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LAPLACE BOUNDS APPROXIMATION FOR AMERICAN OPTIONS

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In this paper, we develop the lower–upper-bound approximation in the space of Laplace transforms for pricing American options. We construct tight lower and upper bounds for the price of a finite-maturity American option when the underlying stock is modeled by a large class of stochastic processes, e.g. a time-homogeneous diffusion process and a jump diffusion process. The novelty of the method is to first take the Laplace transform of the price of the corresponding "capped (barrier) option" with respect to the time to maturity, and then carry out optimization procedures in the Laplace space. Finally, we numerically invert the Laplace transforms to obtain the lower bound of the price of the American option and further utilize the early exercise premium representation in the Laplace space to obtain the upper bound. Numerical examples are conducted to compare the method with a variety of existing methods in the literature as benchmark to demonstrate the accuracy and efficiency.

Keywords: American option pricing, early exercise boundary, jump diffusions, laplace transform, option bounds

1. INTRODUCTION

American options are widely traded in the financial markets, and they are frequently written on a range of underlying assets, which include stocks, commodities, interest rates, and exchange rates between two currencies. From empirical studies in Barraclough and Whaley [4] and Jensen and Pedersen [30], a large portion of actual options traded in the market are American-style. While it is standard in the literature to use the Black–Scholes model for the stock price, it is more appropriate to use alternative stochastic processes to model commodities, interest rates, or foreign currency exchange rates because they usually exhibit mean-reverting features. It has also been empirically documented that the stock prices exhibit jump behaviors (e.g. the *Flash Crash* on May 6th, 2010). Hence, it is of interest to develop efficient pricing methods for (finite-maturity) American options when the underlying is modeled by a general time-homogeneous diffusion or a jump diffusion.

Since early exercise is allowed, the price of a finite-maturity American option is a solution to the associated finite-horizon free-boundary problem, and there is no analytical solution even in the Black–Scholes model except in the case of an American call option with no dividend. In the literature, numerous numerical methods have been developed. The following is a partial list of the related literature, and we refer the readers to two survey papers by Broadie and Detemple [10], Detemple [24], the book by Detemple [23], and the references therein. The numerical methods mainly include binomial methods (see, e.g., [6,21,29]), finite difference methods (see, e.g., [7]), analytical partial differential equation (PDE) methods (see, e.g., [45]), integral equation methods (see, e.g., [13,14,25,28,31,32,37]), method of lines [17], Fourier transform approach ([15,16]) least-squares Monte Carlo (LSMC) method (see, e.g., [38]), duality methods (see, e.g., [2,8,27]), static hedging portfolio approach (see, e.g., [18,20,40,41]), and lower–upper-bound methods (see, e.g., [9,19]).

Although there exist many papers on pricing (finite-maturity) American call/put options under the Black–Scholes model, there is relatively few literature on the pricing of American options for more general underlying processes. In the literature, Detemple and Tian [25] consider the time-homogeneous diffusion case, express the American option price using the early exercise premium (EEP) decomposition, and derive a system of nonlinear integral equations for the early exercise boundaries. For the finite-maturity American option under the double-exponential jump diffusion (DEJD) model, numerical methods are employed in the literature to solve the associated free boundary problem. Lattice or differential equation methods are used in Amin [1] and Zhang [44]. In Kou and Wang [34], they extend the approximation technique in Barone-Adesi and Whaley [3] from the geometric Brownian motion (GBM) setting to the double-exponential jump diffusion setting, and it is later extended to the case of hyperexponential jump diffusions in Cai and Sun [11]. There is the following discussion on page 1181 of Kou and Wang [34]: “We want to point out that there exist other more elaborate but more accurate approximations (such as [9,12,31]) for GBM models, and whether these algorithms can be effectively extended to jump diffusion models invites further investigation.” Our paper aims at filling this gap in the literature by proposing a generalized lower–upper-bound approximation (LUBA) framework following the spirit of Broadie and Detemple [9], which applies to general underlying stochastic processes where the Laplace transforms of capped (barrier) options are available. In a recent paper, Leippold and Vasiljevic [36] utilize a maturity randomization approach to obtain a tight lower bound of the finite-maturity American option price under the hyperexponential jump diffusion (HEJD) model. Our work is distinct from their work, since we use the optimization technique in Broadie and Detemple [9] instead of the randomization technique (called “Canadization”) of Carr [12] as utilized in Leippold and Vasiljevic [36]. Intuitively speaking, the idea in Leippold and Vasiljevic [36] (or in [12]) is to “exactly solve the approximate problem with exponential maturity,” and the idea of our approach is along the lines of Broadie and Detemple [9], which is to “approximately solve the exact problem.” In the constant elasticity of variance (CEV) case, our method is distinct from the Laplace–Carson approach in Wong and Zhao [43]. They apply Laplace–Carson transform directly to the corresponding free-boundary problem and derive the functional equation governing the early exercise boundary, which they then solve using quadrature methods. Then they utilize the

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numerically determined early exercise boundary to price American options through Laplace inversion. It is not clear how their method can be generalized to jump diffusions, and it seems that their methods are not very accurate (see the numerical comparisons and the detailed discussion in Remark 5.1). On the other hand, our method provides theoretical lower and upper bounds, which are tight through numerical studies in Section 4, and can be applied to more general underlying stock dynamics such as HEJD.

The contributions of the paper are threefold:

1. First, we generalize the idea of Broadie and Detemple [9] to the Laplace space and propose the theoretical framework for constructing tight lower and upper bounds for (finite-maturity) American options for a large class of stochastic processes including time-homogeneous diffusions and jump diffusions with double-exponential jumps. To the best of authors' knowledge, it is the first time that the LUBA idea is applied to a model with jumps.
2. Second, we obtain explicit expressions of capped options in the CEV and double-exponential jump diffusion models and then explicitly carry out the optimization procedure to arrive at accurate explicit lower and upper bounds.
3. Third, our theoretical framework in the Laplace space is flexible enough to be combined with other improvements in approximating the early exercise boundary (e.g. using an exponential function [19] or a multi-piece exponential function [31]) to arrive at more accurate results (see a brief discussion in Section 6).

The rest of the paper is organized as follows. Section 2 presents the general idea of our 'lower–upper-bound approximation in the Laplace space' (Lap-LUBA 1 and Lap-LUBA 2). Section 3 considers the valuation of American call options written on diffusions. We illustrate with explicit expressions for the GBM and CEV models. Section 4 considers the application of our method to the DEJD model. Section 5 presents numerical examples in the case of diffusion and DEJD settings. Section 6 illustrates the main steps of using an exponential function to approximate the early exercise boundary in our Laplace space framework and concludes the paper with future research directions. Appendix collects formulas related to the benchmark case of geometric Brownian motions.

2. MAIN IDEAS FOR LAP-LUBAS

In this section, we describe the main ideas of Laplace lower–upper-bound approaches (Lap-LUBA 1 and Lap-LUBA 2). This approach extends the idea of using the capped options to approximate the American options for the GBM model in Broadie and Detemple [9] to some more general models. Unlike the case of GBM model, there is no closed-form formula for the capped option for general models. However, for some general models, the closed-form formulas of the capped options can be derived in the Laplace space. This enables us to carry out optimization procedures in the Laplace space. Then, the Laplace inversion and the EEP representation facilitate obtaining the lower and upper bounds on American option. The work flow is summarized in Figure 1.

2.1. Lap-LUBA 1

Given a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{Q})$, where $\mathbb{Q}$ is the risk-neutral probability measure, we first consider the case when the stock price is modeled by a positive time-homogeneous

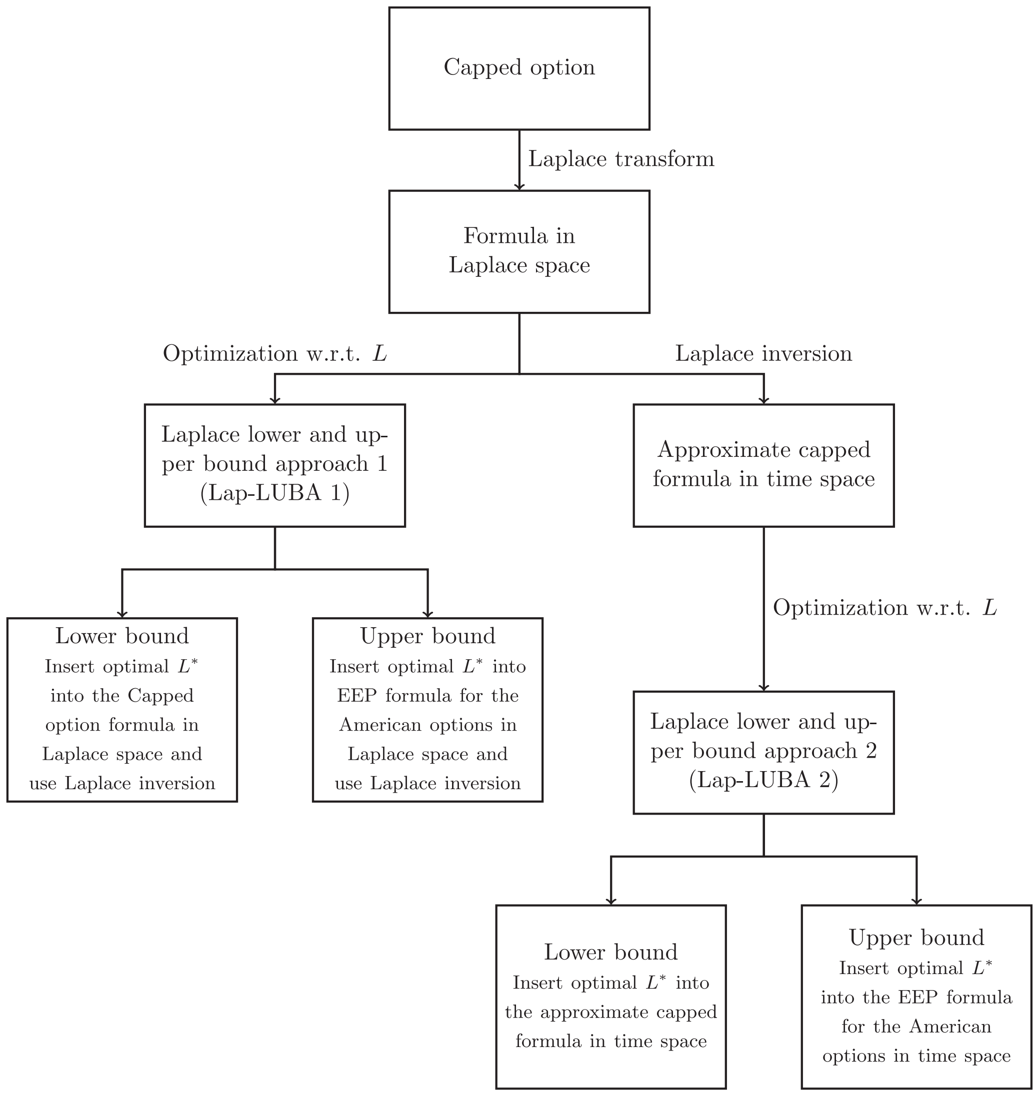

FIGURE 1. Work flow of Laplace lower- and upper-bound approach.

diffusion governed by the following stochastic differential equation (SDE):

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma(S_t)\, dW_t, \quad t \geq 0,\ S_0 > 0. \tag{1}$$

Here $\mu := r - \delta$, where $r \geq 0$ and $\delta \geq 0$ are, respectively, the risk-free interest rate and the dividend rate. The same underlying dynamic is also assumed in Davydov and Linetsky [22] and Detemple and Tian [25].

We first discuss the case of the American call option with dividends under the model (1) and illustrate the main idea of our approach. Note that the following results (such as Eq. (2) and Lemmas 2.1 and 2.2) hold for more general underlying processes beyond diffusions.

Consider the case of a constant cap $L > 0$ which is, in fact, dependent on t but remains unchanged on $[t, T]$ and denote $C(S_t, t, L)$ as the value of a capped call with automatic

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exercise at the cap L, and it has the same strike K and maturity T as the American call option to be priced. Define $\tau_L := \inf\{u \geq t : S_u \geq L\}$, $\tau := \tau_L \wedge T$. Then, the value of the capped call option is written as

$$C(S_t, t, L) = E_t[e^{-r(\tau - t)}(S_\tau \wedge L - K)^+].$$

From the analysis in Section 7.2 of Detemple and Tian [25], we have that $C(S_t, t) \geq C(S_t, t, L)$ for all $L > 0$, or equivalently their Laplace transforms with respect to the time-to-maturity $u := T - t$ also satisfy

$$\mathcal{L}(C(S_t, t)) := \int_0^\infty e^{-\lambda u} C(S_t, T - u)\, du \geq \int_0^\infty e^{-\lambda u} C(S_t, T - u, L)\, du =: \mathcal{L}(C(S_t, t, L)). \tag{2}$$

Denote $C^*(S_t, \lambda, L) := \mathcal{L}(C(S_t, t, L))$, and the main idea of our LUBA in the Laplace space is: instead of maximizing the expression $C(S_t, t, L)$ (see Eq. (44) on p. 933 of [25]) with respect to L in the original time space, we aim to maximize $C^*(S_t, \lambda, L)$ with respect to L in the Laplace space by solving the following nonlinear equation:

$$D(L, \lambda) = 0, \tag{3}$$

where

$$D(L, \lambda) := \lim_{S_t \uparrow L} \frac{\partial C^*(S_t, \lambda, L)}{\partial L}. \tag{4}$$

The explicit formula of $D(L, \lambda)$ for the time-homogeneous diffusion case in (1) is given in (20) and it is expressed in terms of solutions to the associated *Sturm–Liouville* ordinary differential equation (ODE). Denote the solution of the above optimization problem as L^*, then we insert it into the formula (11) below and apply the Laplace inversion to obtain the lower bound on the price of the American call option. Next we insert L^* into the EEP expression for the American option in the Laplace space (see the following (A.10), (35), and (53)) and apply Laplace inversion to obtain the corresponding upper bound on the price of the American call option. Recall the following lemma.

Lemma 1 (Lemma 1 of [9]): *Suppose that $L_s^{(1)}$ and $L_s^{(2)}$ are any continuous time-dependent boundaries satisfying $L_s^{(2)} > L_s^{(1)} \geq B_s^*$ for all $s \in [t, T]$, where B_s^* is the optimal exercise boundary. Then $C_t(S_t, t, L_t^{(2)}) < C_t(S_t, t, L_t^{(1)})$.*

Note that the proof of Lemma 2.1 holds for more general underlying processes, then we have the following result.

Lemma 2 (Laplace space version of the Proposition 10 of [25]): *Suppose that the underlying stock price follows the diffusion process in (1) and $S\sigma(S)$ is a function of S and satisfies the Lipschitz condition. Let B_t^* be the optimal exercise boundary for the American call option, and $L^*(\lambda)$ be the exercise boundary for a capped call option in the Laplace space, and it solves $D(L^*(\lambda), \lambda) = 0$, where $D(\cdot, \cdot)$ is defined in (4). Then $\mathcal{L}^{-1}(L^*(\lambda)) \leq B_t^*$, for all $t \in [0, T]$, where $\mathcal{L}^{-1}$ denotes the inverse Laplace operator.*

Proof: Since $C_t(S_t, t, L^{(2)}) < C_t(S_t, t, L^{(1)})$ is equivalent to $\mathcal{L}(C_t(S_t, t, L^{(2)})) < \mathcal{L}(C_t(S_t, t, L^{(1)}))$, the result follows as an easy consequence of Lemma 2.1. ■

On p. 934, Detemple and Tian [25] mentioned

> This approach (i.e. the lower-upper bound approach) becomes particularly attractive when the capped call option $C(S_t, t, L)$ has a closed-form expression.

Now, we have weakened the assumptions of the approach to only requiring that we have a closed-form expression for the Laplace transform $C^*(S_t, \lambda, L) := \mathcal{L}(C(S_t, t, L))$ with respect to the time-to-maturity $u := T - t$. This can be applied to more situations, since most of the time, the density of the first passage time of the diffusion to a constant boundary is not available, but its Laplace transform is tractable if we have the explicit eigenfunctions from the associated Sturm–Liouville ODE.

2.2. Lap-LUBA 2

For the Lap-LUBA 2 method, we first derive the Laplace transform formula for the capped option in the Laplace space, then use numerical Laplace inversion to get the approximation of the capped option price and the associated derivatives. After optimizing the approximate formula for the capped option w.r.t. L in the time domain, we get the optimal value L^*. Inserting the optimal value L^* into the approximation of the capped option in the time space, we obtain the lower bound on the American option value and inserting the optimal value L^* into the EEP formula for the American options in the time space, we obtain the upper bound on the American option value.

Denote $\lim_{S_t \to L} (\partial C(S_t, L, t))/\partial L$ by $(\partial C(L, t))/\partial L$, and $\mathcal{L}(C(S_t, L, t))$ by $C^*(S_t, L, \lambda)$, then we have the approximate capped option formula in the time domain using the Gaver–Stehfest Laplace inversion formula (see [35]):

$$C(S_t, L, t) \approx \frac{\ln 2}{T - t} \sum_{k=1}^{N} V_k C^* \left(S_t, L, \frac{k \ln 2}{T - t} \right), \tag{5}$$

where

$$V_k = (-1)^{k+N/2} \sum_{j=[(k+1)/2]}^{\min(k,N/2)} \frac{j^{N/2}(2j)!}{(\frac{N}{2} - j)!j!(j-1)!(k-j)!(2j-k)!}. \tag{6}$$

Similarly, the first- and second-order partial derivatives are given by

$$\frac{\partial C(L, t)}{\partial L} \approx \frac{\ln 2}{T - t} \sum_{k=1}^{N} V_k D \left(L, \frac{k \ln 2}{T - t} \right), \tag{7}$$

and

$$\frac{\partial^2 C(L, t)}{\partial L^2} \approx \frac{\ln 2}{T - t} \sum_{k=1}^{N} V_k \frac{\partial D\left(L, \frac{k \ln 2}{T - t} \right)}{\partial L}, \tag{8}$$

where V_k is defined in (6).

Therefore, we solve the following algebraic equation

$$\frac{\partial C(L, t)}{\partial L} \approx \frac{\ln 2}{T - t} \sum_{k=1}^{N} V_k D \left(L, \frac{k \ln 2}{T - t} \right) = 0 \tag{9}$$

using the Newton's method to get the value L^* and then use the work flow of Lap-LUBA 2 to obtain the lower and upper bounds on the American option value.

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3. AMERICAN OPTIONS UNDER TIME-HOMOGENEOUS DIFFUSIONS

3.1. General Formulas

In this section, for the general form of time-homogeneous diffusion (1), we derive the formulas to be used in Lap-LUBAs. Define $\tau_L := \inf\{u \geq t : S_u \geq L\}$, $\tau := \tau_L \wedge T$, and recall the following general decomposition for the capped call option

$$\begin{aligned} C(S_t, t, L) &= E_t[e^{-r(\tau-t)}(S_\tau \wedge L - K)^+] \\ &= E_t[e^{-r(\tau-t)}(S_\tau \wedge L - K)^+ \mathbf{1}_{\{\tau_L < T\}}] + E_t[e^{-r(\tau-t)}(S_\tau \wedge L - K)^+ \mathbf{1}_{\{\tau_L \geq T\}}] \\ &= (L-K)E_t[e^{-r(\tau_L-t)}\mathbf{1}_{\{\tau_L<T\}}] + E_t[e^{-r(T-t)}(S_T-K)^+\mathbf{1}_{\{\tau_L \geq T\}}]. \end{aligned} \quad \mathbf{(10)}$$

Then we have the following results.

Proposition 1: *Under the model (1), the Laplace transform of the value of the capped call option with respect to the time-to-maturity $u := T - t$ is given by*

$$\begin{aligned} C^*(S_t, \lambda, L) &:= \mathcal{L}(C(S_t, t, L)) = (L-K)\frac{1}{\lambda}\frac{\psi_{r+\lambda}(S_t)}{\psi_{r+\lambda}(L)} \\ &\quad + \frac{\psi_{\lambda+r}(S_t)}{\omega_{\lambda+r}\psi_{\lambda+r}(L)}[\psi_{\lambda+r}(L)(J_{\lambda+r}(K,K,L)\mathbf{1}_{\{S_t\leq K\}} + J_{\lambda+r}(K,S_t,L)\mathbf{1}_{\{S_t>K\}}) \\ &\quad - \phi_{\lambda+r}(L)(I_{\lambda+r}(K,K,L)\mathbf{1}_{\{S_t\leq K\}} + I_{\lambda+r}(K,S_t,L)\mathbf{1}_{\{S_t>K\}})] \\ &\quad + \mathbf{1}_{\{S_t>K\}}\frac{\Delta_{\lambda+r}(S_t,L)}{\omega_{\lambda+r}\psi_{\lambda+r}(L)}I_{\lambda+r}(K,K,S_t), \end{aligned} \quad \mathbf{(11)}$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function, and the functions $\psi_{\lambda+r}(y)$ and $\phi_{\lambda+r}(y)$ are, respectively, the unique (up to a multiplicative constant) increasing and decreasing solution of the following Sturm–Liouville ODE

$$\mu y \frac{du}{dy} + \frac{1}{2}y^2\sigma^2(y)\frac{d^2u}{dy^2} = (\lambda + r)u, \quad y \in (0, \infty). \quad \mathbf{(12)}$$

The other auxiliary functions are defined for $0 < K \leq A < B < \infty$ and $\lambda > 0$:

$$\begin{aligned} I_{\lambda+r}(K,A,B) &:= \int_A^B (y-K)\psi_{\lambda+r}(y)\zeta(y)\,dy, \\ J_{\lambda+r}(K,A,B) &:= \int_A^B (y-K)\phi_{\lambda+r}(y)\zeta(y)\,dy, \\ \zeta(y) &:= \frac{2}{\sigma^2(y)y^2 s(y)}, \\ s(y) &:= \exp\left(-\int_\cdot^y \frac{2\mu\,dx}{\sigma^2(x)x}\right), \\ \Delta_{\lambda+r}(A,B) &:= \phi_{\lambda+r}(A)\psi_{\lambda+r}(B) - \psi_{\lambda+r}(A)\phi_{\lambda+r}(B), \end{aligned} \quad \mathbf{(13)}$$

and $\omega_{\lambda+r}$ is the Wronskian, which is a constant satisfying

$$\phi_{\lambda+r}(x)\frac{d\psi_{\lambda+r}}{dx}(x) - \psi_{\lambda+r}(x)\frac{d\phi_{\lambda+r}}{dx}(x) = s(x)\omega_{\lambda+r}. \quad \mathbf{(14)}$$

PROOF: First, we analyze the two terms in (10) separately utilizing results from Davydov and Linetsky [22]. We take the Laplace transform of both sides of (10) with respect to the time-to-maturity $u := T - t$. Note that we assume that $S_t < L$, then from Davydov and Linetsky [22 Eq. (16)], we have

$$\begin{aligned}\int_0^\infty e^{-\lambda u} E_t[e^{-r(\tau_L - t)} \mathbf{1}_{\{\tau_L < T\}}]\, du &= \int_0^\infty e^{-\lambda u} E_t[e^{-r(\tau_L - t)} \mathbf{1}_{\{\tau_L - t < u\}}]\, du \\ &= \frac{1}{\lambda} E_t[e^{-(r+\lambda)(\tau_L - t)}] = \frac{1}{\lambda} \frac{\psi_{r+\lambda}(S_t)}{\psi_{r+\lambda}(L)}. \end{aligned} \tag{15}$$

From Davydov and Linetsky [22 Eq. (21)], we have that if $S_t \le K$, then

$$\begin{aligned}&\int_0^\infty e^{-\lambda u} E_t[e^{-ru}(S_T - K)^+ \mathbf{1}_{\{\tau_L \ge T\}}]\, du \\ &= \int_0^\infty e^{-(\lambda + r)u} E_t[(S_T - K)^+ \mathbf{1}_{\{u \le \tau_L - t\}}]\, du \\ &= \frac{\Delta_{\lambda+r}(0, S_t)}{\omega_{\lambda+r}\Delta_{\lambda+r}(0, L)} [\psi_{\lambda+r}(L) J_{\lambda+r}(K, K, L) - \phi_{\lambda+r}(L) I_{\lambda+r}(K, K, L)]; \end{aligned} \tag{16}$$

and if $S_t > K$, then

$$\begin{aligned}&\int_0^\infty e^{-\lambda u} E_t[e^{-ru}(S_T - K)^+ \mathbf{1}_{\{\tau_L \ge T\}}]\, du \\ &= \int_0^\infty e^{-(\lambda + r)u} E_t[(S_T - K)^+ \mathbf{1}_{\{u \le \tau_L - t\}}]\, du \\ &= \frac{\Delta_{\lambda+r}(0, S_t)}{\omega_{\lambda+r}\Delta_{\lambda+r}(0, L)} [\psi_{\lambda+r}(L) J_{\lambda+r}(K, S_t, L) - \phi_{\lambda+r}(L) I_{\lambda+r}(K, S_t, L)] \\ &\quad + \frac{\Delta_{\lambda+r}(S_t, L)}{\omega_{\lambda+r}\Delta_{\lambda+r}(0, L)} [\phi_{\lambda+r}(0) I_{\lambda+r}(K, K, S_t) - \psi_{\lambda+r}(0) J_{\lambda+r}(K, K, S_t)]. \end{aligned} \tag{17}$$

Finally, we obtain the formula of the Laplace transform for the capped call

$$\begin{aligned}&C^*(S_t, \lambda, L) \\ &:= \mathcal{L}(C(S_t, t, L)) = (L - K)\frac{1}{\lambda}\frac{\psi_{r+\lambda}(S_t)}{\psi_{r+\lambda}(L)} \\ &\quad + \frac{\Delta_{\lambda+r}(0, S_t)}{\omega_{\lambda+r}\Delta_{\lambda+r}(0, L)} [\psi_{\lambda+r}(L)(J_{\lambda+r}(K, K, L)\mathbf{1}_{\{S_t \le K\}} + J_{\lambda+r}(K, S_t, L)\mathbf{1}_{\{S_t > K\}}) \\ &\quad - \phi_{\lambda+r}(L)(I_{\lambda+r}(K, K, L)\mathbf{1}_{\{S_t \le K\}} + I_{\lambda+r}(K, S_t, L)\mathbf{1}_{\{S_t > K\}})] \\ &\quad + \mathbf{1}_{\{S_t > K\}} \frac{\Delta_{\lambda+r}(S_t, L)}{\omega_{\lambda+r}\Delta_{\lambda+r}(0, L)} [\phi_{\lambda+r}(0) I_{\lambda+r}(K, K, S_t) - \psi_{\lambda+r}(0) J_{\lambda+r}(K, K, S_t)]. \end{aligned} \tag{18}$$

Using the boundary conditions $\psi_{\lambda+r}(0+) = 0$ as in Davydov and Linetsky [22 Eqs. (10) and (11)], we simplify (18) to arrive at (11). This completes the proof. ■

To start the optimization w.r.t. L, we shall first carry out the computation for general diffusions and express it using the associated eigenfunctions. We take the first-order derivative of (11) w.r.t. L and derive that

$$
\begin{aligned}
&\frac{\partial C^*(S_t,\lambda,L)}{\partial L}\\
&\quad= \frac{1}{\lambda}\frac{\psi_{r+\lambda}(S_t)}{\psi_{r+\lambda}(L)} - \frac{L-K}{\lambda}\frac{\psi_{r+\lambda}(S_t)\psi'_{r+\lambda}(L)}{\psi^2_{r+\lambda}(L)}\\
&\qquad - \frac{\psi_{\lambda+r}(S_t)\psi'_{\lambda+r}(L)}{\omega_{\lambda+r}\psi^2_{\lambda+r}(L)}[\psi_{\lambda+r}(L)(J_{\lambda+r}(K,K,L)\mathbf{1}_{\{S_t\le K\}} + J_{\lambda+r}(K,S_t,L)\mathbf{1}_{\{S_t>K\}})\\
&\qquad - \phi_{\lambda+r}(L)(I_{\lambda+r}(K,K,L)\mathbf{1}_{\{S_t\le K\}} + I_{\lambda+r}(K,S_t,L)\mathbf{1}_{\{S_t>K\}})]\\
&\qquad + \frac{\psi_{\lambda+r}(S_t)}{\omega_{\lambda+r}\psi_{\lambda+r}(L)}[\psi'_{\lambda+r}(L)(J_{\lambda+r}(K,K,L)\mathbf{1}_{\{S_t\le K\}} + J_{\lambda+r}(K,S_t,L)\mathbf{1}_{\{S_t>K\}})\\
&\qquad - \phi'_{\lambda+r}(L)(I_{\lambda+r}(K,K,L)\mathbf{1}_{\{S_t\le K\}} + I_{\lambda+r}(K,S_t,L)\mathbf{1}_{\{S_t>K\}})]\\
&\qquad + \mathbf{1}_{\{S_t>K\}}I_{\lambda+r}(K,K,S_t)\\
&\qquad\quad \cdot\left(\frac{\frac{\partial\Delta_{\lambda+r}(S_t,L)}{\partial L}\psi_{\lambda+r}(L) - \Delta_{\lambda+r}(S_t,L)\psi'_{\lambda+r}(L)}{\omega_{\lambda+r}\psi^2_{\lambda+r}(L)}\right). \qquad (19)
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
D(L,\lambda) &= \lim_{S_t\uparrow L}\frac{\partial C^*(S_t,\lambda,L)}{\partial L}\\
&= \frac{1}{\lambda} - \frac{(L-K)}{\lambda}\frac{\psi'_{\lambda+r}(L)}{\psi_{\lambda+r}(L)} + \frac{I_{\lambda+r}(K,K,L)}{\omega_{\lambda+r}}\left[\frac{\phi_{\lambda+r}(L)\psi'_{\lambda+r}(L)}{\psi_{\lambda+r}(L)} - \phi'_{\lambda+r}(L)\right]. \qquad (20)
\end{aligned}
$$

3.2. The Constant Elasticity of Variance Model

Under the risk-neutral measure, the CEV model is given by the following SDE:

$$
dS_t = \mu S_t\,dt + \sigma S_t^{\beta+1}\,dW_t, \quad S_0>0,\ \beta\neq 0, \qquad (21)
$$

with $\mu = r-\delta$. Here, we consider the case when we are at time t; thus, the current stock price is S_t instead of S_0. From Proposition 5 of Davydov and Linetsky [22], the fundamental solutions to the CEV ODE are

$$
\psi_{\lambda+r}(y) = \begin{cases} y^{\beta+1/2}\,e^{\varepsilon x/2}M_{k,m}(x), & \text{if } \beta<0,\ \mu\neq 0,\\ y^{\beta+1/2}\,e^{\varepsilon x/2}W_{k,m}(x), & \text{if } \beta>0,\ \mu\neq 0,\\ y^{1/2}F_\nu(\sqrt{2(\lambda+r)}z), & \text{if } \beta<0,\ \mu=0,\\ y^{1/2}G_\nu(\sqrt{2(\lambda+r)}z), & \text{if } \beta>0,\ \mu=0, \end{cases} \qquad (22)
$$

and

$$\phi_{\lambda+r}(y) = \begin{cases} y^{\beta+1/2}\, e^{\varepsilon x/2} W_{k,m}(x), & \text{if } \beta < 0,\ \mu \neq 0, \\ y^{\beta+1/2}\, e^{\varepsilon x/2} M_{k,m}(x), & \text{if } \beta > 0,\ \mu \neq 0, \\ y^{1/2} G_\nu(\sqrt{2(\lambda+r)}z), & \text{if } \beta < 0,\ \mu = 0, \\ y^{1/2} F_\nu(\sqrt{2(\lambda+r)}z), & \text{if } \beta > 0,\ \mu = 0, \end{cases} \tag{23}$$

where

$$x := \frac{|\mu|}{\sigma^2|\beta|} y^{-2\beta}, \quad z := \frac{1}{\sigma|\beta|} y^{-\beta}, \tag{24}$$

and here the constants are given by

$$\varepsilon := \operatorname{sign}(\mu\beta), \quad m := \frac{1}{4|\beta|}, \tag{25}$$

$$k := \varepsilon\left(\frac{1}{2} + \frac{1}{4\beta}\right) - \frac{\lambda+r}{2|\mu\beta|}, \quad \nu := \frac{1}{2|\beta|}. \tag{26}$$

Here, $M_{k,m}(x)$ and $W_{k,m}(x)$ are Whittaker functions, and $F_\nu(x)$ and $G_\nu(x)$ are the modified Bessel functions. The scale density is

$$s(y) = \exp\left(\frac{\mu}{\sigma^2\beta} y^{-2\beta}\right), \tag{27}$$

and the Wronskian is

$$\omega_{\lambda+r} = \begin{cases} \dfrac{2|\mu|\Gamma(2m+1)}{\sigma^2\Gamma(m-k+1/2)} & \text{if } \mu \neq 0, \\ |\beta| & \text{if } \mu = 0, \end{cases} \tag{28}$$

where $\Gamma(\cdot)$ is the Gamma function.

The auxiliary functions can be calculated as follows:

$$I_{\lambda+r}(K,A,B) = \begin{cases} \displaystyle\int_A^B (y-K) y^{\beta+1/2}\, e^{\varepsilon x/2} M_{k,m}(x)\zeta(y)\, dy, & \text{if } \beta < 0,\ \mu \neq 0, \\ \displaystyle\int_A^B (y-K) y^{\beta+1/2}\, e^{\varepsilon x/2} W_{k,m}(x)\zeta(y)\, dy, & \text{if } \beta > 0,\ \mu \neq 0, \\ \displaystyle\int_A^B (y-K) y^{1/2} F_\nu(\sqrt{2(\lambda+r)}z)\zeta(y)\, dy, & \text{if } \beta < 0,\ \mu = 0, \\ \displaystyle\int_A^B (y-K) y^{1/2} G_\nu(\sqrt{2(\lambda+r)}z)\zeta(y)\, dy, & \text{if } \beta > 0,\ \mu = 0, \end{cases} \tag{29}$$

and

$$J_{\lambda+r}(K,A,B) = \begin{cases} \displaystyle\int_A^B (y-K) y^{\beta+1/2}\, e^{\varepsilon x/2} W_{k,m}(x)\zeta(y)\, dy, & \text{if } \beta < 0,\ \mu \neq 0, \\ \displaystyle\int_A^B (y-K) y^{\beta+1/2}\, e^{\varepsilon x/2} M_{k,m}(x)\zeta(y)\, dy, & \text{if } \beta > 0,\ \mu \neq 0, \\ \displaystyle\int_A^B (y-K) y^{1/2} G_\nu(\sqrt{2(\lambda+r)}z)\zeta(y)\, dy, & \text{if } \beta < 0,\ \mu = 0, \\ \displaystyle\int_A^B (y-K) y^{1/2} F_\nu(\sqrt{2(\lambda+r)}z)\zeta(y)\, dy, & \text{if } \beta > 0,\ \mu = 0, \end{cases} \tag{30}$$

where

$$\zeta(y) = \frac{2}{\sigma^2 y^{2\beta+4} \exp\left(\frac{\mu}{\sigma^2\beta} y^{-2\beta}\right)}. \tag{31}$$

The capped call option in the Laplace space for the CEV model is the formula (11) with the replacement of the corresponding parts by (22)–(31).

We then calculate the derivatives:

$$\psi'_{\lambda+r}(y) = \begin{cases} \left(\beta + \frac{1}{2}\right) y^{\beta-1/2}\, e^{\varepsilon x/2} M_{k,m}(x) + y^{\beta+1/2}\, e^{\varepsilon x/2} \frac{\varepsilon}{2}\frac{dx}{dy} M_{k,m}(x) \\ \quad + y^{\beta+1/2}\, e^{\varepsilon x/2} \frac{dM_{k,m}(x)}{dx}\frac{dx}{dy}, \quad \text{if } \beta < 0,\ \mu \neq 0; \\ (\beta + \frac{1}{2}) y^{\beta-1/2}\, e^{\varepsilon x/2} W_{k,m}(x) + y^{\beta+1/2}\, e^{\varepsilon x/2} \frac{\varepsilon}{2}\frac{dx}{dy} W_{k,m}(x) \\ \quad + y^{\beta+1/2}\, e^{\varepsilon x/2} \frac{dW_{k,m}(x)}{dx}\frac{dx}{dy}, \quad \text{if } \beta > 0,\ \mu \neq 0; \\ \frac{1}{2} y^{-1/2} F_\nu(\sqrt{2(\lambda+r)}z) + y^{1/2} F'_\nu(\sqrt{2(\lambda+r)}z)\sqrt{2(\lambda+r)}\frac{dz}{dy}, \\ \quad \text{if } \beta < 0,\ \mu = 0; \\ \frac{1}{2} y^{-1/2} G_\nu(\sqrt{2(\lambda+r)}z) + y^{1/2} G'_\nu(\sqrt{2(\lambda+r)}z)\sqrt{2(\lambda+r)}\frac{dz}{dy}, \\ \quad \text{if } \beta > 0,\ \mu = 0, \end{cases} \tag{32}$$

and

$$\phi'_{\lambda+r}(y) = \begin{cases} \left(\beta + \frac{1}{2}\right) y^{\beta-1/2}\, e^{\varepsilon x/2} W_{k,m}(x) + y^{\beta+1/2}\, e^{\varepsilon x/2} \frac{\varepsilon}{2}\frac{dx}{dy} W_{k,m}(x) \\ \quad + y^{\beta+1/2}\, e^{\varepsilon x/2} \frac{dW_{k,m}(x)}{dx}\frac{dx}{dy}, \quad \text{if } \beta < 0,\ \mu \neq 0; \\ \left(\beta + \frac{1}{2}\right) y^{\beta-1/2}\, e^{\varepsilon x/2} M_{k,m}(x) + y^{\beta+1/2}\, e^{\varepsilon x/2} \frac{\varepsilon}{2}\frac{dx}{dy} M_{k,m}(x) \\ \quad + y^{\beta+1/2}\, e^{\varepsilon x/2} \frac{dM_{k,m}(x)}{dx}\frac{dx}{dy}, \quad \text{if } \beta > 0,\ \mu \neq 0; \\ \frac{1}{2} y^{-1/2} G_\nu(\sqrt{2(\lambda+r)}z) + y^{1/2} G'_\nu(\sqrt{2(\lambda+r)}z)\sqrt{2(\lambda+r)}\frac{dz}{dy}, \\ \quad \text{if } \beta < 0,\ \mu = 0; \\ \frac{1}{2} y^{-1/2} F_\nu(\sqrt{2(\lambda+r)}z) + y^{1/2} F'_\nu(\sqrt{2(\lambda+r)}z)\sqrt{2(\lambda+r)}\frac{dz}{dy}, \\ \quad \text{if } \beta > 0, \mu = 0. \end{cases} \tag{33}$$

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Using (22), (23), (32), and (33), we simplify (20) for the CEV case as

$$D(L,\lambda)=\begin{cases}\dfrac{1}{\lambda}-\dfrac{(L-K)}{\lambda}\left[\left(\beta+\dfrac{1}{2}\right)L^{-1}+\dfrac{\varepsilon}{2}\dfrac{dx(L)}{dL}+\dfrac{dM_{k,m}(x)}{dx}\dfrac{dx(L)}{dL}\dfrac{1}{M_{k,m}(x)}\right]\\ \quad +L^{\beta+1/2}\,e^{\varepsilon x/2}\dfrac{I_{\lambda+r}(K,K,L)}{\omega_{\lambda+r}}\left[\dfrac{dM_{k,m}(x)}{dx}\dfrac{dx(L)}{dL}\dfrac{W_{k,m}(x)}{M_{k,m}(x)}-\dfrac{dW_{k,m}(x)}{dx}\dfrac{dx(L)}{dL}\right],\\ \quad \text{if } \beta<0,\ \mu\neq 0;\\ \dfrac{1}{\lambda}-\dfrac{(L-K)}{\lambda}\left[\left(\beta+\dfrac{1}{2}\right)L^{-1}+\dfrac{\varepsilon}{2}\dfrac{dx(L)}{dL}+\dfrac{dW_{k,m}(x)}{dx}\dfrac{dx(L)}{dL}\dfrac{1}{W_{k,m}(x)}\right]\\ \quad +L^{\beta+1/2}\,e^{\varepsilon x/2}\dfrac{I_{\lambda+r}(K,K,L)}{\omega_{\lambda+r}}\left[\dfrac{dW_{k,m}(x)}{dx}\dfrac{dx(L)}{dL}\dfrac{M_{k,m}(x)}{W_{k,m}(x)}-\dfrac{dM_{k,m}(x)}{dx}\dfrac{dx(L)}{dL}\right],\\ \quad \text{if } \beta>0,\ \mu\neq 0;\\ \dfrac{1}{\lambda}-\dfrac{(L-K)}{\lambda}\left[\dfrac{1}{2}L^{-1}+\sqrt{2(\lambda+r)}F_v'(\sqrt{2(\lambda+r)}z)\dfrac{dz(L)}{dL}\dfrac{1}{F_v(\sqrt{2(\lambda+r)}z)}\right]\\ \quad +L^{1/2}\dfrac{\sqrt{2(\lambda+r)}I_{\lambda+r}(K,K,L)}{\omega_{\lambda+r}}\left[F_v'(\sqrt{2(\lambda+r)}z)\dfrac{dz(L)}{dL}\dfrac{G_v(\sqrt{2(\lambda+r)}z)}{F_v(\sqrt{2(\lambda+r)}z)}\right.\\ \quad \left.-G_v'(\sqrt{2(\lambda+r)}z)\dfrac{dz(L)}{dL}\right],\quad \text{if } \beta<0,\ \mu=0;\\ \dfrac{1}{\lambda}-\dfrac{(L-K)}{\lambda}\left[\dfrac{1}{2}L^{-1}+\sqrt{2(\lambda+r)}G_v'(\sqrt{2(\lambda+r)}z)\dfrac{dz(L)}{dL}\dfrac{1}{G_v(\sqrt{2(\lambda+r)}z)}\right]\\ \quad +L^{1/2}\dfrac{\sqrt{2(\lambda+r)}I_{\lambda+r}(K,K,L)}{\omega_{\lambda+r}}\left[G_v'(\sqrt{2(\lambda+r)}z)\dfrac{dz(L)}{dL}\dfrac{F_v(\sqrt{2(\lambda+r)}z)}{G_v(\sqrt{2(\lambda+r)}z)}\right.\\ \quad \left.-F_v'(\sqrt{2(\lambda+r)}z)\dfrac{dz(L)}{dL}\right],\quad \text{if } \beta>0,\ \mu=0.\end{cases}\tag{34}$$

Consequently, we use the bisection method to solve the algebraic Eq. (3) with the expression (34) to the get the optimal value L^* and then follow the work flow of Lap-LUBA 1 to get a lower bound on the American call value.

To calculate the upper bound on the American call value using the Lap-LUBA 1 method, we derive the EEP representation in the Laplace space by following Wong and Zhao [43]

$$C^*(S,\lambda,B(\lambda))=\begin{cases}C_{11}\psi_{\lambda+r}(S)+C_{12}\phi_{\lambda+r}(S), & \text{when } S\in(0,K),\\ C_{21}\psi_{\lambda+r}(S)+C_{22}\phi_{\lambda+r}(S)+u_{\lambda+r}(S), & \text{when } S\in[K,B(\lambda)),\end{cases}\tag{35}$$

where $B(\lambda)$ is the early exercise boundary in the Laplace space,

$$\begin{cases}C_{11}=\dfrac{a_5(a_2b_2-a_4b_1)+a_6(a_3b_1-a_1b_2)}{a_5(a_2a_3-a_1a_4)}+\dfrac{b_3}{a_5},\\ C_{12}=0,\\ C_{21}=\dfrac{a_6(a_3b_1-a_1b_2)}{a_5(a_2a_3-a_1a_4)}+\dfrac{b_3}{a_5},\\ C_{22}=\dfrac{a_1b_2-a_3b_1}{a_2a_3-a_1a_4},\\ u_{\lambda+r}(S)=\dfrac{\lambda}{\lambda+\delta}S-\dfrac{\lambda}{\lambda+r}K,\end{cases}$$

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and

$$a_1 = \psi_{\lambda+r}|_{S=K}, \quad a_2 = \phi_{\lambda+r}|_{S=K}, \quad a_3 = \left.\frac{d\psi_{\lambda+r}}{dS}\right|_{S=K},$$
$$a_4 = \left.\frac{d\phi_{\lambda+r}}{dS}\right|_{S=K}, \quad a_5 = \left.\frac{d\psi_{\lambda+r}}{dS}\right|_{S=B(\lambda)}, \quad a_6 = \left.\frac{d\phi_{\lambda+r}}{dS}\right|_{S=B(\lambda)},$$
$$b_1 = u_{\lambda+r}|_{S=K}, \quad b_2 = \left.\frac{du_{\lambda+r}}{dS}\right|_{S=K}, \quad b_3 = 1 - \left.\frac{du_{\lambda+r}}{dS}\right|_{S=B(\lambda)}.$$

Then, we replace $B(\lambda)$ in (35) by L^* and use Laplace inversion to obtain the upper bound on the price of the American call option.

For the Lap-LUBA 2 method, we solve Eq. (9) with replacement of function D by (34) using the bisection method to get L^* and then follow the work flow of Lap-LUBA 2 to get the lower bound on the price of American call option.

Furthermore, we can calculate the upper bound on the American call value by the Lap-LUBA 2 method by using the following EEP representation in the time space from Detemple and Tian [25 Prop. 3],

$$C(S_t, t; B(t)) = C^e(S_t, t) + \Pi(S_t, t; B(t)), \tag{36}$$

with

$$C^e(S_t, t) = S_t e^{-\delta(T-t)}\varphi_1(S_t, K, T) - K e^{-r(T-t)}\varphi_2(S_t, K, T), \tag{37}$$

and

$$\Pi(S_t, t; B(t)) = \int_t^T [\delta S_t\, e^{-\delta(T-t)}\varphi_1(S_t, B(\nu), \nu) - rK\, e^{-r(T-t)}\varphi_2(S_t, B(\nu), \nu)]\, d\nu, \tag{38}$$

where

$$\varphi_1(S_t, B(\nu), \nu) = \begin{cases} \chi^2\left(2y_{B(\nu)}(\nu); 2 - \frac{1}{\beta}, 2x_{S_t}(\nu)\right), & \text{if } \beta < 0, \\ \chi^2\left(2x_{S_t}(\nu); \frac{1}{\beta}, 2y_{B(\nu)}(\nu)\right), & \text{if } \beta > 0, \end{cases}$$
$$\varphi_2(S_t, B(\nu), \nu) = \begin{cases} 1 - \chi^2\left(2x_{S_t}(\nu); -\frac{1}{\beta}, 2y_{B(\nu)}(\nu)\right), & \text{if } \beta < 0, \\ 1 - \chi^2\left(2y_{B(\nu)}(\nu); 2 + \frac{1}{\beta}, 2x_{S_t}(\nu)\right), & \text{if } \beta > 0, \end{cases}$$

and

$$x_D(\nu) = \frac{r-\delta}{\sigma^2\beta\left(1 - e^{-2(r-\delta)\beta(\nu-t)}\right)} D^{-2\beta}\, e^{-2(r-\delta)\beta(\nu-t)},$$
$$y_D(\nu) = \frac{r-\delta}{\sigma^2\beta\left(1 - e^{-2(r-\delta)\beta(\nu-t)}\right)} D^{-2\beta}.$$

The function $\chi^2(x; \nu, y)$ is the complementary noncentral chi-square distribution function evaluated at x, with ν degrees of freedom and nonconcentrality parameter y.

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4. AMERICAN OPTIONS UNDER DOUBLE-EXPONENTIAL JUMP DIFFUSIONS

In this section, we shall extend the Lap-LUBAs to the case of double-exponential jump diffusions [34]. We consider the finite-maturity American put option because we want to compare our results with those in Leippold and Vasiljevic [36], where they only consider American puts. The method developed in this section can be analogously applied to the valuation of (finite-maturity) American call options under the DEJD model.

Under the risk-neutral measure, the double-exponential jump diffusion model is given by

$$\frac{dS_t}{S_{t-}} = (r - \delta - \lambda_2 \zeta)\, dt + \sigma\, dW_t + d\left(\sum_{i=1}^{N(t)} (V_i - 1)\right), \tag{39}$$

where $N(t)$ is an independent Poisson process with the rate λ_2, and $\{Y_i = \ln(V_i) : \ i = 1, 2, \ldots\}$ is a sequence of independent and identically distributed double-exponential random variables with the probability density function

$$f_Y(y) = p\eta_1\, e^{-\eta_1 y} \mathbf{1}_{\{y \geq 0\}} + q\eta_2\, e^{\eta_2 y} \mathbf{1}_{\{y<0\}}, \tag{40}$$

where $\eta_1 > 1,\ \eta_2 > 0,\ p \geq 0,\ q \geq 0$, and $p + q = 1$. The return process is given by

$$X_t := \ln(S_t/S_0) = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \quad X_0 = 0, \tag{41}$$

where $\mu = r - \delta - \sigma^2/2 - \lambda_2 \zeta$, and we can calculate $\zeta := E[e^Y - 1] = p\eta_1/(\eta_1 - 1) + q\eta_2/(\eta_2 + 1) - 1$.

To make the notations consistent, we have the following correspondence between the parameters in Sepp [42] and our parameters. In the following equation, the left-hand side is his notation and the right-hand side corresponds to our notation:

$$d \to \delta, \quad \alpha \to \zeta, \quad J \to Y, \quad q^+ \to p, \quad 1/\eta^+ \to \eta_1, \quad q^- \to q, \quad 1/\eta^- \to \eta_2. \tag{42}$$

Let L_u and L_d be, respectively, the up and down barriers, and denote the value of a double-barrier knock-out option by $F^{\mathrm{DB}}(S,t)$. Then, it satisfies the following boundary conditions for $0 \leq t \leq T$:

$$F^{\mathrm{DB}}(S, T-t) = \phi_u^*(T-t), \quad \text{if } S \geq L_u, \quad F^{\mathrm{DB}}(S, T-t) = \phi_d^*(T-t), \quad \text{if } S \leq L_d, \tag{43}$$

where $\phi_u^*(t)$ and $\phi_d^*(t)$ are contract functions that determine payoffs when the corresponding barrier is reached. The following notations are used:

$$u = T - t, \quad x = \ln\frac{S}{K}, \quad x_u = \ln\frac{L_u}{K}, \quad x_d = \ln\frac{L_d}{K}, \quad \phi_u(t) = \frac{\phi_u^*(t)}{K}, \quad \phi_d(t) = \frac{\phi_d^*(t)}{K}. \tag{44}$$

Define $\bar{\phi}_u := \mathcal{L}(\phi_u(t))$ and $\bar{\phi}_d := \mathcal{L}(\phi_d(t))$, and for standard barrier options, it is usually the case that $\phi_u(\cdot) = \phi_d(\cdot) = 0$. In terms of the new notations, we have the new representation of $F^{\mathrm{DB}}(S,u)$ as $V^{\mathrm{DB}}(x,u)$, where $u = T - t$. The Laplace transform of $V^{\mathrm{DB}}(x,u)$ with respect to the time-to-maturity u is defined by

$$U^{\mathrm{DB}}(x, \lambda) := \int_0^\infty e^{-\lambda u} V^{\mathrm{DB}}(x,u)\, du, \tag{45}$$

where λ is a transform variable with a positive real part.

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We also need the following lemma.

Lemma 3 [42 Lemma 4.1::] *The following characteristic equation,*

$$\frac{1}{2}\sigma^2\psi^2 + \mu\psi - (r+\lambda+\lambda_2) + \lambda_2\left[\frac{p\eta_1}{\eta_1-\psi} + \frac{q\eta_2}{\eta_2-\psi}\right] = 0, \tag{46}$$

has four real roots ψ_i, $i = 0,1,2,3$, such that

$$-\infty < \psi_3 < -\eta_2 < \psi_2 < 0 < \psi_1 < \eta_1 < \psi_0 < \infty. \tag{47}$$

Now we recall the following result on the Laplace transform of the double barrier option under double-exponential jump diffusions using our notation.

Lemma 4 [42 Prop. 5.1::] *In the Laplace space, the value of a double-barrier knockout option under a double-exponential jump diffusion model in (39) is given by the formula*

$$U^{DB}(x,\lambda) = \begin{cases} (C_0+C_4)\,e^{\psi_0 x} + (C_1+C_5)\,e^{\psi_1 x} + C_6\,e^{\psi_2 x} + C_7\,e^{\psi_3 x} & \\ \quad + \dfrac{\phi-1}{2}\left[\dfrac{e^x}{\delta+\lambda} - \dfrac{1}{r+\lambda}\right], & \text{if } x < 0, \\ (C_2+C_6)\,e^{\psi_2 x} + (C_3+C_7)\,e^{\psi_3 x} + C_4\,e^{\psi_0 x} + C_5\,e^{\psi_1 x} & \\ \quad + \dfrac{\phi+1}{2}\left[\dfrac{e^x}{\delta+\lambda} - \dfrac{1}{r+\lambda}\right], & \text{if } x \geq 0, \end{cases} \tag{48}$$

where the constants C_j, $j = 0,1,2,3$ are solutions of the following system

$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ \psi_0 & \psi_1 & -\psi_2 & -\psi_3 \\ \dfrac{\eta_2}{\psi_0+\eta_2} & \dfrac{\eta_2}{\psi_1+\eta_2} & -\dfrac{\eta_2}{\psi_2+\eta_2} & -\dfrac{\eta_2}{\psi_3+\eta_2} \\ \dfrac{\eta_1}{\psi_0-\eta_1} & \dfrac{\eta_1}{\psi_1-\eta_1} & -\dfrac{\eta_1}{\psi_2-\eta_1} & -\dfrac{\eta_1}{\psi_3-\eta_1} \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\delta+\lambda} - \dfrac{1}{r+\lambda} \\ \dfrac{1}{\delta+\lambda} \\ \dfrac{\eta_2}{(\delta+\lambda)(1+\eta_2)} - \dfrac{1}{r+\lambda} \\ \dfrac{\eta_1}{(\delta+\lambda)(1-\eta_1)} + \dfrac{1}{r+\lambda} \end{pmatrix},$$

and the constants C_j, $j = 4,5,6,7$ are solutions of the system

$$\begin{pmatrix} \dfrac{\eta_2\,e^{\psi_0 x_d}}{\psi_0+\eta_2} & \dfrac{\eta_2\,e^{\psi_1 x_d}}{\psi_1+\eta_2} & \dfrac{\eta_2\,e^{\psi_2 x_d}}{\psi_2+\eta_2} & \dfrac{\eta_2\,e^{\psi_3 x_d}}{\psi_3+\eta_2} \\ e^{\psi_0 x_d} & e^{\psi_1 x_d} & e^{\psi_2 x_d} & e^{\psi_3 x_d} \\ e^{\psi_0 x_u} & e^{\psi_1 x_u} & e^{\psi_2 x_u} & e^{\psi_3 x_u} \\ \dfrac{\eta_1\,e^{\psi_0 x_u}}{\psi_0-\eta_1} & \dfrac{\eta_1\,e^{\psi_1 x_u}}{\psi_1-\eta_1} & \dfrac{\eta_1\,e^{\psi_2 x_u}}{\psi_2-\eta_1} & \dfrac{\eta_1\,e^{\psi_3 x_u}}{\psi_3-\eta_1} \end{pmatrix} \begin{pmatrix} C_4 \\ C_5 \\ C_6 \\ C_7 \end{pmatrix}$$

$$= \begin{pmatrix} -\dfrac{\phi-1}{2}\left(\dfrac{\eta_2\,e^{x_d}}{(\delta+\lambda)(1+\eta_2)} - \dfrac{1}{r+\lambda}\right) + \bar{\phi}_d - \dfrac{\eta_2\,e^{\psi_0 x_d}}{\psi_0+\eta_2}C_0 - \dfrac{\eta_2\,e^{\psi_1 x_d}}{\psi_1+\eta_2}C_1 \\ -\dfrac{\phi-1}{2}\left(\dfrac{e^{x_d}}{\delta+\lambda} - \dfrac{1}{r+\lambda}\right) + \bar{\phi}_d - e^{\psi_0 x_d}C_0 - e^{\psi_1 x_d}C_1 \\ -\dfrac{\phi+1}{2}\left(\dfrac{e^{x_u}}{\delta+\lambda} - \dfrac{1}{r+\lambda}\right) + \bar{\phi}_u - e^{\psi_2 x_u}C_2 - e^{\psi_3 x_u}C_3 \\ -\dfrac{\phi+1}{2}\left(\dfrac{\eta_1\,e^{x_u}}{(\delta+\lambda)(1-\eta_1)} + \dfrac{1}{r+\lambda}\right) - \bar{\phi}_u - \dfrac{\eta_1\,e^{\psi_2 x_u}}{\psi_2-\eta_1}C_2 - \dfrac{\eta_1\,e^{\psi_3 x_u}}{\psi_3-\eta_1}C_3 \end{pmatrix}.$$

Note that we only need the price of the single down-and-out barrier option to carry out our optimization procedure. As illustrated in p. 10 of Sepp [42], for the case of down-and-out barrier option, we take $x_u \to \infty$ and set $C_4 = C_5 = 0$ in the linear systems contained in Lemma 4.2. Then we have that in the Laplace space, the value of the single down-and-out barrier option under a double-exponential jump diffusion model is given by the formula:

$$U^{\mathrm{DOB}}(x,\lambda) = \begin{cases} C_0\, e^{\psi_0 x} + C_1\, e^{\psi_1 x} + C_6\, e^{\psi_2 x} + C_7\, e^{\psi_3 x} + \dfrac{\phi - 1}{2}\left[\dfrac{e^x}{\delta+\lambda} - \dfrac{1}{r+\lambda}\right], & \text{if } x < 0, \\ (C_2 + C_6)\, e^{\psi_2 x} + (C_3 + C_7)\, e^{\psi_3 x} + \dfrac{\phi + 1}{2}\left[\dfrac{e^x}{\delta+\lambda} - \dfrac{1}{r+\lambda}\right], & \text{if } x \geq 0, \end{cases} \tag{49}$$

where the constants $C_j,\ j = 0,1,2,3$ are solutions of the system in Lemma 4.2 and the constants $C_j,\ j = 6,7$ are solutions of the following simplified system

$$\begin{pmatrix} e^{\psi_2 x_u} & e^{\psi_3 x_u} \\ \dfrac{\eta_2 e^{\psi_2 x_u}}{\psi_2 + \eta_2} & \dfrac{\eta_2 e^{\psi_3 x_u}}{\psi_3 + \eta_2} \end{pmatrix} \begin{pmatrix} C_6 \\ C_7 \end{pmatrix} = \begin{pmatrix} -\dfrac{\phi-1}{2}\left(\dfrac{e^{x_d}}{\delta+\lambda} - \dfrac{1}{r+\lambda}\right) + \bar{\phi}_d - e^{\psi_0 x_d} C_0 - e^{\psi_1 x_d} C_1 \\ -\dfrac{\phi-1}{2}\left(\dfrac{\eta_2\, e^{x_d}}{(\delta+\lambda)(1+\eta_2)} - \dfrac{1}{r+\lambda}\right) + \bar{\phi}_d - \dfrac{\eta_2\, e^{\psi_0 x_d}}{\psi_0 + \eta_2} C_0 - \dfrac{\eta_2\, e^{\psi_1 x_d}}{\psi_1 + \eta_2} C_1 \end{pmatrix}. \tag{50}$$

From the above discussion, we can obtain the Laplace transform formula for the capped put option under the DEJD model, which is summarized in the following proposition.

PROPOSITION 2: *Denote the capped put option by $P(S_t, t, L)$ and the Laplace transform by $P^*(S_t, \lambda, L)$ under the double-exponential jump diffusion model. Then, the one-dimensional Laplace transform of the value of the capped put option with respect to the time-to-maturity $u := T - t$ is given by*

$$P^*(S_t,\lambda,L) := \int_0^\infty e^{-\lambda u} P(S_t, T-u, L)\, du = \begin{cases} C_0\, e^{\psi_0 x} + C_1\, e^{\psi_1 x} + \widehat{C}_6\, e^{\psi_2 x} + \widehat{C}_7\, e^{\psi_3 x} + \dfrac{\phi - 1}{2}\left[\dfrac{e^x}{\delta+\lambda} - \dfrac{1}{r+\lambda}\right], & \text{if } x < 0, \\ (C_2 + \widehat{C}_6)\, e^{\psi_2 x} + (C_3 + \widehat{C}_7)\, e^{\psi_3 x} + \dfrac{\phi + 1}{2}\left[\dfrac{e^x}{\delta+\lambda} - \dfrac{1}{r+\lambda}\right], & \text{if } x \geq 0, \end{cases} \tag{51}$$

where $C_0,\ C_1,\ C_2,\ C_3$ are given by the system in Lemma 4.2, and $\widehat{C}_6,\ \widehat{C}_7$ given by system (50) with $\bar{\phi}_d = (K-L)^+/(\lambda K)$, $x := \ln(S_t/K)$, $x_d := \ln(L/K) > 0$.

PROOF: We can identify the capped put option as a down-and-out barrier option with a constant rebate at the lower barrier hitting time. The rebate level is $\phi_d^*(t) = (K-L)^+$, after dividing K and Laplace transform acquires $\bar{\phi}_d := \mathcal{L}(\phi_d(t)) = (K-L)^+/\lambda/K$. Therefore, formula (51) follows from (49) by replacing $C_6,\ C_7$ with $\widehat{C}_6,\ \widehat{C}_7$. ■

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For the DEJD model, we can calculate

$$D(L,\lambda) = \lim_{S_t \downarrow L} \frac{\partial P^*(S_t,\lambda,L)}{\partial L} = \frac{d\widehat{C}_6}{dL}\, e^{\psi_2 x_d} + \frac{d\widehat{C}_7}{dL}\, e^{\psi_3 x_d}, \tag{52}$$

where

$$\begin{aligned}
\frac{d\widehat{C}_6}{dL} = &-\frac{(\psi_2+\eta_2)\psi_2}{(\psi_2-\psi_3)L}\, e^{-\psi_2 x_d}\left[-\frac{\phi-1}{2}\left(\frac{e^{x_d}}{\delta+\lambda}-\frac{1}{r+\lambda}\right)+\bar{\phi}_d - e^{\psi_0 x_d}C_0 - e^{\psi_1 x_d}C_1\right]\\
&+\frac{\psi_2+\eta_2}{\psi_2-\psi_3}\, e^{-\psi_2 x_d}\left[-\frac{\phi-1}{2}\frac{e^{x_d}}{L(\delta+\lambda)}-\frac{1}{K\lambda}-\frac{C_0\psi_0}{L}\, e^{\psi_0 x_d}-\frac{C_1\psi_1}{L}\, e^{\psi_1 x_d}\right]\\
&+\frac{(\psi_2+\eta_2)(\psi_3+\eta_2)\psi_2}{\eta_2(\psi_2-\psi_3)L}\, e^{-\psi_2 x_d}\left[-\frac{\phi-1}{2}\left(\frac{\eta_2\, e^{x_d}}{(\delta+\lambda)(1+\eta_2)}-\frac{1}{r+\lambda}\right)+\bar{\phi}_d\right.\\
&\left.-\frac{\eta_2 C_0\, e^{\psi_0 x_d}}{\psi_0+\eta_2}-\frac{\eta_2 C_1\, e^{\psi_1 x_d}}{\psi_1+\eta_2}\right]\\
&-\frac{(\psi_2+\eta_2)(\psi_3+\eta_2)}{\eta_2(\psi_2-\Psi_3)}\, e^{-\psi_2 x_d}\left[-\frac{\phi-1}{2}\frac{\eta_2\, e^{x_d}}{(\delta+\lambda)(1+\eta_2)L}-\frac{1}{K\lambda}\right.\\
&\left.-\frac{\eta_2 C_0\psi_0\, e^{\psi_0 x_d}}{(\psi_0+\eta_2)L}-\frac{\eta_2 C_1\psi_1\, e^{\psi_1 x_d}}{(\psi_1+\eta_2)L}\right],
\end{aligned}$$

and

$$\begin{aligned}
\frac{d\widehat{C}_7}{dL} = &\frac{(\psi_3+\eta_2)\psi_3}{(\psi_2-\psi_3)L}\, e^{-\psi_3 x_d}\left[-\frac{\phi-1}{2}\left(\frac{e^{x_d}}{\delta+\lambda}-\frac{1}{r+\lambda}\right)+\bar{\phi}_d - e^{\psi_0 x_d}C_0 - e^{\psi_1 x_d}C_1\right]\\
&-\frac{\psi_3+\eta_2}{\psi_2-\psi_3}\, e^{-\psi_3 x_d}\left[-\frac{\phi-1}{2}\frac{e^{x_d}}{L(\delta+\lambda)}-\frac{1}{K\lambda}-\frac{C_0\psi_0}{L}\, e^{\psi_0 x_d}-\frac{C_1\psi_1}{L}\, e^{\psi_1 x_d}\right]\\
&-\frac{(\psi_2+\eta_2)(\psi_3+\eta_2)\psi_3}{\eta_2(\psi_2-\psi_3)L}\, e^{-\psi_3 x_d}\left[-\frac{\phi-1}{2}\left(\frac{\eta_2\, e^{x_d}}{(\delta+\lambda)(1+\eta_2)}-\frac{1}{r+\lambda}\right)+\bar{\phi}_d\right.\\
&\left.-\frac{\eta_2 C_0\, e^{\psi_0 x_d}}{\psi_0+\eta_2}-\frac{\eta_2 C_1\, e^{\psi_1 x_d}}{\psi_1+\eta_2}\right]\\
&+\frac{(\psi_2+\eta_2)(\psi_3+\eta_2)}{\eta_2(\psi_2-\psi_3)}\, e^{-\psi_3 x_d}\left[-\frac{\phi-1}{2}\frac{\eta_2\, e^{x_d}}{(\delta+\lambda)(1+\eta_2)L}-\frac{1}{K\lambda}\right.\\
&\left.-\frac{\eta_2 C_0\psi_0\, e^{\psi_0 x_d}}{(\psi_0+\eta_2)L}-\frac{\eta_2 C_1\psi_1\, e^{\psi_1 x_d}}{(\psi_1+\eta_2)L}\right].
\end{aligned}$$

We solve the algebraic equation $D(L,\lambda)=0$ with expression (52) using the bisection method to get the value L^* and then follow the work flow of Lap-LUBA 1 to obtain the lower bound on the price of American options.

To calculate the upper bound of the American options using the Lap-LUBA 1 method, we need the following EEP representation in the Laplace space (see [36]). To make the notations consistent, we have the following correspondence between the parameters in Leippold and Vasiljevic [36] and our parameters. The left-hand side is their notation and the right-hand side corresponds to our notation:

$$\beta_2 \to \psi_0, \quad \beta_1 \to \psi_1, \quad \gamma_1 \to \psi_2, \quad \gamma_2 \to \psi_3, \quad \eta_1 \to \eta_1, \quad \theta_1 \to \eta_2.$$

The EEP representation of the American put option in the Laplace space can be written as

$$P_A^*(S,\lambda) = \begin{cases} p_E^*(S,\lambda) + e_p^*(S,\lambda), & \text{if } S > B(\lambda), \\ K - S, & \text{if } S \le B(\lambda), \end{cases} \tag{53}$$

where $B(\lambda)$ is the time-independent early exercise boundary in the Laplace space, and $P_A^*(S,\lambda)$ is the standard American put option price. Here, $p_E^*(S,\lambda)$ is the Canadized European put option price given by

$$p_E^*(S,\lambda) = \begin{cases} \underline{\omega}_1 \left(\frac{S}{K}\right)^{\psi_1} + \underline{\omega}_2 \left(\frac{S}{K}\right)^{\psi_0} + \frac{\lambda K}{\lambda + r} - \frac{\lambda S}{\lambda + \delta}, & \text{if } S \ge K, \\ \overline{\omega}_1 \left(\frac{S}{K}\right)^{\psi_2} + \overline{\omega}_2 \left(\frac{S}{K}\right)^{\psi_3}, & \text{if } S \ge K. \end{cases}$$

The coefficients $\omega = (\underline{\omega}_1, \underline{\omega}_2, \overline{\omega}_1, \overline{\omega}_2)'$ are solutions of the following matrix equation

$$A\omega = J,$$

where

$$A = \begin{pmatrix} 1 & 1 & -1 & -1 \\ \psi_1 & \psi_0 & -\psi_2 & -\psi_3 \\ \frac{1}{\eta_1 - \psi_1} & \frac{1}{\eta_1 - \psi_0} & -\frac{1}{\eta_1 - \psi_2} & -\frac{1}{\eta_1 - \psi_3} \\ \frac{1}{\eta_2 + \psi_1} & \frac{1}{\eta_2 + \psi_0} & -\frac{1}{\eta_2 + \psi_2} & -\frac{1}{\eta_2 + \psi_3} \end{pmatrix},$$

$$J = \begin{pmatrix} \frac{\lambda K}{\lambda + \delta} - \frac{\lambda K}{\lambda + r} & \frac{\lambda K}{\lambda + \delta} & \frac{1}{\eta_1 - 1}\frac{\lambda K}{\lambda + \delta} - \frac{1}{\eta_1}\frac{\lambda K}{\lambda + r} & \frac{1}{\eta_2 + 1}\frac{\lambda K}{\lambda + \delta} - \frac{1}{\eta_2}\frac{\lambda K}{\lambda + r} \end{pmatrix}'.$$

The EEP $e_p^*(S,\lambda)$ is given by

$$e_p^*(S,\lambda) = \begin{cases} \sum_{j=1}^{2} v_j \left(\frac{S}{B(\lambda)}\right)^{\psi_{j+1}}, & \text{if } S > B(\lambda), \\ K - S - p_E^*(S,\lambda), & \text{if } S \le B(\lambda), \end{cases}$$

where $B(\lambda)$ is the time-independent early exercise boundary, and $p_E^*(S,\lambda)$ is the Canadized European put option price. The coefficients $v = (v_1, v_2)'$ can be solved in the matrix equation as follows:

$$\widetilde{A}v = \widetilde{J},$$

where

$$\widetilde{A} = \begin{pmatrix} 1 & 1 \\ \frac{1}{\eta_2 + \psi_2} & \frac{1}{\eta_2 + \psi_3} \end{pmatrix}, \quad \widetilde{J} = -\Omega b + \frac{B(\lambda)}{K}\omega + \epsilon,$$

$$\Omega = \begin{pmatrix} \underline{\omega}_1 & \underline{\omega}_2 \\ \frac{\underline{\omega}_1}{\eta_2 + \psi_1} & \frac{\underline{\omega}_2}{\eta_2 + \psi_0} \end{pmatrix}, \quad b = \begin{pmatrix} \left(\frac{B(\lambda)}{K}\right)^{\psi_1} & \left(\frac{B(\lambda)}{K}\right)^{\psi_0} \end{pmatrix}',$$

$$\omega = \left(-\frac{\delta K}{\lambda + \delta} \quad -\frac{1}{\eta_2 + 1}\frac{\delta K}{\lambda + \delta}\right)', \qquad \epsilon = \left(\frac{rK}{\lambda + r} \quad \frac{1}{\eta_2}\frac{rK}{\lambda + r}\right)',$$

and the detailed calculations can be found in Leippold and Vasiljevic [36].

According to the work flow of Lap-LUBA 1 for calculating the upper bound, we replace $B(\lambda)$ in (53) by L^* and use the Laplace inversion to obtain the upper bound on the price of the American options.

For the Lap-LUBA 2 method, we solve Eq. (9) with the replacement of function D by (52) using the bisection method to get L^* and then follow the work flow of Lap-LUBA 2 to get the lower bound on the price of American options.

To calculate the upper bound using the Lap-LUBA 2 method, we need EEP representation of the American put options under the DEJD model (see [5]). Denote $z = \ln S_t$, and let $F(t, z)$ be the price of the American option and $\{b(\tau) : 0 \le \tau \le t\}$ be the optimal exercise boundary. Then, the EEP for an American put option can be expressed as follows:

$$F(t, z) = F_E(t, z) + e_1(t, z) - e_2(t, z), \tag{54}$$

where $F_E(t, z)$ is the European put option price explicitly expressed in Theorem 2 of Kou [33].

$$\begin{aligned} e_1(t, z) &= rK \int_0^t e^{-rs} P[Z_s(z) \le b(t-s)]\, ds - \lambda_2 \zeta\, e^z \int_0^t \int_{-\infty}^{b(t-s)-z-\mu s} e^w f(w, s)\, dw\, ds, \\ e_2(t, z) &= \int_0^t e^{-rs} \int_{-\infty}^{b(t-s)-z-\mu s} f(w, s) \int_{b(t-s)-z-\mu s-w}^{\infty} [F(t-s, z+\mu s+w+y) \\ &\quad - (K - e^{z+\mu s+w})] p\eta_1\, e^{-\eta_1 y}\, dy\, dw\, ds, \end{aligned}$$

with

$$\begin{aligned} f(w, s) &= \pi_0 \varphi\left(\frac{w}{\sigma\sqrt{s}}\right) + \sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} P_{n,k} (\widetilde{\sigma}\eta_1)^k \frac{e^{(\widetilde{\sigma}\eta_1)^2/2}}{\widetilde{\sigma}\sqrt{2\pi}}\, e^{w\eta_1} \mathcal{H}_{k-1}\left(-\frac{w}{\widetilde{\sigma}} + \widetilde{\sigma}\eta_1\right) \\ &\quad + \sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} Q_{n,k} (\widetilde{\sigma}\eta_2)^k \frac{e^{(\widetilde{\sigma}\eta_2)^2/2}}{\widetilde{\sigma}\sqrt{2\pi}}\, e^{w\eta_2} \mathcal{H}_{k-1}\left(\frac{w}{\widetilde{\sigma}} + \widetilde{\sigma}\eta_2\right). \end{aligned}$$

Here, φ is the density function of the standard normal distribution, and

$$\begin{aligned} \pi_n &= e^{-\lambda_2 s} \frac{(\lambda_2 s)^n}{n!}, \quad \widetilde{\sigma} = \sigma\sqrt{s}, \\ P_{n,k} &= \sum_{i=k}^{n-1} \binom{n-k-1}{i-k} \binom{n}{i} \left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{i-k} \left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{n-i} p^i q^{n-i}, \\ Q_{n,k} &= \sum_{i=k}^{n-1} \binom{n-k-1}{i-k} \binom{n}{i} \left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{n-i} \left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{i-k} p^{n-i} q^i, \end{aligned}$$

for $1 \le k \le n-1$, and $P_{n,n} = p^n$, $Q_{n,n} = q^n$, and $p + q = 1$. The probability $P[Z_s(z) \le b(t-s)]$ in $e_1(t, z)$ can be explicitly expressed using Theorem B.1 of Kou [33]. Indeed,

$$P[Z_s(z) \le b(t-s)] = 1 - Q(t, s, z),$$

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where by letting $a = b(t-s) - z$, we have

$$Q(t,s,z) = \pi_0 \Phi\left(-\frac{a-\mu s}{\sigma\sqrt{s}}\right) + \frac{e^{(\sigma\eta_1)^2 s/2}}{\sigma\sqrt{2\pi s}} \sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} P_{n,k}(\sigma\sqrt{s}\eta_1)^k \times I_{k-1}\left(a-\mu s; -\eta_1, -\frac{1}{\sigma\sqrt{s}}, -\sigma\eta_1\sqrt{s}\right) + \frac{e^{(\sigma\eta_2)^2 s/2}}{\sigma\sqrt{2\pi s}} \sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n} Q_{n,k}(\sigma\sqrt{s}\eta_2)^k \times I_{k-1}\left(a-\mu s; \eta_2, \frac{1}{\sigma\sqrt{s}}, -\sigma\eta_2\sqrt{s}\right)$$

with Φ being the standard normal cumulative distribution function and

$$I_n(c;\alpha,\beta,\delta) = -\frac{e^{\alpha c}}{\alpha}\sum_{i=0}^{n}\left(\frac{\beta}{\alpha}\right)^{n-i}\mathcal{H}_i(\beta c-\delta) + \left(\frac{\beta}{\alpha}\right)^{n+1}\frac{\sqrt{2\pi}}{\beta}e^{\alpha\delta/\beta+\alpha^2/2\beta^2}\Phi\left(-\beta c+\delta+\frac{\alpha}{\beta}\right)$$

for $\beta > 0$, $\alpha \neq 0$, $n \geq -1$; and

$$I_n(c;\alpha,\beta,\delta) = -\frac{e^{\alpha c}}{\alpha}\sum_{i=0}^{n}\left(\frac{\beta}{\alpha}\right)^{n-i}\mathcal{H}_i(\beta c-\delta) - \left(\frac{\beta}{\alpha}\right)^{n+1}\frac{\sqrt{2\pi}}{\beta}e^{\alpha\delta/\beta+\alpha^2/2\beta^2}\Phi\left(\beta c-\delta-\frac{\alpha}{\beta}\right)$$

for $\beta < 0$, $\alpha < 0$, and $n \geq -1$.

For every $n \geq 0$, the $\mathcal{H}_n$ function above is a nonincreasing function defined by:

$$\mathcal{H}_n(x) = \int_x^{\infty}\mathcal{H}_{n-1}(y)\,dy = \frac{1}{n!}\int_x^{\infty}(t-x)^n e^{-t^2/2}\,dt \geq 0, \quad n = 0,1,2,\ldots$$

We can compute $\mathcal{H}_n$ function recursively as

$$\mathcal{H}_{-1}(x) = \sqrt{2\pi}\varphi(x), \quad \mathcal{H}_0(x) = \sqrt{2\pi}\Phi(-x),$$
$$n\mathcal{H}_n(x) = \mathcal{H}_{n-2}(x) - x\mathcal{H}_{n-1}(x), \quad n \geq 1.$$

For more details, refer to Kou [33]. Finally following the work flow of Lap-LUBA 2, we replace $b(\tau)$ in (54) by $L^*(T-\tau)$ to obtain the upper bound on the price of the American put option.

5. NUMERICAL EXAMPLES

This section reports the numerical results using Lap-LUBA 1 and Lap-LUBA 2 and compare them with the methods in the literature, finite difference methods (FCMs) developed by Muthuraman [39], Laplace transform methods (LTMs) by Zhu [46] (for GBM), Wong and Zhao [43] (for CEV) and Leippold-Vasiljevic [36] (for DEJD), Fourier-cosine series methods (FCMs) by Fang and Oosterlee [26]. These comparison of different methods are summarized in Table 1. To make a fair comparison, we implement the algorithms in the literature ourselves and report corresponding computing times.

TABLE 1. Other methods used for comparisons: FDM, LTM, FCM

Methods	FDM	LTM	FCM
Models	CEV, DEJD	CEV, DEJD	DEJD

REMARK 1: *The pricing results are in general very accurate as seen from the tables, and the methods can also produce a very close approximation to the early exercise boundary as seen from the figures. However, for the Lap-LUBA 1, the "upper bounds" reported in some tables are slightly lower than the benchmark values. The main reason is that the EEP representation of the American options in the Laplace space ([46] (for GBM), [43] (for CEV), and [36] (for DEJD)) is not accurate. The EEP representation in the Laplace space is derived by the Laplace transforms for the pricing PDEs defined on the moving region* $[B(T-\tau), +\infty)$ *which make use of the assumption that early exercise boundary moves very slowly in comparison with the "diffusion" of the option price (see [46] for the elaborations). However it is problematic to use this assumption as the early exercise boundary changes considerably near the expiry date. We implement the LTMs developed by Zhu [46] (for GBM), Wong and Zhao [43] (for CEV), and Leippold and Vasiljevic [36] (for DEJD). Indeed, the numerical results show that the early exercise boundaries solved by the LTMs often have notable errors.*

Thus, we recommend carrying out the optimization procedure in the original time space using the closed-form formulas of the capped option price and the EEP representation in the time space if they are available, which is referred as Lap-LUBA 2. Numerical results show that the Lap-LUBA 2 generates correct and tight lower and upper bounds on the American option values and obtains accurate early exercise boundaries for all the models considered in this paper.

Overall from the numerical results for GBM, CEV, and DEJD models, we have common observations that the Lap-LUBA 2 can generate correct lower and upper bounds on the option values and very close early exercise boundaries to the benchmark. Lap-LUBA 1 can generate the correct lower bound, but sometimes not accurate upper bound and early exercise boundaries.

5.1. Numerical Examples for GBM

In this section, we compare our method with that of Broadie and Detemple [9] to check whether the Laplace lower–upper-bound approaches (Lap-LUBA 1 and Lap-LUBA 2) provide accurate lower-bound approximations to the early exercise boundaries and generate tight lower and upper bounds on option values. It can be seen from Table 2 that the Lap-LUBA 2 has similar performance as Broadie and Detemple's LUBA (B&D LUBA). The upper bound generated by the Lap-LUBA 1 is not accurate as the upper bound is smaller than the true option value in many situations. The reason is that the EEP representation in the Laplace space is not accurate, as discussed in detail in Remark 5.1.

In Figure 2–4, we draw the early exercise boundaries for the American call option with $r < \delta$, $r > \delta$, and $r = \delta$, respectively, using the Lap-LUBAs and other approaches including the binomial approach, B&D LUBA, and LTM. From these figures, we can see that the early exercise boundaries by the Lap-LUBA 2 and B&D LUBA are pretty close to the benchmark (the binomial approach) for all the cases. The early exercise boundaries by Lap-LUBA 1 and LTMs deviate largely from the benchmark for $r \geq \delta$ (see Figures 3 and 4).

Table 2. American call option value bounds and approximations (maturity $T = 3$ years) for GBM

Option parameter	Asset price	Lap-LUBA 1		Lap-LUBA 2		B&D-LUBA		LTM value	True value
		LB	UB	LB	UB	LB	UB		
$r = 0.03$	80	2.532	2.532	2.553	2.589	2.553	2.589	2.532	2.580
$\sigma = 0.20$	90	5.068	5.068	5.121	5.187	5.121	5.187	5.068	5.167
$\delta = 0.07$	100	8.892	8.892	9.002	9.103	9.002	9.103	8.892	9.066
	110	14.170	14.170	14.371	14.504	14.371	14.504	14.170	14.443
	120	21.019	20.586	21.354	21.506	21.354	21.506	20.586	21.414
$r = 0.03$	80	11.152	11.152	11.238	11.354	11.238	11.354	11.152	11.326
$\sigma = 0.40$	90	15.475	15.475	15.608	15.763	15.609	15.763	15.475	15.722
$\delta = 0.07$	100	20.461	20.461	20.656	20.850	20.656	20.850	20.461	20.793
	110	26.065	26.065	26.336	26.569	26.337	26.569	26.065	26.495
	120	32.245	32.245	32.607	32.877	32.607	32.876	32.245	32.781
$r = 0.00$	80	5.408	5.408	5.463	5.540	5.463	5.540	5.408	5.518
$\sigma = 0.30$	90	8.666	8.666	8.766	8.879	8.766	8.879	8.666	8.842
$\delta = 0.07$	100	12.882	12.882	13.048	13.199	13.048	13.199	12.882	13.142
	110	18.092	18.092	18.347	18.535	18.347	18.535	18.092	18.453
	120	24.313	24.313	24.685	24.903	24.685	24.903	24.313	24.791
$r = 0.07$	80	12.147	12.147	12.145	12.145	12.145	12.145	12.147	12.145
$\sigma = 0.30$	90	17.369	17.369	17.367	17.368	17.367	17.368	17.369	17.369
$\delta = 0.03$	100	23.347	23.347	23.347	23.349	23.347	23.349	23.347	23.348
	110	29.960	29.960	29.961	29.964	29.961	29.964	29.960	29.964
	120	37.096	37.096	37.099	37.104	37.099	37.104	37.096	37.104

The strikes for all options are taken as $K = 100$. The number of time steps is taken as $n = 200$ for both "Lap-LUBA 2" and "B&D-LUBA." The Gaver-Stehfest method [35] is used for the Laplace inversion with $N = 8$ for "Lap-LUBA 1" and $N = 12$ for "Lap-LUBA 2." The "true value" column is based on the binomial method with $n = 15{,}000$ time steps. In "Lap-LUBA 1," the computation of lower bound (LB) takes on average 0.004 s, and upper bound (UB) on average 0.004 s. In "Lap-LUBA 2," the computation of the "LB" takes on average 0.006 s, and "UB" on average 0.359 s. In "B&D-LUBA," the computation of the "LB" takes on average 0.003 s, and 'UB' on average 0.150 s. The "LTM" takes on average 0.092 s. The "true value" takes on average 47.590 s.

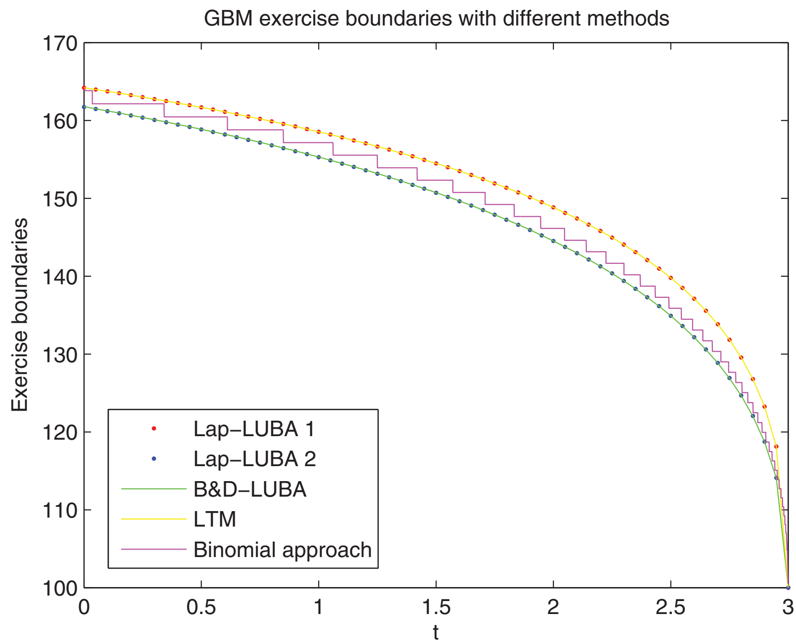

FIGURE **2.** Early exercise boundaries generated by Lap-LUBA 1, Lap-LUBA 2, B&D-LUBA, LTM, and the binomial approach for American call options with GBM. The parameters are $S_0 = 100$, $K = 100$, $r = 0.03$, $\delta = 0.07$, $\sigma = 0.3$, and $T = 3$.

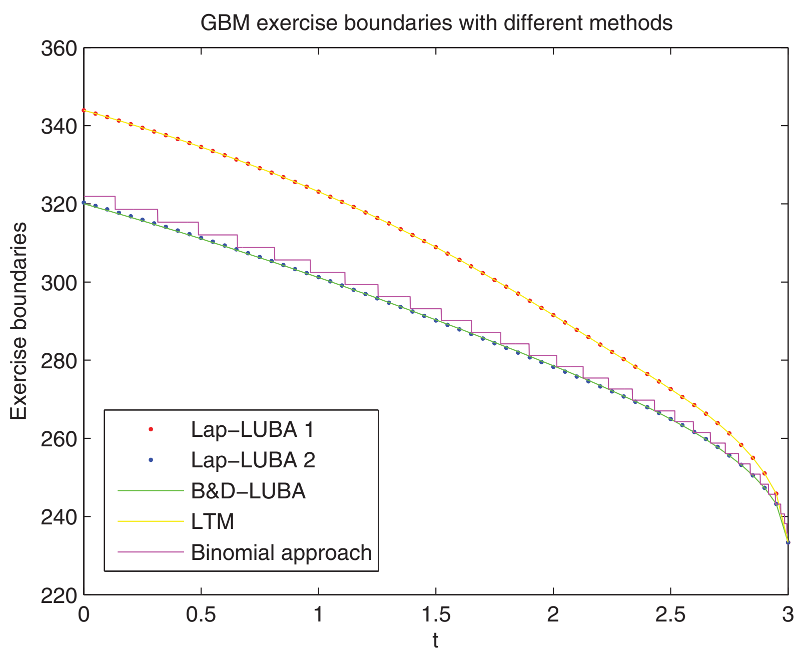

FIGURE **3.** Early exercise boundaries generated by Lap-LUBA 1, Lap-LUBA 2, B&D-LUBA, LTM, and the binomial approach for American call options with GBM. The parameters are $S_0 = 100$, $K = 100$, $r = 0.07$, $\delta = 0.03$, $\sigma = 0.3$, and $T = 3$.

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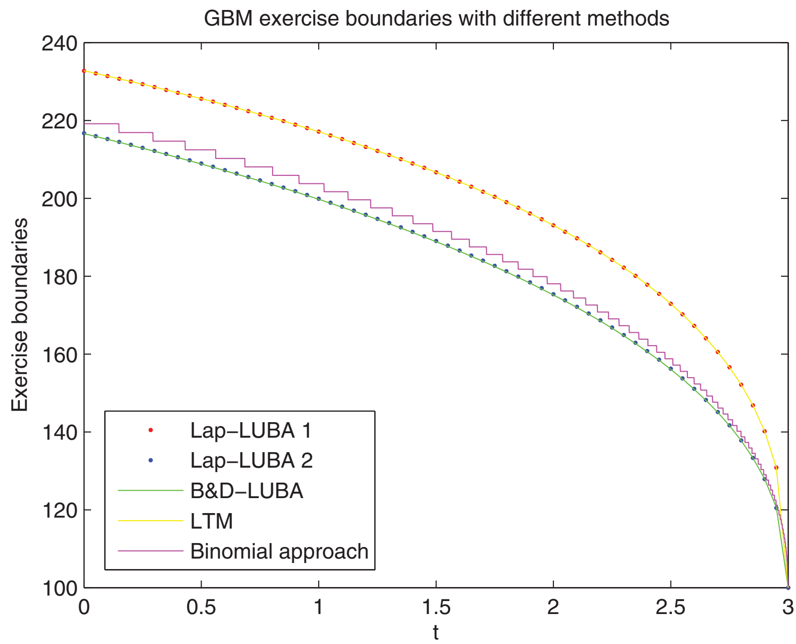

FIGURE 4. Early exercise boundaries generated by Lap-LUBA 1, Lap-LUBA 2, B&D-LUBA, LTM, and the binomial approach for American call options with GBM. The parameters are $S_0 = 100$, $K = 100$, $r = 0.03$, $\delta = 0.03$, $\sigma = 0.3$, and $T = 3$.

5.2. Numerical Examples for CEV

This section reports the numerical results of the Lap-LUBAs methods (Lap-LUBA 1 and Lap-LUBA 2) for solving the American options with underlying asset price following the CEV model. The computed option values are listed in Table 3, and the early exercise boundaries are depicted in Figure 5. It can be seen from Table 3 that the Lap-LUBAs have competitive accurate performance as Wong and Zhao's LTM. However, in occasional cases, the upper bound generated by the Lap-LUBA 1 is larger than the true value as seen at the last rows for $\beta = -1$ and $\beta = -4$ in Table 3. Again, we predict that the reason is that the EEP representation in the Laplace space [43] is not accurate (see Remark 5.1). Due to the inappropriate use of the Laplace transform for free-boundary problems, the early exercise boundaries computed by the Lap-LUBA 1 and LTM of Wong and Zhao [43] have big difference to the benchmark by the FDMs in Figure 5. Since the EEP representation in the time space, which is used in Lap-LUBA 2, is accurate, the early exercise boundary by the Lap-LUBA 2 is close to the benchmark as shown in Figure 5.

5.3. Numerical Examples for DEJD

In this section, we compute the American option price based on DEJD models using the Lap-LUBA methods, the LTMs in Leippold and Vasiljevic [36], FCMs in Fang and Oosterlee [26], FDMs in Muthuraman [39]. We also use regression to obtain more accurate approximation of the option value from the lower bound. The numerical results are listed in Table 4, and the early exercise boundaries are drawn in Figure 6.

From Table 4, we observe that the option values computed by the FCM and FDM are pretty close. Either one can be regarded as the benchmark true value. The LTM is always

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Table 3. American call option value bounds and approximations (maturity $T = 3$ years) for the CEV model

CEV parameter	Asset price	Lap-LUBA 1		Lap-LUBA 2		LTM value	True value
		LB	UB	LB	UB		
$\beta = -1/3$	80	11.6006	11.6006	11.5999	11.6083	11.6006	11.6006
	90	17.0872	17.0872	17.0853	17.0878	17.0871	17.0874
	100	23.3429	23.3429	23.3391	23.3443	23.3429	23.3439
	110	30.2107	30.2107	30.2044	30.2138	30.2107	30.2131
	120	37.5628	37.5628	37.5533	37.5683	37.5628	37.5676
$\beta = -1/2$	80	11.3501	11.3501	11.3498	11.3501	11.3501	11.3500
	90	16.9675	16.9675	16.9664	16.9724	16.9675	16.9675
	100	23.3621	23.3621	23.3596	23.3629	23.3621	23.3625
	110	30.3592	30.3592	30.3546	30.3595	30.3592	30.3607
	120	37.8229	37.8229	37.8153	37.8271	37.8229	37.8263
$\beta = -2/3$	80	11.1119	11.1119	11.1117	11.1118	11.1119	11.1117
	90	16.8602	16.8602	16.8597	16.8603	16.8602	16.8600
	100	23.3951	23.3951	23.3936	23.3956	23.3951	23.3953
	110	30.5241	30.5241	30.5208	30.5254	30.5241	30.5250
	120	38.1032	38.1032	38.0971	38.1062	38.1032	38.1055
$\beta = -1$	80	10.6687	10.6687	10.6687	10.6689	10.6687	10.6687
	90	16.6821	16.6821	16.6820	16.6823	16.6821	16.6820
	100	23.5067	23.5067	23.5062	23.5069	23.5067	23.5067
	110	30.9120	30.9120	30.9105	30.9126	30.9120	30.9123
	120	38.7385	38.7385	38.7351	38.7400	38.7385	38.7396
RMSD		4.4451×10^{-6}		6.1832×10^{-3}			
RMSE		1.6025×10^{-3}	1.6024×10^{-3}	5.4431×10^{-3}	2.1003×10^{-3}	1.6026×10^{-3}	
RMSRE(%)		4.5529×10^{-3}	4.5526×10^{-3}	1.6217×10^{-2}	1.6301×10^{-2}	4.5541×10^{-3}	

The parameters for all the options are taken as $r = 0.07$, $\delta = 0.03$, $K = 100$, $\sigma_0 = 0.3$, and $\sigma_0 = \sigma S_0^{\beta}$. The column “UB” of “Lap-LUBA 1” is computed by the EEP in the laplace space (see [43]). The column “UB” of “Lap-LUBA 2” is computed by the EEP in the time space (see [25]). The “LTM” column is the LTM in Wong and Zhao [43]. The “true value” column is the finite difference method with 15,000 time steps and 4,000 space steps. The Gaver-Stehfest method [35] is used for the Laplace inversion with $N = 12$. The “RMSD” row is the root mean square difference between the “LB” and the “UB” by each method. The “RMSE” row is the root mean square error w.r.t. “True value” by each method. The “RMSRE(%)” row is the root mean square relative error w.r.t. “True value” by each method. In “Lap-LUBA 1,” the computation of the “LB” takes on average 36.893 s and “UB” on average 41.981 s. In “Lap-LUBA 2,” the computation of the “LB” takes on average 78.912 s and “UB” on average 463.136 s. The “LTM” takes on average 20.177 s.

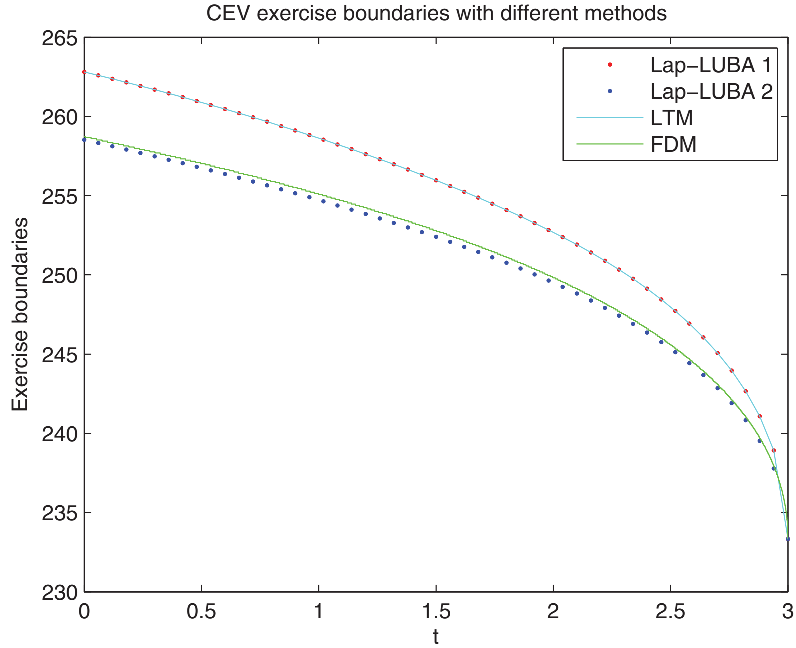

FIGURE 5. Early exercise boundaries generated by the Lap-LUBA 1, Lap-LUBA 2, LTM of Wong and Zhao [43], and FDMs for American call options under the CEV model. The parameters are $S_0 = 100$, $K = 100$, $r = 0.07$, $\delta = 0.03$, $\sigma_0 = 0.3$, $T = 3$, and $\beta = -1$.

lower than the benchmark, which also occurs in the numerical examples of Leippold and Vasiljevic [36]. The Lap-LUBA 2 can generate correct upper bounds. However, the upper bound by the Lap-LUBA 1 is sometimes smaller than the true values by the FCM or FDM. The reasons lie in that the EEP representation in the Laplace space [36] is not accurate (see Remark 5.1). The early exercise boundary generated by Lap-LUBA 2 is closer to the benchmark by the FDM or FCM compared to the other approaches as shown in Figure 6.

Following Broadie and Detemple [9], we use regression to convert the lower bound $P^l(S)$ to the option value approximation $P^1(S)$. The second last column in Table 4 shows that the regression value from the lower bound by the Lap-LUBA 2 is closer to the benchmark values than the lower-bound approximation by the Lap-LUBA 2. The detailed regression approach is presented as follows. The relationship between the lower bound by Lap-LUBA 2 and the regression approximation is

$$P^1(S) = \widehat{\lambda}_1 P^l(S),$$

where $\widehat{\lambda}_1 \geq 1$ is a function of the option parameters $S,\ K,\ T,\ r,\ \delta,\ \sigma,\ \lambda_2,\ \eta_1,\ \eta_2$, and p.

In order to determine $\widehat{\lambda}_1 = \widehat{\lambda}_1(S, K, T, r, \delta, \sigma, \lambda_2, \eta_1, \eta_2, p)$, we first introduce some intermediate variables. Let $a \vee b \equiv \max(a, b)$ and $a \wedge b \equiv \min(a, b)$. Define $x_1 = T$, $x_2 = \sqrt{T}$, $x_3 = S/K$, $x_4 = r$, $x_5 = \delta$, $x_6 = \min(r/(\delta \vee 10^{-5}), 5)$, $x_7 = x_6^2$, $x_8 = \sigma$, $x_9 = (P^l(S) - p(S))/K$, $x_{10} = x_9^2$, $x_{11} = P^l(S)/p(S)$, $x_{12} = x_{11}^2$, $x_{13} = x_{11}^3$, $x_{14} = \lambda_2$, $x_{15} = \eta_1$, $x_{16} = \eta_2$, and $x_{17} = p$. Recall that $p(S)$ denotes the European option value.

Assume that y_1 is the linear combination of these intermediate variables $x_1, \ldots, x_{17}$. Then, the coefficients in the formula for y_1 are determined from a regression of 100 options. In the regression, we choose a distribution of parameters that is reasonable reflection of options that are of interest to academics and practitioners. Volatility σ is distributed uniformly between 0.1 and 0.6. Time to maturity is uniform between 0.1 and 1.0 years with

Table 4. American put option value bounds and approximations (maturity $T = 1$ years) for the DEJD model

Poisson intensity	Asset pric,e	Lap-LUBA 1		Lap-LUBA 2		LTM value	FCM value	Regress value	FDM value
		LB	UB	LB	UB				
$\lambda_2 = 0.5$	80	21.1211	21.1242	21.3487	22.0660	21.1243	21.4742	21.4358	21.4746
	90	14.7051	14.7076	14.7716	15.2532	14.7076	14.9276	14.8428	14.9274
	100	9.9748	9.9765	9.9700	10.2837	9.9765	10.1080	10.0536	10.1075
	110	6.6170	6.6182	6.5869	6.7847	6.6182	6.6918	6.6704	6.6912
	120	4.3126	4.3133	4.2753	4.3969	4.3134	4.3486	4.3467	4.3481
$\lambda_2 = 1$	80	21.1612	21.1673	21.3882	22.0979	21.1676	21.5160	21.4689	21.5163
	90	14.7643	14.7693	14.8329	15.3113	14.7695	14.9887	14.8991	14.9885
	100	10.0409	10.0444	10.0384	10.3513	10.0446	10.1758	10.1179	10.1753
	110	6.6810	6.6833	6.6524	6.8507	6.6834	6.7570	6.7331	6.7564
	120	4.3689	4.3705	4.3326	4.4551	4.3705	4.4058	4.4023	4.4053
$\lambda_2 = 1.5$	80	21.2012	21.2102	21.4277	22.1270	21.2111	21.5578	21.5019	21.5581
	90	14.8232	14.8305	14.8939	15.3669	14.8311	15.0495	14.9548	15.0493
	100	10.1067	10.1118	10.1064	10.4169	10.1123	10.2433	10.1817	10.2427
	110	6.7446	6.7481	6.7177	6.9151	6.7484	6.8220	6.7954	6.8213
	120	4.4250	4.4274	4.3898	4.5118	4.4276	4.4630	4.4576	4.4625
$\lambda_2 = 2$	80	21.2411	21.2531	21.4672	22.1526	21.2545	21.5997	21.5348	21.6000
	90	14.8817	14.8914	14.9546	15.4185	14.8924	15.1101	15.0102	15.1098
	100	10.1720	10.1789	10.1740	10.4775	10.1796	10.3104	10.2450	10.3097
	110	6.8079	6.8126	6.7827	6.9733	6.8131	6.8867	6.8573	6.8860
	120	4.4810	4.4842	4.4469	4.5611	4.4845	4.5200	4.5127	4.5195

The parameters for the options are taken as $r = 0.07,\ \delta = 0.03,\ \sigma = 0.3,\ K = 100,\ p = 0.7,\ q = 0.3,\ \eta_1 = \eta_2 = 30$. The "UB" of "Lap-LUBA 1" is computed by the EEP in the Laplace space (see [36]). The "UB" of "Lap-LUBA 2" is computed by the EEP in the time space (see [5]). The "LTM" is based on Leippold and Vasiljevic [36]. The "FCM" is the value of Bermudan option by Fang and Oosterlee [26], which is an approximation to the exact value of American options with a large number of time steps. The Gaver-Stehfest method [35] is used for Laplace inversion with $N = 8$ for "Lap-LUBA 2," $N = 6$ for both "Lap-LUBA 1" and "LTM." The "FCM" is used with $M = 1,000$ time steps and $N = 1,024$.. The "FDM" column is the finite difference method with 3,000 time steps and $3,000$ space steps. In "Lap-LUBA 1," the computation of the "LB" takes on average 0.016 s and the "UB" on average 0.016 s. In "Lap-LUBA 2," the computation of "LB" takes on average 0.029 s and the "UB" on average 2303.686 s. The "LTM" takes on average 0.011 s. The "FCM" takes on average 53.976 s. The "FDM" takes on average 283.827 s.

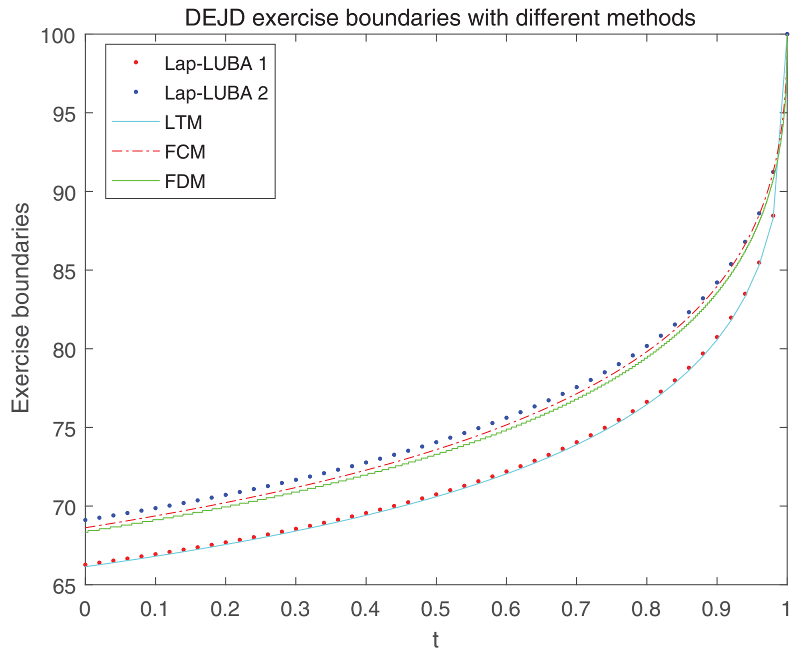

FIGURE 6. Early exercise boundaries generated by the Lap-LUBA 1, Lap-LUBA 2, LTM [36], FCM [26], and the FDM for American put options under the DEJD model. The parameters are $S_0 = 100$, $K = 100$, $r = 0.07$, $\delta = 0.03$, $\sigma_0 = 0.3$, $T = 1$, $\eta_1 = \eta_2 = 30$, $p = 0.7$, $q = 0.3$, and $\lambda_2 = 1$.

probability 0.75, and uniform between 1.0 and 5.0 years with probability 0.25. We fix the strike price at $K = 100$ and take the initial asset price $S \equiv S_0$ to be uniform between 70 and 130. Relative errors do not change if S and K are scaled by the same factor, that is, only the ratio S/K is of interest. The interest rate r and the dividend rate δ are both uniform between 0.0 and 0.10. As for the parameters of Poisson process, λ_2 is distributed uniformly between 0.5 and 5, η_1 and η_2 are independent and both uniform between 10 and 50, and p is uniform between 0.05 and 0.95. Each parameter is selected independently of the others. The regression results are

$$\begin{aligned} y_1 = {} & 4.234 \times 10^0 - 1.403 \times 10^{-2} x_1 + 4.518 \times 10^{-2} x_2 - 7.826 \times 10^{-3} x_3 \\ & + 2.779 \times 10^{-1} x_4 - 5.466 \times 10^{-2} x_5 + 1.098 \times 10^{-2} x_6 \\ & - 1.626 \times 10^{-3} x_7 - 2.963 \times 10^{-2} x_8 + 2.454 \times 10^0 x_9 \\ & - 12.643 \times 10^0 x_{10} - 6.007 \times 10^0 x_{11} + 3.282 \times 10^0 x_{12} \\ & - 5.235 \times 10^{-1} x_{13} - 1.414 \times 10^{-3} x_{14} - 1.979 \times 10^{-4} x_{15} \\ & - 1.140 \times 10^{-4} x_{16} - 1.907 \times 10^{-3} x_{17}. \end{aligned}$$

Finally, define $\widehat{\lambda}_1$ by

$$\widehat{\lambda}_1 = \begin{cases} 1, & \text{if } P^l(S) = p(S) \text{ or } P^l(S) \leq K - S, \\ \max(y_1, 1), & \text{otherwise.} \end{cases} \tag{55}$$

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6. CONCLUSIONS AND DISCUSSIONS

In this paper, we propose a theoretical framework to develop tight lower and upper bounds for the prices of finite-maturity American options when the Laplace transforms of the corresponding "capped (barrier) options" written on the underlying stochastic process are available. We derive explicit expressions when the underlying asset follows a general time-homogeneous diffusion, or a jump diffusion with double-exponential jumps. The study extends the scope of models in the previous literature and proposes an efficient and accurate method for the pricing of (finite-maturity) American options in a rich class of stochastic processes. We foresee this Laplace space framework to have practical applications in the valuation of American options on stocks, commodities, interest rates and exchange rates, where stochastic processes alternative to Black–Scholes framework are desired.

Based on our Laplace space framework, future research direction is to incorporate other improvements in approximating the early exercise boundary in the literature (e.g. using an exponential function ([19]) or a multi-piece exponential function ([31])) to arrive at more accurate lower and upper bounds. The original approach of Broadie and Detemple [9] is based on using a constant L to approximate the early exercise boundary, and a recent paper of Chung *et al.* [19] considers using an exponential function to approximate the early exercise boundary and obtains tighter lower and upper bounds than those in Broadie and Detemple [9].

We illustrate the main ideas and outline the main steps. Specifically, we approximate the early exercise boundary of an American call using an exponential function $B_t := Le^{a(T-t)}$, where we have introduced a new parameter $r \geq a \geq 0$ which controls the curvature of the exponential function. When $a = 0$, it reduces to the constant boundary that we have considered in previous sections.

Define $\tau_{L,a} := \inf\{u \geq t : S_u \geq Le^{au}\} = \inf\{u \geq t : \tilde{S}_u \geq L\}$, where $\tilde{S}_u = e^{-au}S_u$. Thus, we can see that we have translated the problem of valuing the 'capped option' with an exponential boundary to an equivalent problem of valuing the 'capped option' with a constant boundary. The only difference is that we have to adjust the drift parameter of the corresponding stochastic process. Specifically, in the time-homogeneous diffusion setting, the adjusted $\tilde{S}_t$ satisfies

$$\frac{dS_t}{S_t} = (\tilde{r} - \delta)\, dt + \sigma(S_t)\, dW_t, \quad t \geq 0,\ S_0 > 0, \tag{56}$$

where $\tilde{r} = r - a$.

In the double-exponential jump diffusion setting, the corresponding $\tilde{S}_t$ satisfies

$$\frac{d\tilde{S}_t}{\tilde{S}_{t-}} = (\tilde{r} - \delta - \lambda_2\zeta)\, dt + \sigma\, dW_t + d\left(\sum_{i=1}^{N(t)} (V_i - 1)\right), \tag{57}$$

where $\tilde{r} = r - a$.

Then, we introduce

$$\begin{aligned} D_L(L, \lambda, a) &:= \lim_{S_t \uparrow L} \frac{\partial C^*(S_t, \lambda, L, a)}{\partial L} \\ D_a(L, \lambda, a) &:= \lim_{S_t \uparrow L} \frac{\partial C^*(S_t, \lambda, L, a)}{\partial a}. \end{aligned} \tag{58}$$

Similar as Eq. (7) on p. 81 of Chung *et al.* [19], we define

$$H(L, a, t) = (D_L(L, \lambda, a))^2 + (D_a(L, \lambda, a))^2, \tag{59}$$

and denote L^* and a^* as the solution to the equation $H(L, a, t) = 0$. The details involve lengthy calculations and for the brevity of this paper, and they are left to future research.

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APPENDIX A. GEOMETRIC BROWNIAN MOTION

Our benchmark model is the Black–Scholes model, where the stock price is modeled by a GBM, and we shall compare our result with that of Broadie and Detemple [9], where they have obtained a closed-form expression of the capped call option. We are interested in comparing the accuracy of the Lap-LUBA 1 and Lap-LUBA 2 with the B&D LUBA in Broadie and Detemple [9] for computing the finite-maturity American call option prices.

Under the risk-neutral measure, consider the GBM model

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \quad S_0 > 0, \tag{A.1}$$

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with $\mu = r - \delta$ in the risk-neutral sense. The model is a special case of the general diffusion models (1). Therefore, the formulas in Section 3.1 can be applied to the GBM model with a simpler form. The fundamental solutions to the GBM ODE are given in Eq. (15) of Davydov and Linetsky [22]

$$\psi_{\lambda+r}(y) = y^{\gamma_+}, \quad \phi_{\lambda+r}(y) = y^{\gamma_-}, \quad \gamma_\pm = -\gamma \pm \sqrt{\gamma^2 + \frac{2(\lambda+r)}{\sigma^2}}, \quad \gamma = \frac{\mu}{\sigma^2} - \frac{1}{2}, \tag{A.2}$$

and the Wronskian is

$$\omega_{\lambda+r} = 2\sqrt{\gamma^2 + \frac{2(\lambda+r)}{\sigma^2}}. \tag{A.3}$$

The auxiliary functions can be computed as follows:

$$\begin{aligned} I_{\lambda+r}(K,A,B) &= \int_A^B (y-K) y^{\gamma_+} \frac{2}{\sigma^2 y^2 y^{-2\mu/\sigma^2}}\, dy \\ &= \frac{2}{\sigma^2(1-\gamma_-)}(B^{1-\gamma_-} - A^{1-\gamma_-}) + \frac{2K}{\sigma^2\gamma_-}(B^{-\gamma_-} - A^{-\gamma_-}), \end{aligned} \tag{A.4}$$

and

$$\begin{aligned} J_{\lambda+r}(K,A,B) &= \int_A^B (y-K) y^{\gamma_-} \frac{2}{\sigma^2 y^2 y^{-2\mu/\sigma^2}}\, dy \\ &= \frac{2}{\sigma^2(1-\gamma_+)}(B^{1-\gamma_+} - A^{1-\gamma_+}) + \frac{2K}{\sigma^2\gamma_+}(B^{-\gamma_+} - A^{-\gamma_+}), \end{aligned} \tag{A.5}$$

where

$$\gamma_\pm = -\gamma \pm \sqrt{\gamma^2 + \frac{2(\lambda+r)}{\sigma^2}}, \quad \gamma = \frac{\mu}{\sigma^2} - \frac{1}{2}. \tag{A.6}$$

The capped call option for GBM is derived by replacing the corresponding parts in (11) by (A.2) – (A.6),

$$\begin{aligned} C^*(S_t,\lambda,L) \equiv \mathcal{L}(C(S_t,t,L)) &= \frac{L-K}{\lambda}(S_t/L)^{\gamma_+} \\ &+ \frac{1}{\omega_{\lambda+r}}(S_t/L)^{\gamma_+}\Bigg\{\mathbf{1}_{\{S_t\le K\}}\left[\frac{2}{\sigma^2(1-\gamma_+)}(L - K(L/K)^{\gamma_+}) + \frac{2K}{\sigma^2\gamma_+}(1-(L/K)^{\gamma_+})\right] \\ &+\mathbf{1}_{\{S_t>K\}}\left[\frac{2}{\sigma^2(1-\gamma_+)}(L - S_t(L/S_t)^{\gamma_+}) + \frac{2K}{\sigma^2\gamma_+}(1-(L/S_t)^{\gamma_+})\right] \\ &-\mathbf{1}_{\{S_t\le K\}}\left[\frac{2}{\sigma^2(1-\gamma_-)}(L - K(L/K)^{\gamma_-}) + \frac{2K}{\sigma^2\gamma_-}(1-(L/K)^{\gamma_-})\right] \\ &-\mathbf{1}_{\{S_t>K\}}\left[\frac{2}{\sigma^2(1-\gamma_-)}(L - S_t(L/S_t)^{\gamma_-}) + \frac{2K}{\sigma^2\gamma_-}(1-(L/S_t)^{\gamma_-})\right]\Bigg\} \\ &+\mathbf{1}_{\{S_t>K\}}\frac{1}{\omega_{\lambda+r}}(S_t^{\gamma_-} - L^{\gamma_-}(S_t/L)^{\gamma_+})\left[\frac{2}{\sigma^2(1-\gamma_-)}(S_t^{1-\gamma_-} - K^{1-\gamma_-}) + \frac{2K}{\sigma^2\gamma_-}(S_t^{-\gamma_-} - K^{-\gamma_-})\right]. \end{aligned} \tag{A.7}$$

Through direct calculation, we obtain

$$\begin{aligned} D(L,\lambda) &= \frac{1-\gamma_+}{\lambda} + \frac{\gamma_+ K}{\lambda L} \\ &+ \frac{2}{\sigma^2}\left[\frac{1}{1-\gamma_-} + \frac{K}{\gamma_- L}\left[1 - \frac{1}{1-\gamma_-}\left(\frac{K}{L}\right)^{-\gamma_-}\right]\right], \end{aligned} \tag{A.8}$$

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and

$$\frac{\partial D(L,\lambda)}{\partial L} = -\frac{\gamma_+ K}{\lambda L^2} + \frac{2K}{\sigma^2\gamma_- L^2}\left[\left(\frac{K}{L}\right)^{-\gamma_-} - 1\right]. \tag{A.9}$$

Consequently, we use Newton's method to solve the algebraic Eq. (3) with the expression (A.8) to the get the optimal value L^* and then follow the work flow of Lap-LUBA 1 to get a lower bound on the American call value.

To calculate the upper bound on the American call value using the Lap-LUBA 1 method, we derive the EEP representation in the Laplace space by following [43],

$$C^*(S,\lambda,B(\lambda)) = \begin{cases} C_{11}\psi_{\lambda+r}(S) + C_{12}\phi_{\lambda+r}(S), & \text{when } S \in (0,K),\\ C_{21}\psi_{\lambda+r}(S) + C_{22}\phi_{\lambda+r}(S) + u_{\lambda+r}(S), & \text{when } S \in [K,B(\lambda)), \end{cases} \tag{A.10}$$

where $B(\lambda)$ is the early exercise boundary in the Laplace space,

$$\begin{cases} C_{11} = \dfrac{a_5(a_2b_2 - a_4b_1) + a_6(a_3b_1 - a_1b_2)}{a_5(a_2a_3 - a_1a_4)} + \dfrac{b_3}{a_5},\\ C_{12} = 0,\\ C_{21} = \dfrac{a_6(a_3b_1 - a_1b_2)}{a_5(a_2a_3 - a_1a_4)} + \dfrac{b_3}{a_5},\\ C_{22} = \dfrac{a_1b_2 - a_3b_1}{a_2a_3 - a_1a_4},\\ u_{\lambda+r}(S) = \dfrac{\lambda}{\lambda+\delta}S - \dfrac{\lambda}{\lambda+r}K, \end{cases}$$

and

$$\begin{aligned} &a_1 = \psi_{\lambda+r}|_{S=K}, \quad a_2 = \phi_{\lambda+r}|_{S=K}, \quad a_3 = \left.\frac{d\psi_{\lambda+r}}{dS}\right|_{S=K},\\ &a_4 = \left.\frac{d\phi_{\lambda+r}}{dS}\right|_{S=K}, \quad a_5 = \left.\frac{d\psi_{\lambda+r}}{dS}\right|_{S=B(\lambda)}, \quad a_6 = \left.\frac{d\phi_{\lambda+r}}{dS}\right|_{S=B(\lambda)},\\ &b_1 = u_{\lambda+r}|_{S=K}, \quad b_2 = \left.\frac{du_{\lambda+r}}{dS}\right|_{S=K}, \quad b_3 = 1 - \left.\frac{du_{\lambda+r}}{dS}\right|_{S=B(\lambda)}. \end{aligned}$$

Thus, we replace $B(\lambda)$ in (A.10) by L^* and use the Laplace inversion to obtain the upper bound on the American call value.

Also, we can use the Lap-LUBA 2 to calculate the lower bound on the American call value. By replacing the corresponding parts in (5), (7), and (8) by (A.7), (A.8), and (A.9), we solve Eq. (9) by Newton's method and then follow the work flow of Lap-LUBA 2 to get the lower bound on the price of American call option.

Furthermore, we can calculate the upper bound on the American call value by the Lap-LUBA 2 method by using the following EEP representation in the time space from Broadie and Detemple [10],

$$C(S_t,t;B(t)) = C^e(S_t,t) + \Pi(S_t,t;B(t)), \tag{A.11}$$

with

$$C^e(S_t,t) = S_t\,e^{-\delta(T-t)}N(d_2(S_t,K,T)) - Ke^{-r(T-t)}N(d_3(S_t,K,T)),$$

and

$$\Pi(S_t,t;B(t)) = \int_t^T [\delta S_t\,e^{-\delta(T-t)}N(d_2(S_t,B(\nu),\nu)) - rKe^{-r(T-t)}N(d_3(S_t,B(\nu),\nu))]\,d\nu,$$

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where

$$d_2(S_t, B(\nu), \nu) = \frac{1}{\sigma\sqrt{\nu - t}} \times \left[\log(S_t/B(\nu)) + \left(r - \delta + \frac{1}{2}\sigma^2\right)(\nu - t)\right],$$
$$d_3(S_t, B(\nu), \nu) = d_2(S_t, B(\nu), \nu - t) - \sigma\sqrt{\nu - t}.$$

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