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A Garden of Eden theorem for linear subshifts

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Abstract. Let G be an amenable group and let V be a finite-dimensional vector space over an arbitrary field $\mathbb{K}$. We prove that if $X \subset V^G$ is a strongly irreducible linear subshift of finite type and $\tau : X \to X$ is a linear cellular automaton, then τ is surjective if and only if it is pre-injective. We also prove that if G is countable and $X \subset V^G$ is a strongly irreducible linear subshift, then every injective linear cellular automaton $\tau : X \to X$ is surjective.

1. *Introduction*

The goal of this article is to give a version of the Moore–Myhill Garden of Eden theorem for linear cellular automata defined over certain linear subshifts. Before stating our main results, let us briefly recall some basic notions from symbolic dynamics.

Consider a group G and a set A. The set $A^G = \{x : G \to A\}$ is called the set of *configurations* over the group G and the *alphabet* A. We equip $A^G = \prod_{g \in G} A$ with its *prodiscrete* topology, i.e., with the product topology obtained by taking the discrete topology on each factor A of A^G. We also endow A^G with the left action of G defined by $gx(h) = x(g^{-1}h)$ for all $g, h \in G$ and $x \in A^G$. This action is continuous with respect to the prodiscrete topology and is called the *G-shift* action on A^G. It is customary to refer to the G-space A^G as the *full shift* over the group G and the alphabet A.

A closed G-invariant subset of A^G is called a *subshift*.

For $x \in A^G$ and $\Omega \subset G$, let $x|_\Omega$ denote the restriction of x to Ω. One says that a subshift $X \subset A^G$ is *irreducible* if, for every finite subset $\Omega \subset G$ and any two configurations x_1 and x_2 in X, there exist an element $g \in G$ and a configuration $x \in X$ such that $x|_\Omega = x_1|_\Omega$ and $x|_{g\Omega} = x_2|_{g\Omega}$.

Given a finite subset $\Delta \subset G$, one says that a subshift $X \subset A^G$ is Δ-irreducible if the following condition is satisfied: if Ω_1 and Ω_2 are finite subsets of G such that there is no element $g \in \Omega_2$ such that the set $g\Delta$ meets Ω_1 then, given any two configurations $x_1, x_2 \in X$, there exists a configuration $x \in X$ such that $x|_{\Omega_1} = x_1|_{\Omega_1}$ and $x|_{\Omega_2} = x_2|_{\Omega_2}$. A subshift $X \subset A^G$ is said to be *strongly irreducible* if there exists a finite subset $\Delta \subset G$ such that X is Δ-irreducible (cf. [**12**, Definition 4.1]). Note that if G is infinite then every strongly irreducible subshift is irreducible. A trivial example of a strongly irreducible subshift is provided by the full shift A^G, which is Δ-irreducible for $\Delta = \{1_G\}$.

If D is a finite subset of G and L is a subset of A^D, then

$$X_G(D, L) = \{x \in A^G : (g^{-1}x)|_D \in L \text{ for all } g \in G\} \tag{1.1}$$

is clearly a subshift of A^G. A subshift $X \subset A^G$ is said to be *of finite type* if there exist a finite subset $D \subset G$ and a subset $L \subset A^D$ such that $X = X_G(D, L)$. One then says that the finite subset $D \subset G$ is a *defining window* and that $L \subset A^D$ is a *defining law*, relative to the defining window D, for the subshift X. Note that the full shift A^G is a subshift of finite type of itself admitting $D = \{1_G\}$ as a defining window with defining law $L = A^D \cong A$.

A map $\tau : X \to Y$ between subshifts $X, Y \subset A^G$ is called a *cellular automaton* if there exist a finite subset $M \subset G$ and a map $\mu : A^M \to A$ such that

$$\tau(x)(g) = \mu((g^{-1}x)|_M) \quad \text{for all } x \in X \text{ and } g \in G. \tag{1.2}$$

Such a set M is then called a *memory set* and μ is called a *local defining map* for τ. It immediately follows from the above definition that every cellular automaton $\tau : X \to Y$ is continuous and G-equivariant.

If $\tau : X \to X$ is a cellular automaton from a subshift $X \subset A^G$ into itself, a configuration $x_0 \in X$ is called a *Garden of Eden configuration* for τ if x_0 is not in the image of τ. The origin of this biblical terminology comes from the fact that a configuration $x_0 \in X$ is a Garden of Eden configuration for τ if and only if, whatever the choice of an initial configuration $x \in X$, the sequence of its iterates $x, \tau(x), \tau^2(x), \ldots, \tau^n(x), \ldots$ can only take the value x_0 at time $n = 0$.

Two configurations in A^G are said to be *almost equal* if they coincide outside a finite subset of G. One says that a cellular automaton $\tau : X \to Y$ between subshifts $X, Y \subset A^G$ is *pre-injective* if whenever two configurations $x_1, x_2 \in X$ are almost equal and satisfy $\tau(x_1) = \tau(x_2)$ then one has $x_1 = x_2$. Injectivity clearly implies pre-injectivity but there are pre-injective cellular automata that are not injective.

The classical *Garden of Eden theorem* [**8**] states that if $\tau : A^G \to A^G$ is a cellular automaton defined on the full shift over an amenable group G and a finite alphabet A, then the surjectivity of τ (i.e., the absence of Garden of Eden configurations for τ) is equivalent to its pre-injectivity (see §2.2 for the definition of amenability).

The Garden of Eden theorem was extended by Fiorenzi to cellular automata $\tau : X \to X$ for subshifts $X \subset A^G$ with A finite in the following two cases: (1) $G = \mathbb{Z}$ and $X \subset A^{\mathbb{Z}}$ is an irreducible subshift of finite type [**11**, Corollary 2.19]; and (2) G is a finitely generated amenable group and $X \subset A^G$ is a strongly irreducible subshift of finite type [**12**, Corollary 4.8].

Now let G be a group, $\mathbb{K}$ a field, and V a vector space over $\mathbb{K}$. Then there is a natural product vector space structure on V^G and the shift action of G on V^G is clearly $\mathbb{K}$-linear with respect to this vector space structure. One says that a subshift $X \subset V^G$ is a *linear subshift* if X is a vector subspace of V^G. Given linear subshifts $X, Y \subset V^G$, a cellular automaton $\tau : X \to Y$ is called a *linear cellular automaton* if the map τ is $\mathbb{K}$-linear. Note that if $X, Y \subset V^G$ are linear subshifts and $\tau : X \to Y$ is a linear cellular automaton, then the pre-injectivity of τ is equivalent to the fact that the zero configuration is the unique configuration with finite support lying in the kernel of τ.

In [**2**, Theorem 1.2] and [**5**, Corollary 1.4], we proved the following linear version of the Garden of Eden theorem.

THEOREM 1.1. *Let G be an amenable group, $\mathbb{K}$ a field, and V a finite-dimensional vector space over $\mathbb{K}$. Let $\tau : V^G \to V^G$ be a linear cellular automaton. Then τ is surjective if and only if it is pre-injective.*

The main result of the present paper is the following theorem.

THEOREM 1.2. *Let G be an amenable group, $\mathbb{K}$ a field, and V a finite-dimensional vector space over $\mathbb{K}$. Let $X \subset V^G$ be a strongly irreducible linear subshift of finite type and let $\tau : X \to X$ be a linear cellular automaton. Then τ is surjective if and only if it is pre-injective.*

Note that Theorem 1.1 may be recovered from Theorem 1.2 by taking $X = V^G$.

A group G is said to be *surjunctive* [**14**] if, for any finite alphabet A, every injective cellular automaton $\tau : A^G \to A^G$ over G is surjective. It was shown by Lawton (cf. [**14**]) that all residually finite groups are surjunctive. On the other hand, as injectivity implies pre-injectivity, it immediately follows from the Garden of Eden theorem [**8**] that all amenable groups are surjunctive. More generally, Gromov [**16**] and Weiss [**23**] proved that all *sofic* groups are surjunctive. The class of sofic groups includes in particular all residually amenable groups and therefore all residually finite groups as well as all amenable groups. As far as we know, there is no example of a non-surjunctive nor even of a non-sofic group in the literature up to now.

By analogy with the classical finite alphabet case, the following definition was introduced in [**3**, Definition 1.1]. A group G is said to be *L-surjunctive* if, for any field $\mathbb{K}$ and any finite-dimensional vector space V over $\mathbb{K}$, every injective linear cellular automaton $\tau : V^G \to V^G$ is surjective. It turns out (see [**3**]) that a group G is L-surjunctive if and only if G satisfies Kaplansky's stable finiteness conjecture, that is, the group algebra $\mathbb{K}[G]$ is stably finite for any field $\mathbb{K}$ (recall that a ring R is said to be *stably finite* if every one-sided invertible square matrix over R is also two-sided invertible). A linear analogue of the Gromov–Weiss theorem, namely that all sofic groups are L-surjunctive, was established in [**3**, Theorem 1.2]. From this result we deduced that sofic groups satisfy the Kaplansky conjecture on the stable finiteness of group algebras, a result previously established—with completely different methods involving embeddings of the group rings into continuous von Neumann regular rings—by Elek and Szabó [**10**].

Now, given a group G and a vector space V over a field $\mathbb{K}$, let us say that a linear subshift $X \subset V^G$ is *L-surjunctive* if every injective linear cellular automaton $\tau : X \to X$ is surjective. The following result is an immediate consequence of Theorem 1.2.

COROLLARY 1.3. *Let G be an amenable group, $\mathbb{K}$ a field, and V a finite-dimensional vector space over $\mathbb{K}$. Then every strongly irreducible linear subshift of finite type $X \subset V^G$ is L-surjunctive.*

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In fact, when the amenable group G is assumed to be countable, we can remove the hypothesis that the subshift X is of finite type in the previous statement so that we get the following theorem.

THEOREM 1.4. *Let G be a countable amenable group, $\mathbb{K}$ a field, and V a finite-dimensional vector space over $\mathbb{K}$. Then every strongly irreducible linear subshift $X \subset V^G$ is L-surjunctive.*

We do not know whether Theorem 1.4 remains true if the countability assumption is removed.

The paper is organized as follows. Section 2 contains the necessary preliminaries and background material. We recall in particular the definition and main properties of mean dimension for vector subspaces of configurations $X \subset V^G$, where G is an amenable group and V a finite-dimensional vector space. In §3 we study mean dimension of strongly irreducible linear subshifts. We prove that if $X \subset V^G$ is a strongly irreducible linear subshift then the mean dimension of X is greater than the mean dimension of any proper linear subshift $Y \subsetneqq X$ (Proposition 3.2). This result implies in particular that every non-zero strongly irreducible linear subshift has positive mean dimension (Corollary 3.3). In §4 we use the Mittag–Leffler lemma for projective sequences of sets to prove that if G is a countable group, V a finite-dimensional vector space, and $X \subset V^G$ a linear subshift, then every linear cellular automaton $\tau : X \to V^G$ has a closed image in V^G for the prodiscrete topology. This enables us to prove Theorem 1.4. The closed image property of linear cellular automata is extended to possibly uncountable groups in §5 under the additional hypothesis that the source linear subshift X has finite type. The proof of our Garden of Eden theorem (Theorem 1.2) is given in §6. It consists in showing that both the surjectivity and the pre-injectivity of τ are equivalent to the fact that the linear subshifts X and $\tau(X)$ have the same mean dimension (cf. Corollary 6.4). In the last two sections we describe some examples of linear cellular automata which are either pre-injective but not surjective or surjective but not pre-injective.

2. *Preliminaries and background*

In this section we collect some preliminaries and background material that will be needed in the following (for more details the reader is referred to [**6**]).

2.1. *Neighborhoods.*

(See [**2**, §2] and [**6**, §5.4].) Let G be a group. Let E and Ω be subsets of G. The *E-neighborhood* of Ω is the subset $\Omega^{+E} \subset G$ consisting of all elements $g \in G$ such that the set gE meets Ω. Thus, one has

$$\Omega^{+E} = \{g \in G : gE \cap \Omega \neq \varnothing\} = \bigcup_{e \in E} \Omega e^{-1} = \Omega E^{-1}.$$

Remark. The definition of Δ-irreducibility given in the introduction may be reformulated by saying that, given a group G, a set A, and a finite subset $\Delta \subset G$, a subshift $X \subset A^G$ is *Δ-irreducible* if the following condition is satisfied: if Ω_1 and Ω_2 are finite subsets of G such that $\Omega_1^{+\Delta} \cap \Omega_2 = \varnothing$ then, given any two configurations $x_1, x_2 \in X$, there exists a configuration $x \in X$ such that x coincides with x_1 on Ω_1 and with x_2 on Ω_2.

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The following facts will be frequently used in the following proposition.

PROPOSITION 2.1. *Let G be a group. Let E and Ω be subsets of G. Then the following hold:*

(i) *if $1_G \in E$, then $\Omega \subset \Omega^{+E}$;*

(ii) *if $g \in G$, then $g(\Omega^{+E}) = (g\Omega)^{+E}$ so that we can omit parentheses and simply write $g\Omega^{+E}$ instead; and*

(iii) *if Ω and E are finite, then Ω^{+E} is finite.*

Proof. This immediately follows from the definition of Ω^+E. □

PROPOSITION 2.2. *Let G be a group and let A be a set. Let $\tau : A^G \to A^G$ be a cellular automaton with memory set M. Suppose that there is a subset $\Omega \subset G$ and two configurations $x, x' \in A^G$ such that x and x' coincide on Ω. Then the configurations $\tau(x)$ and $\tau(x')$ coincide outside $(G \backslash \Omega)^{+M}$.*

Proof. It suffices to observe that (1.2) implies that $\tau(x)(g)$ depends only on the restriction of x to gM. □

2.2. *Amenable groups.* (See for example [**6**, Ch. 4] and [**15, 22**].) A group G is said to be *amenable* if there exists a left-invariant finitely additive probability measure defined on the set $\mathcal{P}(G)$ of all subsets of G, that is, a map $m : \mathcal{P}(G) \to [0, 1]$ satisfying the following conditions:

(A-1) $m(A \cup B) = m(A) + m(B) - m(A \cap B)$ for all $A, B \in \mathcal{P}(G)$ (finite additivity);

(A-2) $m(G) = 1$ (normalization); and

(A-3) $m(gA) = m(A)$ for all $g \in G$ and $A \in \mathcal{P}(G)$ (left-invariance).

By a fundamental result of Følner [**13**], a group G is amenable if and only if it admits a net $\mathcal{F} = (F_j)_{j \in J}$ consisting of non-empty finite subsets $F_j \subset G$ indexed by a directed set J such that

$$\lim_j \frac{|F_j^{+E} \backslash F_j|}{|F_j|} = 0 \quad \text{for every finite subset } E \subset G, \tag{2.1}$$

where we use $|\cdot|$ to denote cardinality of finite sets. Such a net $\mathcal{F}$ is called a *right Følner net* for G.

All finite groups, all solvable groups, and all finitely generated groups of subexponential growth are amenable. On the other hand, if a group G contains a non-abelian free subgroup then G is not amenable.

2.3. *Tilings.* (See [**2**, §2] and [**6**, §5.6].) Let G be a group. Let E and F be subsets of G. A subset $T \subset G$ is called an (E, F)-*tiling* if it satisfies the following conditions:

(T-1) the subsets gE, $g \in T$, are pairwise disjoint; and

(T-2) $G = \bigcup_{g \in T} gF$.

Note that if T is an (E, F)-tiling then it is also an (E', F')-tiling for all E', F' such that $E' \subset E$ and $F \subset F' \subset G$.

An easy consequence of Zorn's lemma is the following.

LEMMA 2.3. *Let G be a group. Let E be a non-empty subset of G and let $F = EE^{-1} = \{ab^{-1} : a, b \in E\}$. Then G contains an (E, F)-tiling.*

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Proof. See [**2**, Lemma 2.2]. □

In amenable groups we shall use the following lower estimate for the asymptotic growth of tilings with respect to Følner nets.

LEMMA 2.4. *Let G be an amenable group and let $(F_j)_{j\in J}$ be a right Følner net for G. Let E and F be finite subsets of G and suppose that $T \subset G$ is an (E, F)-tiling. For each $j \in J$, let T_j be the subset of T defined by $T_j = \{g \in T : gE \subset F_j\}$. Then there exist a real number $\alpha > 0$ and an element $j_0 \in J$ such that $|T_j| \geq \alpha |F_j|$ for all $j \geq j_0$.*

Proof. See [**2**, Lemma 4.3]. □

2.4. *Mean dimension.* Let G be an amenable group, $\mathcal{F} = (F_j)_{j\in J}$ a right Følner net for G, and V a finite-dimensional vector space over some field $\mathbb{K}$. Given a subset $\Omega \subset G$, we shall denote by $\pi_\Omega : V^G \to V^\Omega$ the projection map. Observe that π_Ω is $\mathbb{K}$-linear for every $\Omega \subset G$. Observe also that the vector space V^Ω is finite-dimensional if Ω is a finite subset of G.

Let X be a vector subspace of V^G. The *mean dimension* $\mathrm{mdim}_{\mathcal{F}}(X)$ of X with respect to the right Følner net $\mathcal{F}$ is defined by

$$\mathrm{mdim}_{\mathcal{F}}(X) = \limsup_j \frac{\dim(\pi_{F_j}(X))}{|F_j|}, \tag{2.2}$$

where we use $\dim(\cdot)$ to denote dimension of finite-dimensional $\mathbb{K}$-vector spaces.

It immediately follows from this definition that $\mathrm{mdim}_{\mathcal{F}}(V^G) = \dim(V)$ and that $\mathrm{mdim}_{\mathcal{F}}(X) \leq \mathrm{mdim}_{\mathcal{F}}(Y)$ whenever X and Y are vector subspaces of V^G such that $X \subset Y$. In particular, we have $0 \leq \mathrm{mdim}_{\mathcal{F}}(X) \leq \dim(V)$ for every vector subspace $X \subset V^G$.

An important property of linear cellular automata is the fact that they cannot increase mean dimension of vector subspaces.

PROPOSITION 2.5. *Let G be an amenable group, $\mathcal{F} = (F_j)_{j\in J}$ a right Følner net for G, and V a finite-dimensional vector space over a field $\mathbb{K}$. Let $\tau : V^G \to V^G$ be a linear cellular automaton and let $X \subset V^G$ be a vector subspace. Then one has* $\mathrm{mdim}_{\mathcal{F}}(\tau(X)) \leq \mathrm{mdim}_{\mathcal{F}}(X)$.

Proof. See [**2**, Proposition 4.7]. □

Remark. It may be shown that if G is an amenable group, $\mathcal{F}$ a right Følner net, V a finite-dimensional vector space, and $X \subset V^G$ a linear subshift, then the lim sup in the definition of $\mathrm{mdim}_{\mathcal{F}}(X)$ is in fact a true limit and that $\mathrm{mdim}_{\mathcal{F}}(X)$ is independent of the choice of the right Følner net $\mathcal{F}$ for G. These two important facts can be deduced from the theory of quasi-tiles in amenable groups developed by Ornstein and Weiss in [**21**] (see [**9, 17, 20**]). However, we do not need them in the present paper.

2.5. *Reversible linear cellular automata.* Let G be a group and let A be a set. A cellular automaton $\tau : X \to Y$ between subshifts $X, Y \subset A^G$ is said to be *reversible* if τ is bijective and the inverse map $\tau^{-1} : Y \to X$ is also a cellular automaton.

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It is well known that every bijective linear cellular automaton $\tau : X \to Y$ between subshifts $X, Y \subset A^G$ is reversible when the alphabet A is finite (this may be easily deduced from the compactness of A^G and the Curtis–Hedlund theorem [**19**], which says that, when the alphabet A is finite, a map between subshifts of A^G is a cellular automaton if and only if it is continuous and G-equivariant). On the other hand, if G contains an element of infinite order and A is infinite then one can construct a bijective cellular automaton $\tau : A^G \to A^G$ that is not reversible (see [**7**, Corollary 1.2]). Similarly, if G contains an element of infinite order and V is an infinite-dimensional vector space then one can construct a bijective linear cellular automaton $\tau : V^G \to V^G$ that is not reversible (see [**7**, Theorem 1.1]).

The following result is proved in [**3**].

THEOREM 2.6. *Let G be a countable group, V a finite-dimensional vector space over a field $\mathbb{K}$, and $X, Y \subset V^G$ two linear subshifts. Then every bijective linear cellular automaton $\tau : X \to Y$ is reversible.*

Proof. See [**3**, Theorem 3.1]. □

We will use the fact that mean dimension of linear subshifts is preserved by reversible linear cellular automata.

PROPOSITION 2.7. *Let G be an amenable group, $\mathcal{F} = (F_j)_{j\in J}$ a right Følner net for G, and V a finite-dimensional vector space over a field $\mathbb{K}$. Let $X, Y \subset V^G$ be two linear subshifts. Suppose that there exists a reversible linear cellular automaton $\tau : X \to Y$. Then one has* $\operatorname{mdim}_{\mathcal{F}}(X) = \operatorname{mdim}_{\mathcal{F}}(Y)$.

Proof. As $\tau : X \to Y$ is a surjective linear cellular automaton, we have $\operatorname{mdim}_{\mathcal{F}}(Y) \le \operatorname{mdim}_{\mathcal{F}}(X)$ by Proposition 2.5. Similarly, we have $\operatorname{mdim}_{\mathcal{F}}(X) \le \operatorname{mdim}_{\mathcal{F}}(Y)$ since $\tau^{-1} : Y \to X$ is a surjective linear cellular automaton. Thus we have $\operatorname{mdim}_{\mathcal{F}}(X) = \operatorname{mdim}_{\mathcal{F}}(Y)$. □

By combining Theorem 2.6 and Proposition 2.7, we get the following corollary.

COROLLARY 2.8. *Let G be a countable amenable group, $\mathcal{F} = (F_j)_{j\in J}$ a right Følner net for G, and V a finite-dimensional vector space over a field $\mathbb{K}$. Let $X, Y \subset V^G$ be two linear subshifts. Suppose that there exists a bijective linear cellular automaton $\tau : X \to Y$. Then one has* $\operatorname{mdim}_{\mathcal{F}}(X) = \operatorname{mdim}_{\mathcal{F}}(Y)$.

3. *Mean dimension of strongly irreducible linear subshifts*

This section contains results on mean dimension of strongly irreducible linear subshifts. We start with a slightly technical lemma, which will also be used in the next section.

LEMMA 3.1. *Let G be an amenable group, $\mathcal{F} = (F_j)_{j\in J}$ a right Følner net for G, and V a finite-dimensional vector space over a field $\mathbb{K}$. Let $X \subset V^G$ be a strongly irreducible linear subshift and let Δ be a finite subset of G such that $1_G \in \Delta$ and X is Δ-irreducible. Let D, E and F be finite subsets of G with $D^{+\Delta} \subset E$. Suppose that $T \subset G$ is an (E, F)-tiling and that Z is a vector subspace of X such that*

$$\pi_{gD}(Z) \subsetneqq \pi_{gD}(X) \tag{3.1}$$

for all $g \in T$. Then one has $\operatorname{mdim}_{\mathcal{F}}(Z) < \operatorname{mdim}_{\mathcal{F}}(X)$.

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Proof. As in Lemma 2.4, let us define, for each $j \in J$, the subset $T_j \subset T$ by $T_j = \{g \in T : gE \subset F_j\}$. Observe that, for all $j \in J$ and $g \in T_j$, we have the inclusions $gD \subset gD^{+\Delta} \subset gE \subset F_j$. Denote, for $j \in J$ and $g \in T_j$, by $\pi_{gD}^{F_j} : V^{F_j} \to V^{gD}$ the natural projection map. Consider, for each $j \in J$, the vector subspace $\pi_{F_j}^*(X) \subset \pi_{F_j}(X)$ defined by

$$\pi_{F_j}^*(X) = \{q \in \pi_{F_j}(X) : \pi_{gD}^{F_j}(q) \in \pi_{gD}(Z) \text{ for all } g \in T_j\}.$$

We claim that

$$\dim(\pi_{F_j}^*(X)) \leq \dim(\pi_{F_j}(X)) - |T_j| \tag{3.2}$$

for all $j \in J$.

To prove our claim, let us fix an element $j \in J$ and suppose that $T_j = \{g_1, g_2, \ldots, g_m\}$, where $m = |T_j|$. Consider, for each $i \in \{0, 1, \ldots, m\}$, the vector subspace $\pi_{F_j}^{(i)}(X) \subset \pi_{F_j}(X)$ defined by

$$\pi_{F_j}^{(i)}(X) = \{q \in \pi_{F_j}(X) : \pi_{g_k D}^{F_j}(q) \in \pi_{g_k D}(Z) \text{ for all } 1 \leq k \leq i\}.$$

Note that

$$\pi_{F_j}^{(i)}(X) \subset \pi_{F_j}^{(i-1)}(X)$$

for all $i = 1, 2, \ldots, m$. Let us show that

$$\dim(\pi_{F_j}^{(i)}(X)) \leq \dim(\pi_{F_j}(X)) - i \tag{3.3}$$

for all $i \in \{0, 1, \ldots, m\}$. Since $\pi_{F_j}^{(m)}(X) = \pi_{F_j}^*(X)$, this will prove (3.2).

To establish (3.3), we use induction on i. For $i = 0$, we have $\pi_{F_j}^{(i)}(X) = \pi_{F_j}(X)$ so that there is nothing to prove. Suppose now that

$$\dim(\pi_{F_j}^{(i-1)}(X)) \leq \dim(\pi_{F_j}(X)) - (i-1)$$

for some $i \leq m - 1$. By hypothesis (3.1), we can find an element $p \in \pi_{g_i D}(X) \setminus \pi_{g_i D}(Z)$. As $(g_i D)^{+\Delta} = g_i D^{+\Delta} \subset g_i E$ and X is Δ-irreducible, there exists an element $x \in X$ such that $\pi_{g_i D}(x) = p$ and x is identically zero on $F_j \setminus g_i E$. Now observe that $\pi_{F_j}(x) \in \pi_{F_j}^{(i-1)}(X)$ since the sets $g_1 D, g_2 D, \ldots, g_{i-1} D$ are all contained in $F_j \setminus g_i E$. On the other hand, we have $\pi_{F_j}(x) \notin \pi_{F_j}^{(i)}(X)$ as $\pi_{g_i D}(x) = p \notin \pi_{g_i D}(Z)$. This shows that $\pi_{F_j}^{(i)}(X)$ is strictly contained in $\pi_{F_j}^{(i-1)}(X)$. Hence we have

$$\dim(\pi_{F_j}^{(i)}(X)) \leq \dim(\pi_{F_j}^{(i-1)}(X)) - 1 \leq (\dim(\pi_{F_j}(X)) - (i-1)) - 1 = \dim(\pi_{F_j}(X)) - i,$$

by using our induction hypothesis. This establishes (3.3) and therefore (3.2).

By Lemma 2.4, we can find a real number $\alpha > 0$ and an element $j_0 \in J$ such that $|T_j| \geq \alpha |F_j|$ for all $j \geq j_0$. Since $\pi_{F_j}(Z) \subset \pi_{F_j}^*(X)$, we deduce from (3.2) that

$$\dim(\pi_{F_j}(Z)) \leq \dim(\pi_{F_j}(X)) - \alpha |F_j| \quad \text{for all } j \geq j_0,$$

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so that

$$\begin{aligned}\operatorname{mdim}_{\mathcal{F}}(Z) &= \limsup_j \frac{\dim(\pi_{F_j}(Z))}{|F_j|}\\ &\leq \limsup_j \frac{\dim(\pi_{F_j}(X))}{|F_j|} - \alpha\\ &= \operatorname{mdim}_{\mathcal{F}}(X) - \alpha\\ &< \operatorname{mdim}_{\mathcal{F}}(X).\end{aligned}$$

□

PROPOSITION 3.2. *Let G be an amenable group, $\mathcal{F} = (F_j)_{j \in J}$ a right Følner net for G, and V a finite-dimensional vector space over a field $\mathbb{K}$. Let $X \subset V^G$ be a strongly irreducible linear subshift and $Y \subset V^G$ a linear subshift such that $Y \subsetneqq X$. Then one has* $\operatorname{mdim}_{\mathcal{F}}(Y) < \operatorname{mdim}_{\mathcal{F}}(X)$.

Proof. As $Y \subsetneqq X$ and Y is closed in V^G for the prodiscrete topology, we can find a finite subset $D \subset G$ such that $\pi_D(Y) \subsetneqq \pi_D(X)$. By the G-invariance of X and Y, this implies that

$$\pi_{gD}(Y) \subsetneqq \pi_{gD}(X)$$

for all $g \in G$.

Let Δ be a finite subset of G such that $1_G \in \Delta$ and X is Δ-irreducible, and take $E = D^{+\Delta}$. By virtue of Lemma 2.3, we can find a finite subset $F \subset G$ and an (E, F)-tiling $T \subset G$. Then, by taking $Z = Y$, all the hypotheses in Lemma 3.1 are satisfied so that we get $\operatorname{mdim}_{\mathcal{F}}(Y) < \operatorname{mdim}_{\mathcal{F}}(X)$. □

COROLLARY 3.3. *Let G be an amenable group, $\mathcal{F} = (F_j)_{j \in J}$ a right Følner net for G, and V a finite-dimensional vector space over a field $\mathbb{K}$. Let $X \subset V^G$ be a non-zero strongly irreducible linear subshift. Then one has* $\operatorname{mdim}_{\mathcal{F}}(X) > 0$.

Proof. It suffices to apply Proposition 3.2 by taking $Y = \{0\}$. □

Corollary 3.3 becomes false if we suppress the hypothesis that X is strongly irreducible even for irreducible linear subshifts of finite type, as the following example shows.

Example 3.1. Take $G = \mathbb{Z}^2$ and the Følner sequence $\mathcal{F} = (F_n)_{n \geq 1}$ given by $F_n = \{0, 1, \ldots, n-1\}^2$ for all $n \geq 1$. Let $\mathbb{K}$ be a field, V a non-zero finite-dimensional vector space over $\mathbb{K}$, and consider the subset $X \subset V^G$ defined by

$$X = \{x \in V^G : x(g) = x(h) \text{ for all } g, h \in G \text{ such that } \rho(g) = \rho(h)\},$$

where $\rho : \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ denotes the projection onto the second factor. In other words, X consists of the configurations that are constant on each horizontal line in $\mathbb{Z}^2$. Observe that X is a linear subshift of finite type with defining window $D = \{(0, 0), (1, 0)\}$ and defining law $L = \{y \in V^D : y(0, 0) = y(1, 0)\}$. On the other hand, X is irreducible. Indeed, this immediately follows from the fact that if Ω is a finite subset of G, then we can translate Ω vertically to get a subset $\Omega' \subset \mathbb{Z}^2$ such that Ω and Ω' have disjoint images under the projection ρ.

However, we have $\dim(\pi_{F_n}(X)) = n \dim(V)$ and $|F_n| = n^2$ for all $n \geq 1$ so that $\operatorname{mdim}_{\mathcal{F}}(X) = \lim_{n \to \infty} n^{-1} \dim(V) = 0$.

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4. *The Mittag–Leffler lemma and the closed image property for linear subshifts*

This section contains the proof of Theorem 1.4.

Let G be a group and let A be a set.

Suppose first that A is finite and let $\tau : X \to A^G$ be a cellular automaton, where $X \subset A^G$ is a subshift. It immediately follows from the compactness of X and the continuity of τ that the image $\tau(X)$ is closed in A^G for the prodiscrete topology. As τ is G-equivariant, we deduce that $\tau(X)$ is a subshift of A^G.

If G contains an element of infinite order and A is infinite then one can construct a cellular automaton $\tau : A^G \to A^G$ whose image is not closed in A^G (see [**7**, Corollary 1.4]). Similarly, if G contains an element of infinite order and V is an infinite-dimensional vector space then one can construct a linear cellular automaton $\tau : V^G \to V^G$ whose image is not closed in V^G (see [**7**, Theorem 1.3]).

On the other hand, if $A = V$ is a finite-dimensional vector space over a field $\mathbb{K}$ and $\tau : V^G \to V^G$ is a linear cellular automaton, then the image of τ is closed in V^G (see [**2**, Lemma 3.1] for G countable and [**5**, Corollary 1.6] in the general case; see also [**16**, §4.D]). As τ is G-equivariant and $\mathbb{K}$-linear, this implies that $\tau(V^G)$ is a linear subshift of V^G.

In this section we extend this last result to linear cellular automata $\tau : X \to V^G$, where G is a countable group, V is a finite-dimensional vector space, and $X \subset V^G$ is a linear subshift. The key point in the proof relies on a general well-known result, namely the Mittag–Leffler lemma for projective sequences of sets. This version of the Mittag–Leffler lemma may be easily deduced from Theorem 1 in [**1**, TG II, §5] (see also [**18**, §I.3]). We give an independent proof here for the convenience of the reader. Let us first recall a few facts about projective limits of projective sequences in the category of sets.

Let $\mathbb{N}$ denote the set of non-negative integers. A *projective sequence* of sets consists of a sequence $(X_n)_{n\in\mathbb{N}}$ of sets together with maps $f_{nm} : X_m \to X_n$ defined for all $m \geq n$ that satisfy the following conditions:

(PS-1) f_{nn} is the identity map on X_n for all $n \in \mathbb{N}$; and

(PS-2) $f_{nk} = f_{nm} \circ f_{mk}$ for all $n, m, k \in \mathbb{N}$ such that $k \geq m \geq n$.

Such a projective sequence will be denoted (X_n, f_{nm}) or simply (X_n). The *projective limit* $\varprojlim X_n$ of the projective sequence (X_n, f_{nm}) is the subset of $\prod_{n\in\mathbb{N}} X_n$ consisting of the sequences $(x_n)_{n\in\mathbb{N}}$ satisfying $x_n = f_{nm}(x_m)$ for all $n, m \in \mathbb{N}$ such that $n \leq m$.

We say that the projective sequence (X_n) satisfies the *Mittag–Leffler condition* if, for each $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $f_{nk}(X_k) = f_{nm}(X_m)$ for all $k \geq m$.

LEMMA 4.1. (Mittag–Leffler) *If (X_n, f_{nm}) is a projective system of non-empty sets that satisfies the Mittag–Leffler condition then its projective limit $X = \varprojlim X_n$ is not empty.*

Proof. First observe that if (X_n, f_{nm}) is an arbitrary projective sequence of sets, then property (PS-2) implies that, for each $n \in \mathbb{N}$, the sequence of sets $f_{nm}(X_m)$, $m \geq n$, is non-increasing. The set $X'_n = \bigcap_{m\geq n} f_{nm}(X_m)$ is called the set of *universal elements* in X_n (cf. [**18**]). It is clear that the map f_{nm} induces by restriction a map $g_{nm} : X'_m \to X'_n$ for all $n \leq m$ and that (X'_n, g_{nm}) is a projective sequence having the same projective limit as the projective sequence (X_n, f_{nm}).

Suppose now that all the sets X_n are non-empty and that the projective sequence (X_n, f_{nm}) satisfies the Mittag–Leffler condition. This means that, for each $n \in \mathbb{N}$, there is an integer $m \geq n$ such that $f_{nk}(X_k) = f_{nm}(X_m)$ for all $k \geq m$. This implies that $X'_n = f_{nm}(X_m)$ so that, in particular, the set X'_n is not empty. We claim that the map $g_{n,n+1} : X'_{n+1} \to X'_n$ is surjective for every $n \in \mathbb{N}$. To see this, let $n \in \mathbb{N}$ and $x'_n \in X'_n$. By the Mittag–Leffler condition, we can find an integer $p \geq n+1$ such that $f_{nk}(X_k) = f_{np}(X_p)$ and $f_{n+1,k}(X_k) = f_{n+1,p}(X_p)$ for all $k \geq p$. It follows that $X'_n = f_{np}(X_p)$ and $X'_{n+1} = f_{n+1,p}(X_p)$. Consequently, we can find $x_p \in X_p$ such that $x'_n = f_{np}(x_p)$. Setting $x'_{n+1} = f_{n+1,p}(x_p)$, we have $x'_{n+1} \in X'_{n+1}$ and

$$g_{n,n+1}(x'_{n+1}) = f_{n,n+1}(x'_{n+1}) = f_{n,n+1} \circ f_{n+1,p}(x_p) = f_{np}(x_p) = x'_n.$$

This proves our claim that $g_{n,n+1}$ is onto. Now, as the sets X'_n are non-empty, we can construct by induction a sequence $(x'_n)_{n\in\mathbb{N}}$ such that $x'_n = g_{n,n+1}(x'_{n+1})$ for all $n \in \mathbb{N}$. This sequence is in the projective limit $\varprojlim X'_n = \varprojlim X_n$. This shows that $\varprojlim X_n$ is not empty. □

THEOREM 4.2. *Let G be a countable group and let V be a finite-dimensional vector space over a field $\mathbb{K}$. Let $\tau : X \to V^G$ be a linear cellular automaton, where $X \subset V^G$ is a linear subshift. Then $\tau(X)$ is closed in V^G for the prodiscrete topology and is therefore a linear subshift of V^G.*

Proof. Since G is countable, we can find a sequence $(A_n)_{n\in\mathbb{N}}$ of finite subsets of G such that $G = \bigcup_{n\in\mathbb{N}} A_n$ and $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$. Let M be a memory set for τ. Let $B_n = \{g \in G : gM \subset A_n\}$. Note that $G = \bigcup_{n\in\mathbb{N}} B_n$ and $B_n \subset B_{n+1}$ for all $n \in \mathbb{N}$. Denote by $\pi_{A_n} : V^G \to V^{A_n}$ and $\pi_{B_n} : V^G \to V^{B_n}$, $n \in \mathbb{N}$, the corresponding projection maps.

Since M is a memory set for τ, it follows from (1.2) that if x and x' are elements in X such that $\pi_{A_n}(x) = \pi_{A_n}(x')$ then $\pi_{B_n}(\tau(x)) = \pi_{B_n}(\tau(x'))$. Therefore, given $x_n \in \pi_{A_n}(X)$ and denoting by $\widetilde{x}_n$ any configuration in X such that $\pi_{A_n}(\widetilde{x}_n) = x_n$, the element

$$y_n = \pi_{B_n}(\tau(\widetilde{x}_n)) \in V^{B_n}$$

does not depend on the particular choice of the extension $\widetilde{x}_n$. Thus we can define a map $\tau_n : \pi_{A_n}(X) \to V^{B_n}$ by setting $\tau_n(x_n) = y_n$ for all $x_n \in \pi_{A_n}(X)$. It is clear that τ_n is $\mathbb{K}$-linear.

Let now $y \in V^G$ and suppose that y is in the closure of $\tau(X)$. Then, for all $n \in \mathbb{N}$, there exists $z_n \in X$ such that

$$\pi_{B_n}(y) = \pi_{B_n}(\tau(z_n)). \tag{4.1}$$

Consider, for each $n \in \mathbb{N}$, the affine subspace $X_n \subset \pi_{A_n}(X)$ defined by $X_n = \tau_n^{-1}(\pi_{B_n}(y))$. We have $X_n \neq \varnothing$ for all n by (4.1). For $n \leq m$, the restriction map $\pi_{A_m}(X) \to \pi_{A_n}(X)$ induces an affine map $f_{nm} : X_m \to X_n$. Conditions (PS-1) and (PS-2) are trivially satisfied so that (X_n, f_{nm}) is a projective sequence. We claim that (X_n, f_{nm}) satisfies the Mittag–Leffler condition. Indeed, consider, for all $n \leq m$, the affine subspace $f_{nm}(X_m) \subset X_n$. We have $f_{nm'}(X_{m'}) \subset f_{nm}(X_m)$ for all $n \leq m \leq m'$ since $f_{nm'} = f_{nm} \circ f_{mm'}$. As the sequence $f_{nm}(X_m)$ $(m = n, n+1, \ldots)$ is a non-increasing sequence of finite-dimensional affine subspaces, it stabilizes, i.e., for each $n \in \mathbb{N}$ there exists an integer $m \geq n$ such that $f_{nk}(X_k) = f_{nm}(X_m)$ if $k \geq m$. Thus, the Mittag–Leffler condition is satisfied. It follows from Lemma 4.1 that the projective limit $\varprojlim X_n$ is non-empty. Choose an element $(x_n)_{n\in\mathbb{N}} \in \varprojlim X_n$. We have that x_{n+1} coincides with x_n on A_n and that $x_n \in \pi_{A_n}(X)$

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for all $n \in \mathbb{N}$. As X is closed in V^G and $G = \bigcup_{n\in\mathbb{N}} A_n$, we deduce that there exists a (unique) configuration $x \in X$ such that $x|_{A_n} = x_n$ for all n. We have $\tau(x)|_{B_n} = \tau_n(x_n) = y_n = y|_{B_n}$ for all n. Since $G = \bigcup_{n\in\mathbb{N}} B_n$, this shows that $\tau(x) = y$. □

COROLLARY 4.3. *Let G be a countable amenable group, $\mathcal{F} = (F_j)_{j\in J}$ a right Følner net for G, and V a finite-dimensional vector space over a field $\mathbb{K}$. Let $\tau : X \to Y$ be a linear cellular automaton, where $X, Y \subset V^G$ are linear subshifts such that* $\mathrm{mdim}_{\mathcal{F}}(X) = \mathrm{mdim}_{\mathcal{F}}(Y)$ *and Y is strongly irreducible. Then the following conditions are equivalent:*
(a) *τ is surjective; and*
(b) $\mathrm{mdim}_{\mathcal{F}}(\tau(X)) = \mathrm{mdim}_{\mathcal{F}}(X)$.

Proof. The implication (a) ⇒ (b) is trivial. Conversely, suppose that $\mathrm{mdim}_{\mathcal{F}}(\tau(X)) = \mathrm{mdim}_{\mathcal{F}}(X)$. Theorem 4.2 implies that $\tau(X)$ is a linear subshift of V^G. As $\tau(X) \subset Y$, it then follows from Proposition 3.2 that $\tau(X) = Y$. Thus, τ is surjective. □

Proof of Theorem 1.4. Let $X \subset V^G$ be a strongly irreducible linear subshift and suppose that $\tau : X \to X$ is an injective linear cellular automaton. Let us show that τ is surjective. Let $\mathcal{F} = (F_j)_{j\in J}$ be a right Følner net for G. We know that $\tau(X)$ is a linear subshift by Theorem 4.2. As τ induces a bijective linear cellular automaton from X onto $\tau(X)$, we have $\mathrm{mdim}_{\mathcal{F}}(\tau(X)) = \mathrm{mdim}_{\mathcal{F}}(X)$ by using Corollary 2.8. Since X is strongly irreducible, this implies that τ is surjective by Corollary 4.3. Thus X is L-surjunctive. □

5. *The closed image property for linear subshifts of finite type*

In this section we show that Theorem 4.2 remains true for any (possibly uncountable) group G if we add the hypothesis that the linear subshift $X \subset V^G$ is of finite type. The proof relies on the fact that a subshift of finite type can be factorized along the left cosets of any subgroup containing a defining window. In order to state this last result in a more precise way, let us first introduce some notation.

Let G be a group and let A be a set. Let H be a subgroup of G and denote by $G/H = \{gH : g \in G\}$ the set consisting of all left cosets of H in G. For every coset $c \in G/H$, we equip the set $A^c = \prod_{g\in c} A$ with its prodiscrete topology and we denote by $\pi_c : A^G \to A^c$ the projection map. Since the cosets $c \in G/H$ form a partition of G, we have a natural identification of topological spaces

$$A^G = \prod_{c\in G/H} A^c.$$

With this identification, we have $x = (x|_c)_{c\in G/H}$ for each $x \in A^G$, where $x|_c = \pi_c(x) \in A^c$ is the restriction of the configuration x to c.

Given a coset $c \in G/H$ and an element $g \in c$, let $\phi_g : H \to c$ denote the bijective map defined by $\phi_g(h) = gh$ for all $h \in H$. Then ϕ_g induces a homeomorphism $\phi_g^* : A^c \to A^H$ given by $\phi_g^*(y) = y \circ \phi_g$ for all $y \in A^c$.

PROPOSITION 5.1. *Let G be a group and let A be a set. Let $X \subset A^G$ be a subshift of finite type. Let $D \subset G$ be a defining window and $L \subset A^D$ a defining law for X, so that $X = X_G(D, L)$. Suppose that H is a subgroup of G such that $D \subset H$. Then one has:*

(i) $X = \prod_{c \in G/H} X_c$, *where* $X_c = \pi_c(X) \subset A^c$ *denotes the projection of* X *on* A^c;
(ii) $X_H = X_H(D, L)$; *and*
(iii) $\phi_g^*(X_c) = X_H$ *for all* $c \in G/H$ *and* $g \in c$.

Proof. In order to establish (i), it suffices to show that $\prod_{c \in G/H} X_c \subset X$ since the converse inclusion is trivial. Suppose that $\widetilde{x} = (\widetilde{x}|_c)_{c \in G/H} \in \prod_{c \in G/H} X_c$. Let $g \in G$ and consider the left coset $c = gH$. Then we can find $x \in X$ such that $\widetilde{x}|_c = x|_c$. As $gD \subset gH = c$, we have

$$(g^{-1}\widetilde{x})(d) = \widetilde{x}(gd) = x(gd) = (g^{-1}x)(d)$$

for all $d \in D$. It follows that $(g^{-1}\widetilde{x})|_D = (g^{-1}x)|_D \in L$ for all $g \in G$. We deduce that $\widetilde{x} \in X_G(D, L) = X$. This completes the proof of (i).

If $x \in X$ then $(h^{-1}x|_H)|_D = (h^{-1}x)|_D \in L$ for all $h \in H$. Thus, we have $X_H \subset X_H(D, L)$. Conversely, suppose that $y \in X_H(D, L)$. Choose a complete set of representatives $R \subset G$ for the left cosets of H in G and consider the configuration $x \in A^G$ defined by $x(rh) = y(h)$ for all $r \in R$ and $h \in H$. Then we clearly have $x \in X_G(D, L) = X$ and $x|_H = y$. Thus $X_H(D, L) \subset X_H$. This completes the proof of (ii).

Let now $c \in G/H$ and $g \in c$. In order to prove

$$\phi_g^*(X_c) \subset X_H, \tag{5.1}$$

let $y_c \in \phi_g^*(X_c)$. Then there exists a (unique) $x_c \in X_c$ such that $y_c = \phi_g^*(x_c)$. Let $x \in X$ such that $\pi_c(x) = x_c$. For all $h \in H$ and $d \in D$, we have

$$(h^{-1}y_c)(d) = y_c(hd) = x_c(ghd) = x(ghd) = (gh)^{-1}x(d),$$

so that $(h^{-1}y_c)|_D = ((gh)^{-1}x)|_D \in L$ since $x \in X = X_G(D, L)$. This shows that $y_c \in X_H(D, L) = X_H$ and (5.1) follows. Conversely, suppose that $x_H \in X_H$ and consider the configuration $x_c = \phi_g^*(x_H) \in A^c$. Let us show that $x_c \in X_c$. Since $x_H \in X_H$, we can find a configuration $x \in X$ such that $x_H = x|_H$. Setting $y = gx \in X$, we have

$$y(gh) = g^{-1}y(h) = x(h) = x_H(h) = x_c(gh),$$

for all $h \in H$. Thus $x_c = y|_c \in X_c$. This gives $X_H \subset \phi_g^*(X_c)$. From this and (5.1) we finally deduce (iii). □

COROLLARY 5.2. *Suppose that G is a group that is not finitely generated. Then:*
(i) *if* A *is a set and* $X \subset A^G$ *is a subshift of finite type that is not reduced to a single configuration then* X *is infinite; and*
(ii) *if* V *is a vector space over a field* $\mathbb{K}$ *and* $X \subset V^G$ *is a linear subshift of finite type that is not reduced to the zero configuration then* X *is infinite-dimensional (as a vector space over* $\mathbb{K}$*).*

Proof. Let A be a set, $X \subset A^G$ a subshift of finite type, and $D \subset G$ a defining window for X. Let H denote the subgroup of G generated by D. Observe that H is of infinite index in G since G is not finitely generated. With the above notation, we have $X = \prod_{c \in G/H} X_c$ by Proposition 5.1. Moreover, for all $c \in G/H$ and $g \in c$, we have $\phi_g^*(X_c) = X_H$. As all the maps ϕ_g^* are bijective, we deduce that X is either reduced to a single configuration or infinite. This proves (i).

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Suppose now that $A = V$ is a vector space over some field $\mathbb{K}$. Then X_c is a vector subspace of V^c and $\phi_g^* : X_c \to X_H$ is an isomorphism of $\mathbb{K}$-vector spaces for all $c \in G/H$ and $g \in c$. As $X = \prod_{c \in G/H} X_c$, we conclude that X is either reduced to the zero configuration or infinite-dimensional. This shows (ii). □

Example. Let G be a group that is not finitely generated and let V be a non-zero finite-dimensional vector space over a field $\mathbb{K}$. Suppose that G_0 is a finite index subgroup of G. Consider the linear subshift $X \subset V^G$ consisting of the configurations $x \in V^G$ which are fixed by each element of G_0. We clearly have $\dim(X) = [G : G_0] \dim(V) < \infty$. Thus X is not of finite type by Corollary 5.2(ii).

THEOREM 5.3. *Let G be a (possibly uncountable) group and let V be a finite-dimensional vector space over a field $\mathbb{K}$. Let $\tau : X \to V^G$ be a linear cellular automaton, where $X \subset V^G$ is a linear subshift of finite type. Then $\tau(X)$ is closed in V^G for the prodiscrete topology and is therefore a linear subshift of V^G.*

Proof. Let $M \subset G$ be a memory set and $\mu : V^M \to V$ a local defining map for τ. Also let $D \subset G$ be a defining window for X and denote by H the subgroup of G generated by M and D. Note that H is finitely generated since both M and D are finite sets.

Setting $X_c = \pi_c(X)$ for all $c \in G/H$, we have $X = \prod_{c \in G/H} X_c$ by Proposition 5.1. On the other hand, if $x \in X$, $c \in G/H$, and $g \in c$, then $\tau(x)(g)$ depends only on the restriction of x to c, since $gM \subset gH = c$. This implies that τ may be written as a product

$$\tau = \prod_{c \in G/H} \tau_c, \tag{5.2}$$

where $\tau_c : X_c \to V^c$ is the unique map that satisfies $\tau_c(x|_c) = (\tau(x))|_c$ for all $x \in X$. Note that $\tau_H : X_H \to V^H$ is the linear cellular automaton over H with memory set $M \subset H$ and local defining map μ.

Let us show that the maps τ_c and τ_H are conjugate by ϕ_g^*, that is,

$$\tau_c = (\phi_g^*)^{-1} \circ \tau_H \circ \phi_g^*. \tag{5.3}$$

Let $y \in X_c$ and let $x \in X$ extending x. For all $h \in H$, we have

$$\begin{aligned}(\phi_g^* \circ \tau_c)(y)(h) &= \phi_g^*(\tau_c(y))(h)\\ &= (\tau_c(y) \circ \phi_g)(h)\\ &= \tau_c(y)(gh)\\ &= \tau(x)(gh)\\ &= g^{-1}\tau(x)(h)\\ &= \tau(g^{-1}x)(h),\end{aligned}$$

where the last equality follows from the G-equivariance of τ. Now observe that the configuration $g^{-1}\widetilde{x} \in X$ extends $x \circ \phi_g \in X_H$. Thus, we have

$$(\phi_g^* \circ \tau_c)(x)(h) = \tau_H(x \circ \phi_g)(h) = \tau_H(\phi_g^*(x))(h) = (\tau_H \circ \phi_g^*)(x)(h).$$

This shows that $\phi_g^* \circ \tau_c = \tau_H \circ \phi_g^*$, which gives (5.3) since ϕ_g^* is bijective.

As the subgroup $H \subset G$ is finitely generated and therefore countable, we deduce from Theorem 4.2 that $\tau_H(X_H)$ is closed in V^H for the prodiscrete topology. Since ϕ_g^* is a homeomorphism, it follows that

$$\tau_c(X_c) = (\phi_g^*)^{-1}(\tau_H(X_H))$$

is closed in V^c for all $c \in G/H$. Thus,

$$\tau(X) = \prod_{c \in G/H} \pi_c(\tau(X)) = \prod_{c \in G/H} \tau_c(X_c)$$

is a closed subspace of V^G. □

COROLLARY 5.4. *Let G be a (possibly uncountable) amenable group, $\mathcal{F} = (F_j)_{j \in J}$ a right Følner net for G, and V a finite-dimensional vector space over a field $\mathbb{K}$. Let $\tau : X \to Y$ be a linear cellular automaton, where $X, Y \subset V^G$ are linear subshifts satisfying* $\mathrm{mdim}_{\mathcal{F}}(X) = \mathrm{mdim}_{\mathcal{F}}(Y)$. *Suppose that X is of finite type and that Y is strongly irreducible. Then the following conditions are equivalent:*

(a) *τ is surjective; and*

(b) $\mathrm{mdim}_{\mathcal{F}}(\tau(X)) = \mathrm{mdim}_{\mathcal{F}}(X)$.

Proof. The implication (a) $\Rightarrow$ (b) is trivial. Conversely, suppose that $\mathrm{mdim}_{\mathcal{F}}(\tau(X)) = \mathrm{mdim}_{\mathcal{F}}(X)$. Theorem 5.3 implies that $\tau(X)$ is a linear subshift of V^G. As $\tau(X) \subset Y$, it then follows from Proposition 3.2 that $\tau(X) = Y$. Thus, τ is surjective. □

6. *Proof of the Garden of Eden theorem*

This section contains the proof of Theorem 1.2. Let us start with the following theorem.

THEOREM 6.1. *Let G be an amenable group, $\mathcal{F} = (F_j)_{j \in J}$ a right Følner net for G, and V a finite-dimensional vector space over a field $\mathbb{K}$. Let $X \subset V^G$ be a strongly irreducible linear subshift of finite type and let $\tau : X \to V^G$ be a linear cellular automaton. Then the following conditions are equivalent:*

(a) *τ is pre-injective; and*

(b) $\mathrm{mdim}_{\mathcal{F}}(\tau(X)) = \mathrm{mdim}_{\mathcal{F}}(X)$.

For the proof of (a) $\Rightarrow$ (b) in Theorem 6.1, we shall use the following lemma.

LEMMA 6.2. *Let G be a group and let V be a finite-dimensional vector space over a field $\mathbb{K}$. Let $X \subset V^G$ be a strongly irreducible linear subshift of finite type and suppose that M is a finite subset of G such that X is M-irreducible, $1_G \in M$, and M^{-1} is a defining window for X. Then, given any configuration $x \in X$ and any finite subset $\Omega \subset G$, there exists a configuration $z \in X$ that coincides with x on Ω and is identically zero on $G \backslash \Omega^{+M}$.*

Proof. Let $x \in X$ and $\Omega \subset G$ a finite subset. Note that we have the inclusions $\Omega \subset \Omega^{+M} \subset \Omega^{+M^2} \subset \Omega^{+M^3}$ since $1_G \in M$. As both x and the zero configuration belong to X and X is M-irreducible, we can find a configuration $z' \in X$ that coincides with x on Ω and is identically zero on $\Omega^{+M^3} \backslash \Omega^{+M}$. Now consider the configuration $z \in V^G$ that coincides with z' on Ω^{+M^3} and is identically zero on $G \backslash \Omega^{+M^3}$. Observe that if $g \in \Omega^{+M^2}$ then

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$gM^{-1} \subset \Omega^{+M^3}$ and therefore z coincides with z' on gM^{-1}, while if $g \in G \backslash \Omega^{+M^2}$ then $gM^{-1} \subset G \backslash \Omega^{+M}$ and therefore z is identically zero on gM^{-1}. As both z' and the zero configuration belong to X and M^{-1} is a defining window for X, we deduce that $z \in X$. On the other hand, z coincides with x on Ω and is identically zero on $G \backslash \Omega^{+M}$. Consequently, z has the required properties. □

Proof of (a) ⇒ (b) in Theorem 6.1. Suppose that $\mathrm{mdim}_{\mathcal{F}}(\tau(X)) < \mathrm{mdim}_{\mathcal{F}}(X)$. Let $Y = \tau(X)$. Let $M \subset G$ be a memory set for τ. Up to enlarging the subset M if necessary, we can also suppose that $1_G \in M$ and that X is M-irreducible and admits M^{-1} as a defining window.

We first observe that $\pi_{F_j^{+M^2}}(Y)$ is a vector subspace of $\pi_{F_j}(Y) \times V^{F_j^{+M^2} \backslash F_j}$ so that we have

$$\dim(\pi_{F_j^{+M^2}}(Y)) \le \dim(\pi_{F_j}(Y)) + |F_j^{+M^2} \backslash F_j| \dim(V). \tag{6.1}$$

On the other hand, as $(F_j)_{j \in J}$ is a right Følner net for G, we have

$$\lim_j \frac{|F_j^{+M^2} \backslash F_j|}{|F_j|} = 0$$

by (2.1). Therefore, after dividing the two sides of (6.1) by $|F_j|$ and taking the lim sup over j, we get

$$\limsup_j \frac{\dim(\pi_{F_j^{+M^2}}(Y))}{|F_j|} \le \limsup_j \frac{\dim(\pi_{F_j}(Y))}{|F_j|} = \mathrm{mdim}_{\mathcal{F}}(Y).$$

As $\mathrm{mdim}_{\mathcal{F}}(Y) < \mathrm{mdim}_{\mathcal{F}}(X)$ by our assumption, this implies that there exists $j_0 \in J$ such that

$$\dim(\pi_{F_{j_0}^{+M^2}}(Y)) < \dim(\pi_{F_{j_0}}(X)). \tag{6.2}$$

Consider now the finite-dimensional vector subspace $Z \subset X$ consisting of all configurations $z \in X$ whose support $\{g \in G : z(g) \neq 0\}$ is contained in $F_{j_0}^{+M}$. By virtue of Lemma 6.2, we have

$$\pi_{F_{j_0}}(Z) = \pi_{F_{j_0}}(X). \tag{6.3}$$

On the other hand, we deduce from Proposition 2.2 that $\tau(z)$ is identically zero on $G \backslash F_{j_0}^{+M^2}$ for every $z \in Z$. Consequently, we have

$$\begin{aligned}
\dim(\tau(Z)) &= \dim(\pi_{F_{j_0}^{+M^2}}(\tau(Z))) \\
&\le \dim(\pi_{F_{j_0}^{+M^2}}(Y)) \\
&< \dim(\pi_{F_{j_0}}(X)) \quad \text{(by (6.2))} \\
&= \dim(\pi_{F_{j_0}}(Z)) \quad \text{(by (6.3))}.
\end{aligned}$$

As $\dim(\pi_{F_{j_0}}(Z)) \le \dim(Z)$, this implies that $\dim(\tau(Z)) < \dim(Z)$. It follows that we can find two distinct configurations $z_1, z_2 \in Z$ such that $\tau(z_1) = \tau(z_2)$. Since all configurations in Z coincide outside $F_{j_0}^{+M}$, this shows that τ is not pre-injective. □

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For the proof of (b) $\Rightarrow$ (a), we shall use the following lemma.

LEMMA 6.3. *Let G be a group and let V be a finite-dimensional vector space over a field $\mathbb{K}$. Let $X \subset V^G$ be a linear subshift of finite type and let D be a defining window for X with $1_G \in D$. Let $(\Omega_i)_{i \in I}$ be a family of subsets of G such that $\Omega_i^{+D} \cap \Omega_j^{+D} = \varnothing$ for all distinct $i, j \in I$. Also let $(x_i)_{i \in I}$ be a family of configurations in X such that the support of x_i is contained in Ω_i for each $i \in I$. Then the configuration $x \in V^G$ defined by $x(g) = x_i(g)$ if $g \in \Omega_i$ for some (necessarily unique) $i \in I$, and by $x(g) = 0$ otherwise, satisfies $x \in X$.*

Proof. If $g \in \Omega_i^{+D}$ for some (necessarily unique) $i \in I$ then x coincides with x_i on gD. Otherwise, x is identically zero on gD. As D is a defining window for X, this shows that $x \in X$. □

Proof of (b) $\Rightarrow$ (a) in Theorem 6.1. Suppose that τ is not pre-injective. This means that we can find a configuration $x_0 \in X$ with finite support $\Omega = \{g \in G : x_0(g) \neq 0\} \neq \varnothing$ satisfying $\tau(x_0) = 0$. Let M be a memory set for τ. We can also assume that $1_G \in M$, that $M = M^{-1}$, and that M is a defining window for X. Let $E = \Omega^{+M^2}$. Then, by Lemma 2.4, we can find a finite subset $F \subset G$ and an (E, F)-tiling $T \subset G$. Note that, for each $g \in G$, the support of the configuration gx_0 is the set $g\Omega$. As $g\Omega \subset g\Omega^{+M}$, this implies that $\pi_{g\Omega^{+M}}(gx_0) \neq 0$. Let us choose, for each $g \in T$, a hyperplane $H_g \subset \pi_{g\Omega^{+M}}(X)$ such that $\pi_{g\Omega^{+M}}(gx_0) \notin H_g$.

Consider now the vector subspace $Y \subset X$ consisting of all the configurations $y \in X$ that satisfy $\pi_{g\Omega^{+M}}(y) \in H_g$ for all $g \in T$. We claim that $\tau(Y) = \tau(X)$. To see this, let x be an arbitrary configuration in X. Then, for each $g \in T$, there exists a scalar $\lambda_g \in \mathbb{K}$ such that $\pi_{g\Omega^{+M}}(x + \lambda_g g x_0) \in H_g$. Now observe that $(g\Omega)^{+M} \cap (g'\Omega)^{+M} \subset gE \cap g'E = \varnothing$ for all distinct $g, g' \in T$ (cf. the defining property (T-1) of a tiling in §2.3). Since X is of finite type with defining window M and $1_G \in M$, it follows from Lemma 6.3 that we can find a configuration $x_0' \in X$ such that $\pi_{g\Omega}(x_0') = \pi_{g\Omega}(\lambda_g g x_0)$ for all $g \in T$ and x_0' is identically zero outside $\coprod_{g \in T} g\Omega$. Note that in fact we have

$$\pi_{g\Omega^{+M^2}}(x_0') = \pi_{g\Omega^{+M^2}}(\lambda_g g x_0) \tag{6.4}$$

for each $g \in T$, since the configuration gx_0 is identically zero outside $g\Omega$.

Consider the configuration $y = x + x_0'$. By construction we have $y \in Y$. Let us show that $\tau(y) = \tau(x)$. Since $y = x$ outside $\coprod_{g \in T} g\Omega$, we deduce from Proposition 2.5 that $\tau(y)$ and $\tau(x)$ coincide outside $\coprod_{g \in T} g\Omega^{+M}$. Now, if $h \in g\Omega^{+M}$ for some (necessarily unique) $g \in T$, then $hM = hM^{-1} \subset g\Omega^{+M^2}$ and therefore

$$\begin{aligned}
\tau(y)(h) &= \tau(x + x_0')(h) \\
&= \tau(x + \lambda_g g x_0)(h) \quad \text{(by (6.4))} \\
&= \tau(x)(h) + \lambda_g g \tau(x_0) \quad \text{(by linearity and } G\text{-equivariance of } \tau) \\
&= \tau(x)(h) \quad \text{(since } x_0 \text{ is in the kernel of } \tau).
\end{aligned}$$

Thus $\tau(x) = \tau(y)$. This proves our claim that $\tau(X) = \tau(Y)$.

Using Proposition 2.5, we deduce that

$$\operatorname{mdim}_{\mathcal{F}}(\tau(X)) = \operatorname{mdim}_{\mathcal{F}}(\tau(Y)) \leq \operatorname{mdim}_{\mathcal{F}}(Y). \tag{6.5}$$

Now observe that, for all $g \in T$, we have

$$(gx_0)|_{\Omega^{+M}} \in \pi_{g\Omega^{+M}}(X) \setminus \pi_{g\Omega^{+M}}(Y)$$

and hence

$$\pi_{g\Omega^{+M}}(Y) \subsetneq \pi_{g\Omega^{+M}}(X).$$

Therefore, we can apply Lemma 3.1 to the strongly irreducible linear subshift X and the vector subspace $Y \subset X$ by taking $\Delta = M$ and $D = \Omega^{+M}$. This gives us $\text{mdim}_{\mathcal{F}}(Y) < \text{mdim}_{\mathcal{F}}(X)$ which, combined with (6.5), implies that $\text{mdim}_{\mathcal{F}}(\tau(X)) < \text{mdim}_{\mathcal{F}}(X)$. □

This completes the proof of Theorem 6.1.

COROLLARY 6.4. *Let G be an amenable group, $\mathcal{F} = (F_j)_{j \in J}$ a right Følner net for G, and V a finite-dimensional vector space over a field $\mathbb{K}$. Let $\tau : X \to Y$ be a linear cellular automaton, where $X, Y \subset V^G$ are linear subshifts satisfying* $\text{mdim}_{\mathcal{F}}(X) = \text{mdim}_{\mathcal{F}}(Y)$. *Suppose that X is strongly irreducible of finite type and that Y is strongly irreducible. Then the following conditions are equivalent:*

(a) *τ is surjective;*

(b) $\text{mdim}_{\mathcal{F}}(\tau(X)) = \text{mdim}_{\mathcal{F}}(X)$*; and*

(c) *τ is pre-injective.*

Proof. The equivalence of conditions (a) and (b) follows from Corollary 5.4. The equivalence between conditions (b) and (c) follows from Theorem 6.1. □

Proof of Theorem 1.2. This follows immediately from the equivalence between conditions (a) and (c) in Corollary 6.4 by taking $X = Y$. □

7. *Pre-injective but not surjective linear cellular automata*

In this section we give examples of pre-injective but not surjective linear cellular automata $\tau : X \to X$, where G is a group, V is a finite-dimensional vector space, and $X \subset V^G$ is a linear subshift. We recall that Theorem 1.2 implies that there is no such example with X strongly irreducible of finite type, and in particular with $X = V^G$, if the group G is amenable. When G contains a non-abelian free subgroup and $\dim(V) = 2$, one can construct a linear cellular automaton $\tau : V^G \to V^G$ that is pre-injective but not surjective. This was done in [**4**, Example 4.10] for free groups of rank two in a more general setting, namely for linear cellular automata whose alphabet is a module over any non-zero ring.

PROPOSITION 7.1. *Let G be a group and let V be a two-dimensional vector space over a field $\mathbb{K}$. Suppose that G contains a non-abelian free subgroup (e.g. G is a non-abelian free group). Then there exists a linear cellular automaton $\tau : V^G \to V^G$ that is pre-injective but not surjective.*

Proof. We may assume $V = \mathbb{K}^2$. Let p_1 and p_2 be the endomorphisms of V defined respectively by $p_1(v) = (\lambda_1, 0)$ and $p_2(v) = (\lambda_2, 0)$ for all $v = (\lambda_1, \lambda_2) \in V$. Let a and b be two elements in G generating a free subgroup of rank two. Consider the map $\tau : V^G \to V^G$ given by

$$\tau(x)(g) = p_1(x(ga)) + p_2(x(gb)) + p_1(x(ga^{-1})) + p_2(x(gb^{-1}))$$

for all $x \in V^G$ and $g \in G$. Clearly τ is a linear cellular automaton admitting $M = \{a, b, a^{-1}, b^{-1}\}$ as a memory set. We have $\tau(V^G) \subset (\mathbb{K} \times \{0\})^G \subsetneq V^G$ so that τ is not surjective.

Let us show that τ is pre-injective. Suppose it is not. Then there exists a configuration $x_0 \in V^G$ with non-empty finite support $\Omega \subset G$ such that $\tau(x_0) = 0$. Let F denote the free subgroup generated by a and b. Choose a left coset $c_0 \in G/F$ such that c_0 meets Ω. The coset c_0 may be viewed as a regular tree of degree four by joining two elements $g, h \in c_0$ if and only if $h^{-1}g \in M$. Consider now an element $g_0 \in \Omega$ that is an ending point of the minimal tree spanned by $c_0 \cap \Omega$ in the tree c_0. Observe that, among the four elements in c_0 that are adjacent to g_0, there are at least three elements outside Ω. As g_0 is in the support of x_0, we must have $p_1(x_0(g_0)) \neq 0$ or $p_2(x_0(g_0)) \neq 0$. If $p_1(x_0(g_0)) \neq 0$, let us choose g_1 outside Ω such that $g_0 = g_1 a$ or $g_0 = g_1 a^{-1}$. This gives a contradiction since g_0 is then the only element in Ω that is adjacent to g_1 so that (7) implies that $\tau(x)(g_1) = p_1(x(g_0)) \neq 0$. If $p_1(x_0(g_0)) = 0$ then $p_2(x_0(g_0)) \neq 0$. In this case, we choose an element g_2 outside Ω such that $g_0 = g_2 b$ or $g_0 = g_2 b^{-1}$. We then get $\tau(x_0(g_2)) = p_2(x_0(g_0)) \neq 0$, which yields also a contradiction. □

PROPOSITION 7.2. *Let G be a group and let V be a one-dimensional vector space over a field $\mathbb{K}$. Then the following hold:*

(i) *if G is infinite, then there exist a linear subshift $X \subset V^G$ and a linear cellular automaton $\tau : X \to X$ that is pre-injective but not surjective;*

(ii) *if G contains an infinite subgroup of infinite index, then there exist an irreducible linear subshift $X \subset V^G$ and a linear cellular automaton $\tau : X \to X$ that is pre-injective but not surjective;*

(iii) *if G is not locally finite (e.g. $G = \mathbb{Z}$), then there exist a linear subshift of finite type $X \subset V^G$ and a linear cellular automaton $\tau : X \to X$ that is pre-injective but not surjective; and*

(iv) *if G contains an infinite finitely generated subgroup of infinite index (e.g. $G = \mathbb{Z}^2$), then there exist an irreducible linear subshift of finite type $X \subset V^G$ and a linear cellular automaton $\tau : X \to X$ that is pre-injective but not surjective.*

Proof. Suppose that H is an infinite subgroup of G. Consider the subset $X \subset V^G$ consisting of the configurations $x \in V^G$ which are constant on each left coset of H. Clearly X is a non-zero linear subshift of V^G. The linear cellular automaton $\tau : X \to X$ defined by $\tau(x) = 0$ for all $x \in X$ is not surjective. However, τ is pre-injective. Indeed, as every left coset of H is infinite, any two configurations in X that are almost equal must coincide. We obtain (i) by taking $H = G$.

If H is of infinite index in G, we can find, for every finite subset $\Omega \subset G$, an element $g \in G$ so that no left coset of H meets both Ω and $g\Omega$. This shows that X is irreducible and (ii) follows.

If H admits a finite generating subset $D \subset H$, then X is of finite type since $X = X_G(D, L)$, where $L \subset V^D$ denotes the vector subspace of V^D consisting of all constant maps from D to V. This shows (iii).

Finally, if H is both finitely generated and of infinite index in G, then X is an irreducible linear subshift of finite type by the preceding observations. This gives (iv). □

Note that none of the linear subshifts $X \subset V^G$ appearing in the proof of Proposition 7.2 is strongly irreducible (G amenable or not). Indeed, suppose that Δ is a finite subset of G. Then, as H is infinite, we can find an element $h_0 \in H$ that is not in Δ. The sets $\Omega_1 = \{h_0\}$ and $\Omega_2 = \{1_G\}$ satisfy $\Omega_1^{+\Delta} \cap \Omega_2 = \varnothing$. However, if $x_1 \in V^G$ is a non-zero constant configuration, we have $x_1 \in X$ but there is no configuration $x \in X$ that coincides with x_1 on Ω_1 and with the zero configuration on Ω_2.

8. *Surjective but not pre-injective linear cellular automata*

In this section we describe examples of surjective but not pre-injective linear cellular automata $\tau : X \to X$, where G is a group, V is a finite-dimensional vector space, and $X \subset V^G$ is a linear subshift. We recall that Theorem 1.2 implies that there is no such example with X strongly irreducible of finite type, and in particular with $X = V^G$, if the group G is amenable. When G contains a non-abelian free subgroup and $\dim(V) = 2$, one can construct a linear cellular automaton $\tau : V^G \to V^G$ that is surjective but not pre-injective. This was done in [**4**, Example 4.11] for free groups of rank two in a more general setting, namely for linear cellular automata whose alphabet is a module over any non-zero ring.

PROPOSITION 8.1. *Let G be a group and let V be a two-dimensional vector space over a field $\mathbb{K}$. Suppose that G contains a non-abelian free subgroup (e.g. G is a non-abelian free group). Then there exists a linear cellular automaton $\tau : V^G \to V^G$ that is surjective but not pre-injective.*

Proof. We may assume $V = \mathbb{K}^2$. Let q_1 and q_2 be the endomorphisms of V respectively defined by $q_1(v) = (\lambda_1, 0)$ and $q_2(v) = (0, \lambda_1)$ for all $v = (\lambda_1, \lambda_2) \in V$. Let a and b be two elements in G generating a free subgroup of rank two. Consider the map $\tau : V^G \to V^G$ given by

$$\tau(x)(g) = q_1(x(ga)) + q_1(x(ga^{-1})) + q_2(x(gb)) + q_2(x(gb^{-1}))$$

for all $x \in V^G$ and $g \in G$. Clearly τ is a linear cellular automaton admitting $M = \{a, b, a^{-1}, b^{-1}\}$ as a memory set. The configuration that takes the value $(0, 1)$ at 1_G and is identically zero on $G \setminus \{1_G\}$ has non-empty finite support and is in the kernel of τ. Therefore τ is not pre-injective.

Let us show that τ is onto. Let $z = (z_1, z_2) \in V^G$. We have to show the existence of a configuration $x = (x_1, x_2) \in V^G$ such that $z = \tau(x)$. Let F denote the free subgroup of G generated by a and b. For $h \in F$, we denote by $\ell(h)$ the *word length* of h, that is, the smallest integer $n \geq 0$ such that h can be written as a product $h = s_1 s_2 \cdots s_n$, where $s_i \in M$ for $1 \leq i \leq n$. Let $R \subset G$ be a complete set of representatives for the left cosets of F in G so that every element $g \in G$ can be uniquely written in the form $g = rh$ with $r \in R$ and $h \in F$. We define $x(g)$ by induction on $\ell(h)$. If $\ell(h) = 0$, that is, $g \in R$, we set $x(g) = (0, 0)$. If $\ell(h) = 1$, that is, $g = rs$ for some $r \in R$ and $s \in M$, we set

$$x(g) = \begin{cases} (z_1(r), 0) & \text{if } s = a, \\ (z_2(r), 0) & \text{if } s = b, \\ (0, 0) & \text{if } s = a^{-1} \text{ or } s = b^{-1}. \end{cases}$$

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Suppose now that, for some integer $n \geq 2$, the value of x has been defined at each element of the form rh, where $r \in R$ and $h \in F$ satisfies $\ell(h) \leq n-1$. Let $g = rh$, where $r \in R$ and $h \in H$ satisfies $\ell(h) = n$. Then h can be uniquely written in the form $h = kss'$, where $k \in F$ satisfies $\ell(k) = n-2$ and $s, s' \in M$ are such that $ss' \neq 1_G$. We set

$$x(g) = \begin{cases} (z_1(rks) - x_1(rk), 0) & \text{if } s \in \{a, a^{-1}\} \text{ and } s = s', \\ (z_2(rks), 0) & \text{if } s \in \{a, a^{-1}\} \text{ and } s' = b, \\ (z_1(rk), 0) & \text{if } s \in \{b, b^{-1}\} \text{ and } s' = a, \\ (z_2(rk) - x_2(rks), 0) & \text{if } s \in \{b, b^{-1}\} \text{ and } s' = s, \\ (0, 0) & \text{otherwise.} \end{cases}$$

The configuration x defined in this way clearly satisfies $z = \tau(x)$. This shows that τ is surjective. □

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