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LARGE VALUES OF *L*-FUNCTIONS ON THE 1-LINE

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Abstract

We study lower bounds of a general family of L-functions on the 1-line. More precisely, we show that for any $F(s)$ in this family, there exist arbitrarily large t such that $F(1 + it) \geq e^{\gamma_F}(\log_2 t + \log_3 t)^m + O(1)$, where m is the order of the pole of $F(s)$ at $s = 1$. This is a generalisation of the result of Aistleitner, Munsch and Mahatab ['Extreme values of the Riemann zeta function on the 1-line', *Int. Math. Res. Not. IMRN* **2019**(22) (2019), 6924–6932]. As a consequence, we get lower bounds for large values of Dedekind zeta-functions and Rankin-Selberg L-functions of the type $L(s, f \times f)$ on the 1-line.

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1. Introduction

The growth of the Riemann zeta-function $\zeta(s)$ in the critical strip $1/2 < \Re(s) < 1$ has been of interest to number theorists for a long time. The upper bound predicted by the Lindelöf hypothesis is $|\zeta(\sigma + it)| \ll |t|^\epsilon$ for any $\epsilon > 0$ and $1/2 < \sigma < 1$. This is a consequence of the Riemann hypothesis. Although there is significant progress towards this bound, no unconditional proof is known (see [22] for more details).

A more intricate question is to investigate how large $|\zeta(\sigma + it)|$ can be for a fixed $\sigma \in [1/2, 1)$ and $t \in [T, T + H]$. Balasubramanian and Ramachandra [7] showed that

$$\max_{t\in[T,T+H]} \left|\zeta\left(\frac{1}{2} + it\right)\right| \geq \exp\left(c\sqrt{\frac{\log H}{\log_2 H}}\right),$$

where c is a positive constant, $H \ll \log_2 T$ and $\log_2 T$ denotes $\log\log T$. We denote $\underbrace{\log\log\ldots\log T}_{k\text{ times}}$ by $\log_k T$. This result was improved by Bondarenko and Seip [9] in a larger interval and later optimised by de la Bretèche and Tenenbaum [10], who showed that

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$$\max_{t\in[0,T]}\left|\zeta\left(\frac{1}{2}+it\right)\right| \geq \exp\left((\sqrt{2}+o(1))\sqrt{\frac{\log T\log_3 T}{\log_2 T}}\right).$$

For $\sigma \in (1/2, 1)$ and $c_\sigma = 0.18(2\sigma-1)^{1-\sigma}$, Aistleitner [1] proved that

$$\max_{t\in(0,T]}|\zeta(\sigma+it)| \geq \exp\left(\frac{c_\sigma(\log T)^{1-\sigma}}{(\log_2 T)^\sigma}\right).$$

On the other hand, we expect much finer results for large values of L-functions on $\Re(s) = 1$. In [12], Granville and Soundararajan used techniques of diophantine approximation to show that

$$\max_{t\in[0,T]}|\zeta(1+it)| \geq e^\gamma(\log_2 T + \log_3 T - \log_4 T + O(1))$$

for arbitrarily large T. This is an improvement on the previous bounds given by Levinson [14]. Granville and Soundararajan [12] conjectured that

$$\max_{t\in[T,2T]}|\zeta(1+it)| = e^\gamma(\log_2 T + \log_3 T + C_1 + o(1)), \tag{1.1}$$

where C_1 is an explicitly computable constant. In 2017, Aistleitner, Munsch and the second author [2] used the resonance method to prove that there is a constant C such that

$$\max_{t\in[\sqrt{T},T]}|\zeta(1+it)| \geq e^\gamma(\log_2 T + \log_3 T + C). \tag{1.2}$$

This result essentially matches (1.1), but the size of the interval is much larger. Over shorter intervals $[T, T+H]$, very little seems to be known regarding large values of $\zeta(1+it)$ (see [5], [6] for further details).

In this paper, we generalise (1.2) to a large class $\mathbb{G}$ of L-functions, which conjecturally contains the Selberg class $\mathbb{S}$. We establish (1.2) for elements in $\mathbb{G}$ with nonnegative Dirichlet coefficients. The key difference between $\mathbb{G}$ and $\mathbb{S}$ is that elements in $\mathbb{G}$ satisfy a polynomial Euler-product which is a more restrictive condition than that in $\mathbb{S}$. However, the functional equation in $\mathbb{S}$ is replaced by a weaker 'growth condition' in $\mathbb{G}$. This is a significant generalisation because most Euler products, which have an analytic continuation exhibit a growth condition, but perhaps not a functional equation. As applications, we prove the analogue of (1.2) for Dedekind zeta-functions $\zeta_K(s)$ and Rankin-Selberg L-functions given by $L(s, f\times f)$. We also prove a generalised Mertens theorem for $\mathbb{G}$ as a precursor to the proof of our main theorem.

The resonance method with a similar resonator was used by Aistleitner, Munsch, Peyrot and the second author [3] to establish large values of Dirichlet L-functions $L(s,\chi)$ with a given conductor q at $s = 1$. Perhaps, a similar method can also be used to establish large values over more general orthogonal families of L-functions in $\mathbb{G}$.

1.1. The class $\mathbb{G}$. In 1991, Selberg [20] introduced a class of L-functions $\mathbb{S}$, which is expected to encapsulate all naturally occurring L-functions arising from arithmetic and geometry.

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DEFINITION 1.1 (The Selberg class). The Selberg class $\mathbb{S}$ consists of meromorphic functions $F(s)$ satisfying the following properties.

(i) **Dirichlet series**. The Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}$$

is absolutely convergent in the region $\Re(s) > 1$. We normalise the leading coefficient by $a_F(1) = 1$.

(ii) **Analytic continuation**. There exists a nonnegative integer k, such that the function $(s-1)^k F(s)$ is an entire function of finite order.

(iii) **Functional equation**. There exist real numbers $Q > 0$ and $\alpha_i \geq 0$, complex numbers β_i and $w \in \mathbb{C}$, with $\Re(\beta_i) \geq 0$ and $|w| = 1$, such that

$$\Phi(s) := Q^s \prod_i \Gamma(\alpha_i s + \beta_i) F(s) \tag{1.3}$$

satisfies the functional equation $\Phi(s) = w\overline{\Phi}(1 - \overline{s})$.

(iv) **Euler product**. There is an Euler product of the form

$$F(s) = \prod_{p \text{ prime}} F_p(s), \tag{1.4}$$

where

$$\log F_p(s) = \sum_{k=1}^{\infty} \frac{b_{p^k}}{p^{ks}}$$

with $b_{p^k} = O(p^{k\theta})$ for some $\theta < 1/2$.

(v) **Ramanujan hypothesis**. For any $\epsilon > 0$,

$$|a_F(n)| = O_\epsilon(n^\epsilon). \tag{1.5}$$

The Euler product implies that the coefficients $a_F(n)$ are multiplicative, that is, $a_F(mn) = a_F(m)a_F(n)$ when $(m, n) = 1$. Moreover, each Euler factor also has a Dirichlet series representation

$$F_p(s) = \sum_{k=0}^{\infty} \frac{a_F(p^k)}{p^{ks}},$$

which is absolutely convergent on $\Re(s) > 0$ and nonvanishing on $\Re(s) > \theta$, where θ is as in (iv).

For the purpose of this paper, we need a stronger Euler product to ensure that the Euler factors factorise completely. We also require a zero free region near the 1-line, similar to that in the proof of prime number theorem. However, we can replace the functional equation with a weaker condition on the growth of L-functions on vertical lines. This leads to the definition of the class $\mathbb{G}$.

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DEFINITION 1.2 (The class $\mathbb{G}$). The class $\mathbb{G}$ consists of meromorphic functions $F(s)$ satisfying (i) and (ii) in Definition 1.1 and the following properties.

(a) **Complete Euler product decomposition**. The Euler product in (1.4) factorises completely, that is,

$$F(s) := \prod_p \prod_{j=1}^{k} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1} \tag{1.6}$$

with $|\alpha_j| \le 1$ and $\Re(s) > 1$.

(b) **Zero-free region**. There exists a positive constant c_F, depending on F, such that $F(s)$ has no zeros in the region

$$\Re(s) \ge 1 - \frac{c_F}{\log(|\Im(s)| + 2)},$$

except the possible Siegel zero of $F(s)$, that is, the possible real exceptional zero of $F(s)$ in the neighbourhood of 1 which is the only zero of $F(s)$ in the interval $(1 - \epsilon, 1)$.

(c) **Growth condition**, For $s = \sigma + it$, define

$$\mu_F^*(\sigma) := \inf\{\lambda > 0 : |F(s)| \ll (|t| + 2)^{\lambda}\}.$$

Then,

$$\frac{\mu_F^*(\sigma)}{1 - 2\sigma}$$

is bounded for $\sigma < 0$.

One expects $\mathbb{S}$ to satisfy conditions (a) and (b). In fact, the Riemann zeta-function, the Dirichlet L-functions, the Dedekind zeta-functions and the Rankin-Selberg L-functions can all be shown to satisfy (a) and (b). Furthermore, for elements in $\mathbb{S}$ the growth condition (c) is a consequence of the functional equation (1.3). However, it is possible to have L-functions which satisfy the growth condition but do not obey a functional equation. One can consider linear combination of elements in $\mathbb{S}$ to see this. A family of L-functions based on a growth condition was introduced by V. K. Murty in [17] (see [11] for more details). The Igusa zeta-function, and the zeta function of groups have Euler products but may not have a functional equation (see [19]).

1.2. The main theorem. We prove a lower bound for large values of L-functions in $\mathbb{G}$ on the 1-line. For a meromorphic function $F(s)$ having a pole of order m at $s = 1$, define

$$c_{-m}(F) = \lim_{s \to 1} (s - 1)^m F(s). \tag{1.7}$$

THEOREM 1.3. *Let $F \in \mathbb{G}$ have nonnegative Dirichlet coefficients $a_F(n)$ and a pole of order m at $s = 1$. Then, there exists a constant $C_F > 0$ depending on $F(s)$ such that*

$$\max_{t \in [\sqrt{T}, T]} |F(1 + it)| \ge e^{\gamma_F}((\log_2 T + \log_3 T)^m - C_F),$$

where $\gamma_F = m\gamma + \log c_{-m}(F)$ and γ is the Euler-Mascheroni constant.

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Here, since $a_F(n) \geq 0$, we clearly have $m \geq 1$. This is important because if F has no pole at $s = 1$, it is possible for $F(s)$ to grow very slowly on the 1-line.

As an immediate corollary, we get the following result for Dedekind zeta-functions $\zeta_K(s)$. Let $K/\mathbb{Q}$ be a number field. The Dedekind zeta-function $\zeta_K(s)$ is defined on $\Re(s) > 1$ by

$$\zeta_K(s) := \sum_{\mathbf{0}\neq\mathfrak{a}\subseteq O_K} \frac{1}{(\mathbb{N}\mathfrak{a})^s} = \prod_{\mathfrak{p}} \Big(1 - \frac{1}{(\mathbb{N}\mathfrak{p})^s}\Big)^{-1},$$

where $\mathfrak{a}$ runs over all nonzero integral ideals and $\mathfrak{p}$ runs over all nonzero prime ideals of K. The function $\zeta_K(s)$ has an analytic continuation to the complex plane except for a simple pole at $s = 1$. Furthermore, ζ_K satisfies properties (a), (b) and (c) and therefore $\zeta_K \in \mathbb{G}$.

Corollary 1.4. *For a number field K, there exists a constant $C_K > 0$ depending on K such that*

$$\max_{t\in[\sqrt{T},T]} |\zeta_K(1+it)| \geq e^{\gamma_K}(\log_2 T + \log_3 T - C_K),$$

where $\gamma_K = \gamma + \log \rho_K$, with ρ_K being the residue of $\zeta_K(s)$ at $s = 1$.

The L-function associated to the Rankin-Selberg convolution of any two holomorphic newforms f and g, denoted by $L(s, f \times g)$, is in the Selberg class and it can also be shown that $L(s, f \times g) \in \mathbb{G}$. Here f and g are normalised Hecke eigenforms of weight k. It is known that if $L(s, f \times g)$ has a pole at $s = 1$, then $f = g$.

Corollary 1.5. *For a normalised Hecke eigenform f, there exists a constant $C_f > 0$ such that*

$$\max_{t\in[\sqrt{T},T]} |L(1+it, f\times f)| \geq e^{\gamma_f}(\log_2 T + \log_3 T - C_f),$$

where $\gamma_f = \gamma + \log \rho_f$, with ρ_f being the residue of $L(s, f \times f)$ at $s = 1$.

Theorem 1.3 is a refined version of the bound established by Aistleitner–Pańkowski [4], which states that if $F(s)$ is in the Selberg class and satisfies the prime number theorem, that is,

$$\sum_{p\leq x} |a_F(p)| = \kappa \frac{x}{\log x} + O\Big(\frac{x}{\log^2 x}\Big),$$

then for large T,

$$\max_{t\in[T,2T]} |F(1+it)| = \Omega((\log\log T)^\kappa). \tag{1.8}$$

Since we are assuming the zero-free region in $\mathbb{G}$, using [13, Theorem 1], we have $\kappa = m$. Hence, we get a slightly more refined result than (1.8), but on a larger interval $[\sqrt{T}, T]$.

The poles of any element F in the Selberg class $\mathbb{S}$ are expected to arise from the Riemann zeta-function. More precisely, if $F(s)$ has a pole of order m at $s = 1$, then $F(s)/\zeta(s)^m$ is expected to be entire and in $\mathbb{S}$. Thus, it is not surprising to expect the lower bound in Theorem 1.3 to be of the order $(\log\log T)^m$.

It is possible to generalise Theorem 1.3 to the Beurling zeta-function [8] by constructing a suitable resonator over Beurling numbers instead of integers. We plan to return to this in future research.

2. Mertens' theorem for the class $\mathbb{G}$

In 1874, Mertens [15] proved the following estimate for the truncated Euler-product of $\zeta(s)$, also known as Mertens' third theorem:

$$\prod_{p\le x}\Big(1-\frac{1}{p}\Big)^{-1}=e^{\gamma}\log x+O(1).$$

The analogue of Mertens' theorem for number fields was proved by Rosen [18]:

$$\prod_{\mathbb{N}\mathfrak{P}<x}\Big(1-\frac{1}{\mathbb{N}\mathfrak{P}}\Big)^{-1}=\rho_K e^{\gamma}\log x+O(1),$$

where ρ_K denotes the residue of $\zeta_K(s)$ at $s=1$. Mertens' theorem for the extended Selberg class satisfying conditions (a) and (b) was proved by Yashiro [23] in 2013. Following a similar approach, one can establish Mertens' theorem for $\mathbb{G}$.

THEOREM 2.1. *Let $F(s)\in\mathbb{G}$. Suppose that $F(s)$ has a pole of order m at $s=1$ and let $c_{-m}(F)$ be as in (1.7). Then, for a constant C_F with $0<C_F\le 1$,*

$$\prod_{p\le x}\prod_{j=1}^{k}\Big(1-\frac{\alpha_j(p)}{p}\Big)^{-1}=c_{-m}(F)e^{\gamma m}(\log x)^m(1+O(e^{-C_F\sqrt{\log x}})).$$

PROOF. We follow closely the method of Yashiro [23]. Let

$$F(1;x):=\prod_{p\le x}\prod_{j=1}^{k}\Big(1-\frac{\alpha_j(p)}{p}\Big)^{-1}\quad\text{and}\quad \log F(s)=\sum_{n=1}^{\infty}\frac{b_F(n)}{n^s}.$$

By the complete Euler product (1.6), $b_F(n)=0$ if $n\ne p^r$ and $b_F(n)\ll n^{\theta}$ for some $\theta<1/2$. Since

$$b_F(p^r)=\frac{1}{r}\sum_{j=1}^{k}\alpha_j(p)^r,$$

we have $|b_F(p^r)|\le k$. Write

$$\log F(1;x)=\sum_{p\le x}\sum_{r=1}^{\infty}\frac{b_F(p^r)}{p^r}=\sum_{n\le x}\frac{b_F(n)}{n}+\sum_{\sqrt{x}<p\le x}\sum_{p^r>x}\frac{b_F(p^r)}{p^r}+\sum_{p\le\sqrt{x}}\sum_{p^r>x}\frac{b_F(p^r)}{p^r}. \tag{2.1}$$

It is easy to estimate the second and third terms above as follows:

$$\sum_{\sqrt{x}<p\le x}\sum_{p^r>x}\frac{b_F(p^r)}{p^r}\ll\sum_{\sqrt{x}<p\le x}\sum_{r=2}^{\infty}\frac{1}{p^r}\ll\sum_{\sqrt{x}<p\le x}\frac{1}{p^2}\ll\frac{1}{\sqrt{x}}$$

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and

$$\sum_{p\le\sqrt{x}}\sum_{p^r>x}\frac{b_F(p^r)}{p^r}\ll\sum_{p\le\sqrt{x}}\frac{1}{x}\ll\frac{1}{\sqrt{x}}.$$

From (2.1),

$$\log F(1;x)=\sum_{n\le x}\frac{b_F(n)}{n}+O\Big(\frac{1}{\sqrt{x}}\Big).$$

Setting $u=1/\log x$ and $T=e^{\sqrt{\log x}}$ and using Perron's formula,

$$\sum_{n\le x}\frac{b_F(n)}{n}=\frac{1}{2\pi i}\int_{u-iT}^{u+iT}\frac{x^s}{s}\log F(1+s)\,ds+O(e^{-c_F\sqrt{\log x}}).$$

Let $u'=C_F/\log T=C_F/\sqrt{\log x}$. Choosing x sufficiently large, we can ensure that there are no Siegel zeros for $F(1+s)$ in the region $[-u',u]$. Hence from condition (b), $F(1+s)$ has no zeros in the region $-u'\le\Re(s)\le u$ and $|\Im(s)|\le T$ and has a pole of order m at $s=0$.

Consider the contour C joining $u-iT,-u'-iT,-u'+iT$ and $u+iT$. By the residue theorem,

$$\text{Res}_{s=0}\Big(\frac{x^s}{s}\log F(1+s)\Big)=\frac{1}{2\pi i}\int_C\frac{x^s}{s}\log F(1+s)\,ds. \tag{2.2}$$

We now estimate this integral. Suppose $s=\sigma+it$. By the growth condition (c),

$$|F(s)|\ll|t|^{\mu_F(\sigma)},$$

where $\mu(\sigma)\ll(1-2\sigma)$. Thus, for our choice of u and u' and for $\sigma\in[-u',u]$,

$$\log F(1+\sigma+iT)\ll(\log T)^2.$$

Hence,

$$\begin{aligned}\Big|\frac{1}{2\pi i}\int_{u+iT}^{-u'+iT}\frac{x^s}{s}\log F(1+s)\,ds\Big|&\ll\Big|\frac{(\log T)^2}{T}\int_u^{-u'}x^\sigma\,d\sigma\Big|\\&\ll(\log x)e^{-\sqrt{\log x}}\ll e^{-c'_F\sqrt{\log x}},\end{aligned}\tag{2.3}$$

for some c'_F with $0<c'_F<1$. Similarly,

$$\Big|\frac{1}{2\pi i}\int_{-u'+iT}^{u+iT}\frac{x^s}{s}\log F(1+s)\,ds\Big|\ll e^{-c'_F\sqrt{\log x}}. \tag{2.4}$$

We use the following result due to Landau (see [16, page 170, Lemma 6.3]) to esimate the other terms in (2.2).

Lemma 2.2. *Let $f(z)$ be an analytic function in the region containing the disc $|z|\le1$ and suppose $|f(z)|\le M$ for $|z|\le1$ and $f(0)\ne0$. Fix r and R such that $0<r<R<1$.*

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Then, for $|z| \le r$,

$$\frac{f'}{f}(z) = \sum_{|\rho|\le R} \frac{1}{z-\rho} + O\Big(\log \frac{M}{|f(0)|}\Big),$$

where ρ *is a zero of* $f(s)$.

Let $f(z) = (z + 1/2 + it)^m F(1 + z + (1/2 + it))$, $R = 5/6$ and $r = 2/3$ in Lemma 2.2. Using the zero-free region (b),

$$|\log s^m F(1+s)| \ll \begin{cases} \log(|t|+4), & |t| \ge 7/8 \text{ and } \sigma \ge -u', \\ 1 & |t| \le 7/8 \text{ and } \sigma \ge -u'. \end{cases}$$

We now have the estimate

$$\begin{aligned} \Big| \int_{-u'}^{-u'+iT} \frac{x^s}{s} \log F(1+s)\, ds \Big| &\ll \int_0^T \frac{x^{-u'}}{|s|} (|\log s^m| + |\log s^m F(1+s)|)\, dt \\ &\ll e^{-c_F'' \sqrt{\log x}}, \end{aligned} \tag{2.5}$$

for some c_F'' *with* $0 < c_F'' < 1$. Similarly,

$$\Big| \int_{-u'-iT}^{-u'} \frac{x^s}{s} \log F(1+s)\, ds \Big| \ll e^{-c_F'' \sqrt{\log x}}. \tag{2.6}$$

Using(2.3), (2.4), (2.5) and (2.6) in (2.2) and choosing $C_F = \min(c_F, c_F', c_F'')$,

$$\frac{1}{2\pi i} \int_{u-iT}^{u+iT} \frac{x^s}{s} \log F(1+s)\, ds = \mathrm{Res}_{s=0}\Big(\frac{x^s}{s} \log F(1+s)\Big) + O(e^{-C_F \sqrt{\log x}})$$

Let C denote the circle of radius u' centred at 0. Then,

$$\frac{1}{2\pi i} \int_C \frac{x^s}{s} \log F(1+s)\, ds = \mathrm{Res}_{s=0}\Big(\frac{x^s}{s} \log F(1+s)\Big).$$

Hence, it suffices to estimate this integral. Since $F(s)$ has a pole of order m at $s = 1$,

$$c_{-m}(F) = \lim_{s\to 1} (s-1)^m F(s) \ne 0.$$

Writing $F(s+1) = (s^{-m})(s^m F(s+1))$,

$$\frac{1}{2\pi i} \int_C \frac{x^s}{s} \log F(1+s)\, ds = -\frac{m}{2\pi i} \int_C \frac{x^s}{s} \log s\, ds + \log c_{-m}(F). \tag{2.7}$$

The integral on the right hand side is

$$\begin{aligned} \int_C \frac{x^s}{s} \log s\, ds &= \int_{-\pi}^{\pi} \frac{x^{u'e^{i\theta}}}{u'e^{i\theta}} (\log u'e^{i\theta})(iu'e^{i\theta})\, d\theta \\ &= i(\log u') \int_{-\pi}^{\pi} e^{u'e^{i\theta} \log x} d\theta - \int_{-\pi}^{\pi} \theta e^{u'e^{i\theta} \log x}\, d\theta. \end{aligned} \tag{2.8}$$

Using series expansion of the exponential function and interchanging the order of summation and integration, since the sum is absolutely convergent,

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$$\int_{-\pi}^{\pi} e^{u' e^{i\theta} \log x}\, d\theta = \int_{-\pi}^{\pi} d\theta + \sum_{r=1}^{\infty} \frac{(u' \log x)^r}{r!} \int_{-\pi}^{\pi} e^{ir\theta} d\theta = 2\pi.$$

Similarly,

$$\begin{aligned}\int_{-\pi}^{\pi} \theta e^{u' e^{i\theta} \log x}\, d\theta &= \int_{-\pi}^{\pi} \theta\, d\theta + \sum_{r=1}^{\infty} \frac{(u' \log x)^r}{r!} \int_{-\pi}^{\pi} \theta e^{ir\theta}\, d\theta \\ &= \sum_{r=1}^{\infty} \Big(\frac{(u' \log x)^r}{r!}\Big)\Big(\frac{(-1)^r 2\pi}{ir}\Big) \\ &= \frac{2\pi}{i} \sum_{r=1}^{\infty} \frac{(-1)^r}{r!} \int_0^{u' \log x} w^{r-1}\, dw \\ &= \frac{2\pi}{i} \int_0^{u' \log x} \frac{e^{-w} - 1}{w}\, dw.\end{aligned}$$

But the Euler-Mascheroni constant γ satisfies the identity

$$\gamma = \int_0^1 \frac{1 - e^{-w}}{w} dw - \int_1^{\infty} \frac{e^{-w}}{w} dw.$$

Thus,

$$\begin{aligned}\int_0^{u' \log x} \frac{e^{-w} - 1}{w}\, dw &= \gamma + \int_1^{u' \log x} \frac{dw}{w} - \int_{u' \log x}^{\infty} \frac{e^{-w}}{w}\, dw \\ &= \gamma + \log\log x + \log u' + O(e^{-C_F \sqrt{\log x}}). \qquad (2.9)\end{aligned}$$

Combining the estimates (2.8) and (2.9),

$$\log F(1; x) = \log c_{-m}(F) + m\gamma + m \log\log x + O(e^{-C_F \sqrt{\log x}}).$$

The result follows by exponentiating both sides since $e^y = 1 + O(y)$ for $|y| < 1$. □

3. Proof of the main theorem

For $F \in \mathbb{G}$, define

$$F(s; Y) := \prod_{p \le Y} \prod_{j=1}^{k} \Big(1 - \frac{\alpha_j(p)}{p^s}\Big)^{-1}.$$

We use the following approximation lemma.

LEMMA 3.1. *For large T,*

$$F(1 + it) = F(1 + it; Y)\Big(1 + O\Big(\frac{1}{(\log T)^{10}}\Big)\Big),$$

for $Y = \exp((\log T)^{10})$ and $T^{1/10} \le |t| \le T$.

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PROOF. From the Euler product of $F(s)$, for $\Re(s) > 1$,

$$\log F(s) = -\sum_{p}\sum_{j=1}^{k}\log\Big(1-\frac{\alpha_j(p)}{p^s}\Big) = \sum_{p}\sum_{j=1}^{k}\sum_{l}\frac{\alpha_j(p)^l}{lp^{ls}}.$$

Let $t_0 > 0$ and let $\alpha > 0$ be any sufficiently large constant. Define

$$\sigma_0 := \frac{1}{\alpha\log T}, \quad \sigma_1 := \frac{1}{(\log T)^{20}} \quad \text{and} \quad T_0 := \frac{T^{1/10}}{2}.$$

Applying Perron's summation formula as in [21, Theorem II.2.2],

$$\int_{\sigma_1-iT_0}^{\sigma_1+iT_0}\log F(1+it_0+s)\frac{Y^s}{s}ds = -\sum_{p\le Y}\sum_{j=1}^{k}\log\Big(1-\frac{\alpha_j(p)}{p^{1+it_0}}\Big)+O\Big(\frac{1}{(\log T)^{10}}\Big).$$

We shift the path of integration to the left. By the zero-free region of $F \in \mathbb{G}$, the only pole of the above integrand in $\Re(s) \ge -\sigma_0$ and $\Im(s) \le T_0$ is at $s = 0$. Therefore,

$$\log F(1+it_0) = -\sum_{p\le Y}\sum_{j=1}^{k}\log\Big(1-\frac{\alpha_j(p)}{p^{1+it_0}}\Big)+O\Big(\frac{1}{(\log T)^{10}}+\int_C\log F(1+it_0+s)\frac{Y^s}{s}ds\Big), \tag{3.1}$$

where C is the contour joining $-\sigma_0 - iT_0, \sigma_1 - iT_0, \sigma_1 + iT_0$ and $-\sigma_0 + iT_0$. Since, $|\log F(\sigma + it)| \ll \log t$ on C,

$$\int_{\sigma_1-iT_0}^{-\sigma_0-iT_0}\log F(1+it_0+s)\frac{Y^s}{s}ds \ll \frac{\log T}{T^{1/10}}, \quad \int_{-\sigma_0+iT_0}^{\sigma_1+iT_0}\log F(1+it_0+s)\frac{Y^s}{s}ds \ll \frac{\log T}{T^{1/10}}, \tag{3.2}$$

and

$$\int_{-\sigma_0-iT_0}^{-\sigma_0+iT_0}\log F(1+it_0+s)\frac{Y^s}{s}ds \ll \frac{(\log T)^2}{\exp(\alpha^{-1}(\log T)^9)}, \tag{3.3}$$

where all implied constants are absolute. Substituting the bounds from (3.2) and (3.3) into (3.1), for $T^{1/10} \le t_0 \le T$,

$$\log F(1+it_0) = -\sum_{p\le Y}\sum_{j=1}^{k}\log\Big(1-\frac{\alpha_j(p)}{p^{1+it_0}}\Big)+O\Big(\frac{1}{(\log T)^{10}}\Big).$$

We may argue similarly when t_0 is negative. □

By Lemma 3.1, it suffices to show Theorem 1.3 for $F(1 + it; Y)$. We follow closely the argument in [2]. Set

$$X = \tfrac{1}{6}(\log T)(\log_2 T)$$

and for primes $p \le X$ set

$$q_p = \Big(1-\frac{p}{X}\Big).$$

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Also set $q_1 = 1$ and $q_p = 0$ for $p > X$. We now extend the definition completely multiplicatively to all positive integers. If $n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$, set

$$q_n := q_{p_1}^{a_1} q_{p_2}^{a_2} \cdots q_{p_m}^{a_m}$$

and define

$$R(t) = \prod_{p \le X} (1 - q_p p^{it})^{-1}.$$

Then

$$\log(|R(t)|) \le \sum_{p \le X} (\log X - \log p) = \pi(X) \log X - \vartheta(X),$$

where $\pi(X)$ is the prime counting function and $\vartheta(X)$ is the first Chebyshev function. By partial summation,

$$\pi(X) \log X - \vartheta(X) = \int_2^X \frac{\pi(t)}{t}\, dt = (1 + o(1)) \frac{X}{\log X}.$$

By our choice of X,

$$|R(t)|^2 \le T^{1/3 + o(1)}. \tag{3.4}$$

From the Euler product, $R(t)$ has the series representation $R(t) = \sum_{n=1}^{\infty} q_n n^{it}$, so

$$|R(t)|^2 = \left(\sum_{n=1}^{\infty} q_n n^{it} \right) \left(\sum_{n=1}^{\infty} q_n n^{-it} \right) = \sum_{m,n=1}^{\infty} q_m q_n \left(\frac{m}{n} \right)^{it}.$$

Recall that

$$F(1 + it; Y) = \prod_{p \le Y} \prod_{j=1}^{k} \left(1 - \frac{\alpha_j(p) p^{-it}}{p} \right)^{-1}.$$

Since $|\alpha_j(p)| \le 1$,

$$|F(1 + it; Y)| \ll (\log Y)^k \ll (\log T)^{10k}.$$

Set $\Phi(t) := e^{-t^2}$ and recall that its Fourier transform is positive. Using (3.4),

$$\left| \int_{|t| \ge T} F(1 + it; Y) |R(t)|^2 \Phi\left(\frac{\log T}{T} t \right) dt \right| \ll 1,$$

and

$$\left| \int_{|t| \le \sqrt{T}} F(1 + it; Y) |R(t)|^2 \Phi\left(\frac{\log T}{T} t \right) dt \right| \ll T^{5/6 + o(1)}.$$

From the positivity of the Fourier coefficients of Φ and the fact that $q_1 = 1$,

$$\int_{\sqrt{T}}^{T} |R(t)|^2\, \Phi\left(\frac{\log T}{T} t \right) dt \gg T^{1 + o(1)}.$$

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By a similar argument, again using the positivity of the Fourier coefficients,

$$\int_{-\infty}^{\infty} F(1+it;Y)|R(t)|^2\Phi\Big(\frac{\log T}{T}t\Big)\,dt \geq \int_{-\infty}^{\infty} F(1+it;X)|R(t)|^2\Phi\Big(\frac{\log T}{T}t\Big)\,dt.$$

So, we restrict ourselves to primes $p \leq X$ in the truncated Euler-product. This is to ensure both $R(t)$ and $F(1+it;X)$ have terms with the same q's.

Write $F(1+it;X)$ as

$$F(1+it;X) := \sum_{n=1}^{\infty} a_k k^{-it},$$

where $a_k \geq 0$ because the Dirichlet coefficients of $F(s)$ are nonnegative. Now define

$$\begin{aligned} I_1 &:= \int_{-\infty}^{\infty} F(1+it;X)|R(t)|^2\Phi\Big(\frac{\log T}{T}t\Big)\,dt \\ &= \sum_{k=1}^{\infty} a_k \sum_{m,n=1}^{\infty} \int_{-\infty}^{\infty} k^{-it} q_m q_n \Big(\frac{m}{n}\Big)^{it} \Phi\Big(\frac{\log T}{T}t\Big)\,dt. \end{aligned}$$

We also define

$$I_2 := \int_{-\infty}^{\infty} |R(t)|^2\Phi\Big(\frac{\log T}{T}t\Big)\,dt.$$

Since we are working with truncated Euler-products, everything is absolutely convergent. Now, using the fact that the Fourier coefficients of Φ are positive and that the q_n are completely multiplicative, we can bound the inner sum of I_1 from below by

$$\sum_{n=1}^{\infty}\sum_{k|m} \int_{-\infty}^{\infty} k^{-it} q_m q_n \Big(\frac{m}{n}\Big)^{it} \Phi\Big(\frac{\log T}{T}t\Big)\,dt = q_k \sum_{n=1}^{\infty}\sum_{r=1}^{\infty} \int_{-\infty}^{\infty} q_r q_n \Big(\frac{r}{n}\Big)^{it} \Phi\Big(\frac{\log T}{T}t\Big)\,dt.$$

Thus,

$$\begin{aligned} \frac{I_1}{I_2} &\geq \sum_{k=1}^{\infty} a_k q_k = \prod_{p\leq X}\prod_{j=1}^{k}\Big(1 - \frac{\alpha_j(p)}{p}q_p\Big)^{-1} \\ &= \prod_{p\leq X}\prod_{j=1}^{k}\Big(1 - \frac{\alpha_j(p)}{p}\Big)^{-1} \cdot \prod_{p\leq X}\prod_{j=1}^{k}\Big(\frac{p-\alpha_j(p)}{p-\alpha_j(p)q_p}\Big). \end{aligned} \tag{3.5}$$

Using the generalised Mertens Theorem 2.1, the first product in (3.5) is

$$\prod_{p\leq X}\prod_{j=1}^{k}\Big(1 - \frac{\alpha_j(p)}{p}\Big)^{-1} = e^{\gamma_F}(\log X)^m + O(1) = e^{\gamma_F}(\log_2 T + \log_3 T)^m + O(1). \tag{3.6}$$

The second product in (3.5) can be bounded as follows:

$$-\log \prod_{p\leq X}\prod_{j=1}^{k}\Big(\frac{p-\alpha_j(p)}{p-\alpha_j(p)q_p}\Big) = -\sum_{p\leq X}\sum_{j=1}^{k}\log\Big(\frac{p-\alpha_j(p)}{p-\alpha_j(p)q_p}\Big) \ll \sum_{p\leq X}\frac{1}{X} \ll \frac{1}{\log X}. \tag{3.7}$$

From (3.5), (3.6) and (3.7),

$$\frac{I_1}{I_2} \geq e^{\gamma_F}(\log_2 T + \log_3 T)^m + O(1).$$

In other words,

$$\frac{\left|\int_{\sqrt{T}}^{T} F(1+it;X)|R(t)|^2 \Phi\left(\frac{\log T}{T}t\right)\,dt\right|}{\int_{\sqrt{T}}^{T} |R(t)|^2 \Phi\left(\frac{\log T}{T}t\right)\,dt} \geq e^{\gamma_F}(\log_2 T + \log_3 T)^m + O(1).$$

Hence, we conclude

$$\max_{t\in[\sqrt{T},T]} |F(1+it)| \geq e^{\gamma_F}((\log_2 T + \log_3 T)^m - C_F).$$

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References

[1] C. Aistleitner, 'Lower bounds for the maximum of the Riemann zeta function along vertical lines', *Math. Ann.* **365**(1–2) (2016), 473–496.

[2] C. Aistleitner, K. Mahatab and M. Munsch, 'Extreme values of the Riemann zeta function on the 1-line', *Int. Math. Res. Not. IMRN* **2019**(22) (2019), 6924–6932.

[3] C. Aistleitner, K. Mahatab, M. Munsch and A. Peyrot, 'On large values of $L(\sigma,\chi)$', *Q. J. Math.* **70**(3) (2019), 831–848.

[4] C. Aistleitner and Ł. Pańkowski, 'Large values of L-functions from the Selberg class', *J. Math. Anal. Appl.* **446**(1) (2017), 345–364.

[5] L. Arguin, D. Belius, P. Bourgade, M. Radziwiłł and K. Soundararajan, 'Maximum of the Riemann zeta function on a short interval of the critical line', *Comm. Pure Appl. Math.* **72**(3) (2019), 500–535.

[6] L.-P. Arguin, F. Ouimet and M. Radziwiłł, 'Moments of the Riemann zeta function on short intervals of the critical line', Preprint, arXiv:1901.04061.

[7] R. Balasubramanian and K. Ramachandra, 'On the frequency of Titchmarsh's phenomenon for $\zeta(s)$. III', *Proc. Indian Acad. Sci.* **86** (1977), 341–351.

[8] A. Beurling, 'Analyse de la loi asymptotique de la distribution des nombres premiers généralisés. I', *Acta Math.* **68**(1) (1937), 255–291.

[9] A. Bondarenko and K. Seip, 'Large greatest common divisor sums and extreme values of the Riemann zeta function', *Duke Math. J.* **166** (2017), 1685–1701.

[10] R. de la Bretèche and G. Tenenbaum, 'Sommes de Gál et applications', *Proc. Lond. Math. Soc. (3)* **119**(1) (2019), 104–134.

[11] A. B. Dixit and V. K. Murty, 'The Lindelöf class of L-functions, II', *Int. J. Number Theory* **15**(10) (2019), 2201–2221.

[12] A. Granville and K. Soundarajan, 'Extreme vaues of $|\zeta(1+it)|$', *The Riemann Zeta Function and Related Themes: Papers in Honour of Professor K. Ramachandra*, Ramanujan Mathematical Society Lecture Notes Series, 2 (International Press, Boston, MA, 2010), 65–80.

[13] J. Kaczorowski and A. Perelli, 'On the prime number theorem for the Selberg class', *Arch. Math.* **80** (2003), 255–263.

[14] N. Levinson, 'Ω-theorems for the Riemann zeta-function', *Acta Arith.* **20** (1972), 317–330.
[15] F. Mertens, 'Ein Beitrag zur analytischen Zahlentheorie', *J. reine angew. Math.* **78** (1874), 46–62.
[16] H. Montgomery and R. C. Vaughan, *Multiplicative Number Theory. I. Classical Theory*, Cambridge Studies in Advanced Mathematics, 97 (Cambridge University Press, Cambridge, 2007).
[17] V. K. Murty, 'The Lindelöf class of L-functions', in: *The Conference on L-Functions* (eds. L. Weng and M. Kaneko) (World Scientific, Singapore, 2007), 165–174.
[18] M. Rosen, 'A generalization of Mertens' theorem', *J. Ramanujan Math. Soc.* **14** (1999), 1–19.
[19] M. du Sautoy, 'Natural boundaries for Euler products of Igusa zeta functions of elliptic curves', *Int. J. Number Theory* **14**(8) (2018), 2317–2331.
[20] A. Selberg, 'Old and new conjectures and results about a class of Dirichlet series', *Collected Papers*, Vol. 2 (Springer, Berlin–Heidelberg–New York, 1991), 47–63.
[21] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Translated from the second French edition (1995) by C. B. Thomas, Cambridge Studies in Advanced Mathematics, 46 (Cambridge University Press, Cambridge, 1995).
[22] E. C. Titchmarsh, *The Theory of the Riemann-Zeta Function.* 2nd edn, revised by D. R. Heath-Brown (Clarendon Press, Oxford, 1986).
[23] Y. Yashiro, 'Mertens' theorem and prime number theorem for Selberg class', Preprint, 2013, arXiv:1311.0754v4.

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