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POINTWISE FINITE FAMILIES OF MAPPINGS

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In [3], Montgomery proved that if h is a pointwise periodic homeomorphism of a connected manifold without boundary onto itself, then h is periodic. Kaul generalized this result in [2] by showing that if X is a connected metrizable manifold without boundary and if (X, T) is a transformation group with T countable such that T is pointwise periodic, then T is periodic. Yang [6] has shown that Kaul's theorem remains true if the assumption of metrizability is dropped. In this note we prove a fairly general theorem about families of continuous mappings. We then apply this theorem to obtain the theorems of Kaul and Yang.

THEOREM 1. *Suppose X and Y are topological spaces such that X is a locally connected Baire space and Y is a Hausdorff space. If F is a family of continuous maps from X to Y such that for every $x \in X$, $F(x)=\{t(x):t \in F\}$ is finite, then there exists U open in X and $t_1, \ldots, t_n \in F$ such that if $t \in F$, then for some i, $1 \leq i \leq n$, $t=t_i$ on U.*

Proof. If E is a finite set, then we shall let $|E|$ denote the number of elements in E. Now for $K=1, 2, \ldots$ let $E_K=\{x \in X:|F(x)| \leq K\}$. Each E_K is closed and $\bigcup_{k=1}^{\infty} E_K=X$. Thus since X is a Baire space, there exists a positive integer m such that $E_m^0 \neq \varnothing$. Now choose $x \in E_m^0$ such that $|F(x)|=\max\{|F(y)|:y \in E_m^0\}=n \leq m$. Then there exists $t_1, \ldots, t_n \in F$ such that $F(x)=\{t_1(x), \ldots, t_n(x)\}$. Let $V_1, \ldots, V_n$ be pairwise disjoint open sets in Y such that $t_i(x) \in V_i$. Since X is locally connected, there exists an open connected set U such that $x \in U$ and $U \subset \bigcap_{i=1}^{n} t_i^{-1}(V_i) \cap E_m^0$. If $y \in U$, then $|F(y)| \leq n$. Since each $t_i(y) \in V_i$, the $t_i(y)$ are all distinct. Thus $|F(y)|=n$ and in fact $F(y)=\{t_1(y), \ldots, t_n(y)\}$ for every $y \in U$. Hence $F(U) \subset \bigcup_{i=1}^{n} V_i$. Now suppose $t \in F$. Then $t(x)=t_i(x)$ for some i, $1 \leq i \leq n$. But $t(U) \subset \bigcup_{j=1}^{n} V_j$. Since U is connected $t(U) \subset V_i$. Hence if $y \in U$ $t(y)=t_i(y)$. Thus $t=t_i$ on U.

Let (X, T) be a transformation group. In what follows we shall use the terminology and notation in [1]. If $x \in X$, then $T=EA$ is a decomposition of T for x if A is compact and $E \subset \{t:xt=x\}$. In this case T is said to be periodic at x. $T=EA$ is a decomposition of T if it is a decomposition for every $x \in X$. T is said to be periodic when such a decomposition exists. T is pointwise periodic if T is periodic at every point of X. Now suppose that T is a countable Hausdorff topological group and $T=EA$ is a decomposition of T for $x \in X$. We may suppose that

$E=\{t:xt=x\}$. Hence E is a closed subgroup. But then $T/E=A/E$ is a countable compact Hausdorff topological group and is therefore finite. Thus there exists A_0 finite such that $T=EA_0$ is a decomposition of T for x. Similarly it can be shown that if $T=EA$ is a decomposition of T with $E=\{t \in T:xt=x \text{ for every } x \in X\}$, then there exists A_0 finite such that $T=EA_0$. Thus the theorems of Kaul and Yang can be equivalently stated as follows:

THEOREM 2 (Kaul, Yang). *If X is a connected manifold without boundary and (X, T) is a transformation group such that for every $x \in X$, xT is finite, then there exists $A \subset T$ such that A is finite and if $E=\{t \in T:xt=x \text{ for all } x \in X\}$ then $A \cdot E=T$.*

Proof. By theorem 1 there exists U open and $t_1, \ldots, t_n \in T$ such that if $t \in T$, then for some i, $1 \leq i \leq n$, $xt=xt_i$ for all $x \in U$. Now the homeomorphism induced by t_it^{-1} is pointwise periodic by assumption and therefore is periodic by the result of Montgomery [3]. But then $t_it^{-1} \in E$ by Smith [5] (or see Montgomery and Zippin [4]). Thus if we let $A=\{t_1, \ldots, t_n\}$ then $AE=T$.

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