THE FREDHOLM ELEMENTS OF A RING

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Introduction. In (**1**), Atkinson characterized the set of Fredholm operators on a Banach space X as those bounded operators invertible modulo the two-sided ideal of compact operators on X. It follows from this characterization that the Fredholm operators can also be described as those bounded operators which are invertible modulo the two-sided ideal of bounded operators on X which have finite-dimensional range. This ideal is the socle of the algebra of all bounded operators on X. Now, if A is any ring with no nilpotent left or right ideals, then the concept of socle makes sense (the socle of A in this case is the algebraic sum of the minimal left ideals of A, or 0 if A has no minimal left ideals). Also, in this case, the socle is a two-sided ideal of A. In this paper we study the elements in a ring A which are invertible modulo the socle. We call these elements the Fredholm elements of A. Often for technical reasons, rather than dealing directly with the Fredholm elements of A, we study those elements of A which are quasi-regular modulo the socle. We call these elements quasi-Fredholm. In § 2 we give a characterization of the quasi-Fredholm elements of A. In § 3 we define a generalized index on the semigroup of Fredholm elements of A, and we prove a multiplicative property of the index on this set. In § 4, with the assumption that A is a Banach algebra, we prove that the index is a continuous function on the set of Fredholm elements of A. Some examples are given in § 5.

1. Ideals of finite order. Let A be a ring. We call A semi-prime if it has no non-zero left or right nilpotent ideals. Throughout this paper we deal only with semi-prime rings. When A is semi-prime, the socle of A, which we denote by S_A, always exists; see (**8**, p. 46, Lemma (2.1.12)).

If B is any subset of A, we let

$$L[B] = \{a \in A \mid aB = 0\} \quad \text{and} \quad R[B] = \{a \in A \mid Ba = 0\}.$$

We note that when A is semi-prime, then $L[A] = R[A] = 0$. In particular, when $u \in A$, $R[A(1-u)] = \{a \in A \mid ua = a\}$.

A right (left) ideal N of A is of finite order if N is the sum of a finite number of minimal right (left) ideals of A. In this case we define the order of N, denoted $\theta(N)$, to be the smallest number of minimal right (left) ideals which have sum N. By convention, the zero ideal has order 0. Some basic results concerning ideals of finite order are proved in (**3**). We state for convenience a useful result, the proof of which may be found in (**3**, p. 497, Theorem 2.2).

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LEMMA 1.1. *Assume that N is a non-zero right (left) ideal of A of finite order m. Then any maximal orthogonal set of minimal idempotents in N contains m elements, and if $\mathscr{M} = \{e_1, e_2, \ldots, e_m\}$ is such a set, then setting*

$$e = e_1 + e_2 + \ldots + e_m,$$

we have that $N = eA$ $(N = Ae)$.

This result implies that every right (left) ideal N of A of finite order is of the form eA (Ae) for e an idempotent in S_A. In this case, we define $\theta(e) = \theta(N)$. Now we prove several additional results concerning ideals of finite order.

LEMMA 1.2. *Assume that M and N are right (left) ideals of A, that $N + M = A$, and that N is of finite order. If K is any right (left) ideal of finite order such that $K \cap M = 0$, then $\theta(K) \leqq \theta(N)$.*

Proof. If either $K = 0$ or $N = 0$, the conclusion is obvious. Thus, we assume that $K \neq 0$, $N \neq 0$. Choose $\{e_1, e_2, \ldots, e_m\}$ a maximal orthogonal set of minimal idempotents in K, and choose $\{f_1, f_2, \ldots, f_n\}$ a maximal orthogonal set of minimal idempotents in N. By Lemma 1.1, $K = e_1A + \ldots + e_mA$, $N = f_1A + \ldots + f_nA$, and $\theta(K) = m$, $\theta(N) = n$. Since $N + M = A$, there exist $x_k \in A$ and $z \in M$ such that $e_1 = f_1x_1 + f_2x_2 + \ldots + f_nx_n + z$. Since $K \cap M = 0$, $f_jx_j \neq 0$ for some j. Then

$$f_jA = f_jx_jA \subset \left(e_1A + \sum_{k=1, k\neq j}^{n} f_kA + M\right).$$

It follows that the right ideal on the right-hand side of this inclusion must be all of A. Now, the proof proceeds by exchanging at the kth step the ideal e_kA for an ideal f_iA by a procedure similar to that given above.

Suppose that $n < m$. Then, in fact, after n such exchanges we would have $(e_1A + e_2A + \ldots + e_nA) + M = A$. But then there exist $y_i \in A$ and $w \in M$ such that

$$e_m = e_1y_1 + e_2y_2 + \ldots + e_ny_n + w.$$

Therefore, $e_m - (e_1y_1 + e_2y_2 + \ldots + e_ny_n) \in M \cap K = 0$, and hence

$$e_m = e_1y_1 + \ldots + e_ny_n.$$

This implies that $e_m = 0$, a contradiction. We conclude that $n \geqq m$.

LEMMA 1.3. *Assume that M is a right (left) ideal and that N is a right (left) ideal of finite order such that $N + M = A$. If K is a maximal modular right (left) ideal with $M \subset K$, then K has the form $(1 - f)A$ $(A(1 - f))$, where f is a minimal idempotent of A.*

Proof. Assume that N, M, and K are as in the statement of the lemma. Then, either $S_A \subset K$ or $L[K] \neq 0$ by (**9**, p. 38, Lemma 3.3). However, if $S_A \subset K$, then $N \subset K$, in which case $A = N + M \subset K$, a contradiction. Thus $L[K] \neq 0$, and by (**9**, p. 38, Lemma 3.3), K has the required form.

LEMMA 1.4. *Assume that N is a right ideal of finite order n and that M is a modular right ideal such that $N + M = A$ and $N \cap M = 0$. Let*

$$\mathscr{M} = \{e_1, \ldots, e_m\}$$

be a maximal orthogonal set of minimal idempotents having the property that $(e_1A + \ldots + e_mA) \cap M = 0$. Then, setting $e = e_1 + \ldots + e_m$, we have that $eA \oplus M = A$ and $n = m$.

A similar statement can be made concerning left ideals.

Proof. By Lemma 1.2, $m \leqq n$. Suppose that $eA + M \neq A$. Then, since this right ideal is a modular right ideal, there exists a maximal modular right ideal K such that $eA + M \subset K$. By Lemma 1.3, there exists a minimal idempotent $f \in A$ such that $K = (1 - f)A$. Let $g = f - ef$. $g \neq 0$ since $f \notin eA$. Also, $gf = g$ and $g^2 = g$. Then g is a minimal idempotent of A, since $Af = Agf = Ag$. Now, $e_kg = 0$ when $1 \leqq k \leqq m$ (since $e_ke = e_k$), and $ge_k = 0$ when $1 \leqq k \leqq m$ (since $fe_k = 0$). Consider the ideal $J = gA + (e_1A + \ldots + e_mA)$. Suppose that $y \in J \cap M$. Then y has the form $y = gx + e_1x_1 + \ldots + e_mx_m$ and $y \in M$. But then $fy = 0$, and $fe_kx_k = 0$ for $1 \leqq k \leqq m$, and $fgx = fx$. Thus $fx = 0$, so that $gx = 0$ and $y \in eA \cap M = 0$. Finally, this contradicts the maximal property of $\mathscr{M}$. Therefore, $eA \oplus M = A$. Now, applying Lemma 1.2, $n \leqq m$, and this completes the proof.

LEMMA 1.5. *Assume that N is a right ideal of finite order n, that $u \in A$, and that $R[A(1 - u)] \cap N = 0$. Then $\theta((1 - u)N) = n$.*

A similar statement holds when N is a left ideal.

Proof. Choose $\mathscr{M} = \{e_1, e_2, \ldots, e_n\}$ a maximal orthogonal set of minimal idempotents in N (see Lemma 1.1). Then

$$(1 - u)N = (1 - u)e_1A + \ldots + (1 - u)e_nA.$$

Note that $(1 - u)e_kA$ is a non-zero minimal right ideal of A for all k. It follows by definition that $\theta((1 - u)N) \leqq n$. To prove equality here it is sufficient to show that $(1 - u)e_1A + \ldots + (1 - u)e_nA$ is a direct sum by (**3**, p. 497, Theorem 2.1). Therefore, assume that $(1 - u)e_1x_1 + \ldots + (1 - u)e_nx_n = 0$, $x_k \in A$. Then $(e_1x_1 + \ldots + e_nx_n) \in R[A(1 - u)] \cap N = 0$, and hence $e_1x_1 + \ldots + e_nx_n = 0$. Then, since the e_k are orthogonal, $e_kx_k = 0$ for $1 \leqq k \leqq n$. This proves that the sum is direct.

2. Fredholm and quasi-Fredholm elements of a ring. As in § 1, we assume that A is a semi-prime ring. When $u, v \in A$, we let

$$u \circ v = u + v - uv.$$

Definition 2.1. $u \in A$ is left (right) quasi-Fredholm in A, abbreviated l.q.-F. (r.q.-F.), if there exists $v \in A$ such that $v \circ u \in S_A$ ($u \circ v \in S_A$). u is quasi-Fredholm in A, abbreviated q.-F., if u is both l.q.-F. and r.q.-F. When A has an identity, $u \in A$ is left Fredholm, right Fredholm, or Fredholm in A, if u is left invertible, right invertible, or invertible modulo S_A, respectively.

Denote by $k(h(S_A))$ (read "the kernel of the hull of S_A") the intersection of the primitive ideals of A which contain S_A. It is understood that when S_A is contained in no primitive ideal of A, then $k(h(S_A)) = A$. We note, without proof, the following proposition.

PROPOSITION 2.2. *Assume that I is a two-sided ideal of A such that*

$$S_A \subset I \subset k(h(S_A)).$$

Then $u \in A$ is left (right) quasi-regular modulo S_A if and only if u is left (right) quasi-regular modulo I.

Now we apply this proposition to the case where A is the ring of bounded operators on a Banach space. Assume that X is a Banach space, and denote by $\mathscr{B}[X]$, $\mathscr{C}[X]$, and $\mathscr{F}[X]$, the algebra of bounded operators, compact operators, and bounded operators of finite rank on X, respectively. Then the socle of $\mathscr{B}[X]$ is $\mathscr{F}[X]$. Furthermore, by (**7**, p. 864, Theorem 1),

$$\mathscr{F}[X] \subset \mathscr{C}[X] \subset k(h(\mathscr{F}[X])).$$

Thus, by Proposition 2.2, $T \in \mathscr{B}[X]$ is a Fredholm operator (that is, T is invertible modulo $\mathscr{C}[X]$) if and only if T is invertible modulo $\mathscr{F}[X]$, the socle of $\mathscr{B}[X]$. Therefore, T is a Fredholm operator in the usual operator sense if and only if T is a Fredholm element of $\mathscr{B}[X]$ in the algebraic sense.

Next we characterize the quasi-Fredholm elements of a ring.

THEOREM 2.3. *Assume that A is a semi-prime ring. Then $u \in A$ is r.q.-F. (l.q.-F.) in A if and only if there exists an idempotent $e \in S_A$ such that*

$$(1 - u)A = (1 - e)A \qquad (A(1 - u) = A(1 - e)).$$

Proof. Assume first that $(1 - u)A = (1 - e)A$, where e is an idempotent in S_A. Then eA is of finite order, and $(1 - u)A + eA = A$. Then there exist $v, w \in A$ such that $(1 - u)v + ew = -u$. Therefore,

$$u \circ v = u + v - uv = -ew \in S_A,$$

and hence by definition, u is r.q.-F.

Conversely, assume that u is r.q.-F. in A. Then there exists $v \in A$ and $s \in S_A$ such that $u \circ v = s$. Therefore

$$(1 - u)(1 - v)A = (1 - u \circ v)A = (1 - s)A,$$

and hence $(1 - s)A \subset (1 - u)A$. By (**3**, p. 499, Theorem 2.4 (3)), $(1 - s)A$ has the form $(1 - g)A$, where g is an idempotent in S_A (the result cited is stated only for left ideals of the form $A(1 - s)$, but a similar statement holds for right ideals). Then $gA + (1 - u)A = A$. Also

$$L[(1 - u)A] \subset L[(1 - s)A] \subset As,$$

and therefore $L[(1 - u)A]$ has finite order. By Lemma 1.1, $L[(1 - u)A] = Ae$, where e is an idempotent in S_A. Let $M = eA + (1 - u)A$. Suppose that K is a maximal modular right ideal of A containing M. By Lemma 1.3, there exists a minimal idempotent f such that $K = (1 - f)A$. Then $f(1 - u)A = 0$ which implies that $f \in Ae$. But also $f(eA) = 0$, so that $f = fe = 0$. Therefore, M is contained in no maximal modular right ideal of A, and thus $M = A$. Now, clearly $(1 - u)A \subset (1 - e)A$. Suppose that $x \in (1 - e)A$. There exist $y, z \in A$ such that $x = ey + (1 - u)z$. Then $ey = ex = 0$. It follows that $x = (1 - u)z \in (1 - u)A$. Therefore $(1 - u)A = (1 - e)A$.

The proof of this theorem provides us with the following corollary.

COROLLARY. *If u is r.q.-F. in A and $L[(1 - u)A] = Ae$, where e is an idempotent in S_A, then $(1 - u)A = (1 - e)A$. A similar statement holds when u is l.q.-F.*

An immediate consequence of the proof of Theorem 2.3 is the following result which generalizes a theorem of Duncan (**6**, p. 98, Theorem 8). We use the definition of modular annihilator ring as given by Yood (**9**, pp. 37, 39).

THEOREM 2.4. *Let A be a semi-prime modular annihilator ring. Given a proper modular right ideal M of A we have that*

(1) *M is the intersection of a finite number of maximal modular right ideals;*

(2) *there exists an idempotent $e \in S_A$ such that $M = (1 - e)A$;*

(3) $M = R[L[M]]$.

A similar result holds for proper modular left ideals of A.

Proof. Since M is a modular right ideal of A, there exists $v \in A$ such that $(1 - v)A \subset M$. A is a modular annihilator ring, so that by (**9**, p. 38, Theorem 3.4 (3)) v is q.-F. in A. Thus, as in the proof of Theorem 2.3, there exists $s \in S_A$ such that $(1 - s)A \subset (1 - v)A \subset M$. From this point, the proof of (2) proceeds exactly as in the second paragraph of the proof of Theorem 2.3 with M in place of the right ideal $(1 - u)A$. (1) and (3) are easy consequences of (2).

Now we consider the special case where A is a ring with a proper involution $*$. This means that $*$ is a map of A into A with the properties $(u + v)^* = u^* + v^*$, $(uv)^* = v^*u^*$, $(u^*)^* = u$, and $u^*u = 0$ implies $u = 0$, whenever $u, v \in A$. A ring with proper involution is automatically semi-prime. An element u in A is normal if $uu^* = u^*u$.

LEMMA 2.5. *Assume that A is a ring with proper involution $*$ and $u \in A$ is normal. Then*

(1) $(1 - u)x = 0$ *if and only if* $x^*(1 - u) = 0$, *for any* $x \in A$;
(2) *If* u *is right or left quasi-regular, then* u *is quasi-regular.*

Proof. If $(1 - u)x = 0$, then $x^*(1 - u^*)(1 - u)x = 0$, and since u is normal, $x^*(1 - u)(1 - u^*)x = 0$. But $*$ is proper, thus $x^*(1 - u) = 0$. The converse is proved in the same way.

Now suppose that u is right quasi-regular. Then there exists $v \in A$ such that $u \circ v = 0$. Assume also that $u \circ w = 0$ for some $w \in A$. Then the two equations, $u + v - uv = 0$ and $u + w - uw = 0$, imply that $(1 - u)(v - w) = 0$. By part (1), $(v^* - w^*)(1 - u) = 0$. Then

$$0 = (v^* - w^*)(1 - u)(1 - v) = (v^* - w^*)(1 - u \circ v) = v^* - w^*.$$

Thus $v = w$. But a direct computation verifies that $u \circ (v \circ u + v) = 0$. Therefore $v = v \circ u + v$, and thus $v \circ u = 0$. We have proved that when u is right quasi-regular, then u is quasi-regular.

THEOREM 2.6. *Assume that* A *is a ring with proper involution* $*$. *Assume that* u *is a normal element of* A. *Then* u *is q.-F. in* A *if* u *is either l.q.-F. or r.q.-F. in* A. *When* u *is q.-F. in* A, *there is a unique self-adjoint idempotent* $e \in S_A$ *such that* $L[(1 - u)A] = Ae$, $R[A(1 - u)] = eA$, $(1 - u)A = (1 - e)A$, *and*

$$A(1 - u) = A(1 - e).$$

Proof. Assume that u is r.q.-F. in A. As in the proof of Theorem 2.3, we have that $L[(1 - u)A]$ is of finite order. Then by (**2**, p. 286, Lemma 2.3), there exists a unique self-adjoint idempotent $e \in S_A$ such that $L[(1 - u)A] = Ae$. Therefore, the corollary to Theorem 2.3 implies that $(1 - u)A = (1 - e)A$. Also, by Lemma 2.5(1), $R[A(1 - u)] = (Ae)^* = eA$. In particular,

$$eu = ue = e.$$

Then $(u + e)$ is normal. Suppose that $(1 - (u + e))A$ is contained in a maximal right ideal K of A. Then Lemma 1.3 implies that K has the form $(1 - f)A$, where f is a minimal idempotent. Therefore $f(1 - (u + e)) = 0$, so $f(1 - u) = fe$. But then $fe = 0$, and $f(1 - u) = 0$. This contradicts the definition of e. Therefore $(1 - (u + e))A = A$. By Lemma 2.5(2),

$$A(1 - (u + e)) = A.$$

It follows that $A(1 - u) + Ae = A$, and so $A(1 - u) = A(1 - e)$.

3. The index. As before, we make the assumption in this section that A is a semi-prime ring.

Definition 3.1. Assume that u is q.-F. in A. We define

$$\kappa(1 - u) = \theta(L[(1 - u)A]) - \theta(R[A(1 - u)]).$$

$\kappa(1 - u)$ is called the index of $1 - u$ in A. We note that the 1 may be formal here.

Our aim in this section is to prove the following analogue of a classical theorem concerning the index of the product of Fredholm operators.

THEOREM 3.2. *Assume that u and v are q.-F. in A. Then $v \circ u$ is q.-F. in A and* $\kappa(1 - v \circ u) = \kappa(1 - v) + \kappa(1 - u)$.

The fact that $v \circ u$ is q.-F. in A is easily verified. We proceed with the proof of the index formula by proving several lemmas. We make the assumptions that $R[A(1 - u)] = eA$, $L[(1 - u)A] = Af$, $R[A(1 - v)] = gA$,

$$L[(1 - v)A] = Ah, A(1 - u) = A(1 - e),$$

$(1 - u)A = (1 - f)A$, $A(1 - v) = A(1 - g)$, and $(1 - v)A = (1 - h)A$, where e, f, g, and h are idempotents in S_A. All these assumptions are justified by Theorem 2.3.

LEMMA 3.3. (i) $\theta(R[A(1 - v \circ u)]) = \theta(e) + \theta((1 - u)A \cap gA)$ *and* (ii) $\theta(L[(1 - v \circ u)A]) = \theta(h) + \theta(A(1 - v) \cap Af)$.

Proof. We only prove (i). Define T as a linear transformation on $(1 - e)A$ with values in $(1 - u)A$ by $T((1 - e)x) = (1 - u)(1 - e)x$, for any $x \in A$. T is an isomorphism of $(1 - e)A$ onto $(1 - u)A$. Let

$$K = T^{-1}((1 - u)A \cap gA).$$

It is not difficult to verify that K is a right ideal of A contained in $(1 - e)A$. Since $K \subset (1 - e)A$, $K \cap R[A(1 - u)] = K \cap eA = 0$. Then from Lemma 1.5, we conclude that $\theta(K) = \theta((1 - u)A \cap gA)$.

Note that $1 - (v \circ u) = (1 - v)(1 - u)$. Clearly, $eA \subset R[A(1 - v)(1 - u)]$. Also, if $w \in K$, then $(1 - u)w \in gA$, and hence $(1 - v)(1 - u)w = 0$. Therefore $eA + K \subset R[A(1 - v)(1 - u)]$. Now assume that

$$z \in R[A(1 - v)(1 - u)].$$

Then $(1 - u)(1 - e)z = (1 - u)z \in gA \cap (1 - u)A$. It follows that $(1 - e)z \in K$. But then $z = ez + (1 - e)z \in eA + K$. This proves that $eA \oplus K = R[A(1 - v)(1 - u)]$. Then

$$\theta(R[A(1 - v)(1 - u)]) = \theta(e) + \theta(K) = \theta(e) + \theta(gA \cap (1 - u)A).$$

LEMMA 3.4. *Let $J = gA \cap (1 - f)A$, and let $K = A(1 - g) \cap Af$. Then* $\theta(J) - \theta(K) = \theta(g) - \theta(f)$.

Proof. $J = g'A$ for some idempotent $g' \in S_A$. Note that $gg' = g'$. Then $g'A = g'gg'A \subset g'gA \subset g'A$. Therefore $J = g'A = g'gA$. Let $g_2 = g'g$, and let $g_1 = g - g_2$. $J = g_2A$, g_2 is an idempotent, and $fg_2 = 0$. A short computation establishes that $g_1^2 = g_1$ and $g_1g_2 = g_2g_1 = 0$. In a similar fashion we can write $f = f_1 + f_2$, where f_1 and f_2 are idempotents, $K = Af_2$, $f_1f_2 = f_2f_1 = 0$, and $f_2g = 0$.

Assume that $w \in g_1A \cap (1 - f_1)A$. Then $f_1w = 0$, and since $w = gw$, we also have that $f_2w = f_2gw = 0$. Thus $fw = 0$. Now

$$w \in g_1A \cap (1 - f)A \subset gA \cap (1 - f)A = g_2A$$

and $w = g_1w = g_1(g_2w) = 0$. We conclude that $g_1A \cap (1 - f_1)A = 0$. Applying Lemma 1.2, we have that $\theta(g_1) \leqq \theta(f_1)$. Again, if $v \in A(1 - g_1) \cap Af_1$, then $vg_1 = 0$ and $vg_2 = (vf)g_2 = 0$. Then $vg = 0$. Thus,

$$v \in A(1 - g) \cap Af_1 \subset A(1 - g) \cap Af = Af_2,$$

so $v = vf_1 = (vf_2)f_1 = 0$. Therefore $A(1 - g_1) \cap Af_1 = 0$, and by Lemma 1.2, $\theta(f_1) \leqq \theta(g_1)$. Now $\theta(f_1) = \theta(g_1), \theta(g) = \theta(g_1) + \theta(g_2)$, and $\theta(f) = \theta(f_1) + \theta(f_2)$. Therefore $\theta(J) - \theta(K) = \theta(g_2) - \theta(f_2) = \theta(g) - \theta(f)$.

We now complete the proof of the theorem. By Lemma 3.3,

$$\theta(L[(1 - v \circ u)A]) = \theta(h) + \theta(A(1 - v) \cap Af)$$

and $\theta(R[A(1 - v \circ u)]) = \theta(e) + \theta((1 - u)A \cap gA)$. Furthermore,

$$A(1 - v) = A(1 - g)$$

and $(1 - u)A = (1 - f)A$. By Lemma 3.4,

$$\theta(f) - \theta(g) = \theta(A(1 - g) \cap Af) - \theta((1 - f)A \cap gA).$$

Combining these facts, we have that

$$\kappa(1 - v \circ u) = (\theta(h) - \theta(g)) + (\theta(f) - \theta(e)) = \kappa(1 - v) + \kappa(1 - u).$$

We close this section with an example. Let X be an arbitrary vector space. Denote by $\mathscr{L}[X]$ the algebra of all linear transformations on X, and by $\mathscr{F}[X]$ the algebra of all linear transformations on X which have finite-dimensional range. Let I be the identity transformation on X. Assume that A is an irreducible subalgebra of $\mathscr{L}[X]$. Then, in particular, A is a semi-prime ring, and also the socle of A is exactly $\mathscr{F}[X] \cap A$. Furthermore, every minimal idempotent of A has one-dimensional range. These statements can be found in (**8**, pp. 64–66).

Now suppose that $T \in A$ is q.-F. in A. Then by Theorem 2.3, there exists projections E and F in $A \cap \mathscr{F}[X]$ such that $R[A(I - T)] = EA$ and $L[(I - T)A] = AF$. It is not difficult to verify that $\theta(E)$ is the dimension of the range of E and that $\theta(F)$ is the dimension of the range of F. The generalized index of $I - T$ is by definition $\theta(F) - \theta(E)$. This is the difference of the dimension of the range of F and the dimension of the range of E. But the former of these is the defect of $I - T$, and the latter is the nullity of $I - T$. Therefore, in this example, when T is q.-F. in A, then the generalized index of $I - T$ is the usual index of $I - T$ (the difference of the defect and nullity of $I - T$).

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4. The continuity of the index. In this section we assume that A is a real or complex semi-prime Banach algebra. With these assumptions, we prove that the index is a continuous function on the open semi-group of Fredholm elements of A. The proof of this theorem is an adaptation to our particular situation of certain arguments of Dieudonné; see (**5**).

THEOREM 4.1. *Assume that A is a semi-prime Banach algebra. Assume that $\{u_n\}$ is a sequence of q.-F. elements of A which converges to a q.-F. element u of A. Then $\kappa(1 - u_n)$ converges to $\kappa(1 - u)$.*

Assume that $\{u_n\}$, u are q.-F. and $u_n \rightarrow u$ in A. By Theorem 2.3, we may assume that there are idempotents e, f, $\{e_n\}$, and $\{f_n\}$ in S_A such that $A(1 - u) = A(1 - e)$, $(1 - u)A = (1 - f)A$, $A(1 - u_n) = A(1 - e_n)$, and $(1 - u_n)A = (1 - f_n)A$. Then by definition, $\kappa(1 - u) = \theta(f) - \theta(e)$ and $\kappa(1 - u_n) = \theta(f_n) - \theta(e_n)$. We begin the proof of the theorem with two lemmas. Since the arguments to establish parts (1) and (2) of these lemmas are nearly the same, we give the proof only for part (1) in both cases. The operator T defined on $(1 - e)A$ by

$$T((1 - e)x) = (1 - u)(1 - e)x = (1 - u)x$$

is one-to-one on $(1 - e)A$ and has closed range $(1 - f)A = (1 - u)A$. Then by the Closed Graph Theorem there exists a real number $m > 0$ such that $||(1 - u)x|| \geqq m||x||$ for all $x \in (1 - e)A$ ($||\cdot||$ is, of course, the Banach algebra norm on A). We use this fact in the proof of the following lemmas.

LEMMA 4.2. (1) *There exists an integer N such that $n \geqq N$ implies that $e_nA \cap (1 - e)A = 0$;*

(2) *There exists an integer N such that $n \geqq N$ implies that $Af_n \cap A(1 - f) = 0$.*

Proof of (1). For any $y \in A$, $(1 - u_n)y = (1 - u)y + (u - u_n)y$. Therefore, whenever $x \in (1 - e)A$, $||(1 - u_n)x|| \geqq m||x|| - ||u - u_n||\,||x||$. Choose an integer N so large that $n \geqq N$ implies that $||u - u_n|| < m$. Suppose that $n \geqq N$ and that $(1 - u_n)(1 - e)y = 0$ for some $y \in A$. Taking $x = (1 - e)y$ in the inequality above, we have that $(1 - e)y = 0$. Thus, $e_nA \cap (1 - e)A = 0$ whenever $n \geqq N$ (recall that $e_nA = R[A(1 - u_n)]$).

LEMMA 4.3. (1) *There is an integer M such that $n \geqq M$ implies that $fA \cap (1 - u_n)(1 - e)A = 0$;*

(2) *There is an integer M such that $n \geqq M$ implies that*

$$Ae \cap A(1 - f)(1 - u_n) = 0.$$

Proof of (1). Again choose N so large that $n \geqq N$ implies that $||u - u_n|| < m$. Assume that there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $n_k \geqq N$ for all $k \geqq 1$, and $fA \cap (1 - u_{n_k})(1 - e)A \neq 0$ for all $k \geqq 1$. Relable this subsequence as $\{v_k\}$. Then there exist fx_k in this intersection such that $||fx_k|| = 1$, and $fx_k = (1 - v_k)y_k$, where $y_k \in (1 - e)A$. Now, $(1 - v_k)y_k = (1 - u)y_k + (u - v_k)y_k$, and hence

$1 = ||(1 - v_k)y_k|| \geqq (m||y_k|| - ||u - v_k|| \, ||y_k||) = (m - ||u - v_k||)||y_k||$. Also, $||fx_k - (1 - u)y_k|| = ||(v_k - u)y_k|| \leqq ||v_k - u|| \, ||y_k||$. These two inequalities together yield $||fx_k - (1 - u)y_k|| \leqq ||v_k - u||/(m - ||u - v_k||)$. It follows that $||fx_k - (1 - u)y_k|| \to 0$. Therefore, $||f(fx_k - (1 - u)y_k)|| \to 0$, and thus $||fx_k|| \to 0$, which is a contradiction. This proves part (1) of the Lemma.

We now complete the proof of Theorem 4.1. Choose N as in Lemma 4.2(1) and M as in Lemma 4.3(1). Assume that $n \geqq \max(N, M)$. By Lemma 4.2, $e_nA \cap (1 - e)A = 0$. Using Lemma 1.4, we can choose a right ideal J_n of finite order such that $A = J_n \oplus e_nA \oplus (1 - e)A$, and then $\theta(J_n) = \theta(e) - \theta(e_n)$. Now $(1 - f_n)A = (1 - u_n)A = (1 - u_n)J_n \oplus (1 - u_n)(1 - e)A$. Therefore $f_nA \oplus (1 - u_n)J_n \oplus (1 - u_n)(1 - e)A = A$. By Lemma 4.3,

$$fA \cap (1 - u_n)(1 - e)A = 0.$$

Applying Lemma 1.2, we have that $\theta(f_n) + \theta((1 - u_n)J_n) \geqq \theta(f)$. Also by construction, $e_nA \cap J_n = 0$, so that by Lemma 1.5, $\theta(J_n) = \theta((1 - u_n)J_n)$. This establishes that $\theta(f_n) - \theta(e_n) \geqq \theta(f) - \theta(e)$. That $\theta(f_n) - \theta(e_n) \leqq \theta(f) - \theta(e)$ is established in a similar fashion by using Lemma 4.2(2) and Lemma 4.3(2). We outline the steps of the proof. Choose N and M as given by Lemmas 4.2(2) and 4.3(2). Choose K_n a left ideal such that $A = A(1 - f) \oplus Af_n \oplus K_n$. $\theta(K_n) = \theta(f) - \theta(f_n)$. Then

$$A(1 - e_n) = A(1 - u_n) = K_n(1 - u_n) \oplus A(1 - f)(1 - u_n).$$

Therefore $A = Ae_n \oplus K_n(1 - u_n) \oplus A(1 - f)(1 - u_n)$. Then

$$\theta(e_n) + \theta(K_n(1 - u_n)) \geqq \theta(e).$$

Also $\theta(K_n(1 - u_n)) = \theta(K_n)$. Thus $\theta(e_n) - \theta(f_n) \geqq \theta(e) - \theta(f)$.

As before, we assume that A is a semi-prime Banach algebra. Let $F[A]$ denote the set of q.-F. elements of A. We note that $I = k(h(S_A))$ is a closed ideal of A. Let π be the natural homomorphism of A onto A/I. Then $F[A]$ is the inverse image under π of the group of quasi-regular elements in A/I. Therefore, $F[A]$ is an open subset of A. Also, $F[A]$ is a semi-group with multiplication the circle operation ($u \circ v = u + v - uv$). When $u \in F[A]$, let $\tau(u) = \kappa(1 - u)$. By Theorem 3.2, $\tau(u \circ v) = \tau(u) + \tau(v)$. Thus, τ is a semi-group homomorphism of $F[A]$ into the integers. By Theorem 4.1, τ is continuous on $F[A]$. Therefore, τ is constant on the components of $F[A]$. We note that τ is an index in the sense of Coburn and Lebow; see (**4**).

5. Some examples. If A is a semi-prime ring with identity 1, then u is a Fredholm element of A if and only if $(1 - u)$ is a quasi-Fredholm element of A. Given a Fredholm element $u \in A$ we define the generalized index of u to be $\kappa(1 - (1 - u))$, where κ is as in Definition 3.1. We write $\kappa(u)$ for the generalized index of u. Given a Banach space X, the Fredholm elements of the algebra $\mathscr{B}[X]$ are the Fredholm operators on X and the generalized index

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of a Fredholm element of $\mathscr{B}[X]$ is the usual index. In the remainder of this section we give examples of the Fredholm elements and generalized index in some other Banach algebras.

Assume that Ω is a non-empty set, and let A be the Banach algebra of all bounded complex-valued functions on Ω. S_A is the set of all $f \in A$ which take the value zero except at a finite number of points of Ω. $g \in A$ is a Fredholm element of A if and only if g is bounded away from zero on a set whose complement in Ω is finite. The index is trivial in this algebra since the index of any Fredholm element is zero. The index is trivial in any commutative ring.

Now, assume that Ω is a compact Hausdorff space and that X is a Banach space. Define A to be the algebra of all continuous functions defined on Ω with values in $\mathscr{B}[X]$. The norm of $f \in A$ is defined by $||f|| = \sup_{x\in\Omega}||f(x)||'$, where $||\cdot||'$ is the usual norm on $\mathscr{B}[X]$. $e \in A$ is a minimal idempotent of A if and only if there exists x_0, an isolated point of Ω, and $E \in \mathscr{B}[X]$, E a projection on X with one-dimensional range, such that $e(x_0) = E$ and $e(x) = 0$ whenever $x \in \Omega$, $x \neq x_0$. It follows that S_A is the set of all $f \in A$ such that f takes the value zero at all except a finite set of isolated points of Ω and $f(x) \in \mathscr{F}[X]$ for all $x \in \Omega$. Assume that h is a Fredholm element of A. Then there exists $g \in A$ such that $hg - 1 \in S_A$ and $gh - 1 \in S_A$. Thus, $h(x)$ is invertible at all points $x \in \Omega$, except at a finite number of isolated points in Ω. Furthermore, $h(x)$ is a Fredholm operator on X for all $x \in \Omega$. Then $\kappa(h) = \Sigma_{x\in\Omega}\kappa'(h(x))$, where κ' is the usual index in $\mathscr{B}[X]$.

Again, let X be a Banach space. Assume that Λ is an index set and $\{X_\lambda | \lambda \in \Lambda\}$ is a collection of closed subspaces of X with the following properties:

(1) $X_\alpha \cap \overline{(\Sigma_{\lambda\in\Lambda,\lambda\neq\alpha}X_\lambda)} = 0$ for all $\alpha \in \Lambda$ (the bar denotes closure);

(2) $\overline{(\Sigma_{\lambda\in\Lambda}X_\lambda)} = X$.

We define A to be the algebra of all operators $T \in \mathscr{B}[X]$ such that T is invariant on each X_λ, $\lambda \in \Lambda$. A is a semi-simple closed subalgebra of $\mathscr{B}[X]$. Also, $S_A = \mathscr{F}[X] \cap A$. We shall not verify this, but we shall identify the minimal idempotents of A. Let X' be the conjugate space of X, and define for $\alpha \in \Lambda$, $Y_\alpha = \{f \in X' | f(X_\lambda) = 0$ whenever $\lambda \neq \alpha\}$. Fix $\alpha \in \Lambda$. Assume that $u \in X_\alpha$, $u \neq 0$. Using (1) we can choose $f \in Y_\alpha$ such that $f(u) = 1$. Let E be the operator defined on X by $E(x) = f(x) \cdot u$ for all $x \in X$. E is a minimal idempotent of A. Conversely, assume that E is a minimal idempotent of A. Then there exists $\alpha \in \Lambda$ such that $E(X_\alpha) \neq 0$ by (2). Thus, there exists $u \in X_\alpha$, $u \neq 0$, such that $Eu = u$. As before, we can choose $f \in Y_\alpha$ such that $f(u) = 1$. Let F be defined by $F(x) = f(x) \cdot u$, $x \in X$. $EF = F$, and so $FA \subset EA$. Since E is minimal, then $EA = FA$. It follows that E is an operator with one-dimensional range. Then there exists $g \in Y_\alpha$, $g(u) = 1$, such that $E(x) = g(x) \cdot u$ for all $x \in X$. Given $T \in A$, we denote the restriction of T to X_λ by T_λ. If T is a Fredholm element of A, then there exists $R \in A$ and U, $V \in S_A$ such that $TR = 1 - U$ and $RT = 1 - V$. Taking into account the form of the minimal idempotents of A, it follows that $U(X_\lambda) = 0$ and

$V(X_\lambda) = 0$ for all but a finite number of $\lambda \in \Lambda$. Furthermore, each T_λ is a Fredholm element of $\mathscr{B}[X_\lambda]$. The generalized index of T is given by

$$\kappa(T) = \sum_{\lambda \in \Lambda} \kappa_\lambda(T_\lambda),$$

where κ_λ is the usual index of T_λ on X_λ.

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