspace of $V_{\chi,\mathcal{H}}$. Hence Theorem 3.1.6 and Proposition 3.1.8 show that $V_{\chi,\mathcal{H}}^n \subset V_{\chi,\mathcal{H}}^{\mathrm{loc\,alg}}$. □

Corollary 3.1.10. *With the notation of Proposition 3.1.8, $V_{\chi,\mathcal{H}}^{\mathrm{loc\,alg}}$ is the kernel of $(\mathfrak{u}^-)^{c(\chi)+1}$.*

Proof. If we identify $V_{\chi,\mathcal{H}}$ and $V_{\chi',\mathcal{H}}$ with subspaces of $C_E^{\mathrm{an}}(E,K)$, then both

$$(\mathfrak{u}^-)^{c(\chi)+1} : V_{\chi,\mathcal{H}} \to V_{\chi,\mathcal{H}} \quad \text{and} \quad (\mathfrak{u}^-._{\ell})^{c(\chi)+1} : V_{\chi,\mathcal{H}} \to V_{\chi',\mathcal{H}}$$

are given by $(\mathrm{d}/\mathrm{d}z)^{c(\chi)+1}$. Hence the kernels of $(\mathfrak{u}^-._{\ell})^{c(\chi)+1}$ and $(\mathfrak{u}^-)^{c(\chi)+1}$ are equal, and the result follow from Proposition 3.1.8. □

3.2. Another description of the representations

3.2.1. In this section we give another description of the representations $V_{\chi,\mathcal{H}}$ by choosing the parabolic subgroup P to be the group of upper triangular matrices in GL_2. In [**Mo1**, **Mo2**], Morita defines analogous representations for $\mathrm{SL}_2(E)$ in this way, except that he takes P to be the group of *lower* triangular matrices. But because any two E-rational Borel subgroups are conjugate by an element of $\mathrm{GL}_2(E)$, there is no loss of generality if we fix a specific one.

For the element $\mathfrak{u}^-$ which generates the Lie algebra of U^- we choose

$$\mathfrak{u}^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

In the following, we will sometimes identify $P\backslash \mathrm{GL}_2(E)$ with $\mathbb{P}^1(E)$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (c:d).$$

We denote by $i: E \to \mathrm{GL}_2(E)$ the map given by $i(z) = \exp(z\mathfrak{u}^-) = 1 + z\mathfrak{u}^-$. The unique point of $P\backslash \mathrm{GL}_2(E) = \mathbb{P}^1(E)$ not in the image of the composite of i and the projection $\mathrm{GL}_2(E) \to P\backslash \mathrm{GL}_2(E)$ will be denoted by ∞. We regard E as an open subset of $\mathbb{P}^1(E)$ via this composite map.

Write the character χ as a product of characters of $E^\times$ as follows:

$$\chi \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \chi_1(ad)\chi_2(d).$$

Then we have that $\chi_2(z) = \exp(c(\chi)\log(z))$ for z sufficiently close to 1. Put $V_\chi = V_{\chi,P\backslash \mathrm{GL}_2(E)}$. If $f \in V_\chi$, then for $z \neq 0$ we have

$$f\left(\begin{pmatrix}1 & 0\\ z & 1\end{pmatrix}\right) = f\left(\begin{pmatrix}1/z & 1\\ 0 & z\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 1/z\end{pmatrix}\right) = \chi_2(z) f\left(\begin{pmatrix}0 & -1\\ 1 & 1/z\end{pmatrix}\right).$$

Now choose a section s of the projection $\mathrm{GL}_2(E) \to P\backslash \mathrm{GL}_2(E)$ whose restriction to an open neighbourhood of ∞ is given by

$$z \mapsto \begin{pmatrix}0 & -1\\ 1 & 1/z\end{pmatrix}.$$

Then s^* identifies V_χ with $C^{\mathrm{an}}_E(P\backslash \mathrm{GL}_2(E), K)$ and one sees that the image of the map

$$i^* : V_\chi \to C^{\mathrm{an}}_E(E, K)$$

induced by i consists of all locally analytic functions f on E which have the property that for $|z| \gg 0$ the function $z \mapsto \chi_2(z)^{-1} f(z)$ can be expanded into a convergent power series in $1/z$. In particular, we can extend $\chi_2(z)^{-1} f(z)$ to a function on $P\backslash \mathrm{GL}_2(E)$ by defining $\chi_2(\infty)^{-1} f(\infty)$ to be the constant term of this series.

The action of $\mathrm{GL}_2(E)$ on this space of functions is given by the formula

$$g = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \quad \text{maps } f \text{ to } (g._\chi f)(z) = \chi_1(ad - bc)\chi_2(bz + d) f\left(\frac{az + c}{bz + d}\right). \tag{3.2.2}$$

Here, if $(az + c)/(bz + d) = \infty$, then the expression on the right-hand side of the formula is defined to be $\chi_1(\det g)\chi_2(az + c)\chi_2^{-1}(\infty) f(\infty)$.

More generally, for a group $G \subset \mathrm{GL}_2(E)$ which satisfies the assumptions made in 3.1.1 and an open orbit $\mathcal{H}$, we distinguish two cases, namely whether $\mathcal{H}$ is contained in E (first case) or not (second case). In the first case, the image of the map $i^* : V_{\chi,\mathcal{H}} \to C^{\mathrm{an}}_E(E, K)$ consists of locally analytic functions on $\mathcal{H}$ with compact support. When ∞ belongs to $\mathcal{H}$, the image of i^* consists of those locally analytic functions f on $E \cap \mathcal{H}$, which have the property that

(i) there is $z_0 \in E^\times$ such that $\{z \in E \mid |z| \geqslant |z_0|\} \subset \mathcal{H}$ and such that on this subset $\chi_2(z)^{-1} f(z)$ can be expanded into a convergent power series in $1/z$;

(ii) $\mathrm{supp}(f) \cap \{z \in E \mid |z| \leqslant |z_0|\}$ is compact.

For $c(\chi) \in \mathbb{Z}_{\geqslant 0}$ the map i^* identifies $V^{\mathrm{loc\,alg}}_{\chi,\mathcal{H}}$ with the space of functions f on $E \cap \mathcal{H}$ which are locally polynomial of degree less than or equal to $c(\chi)$, have compact support and, if $\infty \in \mathcal{H}$, have the property that for $|z| \gg 0$ the function $z \mapsto \chi_2(z)^{-1} f(z)$ is a polynomial in $1/z$ of degree less than or equal to $c(\chi)$. The differential operator $\mathfrak{u}^-._\ell$ becomes differentiation with respect to z.

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3.2.3. Proof of Proposition 3.1.8. As above, we again denote by i the inclusion $E \cap \mathcal{H} \to \tilde{\mathcal{H}}$ induced by i. We first check that the operator $(\mathrm{d}/\mathrm{d}z)^{c(\chi)+1}$ induces a strict surjection $i^*(V_{\chi,\mathcal{H}}) \to i^*(V_{\chi',\mathcal{H}})$. If $\infty \notin \mathcal{H}$, then this is immediate since we can always integrate locally analytic functions on compact subsets. The strictness can be seen as follows: any function $f \in i^*(V_{\chi',\mathcal{H}}) \xrightarrow{\sim} C^{\mathrm{an}}_{E,\mathrm{c}}(\mathcal{H}, K)$ which lies in $\mathcal{F}_{\mathcal{I}}(K)$ for some K-index $\mathcal{I}$ on $\mathcal{H} \cap E$ as in 2.1.4 can be integrated to a function in $\mathcal{F}_{\mathcal{I}'}(K)$, where $\mathcal{I}'$ depends only on $\mathcal{I}$ and not on f. The explicit description of the inductive limit topology given in [**S**, Lemma 5.1(iii)] now implies that differentiation of functions in $C^{\mathrm{an}}_{E,\mathrm{c}}(\mathcal{H}, K)$ is a strict map.

Suppose that $\infty \notin \mathcal{H}$. Suppose that f is any locally E-analytic function on $\{z \in E \mid |z| \geqslant |z_0|\}$ such that

$$\chi_2(z)^{-1} f(z) = \varphi\left(\frac{1}{z}\right) = \sum_{i=0}^{\infty} a_i z^{-i},$$

where $\varphi(1/z)$ is a power series with coefficients in K, which is convergent for $|z| \geqslant |z_0|$. Differentiating both sides of this equation one finds

$$\chi_2(z)^{-1} f'(z) = \frac{c(\chi)}{z}\varphi\left(\frac{1}{z}\right) - \frac{1}{z^2}\varphi'\left(\frac{1}{z}\right) = \frac{c(\chi)a_0}{z} + \frac{(c(\chi)-1)a_1}{z^2} + \frac{(c(\chi)-2)a_2}{z^3} + \cdots.$$

Hence we see that

$$\chi_2(z)^{-1} f^{(c(\chi)+1)}(z) = \sum_{i=2c(\chi)+2}^{\infty} b_i z^{-i},$$

where

$$b_i = a_{i-c(\chi)-1} \prod_{j=0}^{c(\chi)} (2c(\chi)+1-i-j).$$

Finally,

$$\chi_2'(z)^{-1} f^{(c(\chi)+1)}(z) = z^{2c(\chi)+2}\chi_2(z)^{-1} f^{(c(\chi)+1)}(z) = \sum_{i=0}^{\infty} b_{i+2c(\chi)+2} z^{-i}.$$

Using this we see that $(\mathrm{d}/\mathrm{d}z)^{c(\chi)+1}$ induces a continuous surjection $i^*(V_{\chi,\mathcal{H}}) \to i^*(V_{\chi',\mathcal{H}})$, which is strict by an argument similar to that given in the first case. (Note that if $h \in V_{\chi,\mathcal{H}}$ and $i^*(h) = f$, with $\chi_2(z)^{-1} f(z) = \varphi(1/z)$ for $|z| \geqslant |z_0|$, as above, then the calculation of 3.2.1 shows that $s^*(h) = \varphi(1/z)$ for $|z| \geqslant |z_0|$.)

It remains to show that the map $i^*(V_{\chi,\mathcal{H}}) \to i^*(V_{\chi',\mathcal{H}})$ constructed above is G-equivariant. As observed in the proof of Proposition 3.1.3, there is a natural G-equivariant map $V_{\chi,\mathcal{H}} \to V_\chi$ obtained by extending functions by zero from $\tilde{\mathcal{H}}$ to $\mathrm{GL}_2(E)$. This map is also compatible with the action of $\mathrm{Lie}(G)$. Hence, it suffices to consider the case $G = \mathrm{GL}_2(E)$.

Let $f \in i^*(V_\chi)$. We claim that $(g._\chi f)^{(c(\chi)+1)} = g._{\chi'} f^{(c(\chi)+1)}$ for any $g \in \mathrm{GL}_2(E)$, where $(g._\chi f)^{(n)}$ denotes the nth derivative of $g._\chi f$. It is not difficult to check by induction that the following formula is valid for any non-negative integer n:

$$(g._\chi f)^{(n)}(z) = \chi_1(\det g)\chi_2(bz+d) \sum_{0\leqslant i\leqslant n} \binom{n}{i} \left[\prod_{j=i}^{n-1}(c(\chi)-j)\right] \left(\frac{b}{bz+d}\right)^{n-i} \left(\frac{\det g}{(bz+d)^2}\right)^{i} f^{(i)}(z.g).$$

The claim follows by putting $n = c(\chi)+1$ in this expression.

Now let $\tilde{h} \in V_\chi$. Set $h = (\mathfrak{u}^-._\ell)^{c(\chi)+1}\tilde{h}$, and let $h' \in V_{\chi'}$ satisfy $i^*(h') = i^*(h)$. Then for any $g \in \mathrm{GL}_2(E)$ we compute

$$\begin{aligned} i^*(g.h) &= i^*(g.(\mathfrak{u}^-._\ell)^{c(\chi)+1}\tilde{h}) = i^*((\mathfrak{u}^-._\ell)^{c(\chi)+1}g.\tilde{h}) \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{c(\chi)+1} g._\chi i^*(\tilde{h}) = g._{\chi'}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{c(\chi)+1} i^*(\tilde{h}) = g._{\chi'} i^*((\mathfrak{u}^-._\ell)^{c(\chi)+1}\tilde{h}) \\ &= g._{\chi'} i^*(h) = g._{\chi'} i^*(h') = i^*(g.h'). \end{aligned}$$

Since $\mathrm{GL}_2(E)$ acts transitively on itself by right translation, this implies that $h = h'$, which shows that $(\mathfrak{u}^-._\ell)^{(c(\chi)+1)}$ induces a map $V_\chi \to V_{\chi'}$. Since $\mathfrak{u}^-._\ell$ commutes with right translation, this map is automatically $\mathrm{GL}_2(E)$-equivariant. □

3.3. GL_2: principal series and cuspidal representations

3.3.1. In this section we consider in greater detail the case when G is equal to $\mathrm{GL}_2(F)$ or $\mathrm{SL}_2(F)$. Because the basic set-up requires the existence of an open orbit of G on $P\backslash \mathrm{GL}_2(E)$, the degree of E over F is at most three. We distinguish two cases.

- $E = F$: $\mathrm{GL}_2(F)$ and $\mathrm{SL}_2(F)$ both act transitively on $P\backslash \mathrm{GL}_2(E)$. We call this the *principal series* case, although not all representations are irreducible.
- $2 \leqslant [E:F] \leqslant 3$: because of the similarities with the classical smooth theory (in the case of a quadratic extension), we call the representations one gets in the second case *cuspidal*.

Now suppose that $[E:F]$ is 2 or 3. Identifying $P\backslash \mathrm{GL}_2(E)$ with $\mathbb{P}^1(E)$ as in 3.2.1, we see that the open orbits of $\mathrm{GL}_2(F)$ and $\mathrm{SL}_2(F)$ are contained in $\mathbb{P}^1(E) \setminus \mathbb{P}^1(F)$. To describe the orbits, we will work with the description of the group action given in (3.2.2). If $\mathcal{H}$ denotes an open orbit for G, then $\infty \notin \mathcal{H}$ and the space $V_{\chi,\mathcal{H}}$ can be identified with $C^{\mathrm{an}}_{E,\mathrm{c}}(\mathcal{H}, K)$.

The following proposition gives a description of the orbits of $\mathrm{GL}_2(F)$ and $\mathrm{SL}_2(F)$ on $E \setminus F$.

Proposition 3.3.2.

(1) *If E/F is a quadratic or cubic extension, then $\mathrm{GL}_2(F)$ acts transitively on $E \setminus F$.*

(2) *Suppose $[E:F] = 2$. Then $E \setminus F$ consists of two open orbits for the action of $\mathrm{SL}_2(F)$. Denote by σ the non-trivial Galois automorphism of E over F. Then two elements $z, z' \in E \setminus F$ lie in the same orbit if and only if $(z-\sigma(z))/(z'-\sigma(z'))$ is a norm for E/F.*

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(3) *Suppose* $[E:F]=3$. *Then* $E\setminus F$ *consists of* $\#(F^\times/F^{\times,2})$ *open orbits for the action of* $\mathrm{SL}_2(F)$. *Denote by* $\iota,\sigma,\tau : E\to\bar F$ *the* F*-linear embeddings of* E *into* $\bar F$. *Then two elements* $z,z'\in E\setminus F$ *lie in the same orbit if and only if*

$$\frac{(\iota(z)-\sigma(z))(\sigma(z)-\tau(z))(\tau(z)-\iota(z))}{(\iota(z')-\sigma(z'))(\sigma(z')-\tau(z'))(\tau(z')-\iota(z'))}\in F^{\times,2}.$$

Proof. (1) For a given element $z\in E\setminus F$, one has that

$$z.\begin{pmatrix} a & 0\\ c & 1\end{pmatrix} = az+c,$$

which shows that for $[E:F]=2$ the action is transitive. If $[E:F]=3$ and $t\in E\setminus F$, then the map $\mathrm{M}_2(F)\to E$ given by

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\mapsto (az+c)-t(bz+d)$$

is F-linear and hence has a non-trivial kernel. If $g\in\mathrm{M}_2(F)$ is in this kernel, then $\det g\neq 0$ as $z,t\notin F$. This shows that there is a $g\in\mathrm{GL}_2(F)$ with $z.g=t$. (We are grateful to the referee for this concise argument.)

(2) In the case $[E:F]=2$, another computation shows that for any $z\in E\setminus F$ and $g\in\mathrm{GL}_2(F)$, one has

$$\frac{zg-\sigma(zg)}{z-\sigma(z)}=\frac{\det(g)}{N_{E/F}(bz+d)}.$$

This proves the 'only if' part of (2). Conversely, if $z'\in E\setminus F$ is such that $(z'-\sigma(z'))/(z-\sigma(z))$ is in the image of $N_{E/F}$, and $g\in\mathrm{GL}_2(E)$ satisfies $z.g=z'$, then the above formula shows that $\det(g)=N_{E/F}(u)$ for some $u\in E^\times$. Write $u=\alpha+\beta z$ with $\alpha,\beta\in F$, and set

$$g_0=\begin{pmatrix}\alpha+\mathrm{tr}_{E/F}(\beta z) & 1\\ -N_{E/F}(\beta z) & \alpha\end{pmatrix},$$

where $\mathrm{tr}_{E/F}: E\to F$ denotes the trace. Then $z'=z.(g_0^{-1}g)$, and $\det(g_0^{-1}g)=1$.

(3) Similarly, if $[E:F]=3$, then for any $z\in E\setminus F$ and $g\in\mathrm{GL}_2(F)$, one has that

$$\frac{(\iota(zg)-\sigma(zg))(\sigma(zg)-\tau(zg))(\tau(zg)-\iota(zg))}{(\iota(z)-\sigma(z))(\sigma(z)-\tau(z))(\tau(z)-\iota(z))}=\frac{\det(g)^3}{N_{E/F}(bz+d)^2}.$$

This proves the 'only if' part of (3). Conversely, if $z'\in E\setminus F$ is such that the expression in (3) is in $F^{\times,2}$, then choose $g\in\mathrm{GL}_2(E)$ with $z'=z.g$. The above formula then shows that $\det g\in F^{\times,2}$, so that $z'=z.((\det g)^{-1/2}g)$. This proves the assertion. □

3.3.3. The general setting of 3.1.1 also applies to the case when G is a compact-open subgroup of $\mathrm{GL}_2(F)$ and $[E:F]\leqslant 3$.

Consider, for example, the situation that $F=E$ and $G=\mathrm{GL}_2(\mathfrak{o}_E)$. Because $\mathrm{GL}_2(\mathfrak{o}_E)$ acts transitively on $P\backslash\mathrm{GL}_2(E)$, the irreducibility result Theorem 3.1.6 tells us that if $c(\chi)\notin\mathbb{Z}_{\geqslant 0}$, then the representation of $\mathrm{GL}_2(E)$ attached to χ stays irreducible after restriction to $\mathrm{GL}_2(\mathfrak{o}_E)$. Another case, which will be studied in § 4.2, is given by principal congruence subgroups of $\mathrm{GL}_2(E)$.

When $F=E$, any compact-open subgroup of $\mathrm{GL}_2(E)$ has only a finite number of orbits on the flag variety. In contrast to this case, when $2\leqslant[E:F]\leqslant 3$, any compact-open subgroup of $\mathrm{GL}_2(F)$ has an infinite (countable) number of orbits on the (only) open orbit of $\mathrm{GL}_2(F)$ on $P\backslash\mathrm{GL}_2(E)$. This will be of importance when discussing the strong admissibility of these representations in 7.1.1.

3.3.4. Now suppose that E is a quadratic extension of F and that $G=\mathrm{GL}_2(F)$. Let $\mathcal{H}\subset P\backslash\mathrm{GL}_2(E)$ denote the unique open orbit of G. Suppose that, in the notation of 3.2.1, χ_1 is the trivial character. In this case we will give yet another construction of the representation $V_{\chi,\mathcal{H}}$ (the general case where χ_1 is non-trivial can be reduced to this one by twisting by a character). Although this description will not be used in the rest of the paper, it is suggestive of the construction of *smooth* cuspidal representations of $\mathrm{GL}_2(E)$ in terms of characters of a non-split torus. It thus serves to support the heuristic that the locally analytic representations interpolate the smooth ones.

Choose a non-split torus $\mathbb{T}\subset\mathrm{GL}_{2/F}$ such that $\mathbb{T}(F)\xrightarrow{\sim}E^\times$. Let $g\in\mathrm{GL}_2(E)$ be an element such that $\mathbb{T}(E)\subset g^{-1}Pg$. Set $G^g:=g\,\mathrm{GL}_2(F)g^{-1}$ and $T^g=g\mathbb{T}(F)g^{-1}=P\cap G^g$. Then

$$T^g=\left\{\begin{pmatrix}\sigma(z) & 0\\ 0 & z\end{pmatrix}: z\in E^\times\right\}\subset P.$$

Now let

$$H=\left\{p\in P: p=\begin{pmatrix}* & *\\ 0 & 1\end{pmatrix}\right\}.$$

Since $H\cap G^g=H\cap T^g=\{1\}$, we have a G^g-equivariant, open immersion of locally F-analytic manifolds $G^g\hookrightarrow H\backslash\mathrm{GL}_2(E)$. In particular, this gives G^g the structure of a locally E-analytic manifold.

Now we define a locally analytic G^g-representation $W_{\chi,g}$ as follows. The underlying topological K-vector space of $W_{\chi,g}$ consists of functions $f\in C^{\mathrm{an}}_E(G^g,K)$ such that

- for all $e\in T^g=E^\times$ and $h\in G^g$, $f(eh)=\chi_2(e)f(h)$;
- $E^\times\backslash\operatorname{supp}(f)$ is compact.

G^g acts on $W_{\chi,g}$ by multiplication on the right.

Proposition 3.3.5. *If we regard $W_{\chi,g}$ as a representation of $\mathrm{GL}_2(F)$ via the map*

$$\mathrm{GL}_2(F)\xrightarrow{h\mapsto ghg^{-1}}G^g,$$

then there is an isomorphism of locally analytic $\mathrm{GL}_2(F)$-representations $W_{\chi,g}\xrightarrow{\sim}V_{\chi,\mathcal{H}}$.

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Proof. Write $\mathcal{H}^g = \mathcal{H}g^{-1}$, so that $\mathcal{H}^g$ is the unique orbit of G^g on $P\backslash \mathrm{GL}_2(E)$. As in the proposition, we can regard $V_{\chi,\mathcal{H}^g}$ as a $\mathrm{GL}_2(F)$-representation. Then one checks immediately that we have an isomorphism of locally analytic $\mathrm{GL}_2(F)$-representations

$$V_{\chi,\mathcal{H}} \to V_{\chi,\mathcal{H}^g} : f \mapsto (h \mapsto f(hg),\ h \in \tilde{\mathcal{H}}^g).$$

Thus, it suffices to show that $W_{\chi,g} \xrightarrow{\sim} V_{\chi,\mathcal{H}^g}$ as locally analytic G^g-representations. But this follows from the definitions, once we note that $T^g\backslash G^g \subset P\backslash \mathrm{GL}_2(E)$ is open, and hence coincides with $\mathcal{H}^g$, and that the assumption that χ_1 is trivial, implies that any $f \in V_{\chi,\mathcal{H}^g}$ descends to a function on $H\backslash\tilde{\mathcal{H}}^g = G^g$. □

3.4. Quaternion division algebras and unitary groups

3.4.1. Let F be a finite extension of $\boldsymbol{Q}_p$, and let D be a central division algebra of dimension four over F. Given a quadratic extension E of F, there is an embedding $E \hookrightarrow D$, and there are elements $\tau \in D^\times$, $\iota \in F^\times \setminus N_{E/F}(E^\times)$ such that $D = E \oplus E\tau$, $\tau^2 = \iota$, and conjugation by τ leaves E stable and induces the non-trivial Galois automorphism $x \mapsto \bar{x}$ of E over F. Thus, the map

$$D \to M_2(E), \quad a + b\tau \mapsto \begin{pmatrix} a & \iota b \\ \bar{b} & \bar{a} \end{pmatrix}$$

is an embedding of F-algebras. We let $G := D^\times \subset \mathrm{GL}_2(E)$ be the group of units of D, considered as a locally F-analytic subgroup of $\mathrm{GL}_2(E)$. The map $D \otimes_F E \to M_2(E)$ induced by the map above is an isomorphism. Moreover, G acts transitively on $P\backslash \mathrm{GL}_2(E)$. Hence, the conditions in 3.1.1 are satisfied, and we get, for any locally E-analytic character χ of the diagonal torus T, a locally analytic representation of $D^\times$ on the space $V_\chi = V_{\chi,\mathcal{H}}$ defined in 3.1.1. On functions $f \in V_\chi$ the action is given by the formula in (3.2.2): $g._\chi f$ is given by

$$(g._\chi f)(z) = \chi_1(a\bar{a} - \iota b\bar{b})\chi_2(\iota bz + \bar{a}) f\left(\frac{az + \bar{b}}{\iota bz + \bar{a}}\right), \quad \text{where } g = \begin{pmatrix} a & \iota b \\ \bar{b} & \bar{a} \end{pmatrix} \in D^\times.$$

Let $D^{\times,1} = D^\times \cap \mathrm{SL}_2(E) \subset \mathrm{GL}_2(E)$. Then $G = D^{\times,1}$ also satisfies the conditions of 3.1.1. There are two orbits of $D^{\times,1}$ on $P\backslash \mathrm{GL}_2(E)$. This can be seen by using the bijections

$$P\backslash \mathrm{GL}_2(E)/D^{1,\times} \xrightarrow{\sim} E^\times\backslash D^\times/D^{1\times} \xrightarrow{\sim} F^\times/N_{E/F}(E^\times),$$

where the first map is induced by the inclusion $D^\times \subset \mathrm{GL}_2(E)$, and the second by the reduced norm $\mathrm{Nrd} : D^\times \to F^\times$. We will makes use of this in our discussion of unitary groups below, where we also provide a more explicit description of the two orbits.

3.4.2. Let $\langle \cdot\,, \cdot \rangle$ denote a non-degenerate hermitian form on $E \oplus E$ (i.e. $\langle x, y\rangle = \overline{\langle y, x\rangle}$, $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$ for $\alpha \in E$, and for any $y \in E^2$ the form $x \mapsto \langle x, y\rangle$ is non-zero). We think of elements of E^2 as rows, on which $\mathrm{GL}_2(E)$ acts from the right:

$$(x_1, x_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (x_1a + x_2c, x_1b + x_2d).$$

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Consider the unitary group G attached to this hermitian form:

$$G = \{g \in \mathrm{GL}_2(E) \mid \langle xg, yg\rangle = \langle x, y\rangle\}.$$

There is a hermitian 2×2 matrix A such that $\langle x, y\rangle = xA\overline{{}^{\mathrm{t}}y}$. Using this description we find that

$$\mathrm{Lie}(G) = \{X \in M_2(E) \mid XA + A\overline{{}^{\mathrm{t}}X} = 0\},$$

and an easy calculation (assuming A to be a diagonal matrix) shows that $\mathrm{Lie}(G)$ spans $\mathrm{Lie}(\mathrm{GL}_{2/E})$ over E. Of course, one could also use the general fact that G is a form of $\mathrm{GL}_{2/F}$.

We now describe the orbits of G on $P\backslash \mathrm{GL}_2(E)$. There are two cases to distinguish: either the hermitian form is anisotropic (i.e. $\langle x, x\rangle \neq 0$ for any $x \neq 0$) or it is not.

For the first case, let $\iota \in F^\times \backslash N_{E/F}(E^\times)$ and $D \subset M_2(E)$ be as in 3.4.1. Let $* : D \to D$ denote the canonical anti-involution of D which induces the non-trivial Galois automorphism of any subfield $E' \subset D$ which is a quadratic extension of F. If we identify E^2 with $E \oplus E\tau = D$, by $(x_1, x_2) \mapsto x_2 + x_1\tau$, then an anisotropic form is given by $\langle x, y\rangle = \varepsilon(xy^*)$ where $\varepsilon : D \to E$ is the map which sends $a + b\tau$ to a. Any anisotropic form is equivalent to this one. If $x = (x_1, y_1)$ and $y = (y_1, y_2)$, then the form is given explicitly by

$$\langle (x_1, x_2), (y_1, y_2)\rangle = -\iota x_1\bar{y}_1 + x_2\bar{y}_2.$$

It is easily checked that $D^{\times,1}$ is a subgroup of G. Denote by $F^{\times,+}$ (respectively, $F^{\times,-}$) the subgroup $N_{E/F}(E^\times)$ (respectively, $F^\times \backslash N_{E/F}(E^\times)$) of $F^\times$. Put

$$\mathcal{H}^\pm = P\backslash \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(E) \;\middle|\; \langle (0,1)g, (0,1)g\rangle = -\iota|c|^2 + |d|^2 \in F^{\times,\pm} \right\}.$$

These subsets are open in $P\backslash \mathrm{GL}_2(E)$, and stable under the action of right multiplication by G, because Pg is in $\mathcal{H}^\pm$ if and only if $\langle (0,1)g, (0,1)g\rangle \in F^{\times,\pm}$. In particular they are stable by $D^{\times,1}$. Since we saw in 3.4.1 that $D^{\times,1}$ has exactly two open orbits on $P\backslash \mathrm{GL}_2(E)$, $\mathcal{H}^\pm$ must coincide with these orbits. It also follows that $\mathcal{H}^\pm$ are precisely the orbits of G on $P\backslash \mathrm{GL}_2(E)$. As the orbits are quotients of the compact group $D^{\times,1}$ they are also compact, and any compact-open subgroup of G therefore has only finitely many orbits on $\mathcal{H}$.

In the case where the hermitian form is isotropic there exists a basis (v_1, v_2) of E^2 such that $\langle v_1, v_1\rangle = \langle v_2, v_2\rangle = 0$ and $\langle v_1, v_2\rangle = \theta$ for some generator θ of E over F such that $\theta^2 \in F$. One checks easily that G contains $\mathrm{SL}(Fv_1 \oplus Fv_2)$. The quotient $P\backslash \mathrm{GL}_2(E)$ is the disjoint union of three sets of elements $v \bmod P$. Namely, the preimages of $\{0\}$, $F^{\times,+}$ and $F^{\times,-}$ under the map $g \mapsto \langle (0,1)g, (0,1)g\rangle$. Each of these subsets is stable under G. The first of these sets is clearly closed and compact, and corresponds to $\mathbb{P}^1(F) = \mathbb{P}(Fv_1 \oplus Fv_2)$. The other two sets are open and non-compact. Since $\mathrm{SL}_2(F) = \mathrm{SL}(Fv_1 \oplus Fv_2)$ acting on $P\backslash \mathrm{GL}_2(E)$ has two open orbits by Proposition 3.3.2, these two sets are exactly the open orbits of G on $P\backslash \mathrm{GL}_2(E)$. Consequently, any compact-open subgroup has an infinite number of orbits on each of the open subsets stable under G.

As we have seen, the first case is similar to the case of the quaternion division algebra, whereas the second case is close to that of $\mathrm{SL}_2(F)$. For this reason we will not explicitly work out the results of §§ 5–7 for the case of unitary groups, but leave this as an exercise for the reader.

4. Irreducibility of generic representations

The aim of this section is to prove the main result on the irreducibility of the representations constructed above. After recalling some formulae for the action of the Lie algebra in § 4.1, we prove in § 4.2 a result on the irreducibility of certain representations of principal congruence subgroups of $\mathrm{GL}_2(\mathfrak{o}_E)$. This is a generalization of the main result of [**S-T1**], and the proofs mostly follow the methods of that paper. What may be more surprising is that this result can be applied to prove irreducibility of cuspidal representations, which is what we do in § 4.3.

4.1. Action of the Lie algebra

4.1.1. In this section we consider the action of certain elements of the Lie algebra of $\mathrm{GL}_2(E)$ on $V_\chi = V_{\chi,\mathcal{H}}$. We assume that P is the group of upper triangular matrices and work with the model of $V_\chi = V_{\chi,\mathcal{H}}$ described in 3.2.1. That is, we identify $V_\chi = V_{\chi,\mathcal{H}}$ with a space of functions on E. The elements we will consider are

$$\mathfrak{u}^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathfrak{u}^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Define a linear endomorphism Θ of $C_E^{\mathrm{an}}(E,K)$ by $(\Theta f)(z) = zf(z)$.

The following lemma is a variant of [**S-T1**, Lemmas 5.2, 5.3].

Lemma 4.1.2. *We have*

(1) $\mathfrak{u}^- = \mathrm{d}/\mathrm{d}z$*;*

(2) $\mathfrak{u}^+ = c(\chi)\Theta - \Theta^2(\mathrm{d}/\mathrm{d}z) = c(\chi)\Theta - \Theta^2\mathfrak{u}^-$*;*

(3) *for any* $m \geqslant 1$, $\mathfrak{u}^-\Theta^m = m\Theta^{m-1} + \Theta^m\mathfrak{u}^-$*;*

(4) *for any* $m \geqslant 1$,

$$(\mathfrak{u}^+)^m = m!\binom{c(\chi)}{m}\Theta^m + Q_m\mathfrak{u}^-,$$

where Q_m *is a sum of terms of the form* $\Theta^a(\mathfrak{u}^-)^b$ $(a, b \in \boldsymbol{Z}_{\geqslant 0})$ *with coefficients in* K.

Proof. Parts (1) and (2) follow from a simple computation and part (3) follows immediately from part (1). For $m = 1$ the assertion in part (4) is true by part (2). Assuming the assertion is true for a given $m \geqslant 1$, we can calculate

$$\begin{aligned}(\mathfrak{u}^+)^{m+1} &= \mathfrak{u}^+\left(m!\binom{c(\chi)}{m}\Theta^m + Q_m\mathfrak{u}^-\right)\\ &= (c(\chi)\Theta - \Theta^2\mathfrak{u}^-)\left(m!\binom{c(\chi)}{m}\Theta^m + Q_m\mathfrak{u}^-\right)\\ &= c(\chi)m!\binom{c(\chi)}{m}\Theta^{m+1} + c(\chi)\Theta Q_m\mathfrak{u}^- - m!\binom{c(\chi)}{m}\Theta^2\mathfrak{u}^-\Theta^m - \Theta^2\mathfrak{u}^-Q_m\mathfrak{u}^-\\ &= m!\binom{c(\chi)}{m}(c(\chi)-m)\Theta^{m+1} + (c(\chi)\Theta Q_m - m!\binom{c(\chi)}{m}\Theta^{m+2} - \Theta^2\mathfrak{u}^-Q_m)\mathfrak{u}^-,\end{aligned}$$

where in the last equality we have used (3). This proves (4). □

4.2. Representations of principal congruence subgroups

4.2.1. Before proving in Theorem 4.2.5 what we call the local irreducibility result, we begin with a simple lemma. Fix a locally analytic section $s : P\backslash \mathrm{GL}_2(E) \to \mathrm{GL}_2(E)$. Let $G \subset \mathrm{SL}_2(E)$ be a compact-open subgroup, and let $\Delta \subset P\backslash \mathrm{GL}_2(E)$ be an orbit of G. For a character χ of T, we identify the underlying space of the representation $V_{\chi,\Delta}$ with $C^{\mathrm{an}}_{E,\mathrm{c}}(\Delta, K)$ using the section s, as in 3.1.2.

The result of applying an element $g \in G$ to a function $f \in C^{\mathrm{an}}_{E,\mathrm{c}}(\Delta, K)$ will be denoted by $g._\chi f$. Similarly, for $\delta \in D_E(G, K)$ and $\lambda \in D_E(\Delta, K)$, we denote the image of λ under δ by $\delta._\chi\lambda$. We will work with different groups, and it is important that the section s is fixed, independent of G, Δ and χ.

Fix $g_0 \in \mathrm{GL}_2(E)$ and define isomorphisms

$$\begin{gathered} g_0._\chi : C^{\mathrm{an}}_{E,\mathrm{c}}(\Delta g_0, K) \to C^{\mathrm{an}}_{E,\mathrm{c}}(\Delta, K), \quad f \mapsto g_0._\chi f,\\ D_E(\Delta, K) \xrightarrow{\lambda\mapsto\lambda^{g_0}} D_E(\Delta.g_0, K), \quad \lambda^{g_0}(f) = \lambda(g_0._\chi f),\\ D_E(g_0^{-1}Gg_0, K) \xrightarrow{\delta\mapsto\delta^{g_0}} D_E(G, K), \quad \delta^{g_0}(f) = \delta(g \mapsto f(g_0gg_0^{-1})).\end{gathered}$$

Lemma 4.2.2. *For $\lambda \in D_E(\Delta, K)$ and $\delta \in D_E(g_0^{-1}Gg_0, K)$ we have $(\delta^{g_0}\cdot_\chi\lambda)^{g_0} = \delta\cdot_\chi\lambda^{g_0}$.*

Proof. For $f \in C^{\mathrm{an}}_{E,\mathrm{c}}(\Delta.g_0, K)$ we have

$$\begin{aligned}(\delta^{g_0}\cdot_\chi\lambda)^{g_0}(f) &= (\delta^{g_0}\cdot_\chi\lambda)(g_0._\chi f) = \delta^{g_0}(g \mapsto \lambda(g^{-1}._\chi(g_0._\chi f)))\\ &= \delta(g \mapsto \lambda(g_0._\chi(g^{-1}._\chi f))) = \delta(g \mapsto \lambda^{g_0}(g^{-1}._\chi f)) = (\delta\cdot_\chi\lambda^{g_0})(f).\end{aligned}$$

□

Corollary 4.2.3. *Suppose that $D_E(\Delta.g_0, K)$ is a simple $D_E(g_0^{-1}Gg_0, K)$-module. Then $D_E(\Delta, K)$ is a simple $D_E(G, K)$-module.*

Proof. Let $M \subset D_E(\Delta, K)$ be a $D_E(G, K)$-submodule. Then

$$M^{g_0} = \{\lambda^{g_0} \mid \lambda \in M\} \subset D_E(\Delta.g_0, K)$$

is a module under $D_E(g_0^{-1}Gg_0, K)$, because

$$\delta \cdot_\chi \lambda^{g_0} = (\delta^{g_0} \cdot_\chi \lambda)^{g_0} \in M^{g_0}.$$

Hence M^{g_0} is zero or $D_E(\Delta.g_0, K)$. So M is zero or $D_E(\Delta, K)$. □

4.2.4. We now consider a principal congruence subgroup $G_r := 1 + \mathfrak{p}_E^r M_2(\mathfrak{o}_E) \cap \mathrm{SL}_2(E)$ for some $r \in \mathbb{Z}_{>0}$. Moreover, we work with the model described in § 3.2.1. That is, we choose P to be the group of upper triangular matrices, and we identify E with a subset of $P\backslash \mathrm{GL}_2(E)$. It is easy to see that any orbit of G_r can be mapped into $\mathfrak{o}_E \subset E$ by some element of $\mathrm{GL}_2(\mathfrak{o}_E)$, and that any orbit contained in $\mathfrak{o}_E$ is of the form $z_0 + \mathfrak{p}_E^r$ for some $z_0 \in \mathfrak{o}_E$. Moreover, after translation by $\left(\begin{smallmatrix} 1 & 0 \\ -z_0 & 1 \end{smallmatrix}\right)$ we may assume that the orbit is $\mathfrak{p}_E^r$. Then we put

$$g_0 := \begin{pmatrix} \varpi_E^{-r} & 0 \\ 0 & 1 \end{pmatrix},$$

so that the image of this orbit under g_0 is $\mathfrak{o}_E$, and

$$G_r' := g_0^{-1} G_r g_0 = \left\{ \begin{pmatrix} 1 + \mathfrak{p}_E^r & \mathfrak{p}_E^{2r} \\ \mathfrak{o}_E & 1 + \mathfrak{p}_E^r \end{pmatrix} \right\} \cap \mathrm{SL}_2(E).$$

Then, $V_{\chi,\mathfrak{o}_E}$ is a locally analytic representation of G_r', and hence $D_E(\mathfrak{o}_E, K) \simeq (V_{\chi,\mathfrak{o}_E})_b'$ is a module under $D_E(G_r', K)$. Now fix a Lubin–Tate group $\mathcal{G}$ over E.

Theorem 4.2.5. *Assume that K contains all torsion points of $\mathcal{G}$ and that it is spherically complete. Suppose that $c(\chi) \notin \boldsymbol{Z}_{\geqslant 0}$. Then $D_E(\mathfrak{o}_E, K)$ is a simple $D_E(G_r', K)$-module. Consequently, $D_E(\Delta, K)$ is a simple $D_E(G_r, K)$-module, for any orbit Δ of G_r.*

Proof (cf. the proof of Theorem 5.4 in [S-T1]). The second statement follows from Corollary 4.2.3.

Recall the pairing of Theorem 2.2.7:

$$\{\cdot, \cdot\} : \mathcal{O}(\mathcal{G}_K^{\mathrm{rig}}) \times C_E^{\mathrm{an}}(\mathfrak{o}_E, K) \to K,$$

with which we identify $D_E(\mathfrak{o}_E, K)$ with $\mathcal{O}(\mathcal{G}_K^{\mathrm{rig}})$. That is, for each $\lambda \in D_E(\mathfrak{o}_E, K)$ there is a unique $\Phi_\lambda \in \mathcal{O}(\mathcal{G}_K^{\mathrm{rig}})$, such that $\{\Phi_\lambda, f\} = \lambda(f)$ for any $f \in C_E^{\mathrm{an}}(\mathfrak{o}_E, K)$. Let $I \subset D_E(\mathfrak{o}_E, K) \simeq \mathcal{O}(\mathcal{G}_K^{\mathrm{rig}})$ be a $D_E(G_r', K)$-stable non-zero submodule. Because the action of G_r' on $\mathfrak{o}_E$ contains all translations $z \mapsto z + a$, $a \in \mathfrak{o}_E$, the submodule I is an ideal when considered as a subspace of $\mathcal{O}(\mathcal{G}_K^{\mathrm{rig}})$. We want to show that $I = \mathcal{O}(\mathcal{G}_K^{\mathrm{rig}})$. Since K is spherically complete, a result of Lazard [**Laz**] implies that any finitely generated ideal of the ring $\mathcal{O}(\mathcal{G}_K^{\mathrm{rig}})$ is principal, and that the principal ideals are in bijection with the effective divisors on $\mathcal{G}_K^{\mathrm{rig}}$, which are formal sums of closed points, having only finite support on any affinoid subdomain.

The first step is to start with an arbitrary non-zero function $\Phi \in I$ and to construct another function $\Phi_1 \in I$, such that the common zeros of Φ and Φ_1 are contained in the set of torsion points $\mathcal{G}_{\mathrm{tors}}$ of $\mathcal{G}$. By Lazard's result, this implies that there is a function Φ_2 contained in I such that all of its zeros are in $\mathcal{G}_{\mathrm{tors}}$. The second step is to use Φ_2 to construct another function $\Phi_3 \in I$ which does not vanish at any torsion point of $\mathcal{G}$, so that Φ_2 and Φ_3 generate the unit ideal.

First step. Let $\Phi \in I$ be a non-zero function. Let S be the set of zeros of Φ on $\mathcal{G}^{\mathrm{rig}}(\bar{K})$, and put $S' = S \setminus (S \cap \mathcal{G}_{\mathrm{tors}})$. Because S is countable and $\mathfrak{o}_E^\times$ is uncountable, there exists $a \in \mathfrak{o}_E^\times$ such that $[a^2](S') \cap S = \emptyset$, where $[\cdot] = [\cdot]_{\mathcal{G}} : \mathcal{G}_K \to \mathcal{G}_K$ is the multiplication of $\mathfrak{o}_E$ on $\mathcal{G}$. Thus, the common zeros of $\Phi_1 := \Phi \circ [a^2]$ and Φ are contained in $\mathcal{G}_{\mathrm{tors}}$.

Now we show that Φ_1 is in I. To do this, we may choose a arbitrarily close to 1, and we will assume from now on that $a \in 1 + \mathfrak{p}_E^r$. Put $g_a = \mathrm{diag}(a^{-1}, a)$, and let $\delta_{g_a} \in D_E(G'_r, K)$ be the distribution which is given by evaluating functions at g_a. By Theorem 2.2.7, we compute for any $\Phi_\lambda \in \mathcal{O}(\mathcal{G}_K^{\mathrm{rig}})$ and $f \in C_E^{\mathrm{an}}(\mathfrak{o}_E, K)$:

$$\begin{aligned}\{\Phi_{\delta_{g_a} \cdot_\chi \lambda}, f\} &= (\delta_{g_a} \cdot_\chi \lambda)(f) \\ &= \lambda(\chi_2(a^{-1}) f(a^2 \cdot)) \\ &= \{\Phi_\lambda, \chi_2(a^{-1}) f(a^2 \cdot)\} \\ &= \{\chi_2(a^{-1}) \Phi_\lambda \circ [a^2], f\}.\end{aligned}$$

Thus, if Φ is in I, then $\Phi \circ [a^2] = \chi_2(a) \Phi_{\delta_{g_a} \cdot_\chi \lambda}$ is in I.

Second step. Let $\{x_0, x_1, x_2, \dots\} \subset K$ be the set of torsion points of $\mathcal{G}$. By the first step and Lazard's result alluded to above, there is a function $\Phi_2 \in I$ such that the zeros of Φ_2 are contained in $\mathcal{G}_{\mathrm{tors}}$ and, denoting by m_i the order of vanishing of Φ_2 at x_i, we have $0 < m_0 < m_1 < m_2 < \cdots$.

By Lemma 4.1.2 (4) and Theorem 2.2.7, and with the notation introduced in 2.1.12, we have, for any $\Phi_\lambda \in \mathcal{O}(\mathcal{G}_K^{\mathrm{rig}})$ and $f \in C_E^{\mathrm{an}}(\mathfrak{o}_E, K)$,

$$\{\Phi_{\delta_{(\mathfrak{u}^+)^m} \cdot_\chi \lambda}, f\} = \{\Phi_\lambda, c_m \Theta^m f + Q_m \mathfrak{u}^- f\} = \{c_m \Omega^{-m} \Phi_\lambda^{(m)}, f\} + \{\Omega \log_{\mathcal{G}} \cdot \Phi_{\lambda, Q_m}, f\},$$

where $c_m = m! \binom{c(\chi)}{m}$, Φ_{λ, Q_m} is defined by requiring that

$$\{\Phi_{\lambda, Q_m}, f\} = \{\Phi_\lambda, Q_m f\}$$

for any $f \in C_E^{\mathrm{an}}(\mathfrak{o}_E, K)$, and $\Phi_\lambda^{(m)}$ is the mth derivative of Φ_λ. Therefore,

$$\Phi_{2,m} := c_m \Omega^{-m} \Phi_2^{(m)} + \Omega \log_{\mathcal{G}} \Phi_{2, Q_m}$$

is in I and is congruent to $c_m \Omega^{-m} \Phi_2^{(m)}$ modulo $\log_{\mathcal{G}}$. Moreover, because $\log_{\mathcal{G}}$ vanishes on $\mathcal{G}_{\mathrm{tors}}$ and $c_m \neq 0$ (here we use that $c(\chi) \notin \boldsymbol{Z}_{\geqslant 0}$) the function Φ_{2,m_i} does not vanish at x_i but vanishes at all x_j for $j > i$. Taking a zero sequence $(b_i)_{i \geqslant 0}$, $b_i \in K^\times$, which converges to zero sufficiently fast, we see that

$$\Phi_3 := \sum_{i \geqslant 0} b_i \Phi_{2, m_i}$$

is a well-defined element of $\mathcal{O}(\mathcal{G}_K^{\mathrm{rig}})$, which does not vanish at any torsion point of $\mathcal{G}$. Hence Φ_2 and Φ_3 generate the unit ideal.

On the other hand, by [**S-T1**, Corollary 2.6] the b_i may be chosen so that the series $\sum_{i \geqslant 0} b_i \delta_{(\mathfrak{u}^+)^{m_i}}$ converges to a well-defined element of $D_E(G'_r, K)$. Using the continuity and linearity of the action of $D_E(G'_r, K)$ on $D_E(\mathfrak{o}_E, K)$, we obtain

$$\Phi_3 = \sum_{i \geqslant 0} b_i \Phi_{2, m_i} = \biggl(\sum_{i \geqslant 0} b_i \delta_{(\mathfrak{u}^+)^{m_i}}\biggr) \cdot_\chi \Phi_2.$$

Since I is stable under the action of $D_E(G'_r, K)$, and $\Phi_2 \in I$, this shows that $\Phi_3 \in I$, so that I is the unit ideal. This proves the theorem. $\square$

Proposition 4.2.6. *Suppose $c(\chi) \notin \boldsymbol{Z}_{\geqslant 0}$. Let $\Delta_1 \neq \Delta_2 \subset P\backslash \mathrm{GL}_2(E)$ be two G_r-orbits. Then any continuous G_r-equivariant map $C_E^{\mathrm{an}}(\Delta_1, K) \to C_E^{\mathrm{an}}(\Delta_2, K)$ is the zero map. Hence, $D_E(\Delta_1, K)$ and $D_E(\Delta_2, K)$ are non-isomorphic $D_E(G_r, K)$-modules.*

Proof. Without loss of generality we may assume that P is the subgroup of upper triangular matrices and that $\Delta_1 \cup \Delta_2$ is contained in the open subset $E \subset P\backslash \mathrm{GL}_2(E)$ (because $\Delta_1 \cup \Delta_2$ is properly contained in $P\backslash \mathrm{GL}_2(E)$ and we may conjugate by an element of $\mathrm{GL}_2(\mathfrak{o}_E)$ to map any point in the complement to infinity). Let $\phi : C_E^{\mathrm{an}}(\Delta_1, K) \to C_E^{\mathrm{an}}(\Delta_2, K)$ be a continuous G_r-equivariant map. Denote by 1_{Δ_1} the constant function with value 1 on Δ_1. Because the action of $\mathfrak{u}^-$ is differentiation with respect to z, and because ϕ is G_r-equivariant and continuous, $\mathfrak{u}^-.\phi(1_{\Delta_1}) = \phi(\mathfrak{u}^-.1_{\Delta_1}) = 0$. Hence $\phi(1_{\Delta_1})$ is a locally constant function on Δ_2. For $b \in \mathfrak{p}_E^r$ we have $\exp(b\mathfrak{u}^+) \in G_r$. Put $\mathfrak{x}_b := \exp(-b\mathfrak{u}^+) \circ \mathfrak{u}^- \circ \exp(b\mathfrak{u}^+)$. A straightforward calculation shows that, for any locally constant function f on E, we have

$$(\mathfrak{x}_b^n \cdot_\chi f)(z) = n! \binom{c(\chi)}{n} b^n (1 - bz)^n f(z).$$

It follows that for any polynomial $P(z) \in K[z]$ one has $\phi(P(z) \cdot 1_{\Delta_1}) = P(z) \cdot \phi(1_{\Delta_1})$.

Now choose a sequence of polynomials P_i such that $P_i(z) \cdot 1_{\Delta_1}$ converges to 1_{Δ_1} on $C_E^{\mathrm{an}}(\Delta_1, K)$, but $P_i(z) \cdot 1_{\Delta_2}$ converges to the zero function on $C_E^{\mathrm{an}}(\Delta_2, K)$. Such a sequence exists by Lemma 4.2.8 below. In particular $P_i(z) \cdot \phi(1_{\Delta_1})$ converges to zero. But it also converges to $\phi(1_{\Delta_1})$. Hence $\phi(1_{\Delta_1})$ is zero and therefore $\phi(P(z) \cdot 1_{\Delta_1})$ is zero for any polynomial P. Since the space of functions $P(z) \cdot 1_{\Delta_1}$ with P a polynomial is dense in the space of all locally analytic functions on Δ_1, ϕ is the zero map. $\square$

Corollary 4.2.7. *Assume that K is as in Theorem 4.2.5. Suppose $c(\chi) \notin \boldsymbol{Z}_{\geqslant 0}$. Let $\mathcal{C} = \coprod_{1 \leqslant i \leqslant n} \Delta_i$ be a finite union of disjoint G_r-orbits and set*

$$V_{\chi,\mathcal{C}} = \bigoplus_{1 \leqslant i \leqslant n} V_{\chi,\Delta_i}.$$

Then, the closed G_r-stable subspaces of $V_{\chi,\mathcal{C}}$ are all of the form

$$\bigoplus_{i \in S} V_{\chi,\Delta_i} \xrightarrow{\sim} \bigoplus_{i \in S} C_E^{\mathrm{an}}(\Delta_i, K),$$

where $S \subset \{1, \ldots, n\}$.

Proof. Let V be a closed G_r-stable subspace of $C_E^{\mathrm{an}}(\mathcal{C}, K)$. Then, the kernel I of the map $D_E(\mathcal{C}, K) \to V'$ is a $D_E(G_r, K)$-submodule of

$$D_E(\mathcal{C}, K) = \bigoplus_{1 \leqslant i \leqslant n} D_E(\Delta_i, K).$$

By Theorem 4.2.5 and Proposition 4.2.6, we must have $I = \bigoplus_{i\in S'} D_E(\Delta_i, K)$, for some $S' \subset \{1,\dots,n\}$, since all the $D_E(G_r, K)$-submodules of $D_E(\mathcal{C}, K)$ are of this form. Thus, if S denotes the complement of S' then $V \subset \bigoplus_{i\in S} C_E^{\mathrm{an}}(\Delta_i, K)$, and this closed embedding becomes an isomorphism after passing to duals. Since K is spherically complete, we can apply the Hahn–Banach theorem to conclude that $V = \bigoplus_{i\in S} C_E^{\mathrm{an}}(\Delta_i, K)$. □

Lemma 4.2.8. *Let $U \subset E$ be a compact-open subset, and let $f : U \to K$ be a continuous function. Then there exists a sequence $\{f_i\}_{i\geqslant 1} \subset K[z]$ which converges to f.*

Proof. Since f can evidently be approximated by locally constant functions, it suffices to consider the case when f is locally constant, and then by additivity, even the case where f is the characteristic function of an open subset of U. This last case is a special case of [**S-T2**, Theorem 4.7], which shows that a locally E-analytic, $\mathbb{C}_p$-valued function has a generalized Mahler expansion. □

4.3. The irreducibility result

Let $G \subset \mathrm{GL}_2(E)$ be a subgroup as in 3.1.1, and let $V_{\chi,\mathcal{H}}$ be the locally analytic representation of G defined there. Fixing a locally analytic section $s : \mathcal{H} \to \tilde{\mathcal{H}}$, we identify $V_{\chi,\mathcal{H}}$ with $C_{E,\mathrm{c}}^{\mathrm{an}}(\mathcal{H}, K)$, as in 3.1.2. In this section we prove Theorem 3.1.6 on the irreducibility of this representation. As a first reduction we show the following.

Lemma 4.3.1. *$V_{\chi,\mathcal{H}}$ is topologically irreducible if for any non-zero $f \in V_{\chi,\mathcal{H}}$ the minimal closed G-invariant subspace containing f contains $C_E^{\mathrm{an}}(\mathrm{supp}(f), K)$.*

Proof. Given $f \neq 0$, there are, for any compact subset $\mathcal{C}$ in $\mathcal{H}$, elements $g_1,\dots,g_t \in G$ such that $\mathcal{C}$ is contained in $\bigcup_{1\leqslant j\leqslant t} \mathrm{supp}(g_j f)$. Moreover, there are $b_1,\dots,b_t \in K$ such that

$$\mathrm{supp}(b_1(g_1 f) + \cdots + b_t(g_t f)) = \bigcup_{1\leqslant j\leqslant t} \mathrm{supp}(g_j f).$$

To see this, we may assume (by induction) that $t = 2$ (and $b_1 = 1$, $g_1 = 1$). Decompose $\mathrm{supp}(f) \cup \mathrm{supp}(gf)$ (for an arbitrary $g \in G$) into compact-open subsets on which both f and gf are *rigid-analytic*. For each such subset there is at most one $b \in K$ such that $f + b(gf)$, is zero on this subset, so for all other values of b this subset is contained in the support of $f + b(gf)$. Therefore, if we avoid finitely many values, $f + b(gf)$ will non-zero on any of these subsets. Thus $\mathrm{supp}(f) \cup \mathrm{supp}(gf) = \mathrm{supp}(f + b(gf))$.

A G-stable subspace containing f also contains $b_1(g_1 f) + \cdots + b_t(g_t f)$, and by our assumption

$$C_E^{\mathrm{an}}(\mathcal{C}, K) \subset C_E^{\mathrm{an}}(\mathrm{supp}(b_1(g_1 f) + \cdots + b_t(g_t f)), K)$$

is contained in any closed G-stable subspace containing f. Hence, such a subspace is necessarily the whole space. □

Proof of Theorem 3.1.6. Let $f \in V_{\chi,\mathcal{H}}$ be a non-zero function. We will show that a closed G-stable subspace of $V_{\chi,\mathcal{H}}$ containing f necessarily contains $C_E^{\mathrm{an}}(\mathrm{supp}(f), K)$.

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Since f is locally E-analytic, there is an open neighbourhood $\mathfrak{U}$ of $0 \in \mathrm{Lie}(\mathrm{SL}_2(E))$ such that for all $\mathfrak{x} \in \mathfrak{U}$ we have an equality of locally E-analytic functions on $P\backslash \mathrm{GL}_2(E)$:

$$\exp(\mathfrak{x})f = \sum_{k \geqslant 0} \frac{1}{k!}\mathfrak{x}^k f.$$

Next, let $G_f \subset G$ be a compact-open subgroup such that $\mathrm{Ad}(G_f)$ ('Ad' denoting the action by conjugation on $\mathrm{Lie}(\mathrm{SL}_2(E))$) preserves $\mathfrak{U}$, and which stabilizes $\mathrm{supp}(f)$. Note that the identity above also holds for all gf with $g \in G_f$. Let V_f be the minimal closed G_f-stable subspace of $C_E^{\mathrm{an}}(\mathrm{supp}(f), K)$ which contains f. By Lemma 4.3.1 it suffices to show that $V_f = C_E^{\mathrm{an}}(\mathrm{supp}(f), K)$.

Since the E-linear span of $\mathrm{Lie}(G)$ contains $\mathrm{Lie}(\mathrm{SL}_2(E))$, the right-hand side of the identity above is an element of V_f, and the same is true if we replace f by gf, for any $g \in G_f$. Let $r \in \boldsymbol{Z}_{\geqslant 1}$ be such that G_r is contained in $\exp(\mathfrak{U})$. The observation just made, shows that G_r maps the space spanned by the gf, for $g \in G_f$, into V_f. By the continuity of the action of G_r on $C_E^{\mathrm{an}}(\mathrm{supp}(f), K)$, this implies that V_f is stable under G_r.

In order to apply Corollary 4.2.7, we let $L \supset K$ be a spherically complete extension which contains all torsion points of a Lubin–Tate group $\mathcal{G}$, as in Theorem 4.2.5. Such an extension always exists [**Ro**, Corollary 4.48, Theorem 4.49]. Suppose that V_f is a proper subspace of $C_E^{\mathrm{an}}(\mathrm{supp}(f), K)$. Let $V_{f,L}$ be the closure of $V_f \otimes_K L$ in $C_E^{\mathrm{an}}(\mathrm{supp}(f), L)$. Proposition 8.1.4 implies that $V_{f,L}$ is a proper subspace of $C_E^{\mathrm{an}}(\mathrm{supp}(f), L)$, and by the continuity of the action of G_r on $C_E^{\mathrm{an}}(\mathrm{supp}(f), L)$, $V_{f,L}$ is stable under this group. Decompose $\mathrm{supp}(f) = \coprod_{1 \leqslant i \leqslant n} \Delta_i$ into disjoint G_r-orbits. Since $V_{f,L}$ is a proper subspace of $C_E^{\mathrm{an}}(\mathrm{supp}(f), L)$, all elements of $V_{f,L}$ vanish on at least one of the Δ_i, by Corollary 4.2.7. But this is impossible, as f does not vanish on any of these. □

4.4. The locally algebraic subspace

4.4.1. Suppose now that $c(\chi) \in \mathbb{Z}_{\geqslant 0}$. In 3.1.7 we defined a subspace $V_{\chi,\mathcal{H}}^{\mathrm{loc\,alg}}$. For principal series representations of $\mathrm{GL}_2(F)$ this subspace has a two step filtration whose associated graded pieces are irreducible $\mathrm{GL}_2(F)$-modules (see the remarks following Proposition 6.2 in [**S-T1**]). In this subsection we show that this is not the case for cuspidal representations. Namely, in this case, $V_{\chi,\mathcal{H}}^{\mathrm{loc\,alg}}$ has an infinite increasing G-stable filtration, whose associated graded pieces are all non-zero. This is due to the fact that in these cases the stabilizer $S_G(z_0) \subset G$ of any point z_0 in the orbit $\mathcal{H}$, is compact modulo its subgroup of scalar matrices. This in turn has the effect that there are arbitrarily fine coverings of the orbit which are stable under G.

Let G and $\mathcal{H}$ be as in 3.1.1.

Lemma 4.4.2. *Suppose $S_G(z_0)$ is compact modulo its subgroup of scalar matrices. Then G has a fundamental system of neighbourhoods of the identity consisting of compact-open subgroups which are normalized by $S_G(z_0)$.*

Fix such a subgroup $G_0 \subset G$, and assume $\mathcal{H}_0 = z_0.G_0$ is contained in E. Let $\tilde{\mathcal{H}}_0$ be the preimage of $\mathcal{H}_0$ in $P\backslash \mathrm{GL}_2(E)$. If G_0 is sufficiently small then we have the following.

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(1) *For $g \in G$ denote by $V_{\chi,\mathcal{H}_0}(g) \subset V_{\chi,\mathcal{H}}$ the subspace of functions f with support in $\tilde{\mathcal{H}}_0.g$ such that we have*

$$f\left(q\begin{pmatrix}1 & 0\\ z & 1\end{pmatrix}g\right) = \chi(q)\varphi(z)$$

for all $q \in P$, $z \in \mathcal{H}_0$, and some polynomial $\varphi \in K[z]$ of degree less than or equal to $c(\chi)$. If $\tilde{\mathcal{H}}_0.g_1 = \tilde{\mathcal{H}}_0.g_2$, then $V_{\chi,\mathcal{H}_0}(g_1) = V_{\chi,\mathcal{H}_0}(g_2)$.

(2) *The sum*

$$V_{\chi,\mathcal{H},\mathcal{H}_0} = \sum_{g\in G} V_{\chi,\mathcal{H}_0}(g) = \bigoplus_{g\in S_G(z_0)G_0\backslash G} V_{\chi,\mathcal{H}_0}(g) \subset V_{\chi,\mathcal{H}}$$

is a closed G-invariant subspace of $V_{\chi,\mathcal{H}}^{\mathrm{loc\,alg}}$. Moreover, $V_{\chi,\mathcal{H}}^{\mathrm{loc\,alg}}$ is the union of all $V_{\chi,\mathcal{H},z_0.G_0}$, where G_0 runs over all sufficiently small compact-open subgroups of G which are normalized by $S_G(z_0)$.

Proof. As G is a locally profinite group, there is a compact-open subgroup G^c which has a fundamental system of neighbourhoods of the identity consisting of compact-open subgroups which are normal in G^c. Let $Z \subset S_G(z_0)$ be the subgroup of scalar matrices. Since $S_G(z_0)/Z$ is compact, $S_G(z_0)/(S_G(z_0)\cap G^c)Z$ is finite. Let $s_1,\dots,s_n \in S_G(z_0)$ be representatives for this finite set. Then, for any compact-open normal subgroup $G' \subset G^c$ the intersection $\bigcap_{1\leqslant i\leqslant n} s_i G' s_i^{-1}$ is a compact-open subgroup of G normalized by $S_G(z_0)$.

Now fix a compact-open subgroup G_0 of G which is normalized by $S_G(z_0)$. Put $g_0 = \exp(z_0\mathfrak{u}^-)$. Then, as $S_G(z_0)G_0$ is compact modulo its subgroup of scalar matrices, there is an $r \in \mathbb{Z}$ such that, if $g \in S_G(z_0)G_0$ and

$$g_0 g g_0^{-1} = \begin{pmatrix}a & b\\ c & d\end{pmatrix},$$

then one has $(b/d) \in \mathfrak{p}_E^r$. Let $s > 0$ be such that $\chi_2(1+t) = (1+t)^{c(\chi)}$, for all $t \in \mathfrak{p}_E^s$. After passing to a smaller subgroup inside G_0 we can assume that $\mathcal{H}_0 = z_0.G_0$ is contained in $z_0 + \mathfrak{p}_E^{s-r}$.

(1) Suppose $\tilde{\mathcal{H}}_0 g_1 = \tilde{\mathcal{H}}_0 g_2$. Then $g_1 g_2^{-1} \in S_G(z_0)G_0$. Denote the entries of the matrix $g_0 g_1 g_2^{-1} g_0^{-1}$ by a, b, c, d as above. Then we have $\exp(z\mathfrak{u}^-)g_1 = q\exp(z'\mathfrak{u}^-)g_2$, where z is in $\mathcal{H}_0$ and

$$q = \begin{pmatrix}\dfrac{ad-bc}{b(z-z_0)+d} & b\\ 0 & b(z-z_0)+d\end{pmatrix}, \qquad z' = \frac{a(z-z_0)+c}{b(z-z_0)+d} + z_0 \in \mathcal{H}_0.$$

If $f \in V_{\chi,\mathcal{H}_0}(g_2)$, then

$$\begin{aligned} f(\exp(z\mathfrak{u}^-)g_1) &= f(q\exp(z'\mathfrak{u}^-)g_2) = \chi_1(ad-bc)\chi_2(b(z-z_0)+d)\varphi(z')\\ &= \chi_1(ad-bc)\chi_2(d)\chi_2\left(1+\frac{b}{d}(z-z_0)\right)\varphi\left(\frac{a(z-z_0)+c}{b(z-z_0)+d}+z_0\right). \end{aligned}$$

By assumption,

$$\chi_2\left(1+\frac{b}{d}(z-z_0)\right) = \left(1+\frac{b}{d}(z-z_0)\right)^{c(\chi)}$$

so $f \in V_{\chi,\mathcal{H}_0}(g_1)$, and $V_{\chi,\mathcal{H}_0}(g_1) \subset V_{\chi,\mathcal{H}_0}(g_2)$. As the other inclusion holds by symmetry, this proves (1).

(2) For $g, g' \in G$ one obviously has

$$g.V_{\chi,\mathcal{H}_0}(g') = V_{\chi,\mathcal{H}_0}(g'g^{-1}),$$

and from the definitions one sees that $V_{\chi,\mathcal{H}_0}(g)$ is a subspace of $V_{\chi,\mathcal{H}}^{\mathrm{loc\,alg}}$. Thus, $V_{\chi,\mathcal{H},\mathcal{H}_0}$ is a G-invariant subspace of $V_{\chi,\mathcal{H}}^{\mathrm{loc\,alg}}$. By (1), $V_{\chi,\mathcal{H},\mathcal{H}_0}$ is the direct sum of the spaces $V_{\chi,\mathcal{H}_0}(g)$ where g runs over a set of representatives of $G_0\backslash G$. From this fact and from the very definition of the topology on $V_{\chi,\mathcal{H}}$ it follows that $V_{\chi,\mathcal{H},\mathcal{H}_0}$ is a closed subspace.

Now suppose f is an element of $V_{\chi,\mathcal{H}}^{\mathrm{loc\,alg}}$. Then, for any $h \in \tilde{\mathcal{H}}$, the function $z \mapsto f(\exp(z\mathfrak{u}^-)h)$ is, locally in a neighbourhood of 0, a polynomial in z of degree less than or equal to $c(\chi)$. As the support of f is compact modulo P, we can find a sufficiently small open compact subgroup G_0, and finitely many elements $g_1, g_2, \ldots, g_n \in G$ so that on $z_0.G_0$ the function

$$z \mapsto f(\exp(z\mathfrak{u}^-)g_i) = f(\exp((z-z_0)\mathfrak{u}^-)g_0 g_i)$$

is a polynomial in $z - z_0$ (and hence in z) of degree less than or equal to $c(\chi)$, and such that $\bigcup_{1\leqslant i\leqslant n} z_0.G_0 g_i$ contains $P\backslash\operatorname{supp}(f)$. Thus f is contained in the span of the spaces $V_{\chi,\mathcal{H}_0}(g_i)$. This proves (2). □

Proposition 4.4.3. *With the notation and assumptions of Lemma 4.4.2, the smooth character of* $E^\times$ *given by* $\tilde{\chi}_2(d) = \chi_2(d)d^{-c(\chi)}$ *can be naturally considered as a character of* $S_G(z_0)G_0$*, and there is a canonical isomorphism*

$$V_{\chi,\mathcal{H},\mathcal{H}_0} \simeq (\chi_1 \circ \det) \otimes (\text{c-ind}_{S_G(z_0)G_0}^{G}\, \tilde{\chi}_2) \otimes_E \mathrm{Sym}^{c(\chi)}(E^2). \tag{4.4.4}$$

If $G_0' \subset G_0$ *is a compact-open subgroup normalized by* $S_G(z_0)$*, then the formation of* $\tilde{\chi}_2$ *is compatible with restriction from* G_0 *to* G_0'*, and we have*

$$V_{\chi,\mathcal{H}}^{\mathrm{loc\,alg}} \simeq \varinjlim_{G_0'} (\chi_1 \circ \det) \otimes (\text{c-ind}_{S_G(z_0)G_0'}^{G}\, \tilde{\chi}_2) \otimes_E \mathrm{Sym}^{c(\chi)}(E^2)$$

where G_0' *runs over compact-open subgroups of* G_0 *normalized by* $S_G(z_0)$*.*

Proof. For $q \in P$ we denote by $\chi_2(q)$ and $\tilde{\chi}_2(q)$ the value of χ_2 and $\tilde{\chi}_2$ respectively on the lower right entry of q. We define the character $\tilde{\chi}_2$ of G_0 as follows: for $g \in S_G(z_0)G_0$ write $g_0 g g_0^{-1} = q\exp((z-z_0)\mathfrak{u}^-)$ with some $q \in P$ and $z \in \mathcal{H}_0$. Then $\tilde{\chi}_2(g) = \tilde{\chi}_2(q)$. It is not hard to check that this gives a smooth character of $S_G(z_0)G_0$.

Consider an element of the representation on the right of (4.4.4), and assume it has the form $\varphi_{\mathrm{sm}} \otimes \varphi_{\mathrm{alg}}$ where $\varphi_{\mathrm{sm}} : G \to K$ is a smooth function satisfying $\varphi_{\mathrm{sm}}(g'g) = \tilde{\chi}_2(g')\varphi_{\mathrm{sm}}(g)$ for any $g \in G$ and $g' \in S_G(z_0)G_0$, and φ_{alg} is a homogeneous polynomial of

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degree $c(\chi)$ in two variables. Then the inverse of (4.4.4) maps $\varphi_{\mathrm{sm}} \otimes \varphi_{\mathrm{alg}}$ to the following function on $\tilde{\mathcal{H}}$:

$$q \exp(z\mathfrak{u}^-)g \mapsto \chi_1(\det(g))\chi(q)\varphi_{\mathrm{sm}}(g)\varphi_{\mathrm{alg}}((z,1)g), \tag{4.4.5}$$

where $q \in P$, $z \in \mathcal{H}_0$, $g \in G$.

To check that this is well defined, suppose that we have $q_1 \in P$, $g_1 \in G$ and $z_1 \in \mathcal{H}_0$ with $q \exp(z\mathfrak{u}^-)g = q_1 \exp(z_1\mathfrak{u}^-)g_1$. Then $gg_1^{-1} \in S_G(z_0)G_0$ and we compute

$$\varphi_{\mathrm{sm}}(g) = \varphi_{\mathrm{sm}}(gg_1^{-1}g_1) = \tilde{\chi}_2(gg_1^{-1})\varphi_{\mathrm{sm}}(g_1) = \tilde{\chi}_2(q^{-1}q_1)\varphi_{\mathrm{sm}}(g_1). \tag{4.4.6}$$

Next, an examination of the lower row of $\exp(z\mathfrak{u}^-)g = q^{-1}q_1 \exp(z_1\mathfrak{u}^-)g_1$ shows that $(z,1)g = d(q^{-1}q_1)(z,1)g_1$, where $d(q^{-1}q_1)$ denotes the lower right-hand entry of $q^{-1}q_1$, so that

$$\varphi_{\mathrm{alg}}((z,1)g) = \varphi_{\mathrm{alg}}(d(q^{-1}q_1)(z_1,1)g_1) = d(q^{-1}q_1)^{c(\chi)}\varphi_{\mathrm{alg}}((z_1,1)g_1). \tag{4.4.7}$$

Now using (4.4.6), (4.4.7) and the fact that $\tilde{\chi}_2(q^{-1}q_1)d(q^{-1}q_1)^{c(\chi)} = \chi_2(q^{-1}q_1)$ we obtain

$$\begin{aligned}
&\chi_1(\det(g))\chi(q)\varphi_{\mathrm{sm}}(g)\varphi_{\mathrm{alg}}((z,1)g)\\
&\qquad = \chi_1(\det(g))\chi(q)\chi_2(q^{-1}q_1)\varphi_{\mathrm{sm}}(g_1)\varphi_{\mathrm{alg}}((z,1)g_1)\\
&\qquad = \chi_1(\det(g_1))\chi(q_1)\chi_1(\det(q^{-1}q_1))\chi(qq_1^{-1})\chi_2(q^{-1}q_1)\varphi_{\mathrm{sm}}(g_1)\varphi_{\mathrm{alg}}((z,1)g_0g_1)\\
&\qquad = \chi_1(\det(g_1))\chi(q_1)\varphi_{\mathrm{sm}}(g_1)\varphi_{\mathrm{alg}}((z,1)g_1).
\end{aligned}$$

This shows that the map given by sending $\varphi_{\mathrm{sm}} \otimes \varphi_{\mathrm{alg}}$ to the function given by (4.4.5) is well defined, and its G-equivariance follows from a straightforward computation. Moreover, using Lemma 4.4.2 (2) one sees that (4.4.5) is in fact contained in $V_{\chi,\mathcal{H}}^{\mathrm{loc\,alg}}$.

Finally, it not hard to check that the map of representations we have defined is an isomorphism: for the surjectivity note that if $g \in G$, and $f \in V_{\chi,\mathcal{H}_0}(g)$ is as in Lemma 4.4.2 then f is the image of $\varphi_{\mathrm{sm}} \otimes \varphi_{\mathrm{alg}}$ for suitably chosen φ_{sm} (which is supported on $S_G(z_0)G_0.g$) and φ_{alg}. Similarly, the injectivity follows easily from the fact that if φ_{sm} is supported on $S_G(z_0)G_0.g$, then the image of $\varphi_{\mathrm{sm}} \otimes \varphi_{\mathrm{alg}}$ lies in $V_{\chi,\mathcal{H}_0}(g)$.

This shows that we have an isomorphism as in (4.4.4). That formation of $\tilde{\chi}_2$ is compatible with restriction from G_0 to G_0' is clear from the construction. The rest of the proposition now follows from the final claim in Lemma 4.4.2. □

Corollary 4.4.8. *Suppose that E/F has degree 2 or 3 and G is an open subgroup of $\mathrm{SL}_2(F)$, $\mathrm{GL}_2(F)$ or $D^\times$ (in the latter case $[E:F]=2$). Then for any $z_0 \in \mathcal{H}$ $S_G(z_0)$ is compact modulo centre, and the conclusions of Proposition 4.4.3 hold.*

Proof. Since $S_G(z_0) \subset G$ is necessarily a closed subgroup, it suffices to check that it is contained in a subgroup which is compact modulo centre. *A fortiori* it suffices to check this for G equal to $\mathrm{GL}_2(F)$ or $D^\times$ (since G is always a closed subgroup of one of these groups).

For $\mathrm{GL}_2(F)$ this stabilizer is computed in Corollary 5.2.6. The calculations there show that $S_{\mathrm{GL}_2(F)}(z_0)$ can be conjugated into a group of diagonal matrices whose entries are conjugate elements of $E^\times$. Since $E^\times/F^\times$ is compact, we are done in this case.

For $D^\times$ the result is clear since $D^\times/F^\times$ is already compact. □

5. Intertwining operators

5.1. Intertwiners between principal series and cuspidal representations

5.1.1. In this subsection, we let E and E' be finite extensions of F, which are contained in K. Let G be a locally F-analytic group, equipped with embeddings of locally F-analytic groups ι and ι' into $\mathrm{GL}_2(E)$ and $\mathrm{GL}_2(E')$ respectively. We assume that G satisfies the axioms of 3.1.1 as a subgroup of each of these two groups. Let P (respectively, P') be the subgroup of upper triangular matrices in $\mathrm{GL}_2(E)$ (respectively, $\mathrm{GL}_2(E')$), $\mathcal{H}$ (respectively, $\mathcal{H}'$) an open orbit of G on $P\backslash \mathrm{GL}_2(E)$ (respectively, $P'\backslash \mathrm{GL}_2(E')$), and χ (respectively, δ) a $K^\times$-valued locally analytic character of the reductive quotient T (respectively, T') of P (respectively, P'). Consider the locally analytic representations $V_{\chi,\mathcal{H}}$ and $V_{\delta,\mathcal{H}'}$ of G, and a continuous intertwining operator

$$\phi : V_{\chi,\mathcal{H}} \to V_{\delta,\mathcal{H}'}.$$

We work with the model given in 3.2.1. In particular, we write

$$\chi\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \chi_1(ad)\chi_2(d), \qquad \delta\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \delta_1(ad)\delta_2(d).$$

We write $\mathcal{H}_f = \mathcal{H} \cap E$ and $\mathcal{H}'_f = \mathcal{H}' \cap E'$ respectively, and we identify $V_{\chi,\mathcal{H}}$ and $V_{\delta,\mathcal{H}'}$ with subspaces of $C^{\mathrm{an}}_E(\mathcal{H}_f, K)$ and $C^{\mathrm{an}}_{E'}(\mathcal{H}'_f, K)$ respectively, with the group action given by (3.2.2).

Proposition 5.1.2. *Suppose $c(\chi) \notin \mathbb{Z}_{\geqslant 0}$, and that ϕ is non-zero.*

(1) *Let $\mathcal{C} \subset \mathcal{H}_f$ be a non-empty compact-open subset, and denote by $1_{\mathcal{C}}$ the characteristic function of $\mathcal{C}$. Then $\phi(1_{\mathcal{C}})$ is locally constant, and for any polynomial $P(z) \in K[z]$ one has that $\phi(P(z)1_{\mathcal{C}}) = P(z)\phi(1_{\mathcal{C}})$.*

(2) *We have $c(\chi) = c(\delta)$ and $E' \subset E$.*

(3) *If the embeddings ι and ι' are compatible via the inclusion $\mathrm{GL}_2(E') \subset \mathrm{GL}_2(E)$ induced by $E' \subset E$, then $E' = E$, $\mathcal{H} = \mathcal{H}'$ and $\mathrm{supp}(\phi(1_{\mathcal{C}})) = \mathcal{C}$.*

Proof. Let $\mathfrak{u}^+$ and $\mathfrak{u}^-$ be the Lie algebra elements defined in 4.1.1. Since the induced action of $\mathfrak{u}^-$ on locally analytic functions $f(z)$ on open subsets of E or E' is simply differentiation with respect to z (and is independent of the character), one has $0 = \phi(\mathfrak{u}^- \cdot_\chi 1_{\mathcal{C}}) = \mathfrak{u}^- \cdot_\delta \phi(1_{\mathcal{C}})$. Hence $\phi(1_{\mathcal{C}})$ is locally constant.

For $b \in F^\times$, put $\mathfrak{x}_b = \exp(-b\mathfrak{u}^+) \circ \mathfrak{u}^- \circ \exp(b\mathfrak{u}^+)$. As in the proof of Proposition 4.2.6, we have

$$(\mathfrak{x}_b^n \cdot_\chi f)(z) = n!\binom{c(\chi)}{n} b^n (1 - bz)^n f(z) \tag{5.1.3}$$

for any compactly supported, locally constant function f on E, and any $z \in E$. Hence

$$\phi(\mathfrak{x}_b \cdot_\chi 1_{\mathcal{C}}) = \phi(z \mapsto c(\chi) b (1 - bz) 1_{\mathcal{C}}(z)) = c(\chi) b \phi(z \mapsto (1 - bz) 1_{\mathcal{C}}(z)),$$

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and $\phi(\mathfrak{x}_{b\cdot\chi}1_{\mathcal{C}})(z') = \mathfrak{x}_{b\cdot\delta}\phi(1_{\mathcal{C}})(z') = c(\delta)b(1-bz')\phi(1_{\mathcal{C}})(z')$ for any $z' \in \mathcal{H}'$. Therefore,

$$c(\chi)b\phi(z \mapsto (1-bz)1_{\mathcal{C}}(z))(z') = c(\delta)b(1-bz')\phi(1_{\mathcal{C}})(z').$$

Dividing both sides by b and letting b tend to 0 one gets $c(\chi)\phi(1_{\mathcal{C}}) = c(\delta)\phi(1_{\mathcal{C}})$. Because we assumed that ϕ is non-zero and $V_{\chi,\mathcal{H}}$ is irreducible, $\phi(1_{\mathcal{C}})$ is non-zero, so that $c(\chi) = c(\delta)$. This implies that

$$\phi(\mathfrak{x}_b^n\cdot_\chi 1_{\mathcal{C}})(z') = \mathfrak{x}_b^n\cdot_\delta\phi(1_{\mathcal{C}})(z') = n!\binom{c(\chi)}{n}b^n(1-bz')^n\phi(1_{\mathcal{C}})(z')$$

for $z' \in \mathcal{H}'_f$. Comparing with (5.1.3) we see that $\phi(P(z)1_{\mathcal{C}}) = P(z)\phi(1_{\mathcal{C}})$ for any polynomial $P(z) \in K[z]$.

Now suppose that there is a non-empty compact-open subset $\mathcal{C}_1 \subset \mathcal{H}'_f$ such that $\phi(1_{\mathcal{C}})|_{\mathcal{C}_1} \neq 0$ but $\mathcal{C}_1 \cap \mathcal{C} = \emptyset$, where the intersection is taken inside K. Let $\mathcal{C}_2 = \mathcal{C} \cup (\operatorname{supp}(\phi(1_{\mathcal{C}}))\backslash\mathcal{C}_1)$, and choose disjoint compact-open subsets $\tilde{\mathcal{C}}_1, \tilde{\mathcal{C}}_2$ in $E\cdot E'$ with $\mathcal{C}_1 \subset \tilde{\mathcal{C}}_1$ and $\mathcal{C}_2 \subset \tilde{\mathcal{C}}_2$. By Lemma 4.2.8 there is a sequence of polynomials $P_i(z) \in K[z]$ such that $P_i(z)$ converges to 0 for $z \in \tilde{\mathcal{C}}_2$ and to 1 for $z \in \tilde{\mathcal{C}}_1$. It follows that

$$0 = \phi(0) = \lim_{i\to\infty}\phi(P_i(z)1_{\mathcal{C}}) = \lim_{i\to\infty}P_i(z)\phi(1_{\mathcal{C}}) = \phi(1_{\mathcal{C}})\cdot 1_{\mathcal{C}_1}.$$

This is a contradiction. Thus, every compact-open subset of $\operatorname{supp}(\phi(1_{\mathcal{C}}))$ must meet $\mathcal{H}_f$. Since $\operatorname{supp}\phi(1_{\mathcal{C}})$ is an open subset of E', we must have $E' \subset E$. This shows (2).

Now suppose that ι and ι' are compatible with the inclusion $\mathrm{GL}_2(E') \subset \mathrm{GL}_2(E)$. Then $\mathcal{H}$ and $\mathcal{H}'$, being orbits of G on $P\backslash\mathrm{GL}_2(E)$, must be disjoint unless they are equal. Thus, we must have $\mathcal{H} = \mathcal{H}'$ and $E = E'$.

We have already seen that $\operatorname{supp}(\phi(1_{\mathcal{C}})) \subset \mathcal{C}$, and it remains to prove the opposite inclusion. Suppose then that there is a non-empty compact-open subset $\mathcal{C}_1 \subset \mathcal{C}$ such that $\phi(1_{\mathcal{C}})|_{\mathcal{C}_1} = 0$. Then we can find a sequence of polynomials $P_i(z) \in K[z]$ such that $\lim_{i\to\infty}P_i(z)1_{\mathcal{C}_1} = 1_{\mathcal{C}_1}$ but $\lim_{i\to\infty}P_i(z)|_{\mathcal{C}\backslash\mathcal{C}_1} = 0$. It follows that

$$\phi(1_{\mathcal{C}_1}) = \lim_{i\to\infty}\phi(P_i(z)1_{\mathcal{C}_1}) = \lim_{i\to\infty}\phi(P_i(z)1_{\mathcal{C}}) = \lim_{i\to\infty}P_i(z)\phi(1_{\mathcal{C}}) = 0.$$

Again a contradiction, because we assumed ϕ to be non-zero. □

Corollary 5.1.4. *Keep the assumptions of 5.1.1 and suppose that ϕ is non-zero. We no longer assume that $c(\chi) \notin \mathbb{Z}_{\geqslant 0}$. Then $E' \subset E$. If ι and ι' are compatible with the inclusion $\mathrm{GL}_2(E') \subset \mathrm{GL}_2(E)$ then $E = E'$ and $\mathcal{H} = \mathcal{H}'$. If $c(\chi) \in \mathbb{Z}_{\geqslant 0}$ then $c(\delta) = c(\chi)$ or $-2 - c(\chi)$.*

Proof. By Proposition 5.1.2 we have only to consider the case where $c(\chi) \in \mathbb{Z}_{\geqslant 0}$. If $\phi(V_{\chi,\mathcal{H}}^{\mathrm{loc\,alg}}) = 0$, then ϕ induces a map

$$V_{\chi\alpha^{c(\chi)+1},\mathcal{H}} \to V_{\delta,\mathcal{H}'}$$

by Proposition 3.1.8 and the result follows by Proposition 5.1.2 and the observation that $c(\chi\alpha^{c(\chi)+1}) = -2 - c(\chi)$.

Suppose that $\phi(V_{\chi,\mathcal{H}}^{\mathrm{loc\,alg}}) \neq 0$. Since the image of $V_{\chi,\mathcal{H}}^{\mathrm{loc\,alg}}$ is killed by $(\mathfrak{u}^-)^{c(\chi)+1}$ by Corollary 3.1.10, we must have

$$\phi(V_{\chi,\mathcal{H}}^{\mathrm{loc\,alg}}) \subset V_{\delta,\mathcal{H}'}^{\mathrm{loc\,alg}} \neq 0$$

and $c(\delta) \in \mathbb{Z}_{\geqslant 0}$. Now, if the induced map

$$V_{\chi\alpha^{c(\chi)+1},\mathcal{H}} \xrightarrow{\sim} V_{\chi,\mathcal{H}}/V_{\chi,\mathcal{H}}^{\mathrm{loc\,alg}} \to V_{\delta,\mathcal{H}'}/V_{\delta,\mathcal{H}'}^{\mathrm{loc\,alg}} \xrightarrow{\sim} V_{\chi\alpha^{c(\delta)+1},\mathcal{H}'}$$

is non-zero, then we may again conclude by Proposition 5.1.2. In this case we have $c(\chi) = c(\delta)$.

Suppose that this map is 0. Then we have

$$\phi((\mathfrak{u}^-)^{c(\delta)+1}V_{\chi,\mathcal{H}}) = (\mathfrak{u}^-)^{c(\delta)+1}\phi(V_{\chi,\mathcal{H}}) \subset (\mathfrak{u}^-)^{c(\delta)+1}V_{\delta,\mathcal{H}'}^{\mathrm{loc\,alg}} = \{0\}.$$

Since any element in $f \in V_{\chi,\mathcal{H}}$ with $\operatorname{supp} f \subset \mathcal{H}_f$ is in the image in $\mathfrak{u}^-$, ϕ is 0 on all such elements. However, it is easy to see that such elements span $V_{\chi,\mathcal{H}}$, so that $\phi = 0$. □

Corollary 5.1.5. *Keep the assumptions of 5.1.1 and assume that ϕ is non-zero.*

Suppose G is an open subgroup of $\mathrm{GL}_2(F)$, $\mathrm{SL}_2(F)$ (so ι and ι' are the natural inclusions), or an open subgroup of the group of units $D^\times$ of the quaternion algebra D over F. Then $E = E'$ and $\mathcal{H} = \mathcal{H}'$. If $c(\chi) \notin \mathbb{Z}_{\geqslant 0}$ then $c(\chi) = c(\delta)$. If $c(\chi) \in \mathbb{Z}_{\geqslant 0}$ then $c(\chi)$ is one of $c(\delta)$ and $-2 - c(\delta)$.

In particular there are no intertwiners between the principal series and cuspidal representations of $\mathrm{GL}_2(F)$ or $\mathrm{SL}_2(F)$.

Proof. This follows immediately from Proposition 5.1.2 and Corollary 5.1.4. We remark only that if G is an open subgroup of $D^\times$, then the fields E and E' necessarily have degree 2 over F, so that $E' \subset E$ implies $E = E'$. □

5.1.6. Let G be one of $\mathrm{SL}_2(F)$ or $\mathrm{GL}_2(F)$, and suppose that $E' = F$, and that $[E : F] \geqslant 2$. In this case we can give another proof of part of Proposition 5.1.2, which uses a Jacquet functor argument. To explain this, for any locally F-analytic representation V of G, put

$$J(V) = V/\overline{\langle uv - v;\, u \in U^-,\, v \in V\rangle}.$$

That is, $J(V)$ is the maximal Hausdorff quotient of the coinvariants of $J(V)$ under U^-. (This is the most obvious generalization of the Jacquet module for smooth representations. Another extension of the Jacquet functor to locally analytic representations has been given by Emerton [**Em2**].)

Proposition 5.1.7. *We have $J(V_{\chi,\mathcal{H}}) = \{0\}$, while the K-vector space $J(V_{\delta,\mathcal{H}'})$ is two dimensional if $c(\delta) \in \mathbb{Z}_{\geqslant 0}$ and one dimensional otherwise. In particular $V_{\chi,\mathcal{H}}$ and $V_{\delta,\mathcal{H}'}$ are not isomorphic.*

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Proof. As usual we regard elements of $V_{\chi,\mathcal{H}}$ and $V_{\delta,\mathcal{H}'}$ as locally analytic functions on E and E' respectively. The operator $\mathfrak{u}^-$ then becomes differentiation $\mathrm{d}/\mathrm{d}z$, where z is the tautological coordinate on E or E'. Since $V_{\chi,\mathcal{H}}$ consists of the compactly supported, locally analytic functions on $\mathcal{H}$, we have $\mathfrak{u}^- V_{\chi,\mathcal{H}} = V_{\chi,\mathcal{H}}$, so that *a fortiori* $J(V_{\chi,\mathcal{H}}) = \{0\}$.

Similarly, we have that $V_{\delta,\mathcal{H}'}$ consists of locally analytic functions on E', such that

$$\delta_2(z)^{-1} f(z) = \varphi\left(\frac{1}{z}\right) = \sum_{i=0}^{\infty} a_i z^{-i}$$

for $|z| \gg 0$, where $\varphi(1/z)$ is a power series which is convergent for $|z|$ sufficiently large. As in 3.2.3, differentiating both sides of this equation one finds

$$\delta_2(z)^{-1} f'(z) = \frac{c(\delta)}{z}\varphi\left(\frac{1}{z}\right) - \frac{1}{z^2}\varphi'\left(\frac{1}{z}\right) = \frac{c(\delta)a_0}{z} + \frac{(c(\delta)-1)a_1}{z^2} + \frac{(c(\delta)-2)a_2}{z^3} + \cdots .$$

Now the constant coefficient of this series always vanishes, and the coefficient of $z^{-c(\delta)-1}$ vanishes if $c(\delta) \in \mathbb{Z}_{\geqslant 0}$. On the other hand, since we can always integrate locally analytic functions on compact subsets, and because $\varphi(1/z)$ can be approximated by polynomials in $1/z$, any function f such that $a_0 = 0$ and $a_{c(\delta)+1} = 0$ if $c(\delta) \in \mathbb{Z}_{\geqslant 0}$, is in $\overline{\mathfrak{u}^- V_{\delta,\mathcal{H}'}}$. Thus, it suffices to show that if $u \in U^-$ and $f \in V_{\delta,\mathcal{H}'}$ then $uf - f \in \overline{\mathfrak{u}^- V_{\delta,\mathcal{H}'}}$. Write

$$u = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}.$$

Then for $|z| \gg 0$ we have

$$\begin{aligned} \delta(z)^{-1}((u \cdot_\delta f)(z) - f(z)) &= \delta(z)^{-1}(f(z+c) - f(z)) \\ &= \sum_{i=0}^{\infty} \delta_2\left(\frac{z+c}{z}\right) a_i (z+c)^{-i} - a_i z^{-i} \\ &= \sum_{i=0}^{\infty} a_i z^{-i}\left(\left(1 + \frac{c}{z}\right)^{c(\delta)-i} - 1\right), \end{aligned}$$

and the expression on the right-hand side has constant term 0, and the coefficient of $z^{-c(\delta)-1}$ is equal to 0 when $c(\delta) \in \mathbb{Z}_{\geqslant 0}$. This proves the proposition. □

Remark 5.1.8. In fact the proof of the preceding proposition shows that if V is one of $V_{\delta,\mathcal{H}'}$ or $V_{\chi,\mathcal{H}}$, then $J(V) = V/\overline{\mathfrak{u}^- V}$. Moreover, $\mathfrak{u}^- V$ is closed in V, except if $V = V_{\delta,\mathcal{H}'}$, $c(\delta) \notin \mathbb{Z}_{\geqslant 0}$ and $|c(\delta) - n|^{-1}$ is not bounded by $|z^n|$ for any $z \in E$. In particular, $\mathfrak{u}^- V$ is closed in V if $c(\delta) \notin \mathbb{Z}_p \setminus \mathbb{Z}$.

5.2. Intertwiners between cuspidal representations

We now return to the assumptions of 5.1.1. From now on we assume that $E = E'$, $\mathcal{H} = \mathcal{H}'$ and that the maps ι and ι' are equal. Thus, we view G as a subgroup of $\mathrm{GL}_2(E)$.

Lemma 5.2.1. *Let $\mathcal{C} \subset \mathcal{H}_f$ be a compact-open subset. Then, for any*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

such that $\mathcal{C}g^{-1} \subset \mathcal{H}_f$, one has that

$$(g._\delta \phi(1_\mathcal{C}))(z) = \chi_1(\det(g))\chi_2(bz+d)\phi(1_{\mathcal{C}g^{-1}})(z).$$

Proof. $g._\delta\phi(1_\mathcal{C}) = \phi(g._\chi 1_\mathcal{C})$, and

$$(g._\chi 1_\mathcal{C})(z) = \chi_1(\det(g))\chi_2(bz+d)1_\mathcal{C}(z.g) = \chi_1(\det(g))\chi_2(bz+d)1_{\mathcal{C}g^{-1}}(z).$$

For any $z_0 \in \mathcal{H}$ there is a small disc D_{z_0} around z_0 such that for all z in this disc

$$\chi_2(bz+d) = \chi_2(bz_0+d)\chi_2\left(\frac{bz+d}{bz_0+d}\right) = \chi_2(bz_0+d)\sum_{j\geqslant 0}\binom{c(\chi)}{j}\left(\frac{bz+d}{bz_0+d}-1\right)^j.$$

Since $\mathcal{C}g^{-1}$ is compact, it can be written as a finite union of discs D_{z_0} for various points $z_0 \in \mathcal{H}$, and it suffices to consider the case when $\mathcal{C}g^{-1} \subset D_{z_0}$. Then

$$\phi(\chi_1(\det(g))\chi_2(bz+d)1_{\mathcal{C}g^{-1}}(z)) = \chi_1(\det(g))\chi_2(bz_0+d)\phi\left(\chi_2\left(\frac{bz+d}{bz_0+d}\right)1_{\mathcal{C}g^{-1}}(z)\right).$$

By Proposition 5.1.2 and the continuity of ϕ,

$$\phi\left(\chi_2\left(\frac{bz+d}{bz_0+d}\right)1_{\mathcal{C}g^{-1}}(z)\right) = \chi_2\left(\frac{bz+d}{bz_0+d}\right)\phi(1_{\mathcal{C}g^{-1}}(z)).$$

This proves the assertion. □

Proposition 5.2.2. *Suppose $c(\chi) \notin \mathbb{Z}_{\geqslant 0}$, and let $z_0 \in \mathcal{H}_f$. There exists a non-zero continuous intertwining operator $\phi : V_{\chi,\mathcal{H}} \to V_{\delta,\mathcal{H}}$, if and only if $c(\chi) = c(\delta)$, and for all*

$$g_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \in G$$

such that $z_0.g_0 = z_0$ we have

$$\frac{\chi_1}{\delta_1}(\det(g_0))\frac{\chi_2}{\delta_2}(b_0z_0+d_0) = 1. \tag{5.2.3}$$

If these conditions hold, then $V_{\chi,\mathcal{H}}$ and $V_{\delta,\mathcal{H}}$ are isomorphic.

Proof. We have already seen that the existence of a non-zero ϕ implies that $c(\chi) = c(\delta)$. Let $g_0 \in G$ such that $z_0.g_0 = z_0$, and choose compact-open subsets $\mathcal{C}_1 \subset \mathcal{C} \subset \mathcal{H}_f$ such that $z_0 \in \mathcal{C}_1$ and $\mathcal{C}_1 \subset \mathcal{C}g_0^{-1} \subset \mathcal{H}_f$. Then, by Proposition 5.1.2,

$$\begin{aligned}\phi(1_{\mathcal{C}g_0^{-1}})(z_0) &= \phi(1_{\mathcal{C}g_0^{-1}\setminus\mathcal{C}_1})(z_0) + \phi(1_{\mathcal{C}_1})(z_0) = \phi(1_{\mathcal{C}_1})(z_0) \\ &= \phi(1_{\mathcal{C}\setminus\mathcal{C}_1})(z_0) + \phi(1_{\mathcal{C}_1})(z_0) = \phi(1_\mathcal{C})(z_0).\end{aligned}$$

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Since $c(\chi) \notin \mathbb{Z}_{\geqslant 0}$, $V_{\chi,\mathcal{H}}$ is irreducible, so $\phi \neq 0$ implies $\phi(1_{\mathcal{C}}) \neq 0$ and Lemma 5.2.1 shows that

$$\frac{\chi_1}{\delta_1}(\det(g_0))\frac{\chi_2}{\delta_2}(b_0 z_0 + d_0) = 1.$$

Now suppose that the above identity is fulfilled. Then we can define an intertwining operator ϕ as follows. We consider $V_{\chi,\mathcal{H}}$ and $V_{\delta,\mathcal{H}}$ as subspaces of the space of locally E-analytic functions on $\tilde{\mathcal{H}}$, and we define a function $\varphi : \tilde{\mathcal{H}} \to K^\times$ by

$$\varphi(q \exp(z_0 \mathfrak{u}^-) g) = \frac{\delta}{\chi}(q),$$

for $q \in P$, $g \in G$. To see that φ is well defined, note that if $q \exp(z_0\mathfrak{u}^-)g = q_1 \exp(z_0\mathfrak{u}^-)g_1$ with $q_1 \in P$ and $g_1 \in G$, then $g_1 g^{-1} = \exp(-z_0\mathfrak{u}^-)q_1^{-1} q \exp(z_0\mathfrak{u}^-)$ fixes z_0, so that, writing $g_0 = g_1 g^{-1}$, we get

$$\frac{\delta}{\chi}(q_1^{-1}q) = \frac{\delta}{\chi}(\exp(z_0\mathfrak{u}^-)g_1 g^{-1}\exp(-z_0\mathfrak{u}^-)) = \frac{\delta_1}{\chi_2}(\det(g_0))\frac{\delta_2}{\chi_2}(b_0 z_0 + d_0) = 1,$$

where the second equality follows by writing δ and χ in terms of δ_1 and δ_2, and χ_1 and χ_2 respectively, and computing the lower right entry of the matrix

$$\frac{\delta}{\chi}(\exp(z_0\mathfrak{u}^-)g_1 g^{-1}\exp(-z_0\mathfrak{u}^-)).$$

We will see in a moment that φ is E-analytic. Assuming this, we define an isomorphism of locally E-analytic representations

$$\phi : V_{\chi,\mathcal{H}} \to V_{\delta,\mathcal{H}}, \quad f \mapsto \phi(f) = \varphi \cdot f.$$

It remains to check that φ is locally E-analytic. It suffices to check this in a neighbourhood $\mathcal{U}$ of $\exp(z_0\mathfrak{u}^-)$, as any element h in $\tilde{\mathcal{H}}$ can be written as $h = q\exp(z_0\mathfrak{u}^-)g$ with $q \in P$ and $g \in G$, so that $q\mathcal{U}g$ is an open neighbourhood of h, and we have

$$\varphi(qh'g) = \frac{\delta}{\chi}(q)\varphi(h')$$

for any $h' \in \mathcal{U}$. The composite of the map

$$P \times \mathcal{H}_f \to \tilde{\mathcal{H}}, \quad (q,z) \mapsto q\exp(z\mathfrak{u}^-),$$

and φ is given by

$$(q,z) \mapsto \frac{\delta}{\chi}(q)\varphi(\exp(z\mathfrak{u}^-)),$$

so it suffices to see that the map $z \mapsto \varphi(\exp(z\mathfrak{u}^-))$ is locally E-analytic around z_0. Writing $\exp(z\mathfrak{u}^-) = q\exp(z_0\mathfrak{u}^-)g$, one verifies directly that this map is given by

$$z \mapsto \frac{\chi_1}{\delta_1}(\det(g))\frac{\chi_2}{\delta_2}(bz_0 + d),$$

where $g \in G$ satisfies $z_0.g = z$.

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Now we claim that the two characters $(\chi_1/\delta_1)\circ\det$ and χ_2/δ_2 are locally constant. To see this note that the second character is locally constant since $c(\chi) = c(\delta)$. For the first character, if $\det : G \to E^\times$ has finite image, then there is nothing to prove. If det has infinite image, then Lemma 5.2.4 below implies that any compact-open subgroup $G_0 \subset G$ contains infinitely many elements g_0 which are scalars in $\mathrm{GL}_2(E)$. Since χ_2/δ_2 is a smooth character, if we choose G_0 sufficiently small, then (5.2.3) implies that $(\chi_1/\delta_1)(\det g_0) = 1$. Since this holds for infinitely many scalar g_0, and $(\chi_1/\delta_1)\circ\det$ is a locally E-analytic character, this implies it is locally constant.

Now, if we let g vary in a sufficiently small neighbourhood of the unit element in G, then $z = z_0.g$ will vary in a neighbourhood of z_0, but the function

$$z \mapsto \frac{\chi_1}{\delta_1}(\det(g))\frac{\chi_2}{\delta_2}(bz_0 + d)$$

will then be constant on this neighbourhood. Thus we have seen that φ is locally E-analytic. □

Lemma 5.2.4.

(1) *Let* $\mathfrak{g} \subset \mathrm{Lie}(\mathrm{SL}_2(E))$ *be a Lie* F*-subalgebra, which spans* $\mathrm{Lie}(\mathrm{SL}_2(E))$ *as an* E*-vector space. Then there is a simple Lie* F*-subalgebra* $\mathfrak{g}' \subset \mathfrak{g}$ *which spans* $\mathrm{Lie}(\mathrm{SL}_2(E))$ *as an* E*-vector space.*

(2) *Let* $\mathfrak{g} \subset \mathrm{Lie}(\mathrm{GL}_2(E))$ *be a Lie* F*-subalgebra which spans* $\mathrm{Lie}(\mathrm{GL}_2(E))$ *over* E*. Then* $\mathfrak{g}$ *has non-trivial intersection with* $\mathrm{Lie}(E^\times) \subset \mathrm{Lie}(\mathrm{GL}_2(E))$.

Proof. For any F-Lie algebra $\mathfrak{h}$ we will write $\mathfrak{h}_E = \mathfrak{h}\otimes_F E$.

For (1) suppose that $\mathfrak{i} \subset \mathfrak{g}$ is a non-zero ideal of $\mathfrak{g}$. Since $\mathfrak{g}_E$ surjects onto $\mathrm{Lie}(\mathrm{SL}_2(E))$, and the image of $\mathfrak{i}_E$ in $\mathrm{Lie}(\mathrm{SL}_2(E))$ is not trivial, it must be the whole space. Thus, we may replace $\mathfrak{g}$ by $\mathfrak{i}$. Repeating the process, we eventually arrive at a Lie algebra $\mathfrak{g}'$ with the required properties.

To prove (2), suppose on the contrary that the intersection is trivial. Then the composite of the maps

$$\mathfrak{g} \to \mathrm{Lie}(\mathrm{GL}_2(E)) \xrightarrow{\sim} \mathrm{Lie}(\mathrm{SL}_2(E)) \oplus \mathrm{Lie}(E^\times) \to \mathrm{Lie}(\mathrm{SL}_2(E)) \tag{5.2.5}$$

is an injection, and we may choose a Lie subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ as in (1). Since we have a surjection $\mathfrak{g}\otimes_F E \to \mathrm{Lie}(\mathrm{GL}_2(E)) \to \mathrm{Lie}(E^\times)$, and $\mathfrak{g}'$ is simple, the $\mathfrak{g}'$-module $\mathfrak{g}$ contains a non-trivial submodule $\mathfrak{h}$ on which $\mathfrak{g}'$ acts trivially. Then $\mathrm{Lie}(\mathrm{SL}_2(E))$ acts trivially on the image of $\mathfrak{h}_E$ in $\mathrm{Lie}(\mathrm{SL}_2(E))$. But this implies that the image of $\mathfrak{h}_E$, and hence of $\mathfrak{h}$, in $\mathrm{Lie}(\mathrm{SL}_2(E))$ is trivial, which contradicts the fact that (5.2.5) is an injection. □

For $G = \mathrm{GL}_2(F)$, $\mathrm{SL}_2(F)$ or $D^\times$ we can make the result of Proposition 5.2.2 more explicit.

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Corollary 5.2.6. *Keeping the above notation and assumptions we have the following.*

(1) *If* $[E:F]=2$ *and* $G=\mathrm{GL}_2(F)$ *then a non-zero intertwiner* $\phi: V_{\chi,\mathcal{H}} \to V_{\delta,\mathcal{H}}$ *exists if and only if*
$$\left(\frac{\chi_1}{\delta_1}\circ N_{E/F}\right)\frac{\chi_2}{\delta_2}=1.$$

(2) *If* $[E:F]=2$ *and* $G=\mathrm{SL}_2(F)$ *then a non-zero intertwiner* $\phi: V_{\chi,\mathcal{H}} \to V_{\delta,\mathcal{H}}$ *exists if and only if*
$$\frac{\chi_2}{\delta_2}=\psi\circ N_{E/F}$$
for some locally constant character $\psi: F^\times \to K^\times$.

(3) *If* $[E:F]=2$ *and* $G=D^\times$, *where* D *is the quaternion algebra over* F, *then a non-zero intertwiner* $\phi: V_{\chi,\mathcal{H}} \to V_{\delta,\mathcal{H}}$ *exists if and only if*
$$\left(\frac{\chi_1}{\delta_1}\circ N_{E/F}\right)\frac{\chi_2}{\delta_2}=1.$$

(4) *If* $[E:F]=3$ *and* $G=\mathrm{GL}_2(F)$ *then a non-zero intertwiner* $\phi: V_{\chi,\mathcal{H}} \to V_{\delta,\mathcal{H}}$ *exists if and only if* $c(\chi)=c(\delta)$ *and*
$$\frac{\chi_1}{\delta_1}(x^2)\frac{\chi_2}{\delta_2}(x)=1, \quad x\in E^\times.$$

(5) *If* $[E:F]=3$ *and* $G=\mathrm{SL}_2(F)$ *then a non-zero intertwiner* $\phi: V_{\chi,\mathcal{H}} \to V_{\delta,\mathcal{H}}$ *exists if and only if* $c(\chi)=c(\delta)$ *and*
$$\frac{\chi_2}{\delta_2}(-1)=1.$$

(6) *If* $E=F$ *and* $G=\mathrm{GL}_2(F)$ *then a non-zero intertwiner* $\phi: V_{\chi,\mathcal{H}} \to V_{\delta,\mathcal{H}}$ *exists if and only if* $\chi=\delta$.

(7) *If* $E=F$ *and* $G=\mathrm{SL}_2(F)$ *then a non-zero intertwiner* $\phi: V_{\chi,\mathcal{H}} \to V_{\delta,\mathcal{H}}$ *exists if and only if* $\chi_2=\delta_2$.

In each of these cases if a non-zero ϕ *exists, then* $V_{\chi,\mathcal{H}}$ *and* $V_{\delta,\mathcal{H}}$ *are isomorphic.*

Proof. Note that $z_0.g_0=z_0$ if and only if
$$g_0=\begin{pmatrix}1&0\\-z_0&1\end{pmatrix}\begin{pmatrix}x&y\\0&w\end{pmatrix}\begin{pmatrix}1&0\\z_0&1\end{pmatrix}=\begin{pmatrix}x+z_0y & y\\ z_0(w-x)-z_0^2y & w-z_0y\end{pmatrix},$$
where
$$\begin{pmatrix}x&y\\0&w\end{pmatrix}\in P.$$
A simple calculation shows that, with this notation, we have $b_0z_0+d_0=w$. Observe also that when $G\subset\mathrm{GL}_2(F)$ then $y\in F$, and if $w\in F^\times$ then we must have $x=w$ and $y=0$, as $z_0\notin F$.

Now with the assumptions of (1) we must have $x = \sigma(w)$, where σ denotes the non-trivial automorphism of E over F. Hence

$$\frac{\chi_1}{\delta_1}(\det g_0)\frac{\chi_2}{\delta_2}(b_0 z_0 + d_0) = \frac{\chi_1}{\delta_1}(N_{E/F}(w))\frac{\chi_2}{\delta_2}(w).$$

Thus, if

$$\left(\frac{\chi_1}{\delta_1} \circ N_{E/F}\right)\frac{\chi_2}{\delta_2} = 1$$

then a non-zero ϕ exists by Proposition 5.2.2. (Note that this condition implies that $c(\chi) = c(\delta)$ as χ and δ are locally E-analytic.) Conversely, suppose a non-zero ϕ exists. Given $w \in E^\times$ write $w = b_0 z_0 + d_0$ with b_0 and d_0 in F. Taking

$$g_0 = \begin{pmatrix} b_0(z_0 + \sigma(z_0)) + d_0 & b_0 \\ -b_0 z_0 \sigma(z_0) & d_0 \end{pmatrix}$$

and using Proposition 5.2.2, we see that

$$\frac{\chi_1}{\delta_1} \circ N_{E/F}(w)\frac{\chi_2}{\delta_2}(w) = \frac{\chi_1}{\delta_1}(\det g_0)\frac{\chi_2}{\delta_2}(b_0 z_0 + d) = 1.$$

This proves (1).

Suppose we are in the situation of (2). Then $x = \sigma(w) = w^{-1}$ and we have

$$\frac{\chi_1}{\delta_1}(\det g_0)\frac{\chi_2}{\delta_2}(b_0 z_0 + d_0) = \frac{\chi_2}{\delta_2}(w).$$

If χ_2/δ_2 factors through the norm, then the right-hand side is 1 as $N_{E/F}(w) = 1$. It follows that a non-zero ϕ exists by Proposition 5.2.2. Conversely, suppose a non-zero ϕ exists. If $w \in E^\times$ satisfies $N_{E/F}(w) = 1$, then defining g_0 in the same way as in the proof of (1) we see that $g_0 \in \mathrm{SL}_2(F)$, and

$$\frac{\chi_2}{\delta_2}(w) = \frac{\chi_2}{\delta_2}(b_0 z_0 + d) = 1.$$

This proves (2).

For (3) note that, since $\mathcal{H} = P\backslash \mathrm{GL}_2(E)$ in this case, we can choose $z_0 = 0$, when we must have $x = \sigma(w)$, and (3) follows from Proposition 5.2.2.

In the situation of (4), observe that x and w are roots of a quadratic equation with coefficients in F. Since $[E : F] = 3$, we must have w and x in F, so that $x = w$ and $y = 0$, as observed above. Thus, g_0 is a scalar matrix and

$$\frac{\chi_1}{\delta_1}(\det g_0)\frac{\chi_2}{\delta_2}(b_0 z_0 + d_0) = \frac{\chi_1}{\delta_1}(w^2)\frac{\chi_2}{\delta_2}(w).$$

Now Proposition 5.2.2 implies that a non-zero ϕ exists if and only if $c(\chi) = c(\delta)$ and the above expression is equal to 1 for all $w \in F^\times$.

Under the assumptions of (5), g_0 is a scalar matrix in $\mathrm{SL}_2(F)$, so $g_0 = \{\pm 1\}$. Thus Proposition 5.2.2 implies that a non-zero ϕ exists if and only if $c(\chi) = c(\delta)$ and $(\chi_2/\delta_2)(-1) = 1$.

Finally, (6) and (7) are obvious using Proposition 5.2.2. For GL_2 this result was already contained in [**S-T1**, Proposition 6.2]. □

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Corollary 5.2.7. *Suppose $G = \mathrm{GL}_2(F)$. Let $\psi : F^\times \to K^\times$ be a locally F-analytic character. We also regard ψ as a character of $\mathrm{GL}_2(F)$ by composing it with the determinant.*

(1) *If $[E : F] = 2$ then there exists an isomorphism $V_{\chi,\mathcal{H}} \otimes \psi \xrightarrow{\sim} V_{\chi,\mathcal{H}}$ if and only if $\psi \circ N_{E/F} = 1$.*

(2) *If $[E : F] = 3$ then there exists an isomorphism $V_{\chi,\mathcal{H}} \otimes \psi \xrightarrow{\sim} V_{\chi,\mathcal{H}}$ if and only $\psi(x^2) = 1$ for all $x \in F^\times$.*

In particular ψ is locally constant.

Proof. After replacing K by a finite extension, we may assume that ψ extends to a locally E-analytic character $\tilde{\psi} : E^\times \to K^\times$. Then we have

$$V_{\chi\cdot\tilde{\psi},\mathcal{H}} \xrightarrow{\sim} V_{\chi,\mathcal{H}} \otimes \psi$$

where we regard $\tilde{\psi}$ as a character of $\mathrm{GL}_2(E)$—and hence of the diagonal torus—by composing it with the determinant. For example, using the model of 3.2.1 this isomorphism is just given by $f \mapsto f$. Now the corollary follows from Corollary 5.2.6 once we note that multiplying χ by $\tilde{\psi}$ leaves χ_2 unchanged, and multiplies χ_1 by $\tilde{\psi}$. □

5.2.8. Using Corollary 5.2.7, one can recover in some cases the result of Proposition 5.1.2 that, if $V_{\chi,\mathcal{H}}$ is isomorphic to $V_{\delta,\mathcal{H}}$ then $E = E'$. Namely, if $G = \mathrm{GL}_2(F)$ and E has degree 2, then Corollary 5.2.7 shows that the set of characters ψ such that $V_{\chi,\mathcal{H}} \otimes \psi \xrightarrow{\sim} V_{\chi,\mathcal{H}}$ can only equal the analogous set of characters for $V_{\delta,\mathcal{H}}$ if $E = E'$.

6. Relations with Weil group representations

6.1.1. We recall the definition of the Weil group W_F [**Ta**]. Let E/F be a finite Galois extension. By local class field theory there is a canonical element $c_{E/F} \in H^2(\mathrm{Gal}(E/F), E^\times)$. The relative Weil group $W_{E/F}$ is the corresponding extension

$$0 \to E^\times \to W_{E/F} \to \mathrm{Gal}(E/F) \to 0.$$

Since $E^\times$ is naturally a locally F-analytic group, this gives $W_{E/F}$ the structure of a locally F-analytic group. The Weil group W_F is given by $W_F = \varprojlim_E W_{E/F}$ where E runs through the finite Galois extensions of F. For an extension E'/E of finite Galois extensions of F, the corresponding transition map in the inverse limit induces the natural projection $\mathrm{Gal}(E'/F) \to \mathrm{Gal}(E/F)$, and the norm map $N_{E'/E} : E'^\times \to E^\times$. W_F therefore has a structure of a *pro-locally F-analytic* group. Thus, a locally F-analytic representation of W_F is just a locally F-analytic representation of $W_{E/F}$ for some finite extension E of F.

If E is a finite extension of F, then there is a natural inclusion $W_E \to W_F$ of locally F-analytic groups, which makes W_E a subgroup of W_F of index $[E : F]$.

6.1.2. Suppose that E/F is an extension of degree 1 or 2, and let χ be as in 3.1.1. For any locally E-analytic character $\psi : E^\times \to K^\times$, we may regard ψ as a character of W_E via the projection $W_E \to W_{E/E} = E^\times$, or as a character of W_F via the composite $W_F \to F^\times \hookrightarrow E^\times$.

Now assume that $c(\chi)$ is not a non-negative integer. Suppose that $G = \mathrm{GL}_2(F)$ and that $\mathcal{H}$ is the unique open orbit of G on $P \setminus \mathrm{GL}_2(E)$. If $[E : F] = 2$ then we set

$$\sigma(V_{\chi,\mathcal{H}}) = (\mathrm{Ind}_{W_E}^{W_F} \chi_2) \otimes \chi_1 : W_F \to \mathrm{GL}_2(K),$$

where we regard χ_2 and χ_1 as characters of W_E and W_F respectively, as explained above. (Here we have chosen a basis for $(\mathrm{Ind}_{W_E}^{W_F} \chi_2) \otimes \chi_1$, to get a map into $\mathrm{GL}_2(K)$, but we will only be concerned with the isomorphism class of this representation.)

If $E = F$ then we set $\sigma(V_{\chi,\mathcal{H}})$ equal to the representation of W_F corresponding to the composite

$$W_F \to F^\times \xrightarrow{a \mapsto (\chi_1(a), \chi_1(a)\chi_2(a))} K^\times \times K^\times \hookrightarrow \mathrm{GL}_2(K).$$

Suppose that $G = \mathrm{SL}_2(F)$ and $\mathcal{H}$ is an open orbit of $\mathrm{SL}_2(F)$ on $P \setminus \mathrm{GL}_2(F)$. Then we define $\sigma(V_{\chi,\mathcal{H}})$ in the same way as above, depending on whether $[E : F]$ is 1 or 2, but we compose the resulting representation with the projection $\mathrm{GL}_2(K) \to \mathrm{PGL}_2(K)$.

Proposition 6.1.3. *Let χ and χ' be two locally E-analytic characters of the diagonal torus of $\mathrm{GL}_2(E)$, and suppose that $c(\chi)$ and $c(\chi')$ are not non-negative integers. Let G be one of $\mathrm{GL}_2(F)$ or $\mathrm{SL}_2(F)$. Let $\mathcal{H}$ and $\mathcal{H}'$ be open orbits of G on $P \setminus \mathrm{GL}_2(E)$.*

(1) *If $V_{\chi,\mathcal{H}}$ is isomorphic to $V_{\chi',\mathcal{H}'}$ then $\sigma(V_{\chi,\mathcal{H}})$ is isomorphic to $\sigma(V_{\chi',\mathcal{H}'})$.*

(2) *If $[E : F] = 2$ and $G = \mathrm{GL}_2(F)$ then the converse of (1) also holds.*

Proof. This is a simple exercise using Corollarys 5.2.6 and 5.1.5. □

Remarks 6.1.4.

(1) If $E = F$ then the converse of (1) fails because, if χ' is the composite of χ and the map

$$E^\times \times E^\times \xrightarrow{(a,b) \mapsto (b,a)} E^\times \times E^\times,$$

and $c(\chi') \notin \mathbb{Z}_{\geqslant 0}$, then $\sigma(V_{\chi,\mathcal{H}}) = \sigma(V_{\chi',\mathcal{H}'})$.

(2) If $[E : F] = 2$, $G = \mathrm{SL}_2(F)$, and $\mathcal{H}$ and $\mathcal{H}'$ are the two open orbits of $\mathrm{SL}_2(F)$ on $P \setminus \mathrm{GL}_2(E)$, then $\sigma(V_{\chi,\mathcal{H}}) \xrightarrow{\sim} \sigma(V_{\chi,\mathcal{H}'})$ (and Corollary 5.2.6 (2) shows that this is the only obstruction to the converse of (1) holding). This is completely analogous to the situation in the theory of discrete series representations of $\mathrm{SL}_2(\mathbb{R})$, and the phenomenon of L-indistinguishability which occurs there [**L-L**]. Thus, the pair $\{V_{\chi,\mathcal{H}}, V_{\chi,\mathcal{H}'}\}$ may be thought of as a 'p-adic L-packet'.

(3) We do not know whether one can reasonably attach representations of W_F to a representation $V_{\chi,\mathcal{H}}$ in the case where $[E : F] = 3$.

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6.1.5. Now suppose that $[E : F] = 2$. Let $\mathcal{H}$ be the open orbit of $\mathrm{GL}_2(F)$ on $P\backslash\mathrm{GL}_2(E)$. If $c(\chi) \notin \mathbb{Z}_{\geqslant 0}$ then we denote by $\rho(V_{\chi,\mathcal{H}})$ the representation $V_{\chi,P\backslash\mathrm{GL}_2(E)}$ of $D^\times$, corresponding to the character χ.

Proposition 6.1.6. *Let E and E' be two quadratic extensions of F, and let χ (respectively, χ') be a locally E-analytic character of the diagonal torus in $\mathrm{GL}_2(E)$ (respectively, $\mathrm{GL}_2(E')$). Then $V_{\chi,\mathcal{H}}$ is isomorphic to $V_{\chi',\mathcal{H}'}$ if and only if $\rho(V_{\chi,\mathcal{H}})$ is isomorphic to $\rho(V_{\chi',\mathcal{H}'})$.*

Proof. This follows immediately from Corollary 5.1.5 and parts (1) and (3) of Corollary 5.2.6. □

7. Admissibility

7.1.1. Let us recall that in [**S-T1**] (cf. the definition after Corollary 3.4) a locally analytic representation V of a locally analytic, compact group G is called *strongly admissible*, if V is a semi-compact inductive limit, and V_b' is a countably generated module over the distribution algebra $D(G,K)$. It can be shown that it is then in fact finitely generated. Examples of strongly admissible representations are given by the irreducible principal series representations, when restricted to any compact-open subgroup of $\mathrm{GL}_2(F)$. This follows from the fact that the duals of these representations are simple modules over the distribution algebra of $\mathrm{GL}_2(\mathfrak{o}_F)$ (cf. [**S-T1**, Theorem 6.1]).

In [**S-T3**], Schneider and Teitelbaum introduced the notion of an *admissible* representation of G. This is a generalization of the notion of a strongly admissible representation, though it seems that the admissible representations thus far encountered in practice are strongly admissible. We refer to [**S-T3**] for the definition of an admissible G-representation.

Let G and $\mathcal{H}$ be as in 3.1.1. In this subsection we show that the restrictions of the representations $V_{\chi,\mathcal{H}}$ to compact-open subgroups are admissible if and only if $\mathcal{H}$ is compact, in which case $V_{\chi,\mathcal{H}}$ is strongly admissible.

Lemma 7.1.2. *Suppose that $\mathcal{H}$ is not compact. Then for any compact-open subgroup H of $\mathrm{GL}_2(E)$, there exists a $z_0 \in E$ and a compact-open subgroup $G_0 \subset G$ such that $z_0.H \subset E$ and $z_0.H$ contains infinitely many disjoint orbits of G_0 on $\mathcal{H}$.*

Proof. For any compact-open subgroup G_0 of G, we let $S(G_0) = \{\Delta_i\}_{i\in I}$ be the set of orbits of G_0 on $\mathcal{H}$. We call $z_0 \in P\backslash\mathrm{GL}_2(E)$ a *limit point* of $S(G_0)$ if there is an infinite subset $\Delta_{i_1}, \Delta_{i_2}, \ldots$ of $S(G_0)$ such that for some sequence $\{d_j\}_{j\geqslant 1}$ with $d_j \in \Delta_{i_j}$, we have $\varinjlim d_j = z_0$.

Now for any choice of G_0, $S(G_0)$ has a limit point on $P\backslash\mathrm{GL}_2(E)$, since this space is compact, while $S(G_0)$ is infinite, as $\mathcal{H}$ is non-compact. Moreover, the set of limit points is stable under G_0. If ∞ were the only limit point of G_0, then $G_0 \subset \mathrm{GL}_2(E)$ would have to be contained in the lower triangular matrices, which contradicts the assumption that the E-linear span of its Lie algebra contains $\mathrm{Lie}(\mathrm{SL}_2(E))$. Thus $S(G_0)$ has a limit point $z_0 \in E$.

Choose an infinite subset $\Delta_{i_1}, \Delta_{i_2}, \dots$ of $S(G_0)$ and a sequence $\{d_j\}_{j \geqslant 1}$ with $d_j \in \Delta_{i_j}$, and $\varinjlim d_j = z_0$. Choose a compact-open subgroup H of $\mathrm{GL}_2(E)$ such that $z_0.H \subset E$. After replacing G_0 by a smaller group, we may assume that $G_0 \subset H$. Then Δ_{i_j} meets $z_0.H$ for j sufficiently large, and this implies that $\Delta_{i_j} \subset z_0.H$, as $G_0 \subset H$. □

Proposition 7.1.3. *The representation* $V_{\chi,\mathcal{H}}$ *is admissible when restricted to compact-open subgroups of* G *if and only if* $\mathcal{H}$ *is compact, in which case* $V_{\chi,\mathcal{H}}$ *is strongly admissible.*

Proof. If $G_1 \subset G_2$ are two compact-open subgroups of G, then it is easy to see that $D_F(G_2, K)$ is a finite module over $D_F(G_1, K)$. Hence the proposition is true for one compact-open subgroup if and only if it is true for all of them.

Suppose first that $\mathcal{H}$ is compact, and let G_0 be a compact-open subgroup of G. Then G_0 has only finitely many orbits on $\mathcal{H}$, and it suffices to show that for any such orbit Δ, $V_{\chi,\Delta}$ is strongly admissible as a G_0-representation. Let $z_0 \in \Delta$. Then the orbit map

$$G_0 \xrightarrow{g \mapsto z_0 . g} \Delta$$

induces G_0-equivariant maps

$$D_F(G_0, K) \to D_F(\Delta, K) \to D_E(\Delta, K). \tag{7.1.4}$$

Now since the orbit map above is a surjection with smooth fibres, it admits a locally analytic section. Hence the first map in (7.1.4) admits a section, and is, in particular, surjective. Since the canonical map $C_E^{\mathrm{an}}(\Delta, K) \to C_F^{\mathrm{an}}(\Delta, K)$ is a homeomorphism onto its image, the second map in (7.1.4) is also surjective. Hence the composite of the maps in (7.1.4) is surjective. This composite maps $\delta \in D_F(G_0, K)$ to the linear form $f \mapsto \delta(g \mapsto f(z_0.g))$. We denote by λ_{z_0} the Dirac distribution supported in z_0. For a given distribution $\delta \in D_F(G_0, K)$ define $\delta' \in D_F(G_0, K)$ by

$$\delta'(\varphi) = \delta(g \mapsto \chi_1(\det(g))^{-1} \chi_2(bz_0 + d)^{-1} \varphi(g^{-1})).$$

Then we compute (cf. 2.1.9)

$$\begin{aligned}(\delta' \cdot_\chi \lambda_{z_0})(f) &= \delta'(g \mapsto \lambda_{z_0}(g^{-1} \cdot_\chi f)) \\ &= \delta(g \mapsto \chi_1(\det(g))^{-1} \chi_2(bz_0 + d)^{-1} \lambda_{z_0}(g \cdot_\chi f)) \\ &= \delta(g \mapsto \chi_1(\det(g))^{-1} \chi_2(bz_0 + d)^{-1} (\chi_1(\det(g)) \chi_2(bz_0 + d) f(z_0.g))) \\ &= \delta(g \mapsto f(z_0.g)).\end{aligned}$$

Hence λ_{z_0} is a generator of $D_E(\Delta, K)$ over $D_F(G_0, K)$. It follows that $V_{\chi,\mathcal{H}}$ is a strongly admissible G_0-representation.

Suppose conversely that $\mathcal{H}$ is not compact, but that $V_{\chi,\mathcal{H}}$ is admissible. We identify $V_{\chi,\mathcal{H}}$ with $C_{E,\mathrm{c}}^{\mathrm{an}}(\mathcal{H}, K)$. By Lemma 7.1.2 there is a compact-open subgroup $G_0 \subset G$, a $z_0 \in E$, and a compact-open subgroup $H \subset \mathrm{GL}_2(E)$ such that $z_0.H \subset E$ and $z_0.H$ contains infinitely many disjoint orbits of G_0.

Fix a global chart $G_0 \hookrightarrow F^d$ on G_0. A vector $v \in V_{\chi,\mathcal{H}}$ is called G_0-analytic, if the orbit map

$$G_0 \xrightarrow{h \mapsto h.v} V_{\chi,\mathcal{H}}$$

factors through a BH-subspace W of $V_{\chi,\mathcal{H}}$ and induces a W-valued analytic map on G_0. That is, the map is given by a convergent power series in the chosen global coordinates on G_0, with coefficients in W. Denote by $V_{G_0} \subset V_{\chi,\mathcal{H}}$ the subspace of G_0-analytic vectors. By [**Em1**, Theorem 5.1.15(ii)], if $V_{\chi,\mathcal{H}}$ is admissible, then V_{G_0} is a BH-subspace of $V_{\chi,\mathcal{H}}$.

Now let $\Sigma = \{\Delta_1, \Delta_2, \dots\}$ be an infinite set of orbits of G_0 contained in $z_0.H$. For $i \in \mathbb{N}^+ \cup \{\infty\}$, let $V_{G_0,i} \subset V_{G_0}$ denote the subspace of G_0-analytic functions supported on $\bigcup_{j=1}^{i} \Delta_j$. Then $V_{G_0,i}$ is evidently closed in V_{G_0} with its induced topology as a subspace of $V_{\chi,\mathcal{H}}$. Hence, *a fortiori* it is closed in V_{G_0} with its Banach space topology. This implies that each $V_{G_0,i}$ is a BH-subspace of $V_{\chi,\mathcal{H}}$, and from now on we consider these subspaces with their Banach space topologies. In particular, we have $V_{G_0,\infty} = \bigcup_{i\geqslant 1} V_{G_0,i}$ is a countable union of closed subspaces.

Now choose G_0 so small that $\chi_2(bz+d) = (bz+d)^{c(\chi)}$ for $z \in z_0.H$ and

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_0,$$

and a, b, c, d are analytic functions on G_0 with respect to the fixed global chart. Then it is easy to check that for any $N > 0$ there are non-zero G_0-analytic vectors supported on Δ_N. In fact any function which is analytic (as a function on a subset of E) on Δ_N corresponds to a G_0-analytic vector in $V_{\chi,\mathcal{H}}$. Thus for i an integer, each $V_{G_0,i}$ is a proper subspace of $V_{G_0,\infty}$, and in particular nowhere dense. Since $V_{G_0,\infty}$ is the union of these subspaces, we get a contradiction by the Baire category theorem. $\square$

Corollary 7.1.5. *If $[E:F] > 1$ and G is one of $\mathrm{GL}_2(F)$ or $\mathrm{SL}_2(F)$, then the representation $V_{\chi,\mathcal{H}}$ is not strongly admissible. If $[E:F] = 2$ and $G = D^\times$ then $V_{\chi,\mathcal{H}}$ is strongly admissible.*

Proof. This follows immediately from the description in Proposition 3.3.2 of the open orbits of these groups on $P \setminus \mathrm{GL}_2(E)$. $\square$

7.1.6. Although the representations $V_{\chi,\mathcal{H}}$ are not in general admissible, they do have the following property which seems to be a natural generalization of the notion of admissibility for smooth representations.

Proposition 7.1.7. *Let G, χ and $\mathcal{H}$ be as in 3.1.1, and let G_0 be a compact-open subgroup of G. Then*

$$V_{\chi,\mathcal{H}}|_{G_0} \xrightarrow{\sim} \bigoplus_{i\in I} V_i,$$

where I is a countable index set, V_i is a strongly admissible G_0-representation, and for $i, j \in I$ with $i \neq j$, there exist no non-zero intertwiners $\phi : V_i \to V_j$.

If $V_{\chi,\mathcal{H}}$ is topologically irreducible, then the V_i can be chosen to be topologically irreducible G_0-representations.

Proof. Let I be the set of orbits of G_0 on $\mathcal{H}$. For any $i \in I$ we take $V_i = V_{\chi,i}$. Then we clearly have an isomorphism as required by the proposition. Since G_0 is compact, so is each orbit i, so V_i is strongly admissible by Proposition 7.1.3. The statement on intertwiners between the V_i follows from Proposition 5.1.2.

Finally, if $V_{\chi,\mathcal{H}}$ is irreducible, then $c(\chi) \notin \mathbb{Z}_{\geqslant 0}$ by Proposition 3.1.8, so that each V_i is irreducible by Theorem 3.1.6. □

8. Appendix

8.1. On properties of semi-compact inductive limits

We begin by stating the following useful proposition.

Proposition 8.1.1. *Let A be a compactoid subset of a locally convex Hausdorff K-vector space V, and let $a \in K$ such that $|a| > 1$. Then, for any open lattice $L \subset V$ there exist finitely many $v_1, \dots, v_n \in aA$ such that $A \subset L + \mathfrak{o}_K v_1 + \cdots + \mathfrak{o}_K v_n$.*

Proof (cf. Remark 12.8 in [S]). Note that the standing assumption there that K be spherically complete is not necessary for this assertion. □

Theorem 8.1.2. *Let V be a locally convex Hausdorff K-vector space which is a semi-compact inductive limit. Then any closed subspace U of V is a semi-compact inductive limit. More precisely, if $V = \lim(V_1 \to V_2 \to \cdots)$ is the locally convex inductive limit of Banach spaces V_i with semi-compact transition maps, then U is equal to the union of all $V_i \cap U$, and the subspace topology on U is equal to the locally convex inductive limit topology of the $V_i \cap U$.*

Proof. The last assertion about the topologies is Theorem 3.1.16 in [**GKPS**]. For the sake of completeness we show the simple assertion that the induced maps $V_i \cap U \xrightarrow{\iota_i} V_{i+1} \cap U$ are compact. Put $B_i(\varepsilon) := \{v \in V_i \mid \|v\|_{V_i} \leqslant \varepsilon\}$. Then the image of $B_i(1)$ under ι_i is compactoid in V_{i+1}, and so is $\iota_i(B_i(1) \cap U)$. By Proposition 8.1.1 there are for a given $\varepsilon > 0$ elements $v_1, \dots, v_n \in a\iota_i(B_i(1) \cap U)$, where $a \in K$ is as in Proposition 8.1.1, such that

$$\iota_i(B_i(1) \cap U) \subset B_{i+1}(\varepsilon) + \sum_{j=1}^{n} \mathfrak{o}_K v_j.$$

It follows immediately that

$$\iota_i(B_i(1) \cap U) \subset (B_{i+1}(\varepsilon) \cap U) + \sum_{j=1}^{n} \mathfrak{o}_K v_j,$$

and hence $\iota_i(B_i(1) \cap U)$ is compactoid in $V_{i+1} \cap U$. □

Proposition 8.1.3. *Let V be a locally convex Hausdorff K-vector space which is a semi-compact inductive limit, and let $U \subset V$ be a closed subspace. Then the quotient V/U, equipped with the quotient topology, is a semi-compact inductive limit.*

Proof. Set $W = V/U$. Write V as an inductive limit of Banach spaces V_i with compact transition maps: $V = \lim_i(V_1 \to V_2 \to \cdots)$. Put $U_i = V_i \cap U$, and $W_i = V_i/U_i$, with the quotient topology. If $B \subset W_i$ is an open ball, then it is the image of an open ball $\tilde{B} \subset V_i$. Since the image of $\tilde{B} \subset V_i$ in V_{i+1} is compactoid, so is its image in W_{i+1}. Hence the image of B in W_{i+1} is compactoid, and the map $W_i \to W_{i+1}$ is a compact map of Banach spaces.

We consider $\lim_i(W_1 \to W_2 \to \cdots)$, equipped with the locally convex direct limit topology. The inclusions $W_i \hookrightarrow W$ induce a continuous bijection $\lim_i(W_1 \to W_2 \to \cdots) \to W$. On the other hand, the map

$$V = \lim_i(V_1 \to V_2 \to \cdots) \to \lim_i(W_1 \to W_2 \to \cdots)$$

has kernel U, and hence factors through W, as $W = V/U$ has the quotient topology. This shows that $\lim_i(W_1 \to W_2 \to \cdots) \xrightarrow{\sim} W$. □

The following proposition is used in the proof of Theorem 4.2.5.

Proposition 8.1.4. *Let $F \subset K \subset L$ be a sequence of p-adic fields, with F a finite extension of $\mathbb{Q}_p$, and that L is spherically complete.*

Let M be an analytic manifold and suppose M can be written as an increasing union of a countable number of compact-open subsets. Let U be a closed proper subspace $C^{\mathrm{an}}_{F,\mathrm{c}}(M,K)$ (as introduced in 2.1.4). Then, the closure of $U \otimes_K L$ inside $C^{\mathrm{an}}_{F,\mathrm{c}}(M,L)$ is a proper subspace.

Proof. By Proposition 8.1.3, the quotient V of $C^{\mathrm{an}}_{F,\mathrm{c}}(M,K)$ by U is a semi-compact inductive limit and therefore reflexive (cf. [**GKPS**, Theorem 3.1.7]). Because V is not zero, the dual space of V is non-zero, and hence there is a non-zero continuous linear form, λ on $C^{\mathrm{an}}_{F,\mathrm{c}}(M,K)$, which vanishes on U. Extending λ linearly to the algebraic tensor product $C^{\mathrm{an}}_{F,\mathrm{c}}(M,K) \otimes_K L$ provides us with a linear form on this subspace of $C^{\mathrm{an}}_{F,\mathrm{c}}(M,L)$, which is continuous for the projective tensor product topology on $C^{\mathrm{an}}_{F,\mathrm{c}}(M,K) \otimes_K L$ (cf. [**S**, § 17]).

Now we show that the subspace topology on $C^{\mathrm{an}}_{F,\mathrm{c}}(M,K) \otimes_K L \subset C^{\mathrm{an}}_{F,\mathrm{c}}(M,L)$ is equal to the projective tensor product topology. To see this, it is sufficient to consider the corresponding question for the Banach subspaces $\mathcal{F}_{\mathcal{I}}(K) \otimes_K L \subset \mathcal{F}_{\mathcal{I}}(L)$, where $\mathcal{I}$ is a compact K-index on M, which gives us 'by extension of scalars' a compact L-index on M, again denoted by $\mathcal{I}$, and $\mathcal{F}_{\mathcal{I}}(K) \otimes_K L$ is equipped with the projective tensor product topology. Denote by $\|\cdot\|_1 := \|\cdot\|_{\mathcal{F}_{\mathcal{I}}(K)} \otimes |\cdot|_L$ the corresponding norm on $\mathcal{F}_{\mathcal{I}}(K) \otimes_K L$ (cf. [**S**, Proposition 17.4]) and by $\|\cdot\|_2$ the restriction of the norm on $\mathcal{F}_{\mathcal{I}}(L)$. We may assume that $\mathcal{I}$ is of the form (D, ϕ, K), and that $\phi(D)$ is the unit disc around 0. Hence we may even assume that D is the unit disc around 0 (and ϕ is the identity). From the very definition of $\|\cdot\|_1$ it follows that $\|f\|_2 \leqslant \|f\|_1$ for any $f \in \mathcal{F}_{\mathcal{I}}(K) \otimes_K L \subset \mathcal{F}_{\mathcal{I}}(L)$. And if $f(x) = \sum_{i=0}^N a_i x^i$ is a polynomial, we have $\|f\|_1 \leqslant \max_{0 \leqslant i \leqslant N} |a_i|_L = \|f\|_2$. Hence we have an equality in this case. But any function in $\mathcal{F}_{\mathcal{I}}(K) \otimes_K L$ can be approximated by polynomials, simultaneously for both norms.

Hence the norms and topologies coincide. It follows that λ is continuous for the subspace topology on $C^{\mathrm{an}}_{F,\mathrm{c}}(M,K)\otimes_K L$. By the Hahn–Banach theorem, and because L is spherically complete, λ extends to a continuous non-zero linear form on $C^{\mathrm{an}}_{F,\mathrm{c}}(M,L)$ (cf. [**S**, Corollary 9.4]). It is zero on the subspace $U\otimes_K L$, and therefore zero on the closure of that subspace. Hence the closure of $U\otimes_K L$ in $C^{\mathrm{an}}_{F,\mathrm{c}}(M,L)$ is a proper subspace. □

8.2. A proposition on morphisms between Fréchet spaces

The following proposition is a result for Fréchet spaces over a non-archimedean completely valued field. There are similar statements over $\mathbb{R}$ or $\mathbb{C}$ (cf. [**Bou2**, § 4.2]), however, we could not find an exact reference, and wanted to give a complete proof.

Lemma 8.2.1. *Let $\phi: V\to W$ be a continuous surjective morphism of Fréchet spaces over K. Then any compactoid subset of W is the image of a compactoid subset of V.*

Proof. The topology of V can be defined by a countable family of seminorms p_n, $n=0,1,\dots$ (cf. [**S**, Proposition 8.1]). Replacing p_n by $\max\{p_0,\dots,p_n\}$, we may assume that $p_0(v)\leqslant p_1(v)\leqslant\cdots$ for any $v\in V$. Put

$$\|v\|=\sup_{n\geqslant 0}\frac{1}{2^n}\frac{p_n(v)}{1+p_n(v)}.$$

Then $d(v_1,v_2)=\|v_1-v_2\|$ is a translation-invariant metric on V, as is shown in the proof of Proposition 8.1 in [**S**]. For any $v\in V$ and $\alpha\in\mathfrak{o}_K$ one has $\|\alpha v\|\leqslant\|v\|$, so that for any $\varepsilon\in\mathbb{R}_{>0}$

$$B_V(0,\varepsilon):=\{v\in V \mid \|v\|<\varepsilon\}$$

is an open lattice in V. Note that $B_V(0,1)=V$. By the open mapping theorem, $\phi(B_V(0,\varepsilon))$ is an open lattice in W. Now let $A\subset W$ be a compactoid subset. Suppose we have already shown that A is the image of a compactoid subset $C\subset V$. Then, for any subset $A'\subset A$ we have that $C\cap\phi^{-1}(A')$ is compactoid in V and projects onto A'. Hence we may assume that A is an $\mathfrak{o}_K$-module and it is closed (because the closure of a compactoid subset is compactoid).

Let us first give an outline of the rest of the proof. For $i=1,2,\dots$ we will inductively define vectors $v_{i,1},v_{i,2},\dots,v_{i,n_i}\in B_V(0,1/2^{i-1})$ such that for any $w\in A$ there exist scalars $\alpha_{i,j_i}\in\mathfrak{o}_K$, $1\leqslant i$, $1\leqslant j_i\leqslant n_i$ such that for any $\varepsilon\in\mathbb{R}_{>0}$ there is an $i(\varepsilon)\geqslant 1$ and $v(\varepsilon)\in B_V(0,\varepsilon)$ such that

$$w=\phi\Biggl(v(\varepsilon)+\sum_{1\leqslant i\leqslant i(\varepsilon),\,1\leqslant j_i\leqslant n_i}\alpha_{i,j_i}v_{i,j_i}\Biggr).$$

This means that A is contained in the image of

$$C:=\overline{\sum_{1\leqslant i,\,1\leqslant j_i\leqslant n_i}\mathfrak{o}_K v_{i,j_i}}.$$

On the other hand, we will have that $\phi(v_{i,j})\in A$ for any i, j, so that $\phi(C)=A$, because we assume A to be closed. The set of all $v_{i,j}$ is compactoid because any open lattice

$L \subset V$ contains a ball $B_V(0,\varepsilon)$ and therefore almost all vectors $v_{i,j}$. This implies that C is compactoid.

Now we come to the actual construction of the vectors $v_{i,j}$. Fix $a \in K$ such that $1 < |a| < 2$. For $i = 1$ we find elements $v_{1,1},\dots,v_{1,n_1} \in B_V(0,1) = V$ such that

$$A \subset \phi(B_V(0,\tfrac{1}{4})) + \sum_{j=1,\dots,n_1} \mathfrak{o}_K \phi(v_{1,j}).$$

Now suppose we have defined $v_{i,1}, v_{i,2},\dots,v_{i,n_i} \in B_V(0,1/2^{i-1})$ for $i = 1,\dots,k$ such that

$$A \subset \phi\biggl(B_V\biggl(0,\frac{1}{2^{k+1}}\biggr)\biggr) + \sum_{0\leqslant i\leqslant k,\, j_i=1,\dots,n_i} \mathfrak{o}_K \phi(v_{i,j_i}).$$

For any $w \in A$ fix $v_w \in B_V(0,1/2^{k+1})$ and $\alpha_{i,j_i} \in \mathfrak{o}_K$, $1 \leqslant i \leqslant k$, $1 \leqslant j_i \leqslant n_i$, such that

$$w = \phi(v_w) + \sum_{1\leqslant i\leqslant k,\, 1\leqslant j_i\leqslant n_i} \alpha_{i,j_i}\phi(v_{i,j_i}).$$

The set $\phi(\{v_w \mid w \in A\})$ is compactoid because it is contained in the $\mathfrak{o}_K$-submodule generated by A and the vectors $\phi(v_{i,j_i})$, $1 \leqslant i \leqslant k$, $1 \leqslant j_i \leqslant n_i$. By Proposition 8.1.1 there exist $w_1,\dots,w_{n_{k+1}} \in a\phi(B_V(0,1/2^{k+1}))$ such that

$$\phi(\{v_w \mid w \in A\}) \subset \phi\biggl(B_V\biggl(0,\frac{1}{2^{k+2}}\biggr)\biggr) + \sum_{j=1,\dots,n_{k+1}} \mathfrak{o}_K w_j.$$

For any $v \in V$ one has $\|av\| \leqslant |a|\,\|v\|$. Hence there are

$$v_{k+1,j} \in aB_V\biggl(0,\frac{1}{2^{k+1}}\biggr) \subset B_V\biggl(0,\frac{1}{2^k}\biggr)$$

such that $w_j = \phi(v_{k+1,j})$. This finishes the proof. □

Proposition 8.2.2. *Let $\phi : V \to W$ be a continuous surjective map of Fréchet spaces over K, and assume W is also a Montel space. Then the map on dual spaces $\phi' : W'_b \to V'_b$ is a homeomorphism onto its image.*

Proof. We will show that the image of any open lattice in W'_b under ϕ' is the intersection of an open lattice in V'_b with the image of ϕ'. For a bounded subset B of W and $\varepsilon \in \mathbb{R}_{>0}$, the lattice

$$\Bigl\{l \in W' \Bigm| \sup_{w\in B} |l(w)| < \varepsilon\Bigr\}$$

is open, and by letting B run over all bounded subsets of W and ε over all positive real numbers, one gets a fundamental system of open lattices in W'_b. Because W is Montel, B is compactoid. Hence, by Lemma 8.2.1, there is a compactoid subset $\tilde{B} \subset V$, such that $\phi(\tilde{B}) = B$. Of course, $\tilde{B}$ is bounded in V. But then we have obviously

$$\phi'\Bigl(\Bigl\{l \in W' \Bigm| \sup_{w\in B} |l(w)| < \varepsilon\Bigr\}\Bigr) = \Bigl\{\lambda \in V' \Bigm| \sup_{v\in\tilde{B}} |\lambda(v)| < \varepsilon\Bigr\} \cap \phi'(W').$$

Therefore, ϕ' induces an open map from W'_b onto its image. Because ϕ' is continuous, it is a homeomorphism onto its image. □

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