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RELATIVE AMENABILITY AND THE NON-AMENABILITY OF $B(l^1)$

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Abstract

In this paper we begin with a short, direct proof that the Banach algebra $B(l^1)$ is not amenable. We continue by showing that various direct sums of matrix algebras are not amenable either, for example the direct sum of the finite dimensional algebras $\bigoplus_{n=1}^{\infty} B(l_n^p)$ is not amenable for $1 \le p \le \infty$, $p \ne 2$. Our method of proof naturally involves free group algebras, (by which we mean certain subalgebras of $B(X)$ for some space X with symmetric basis — not necessarily $X = l^2$) and we introduce the notion of 'relative amenability' of these algebras.

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Introduction

It is a classical result that the full matrix algebra $\mathcal{A} = M_n(\mathbb{C})$ has a diagonal. That is, the tensor product $\mathcal{A} \otimes \mathcal{A}$ has an element d $\big($namely $d = (1/n)\sum_{i,j=1}^{n} E_{i,j} \otimes E_{j,i}$, where $E_{i,j}$ is the matrix with a 1 in row i column j and zeros elsewhere$\big)$ such that $d \cdot a = a \cdot d$ for all $a \in \mathcal{A}$ (a statement that makes sense because $\mathcal{A} \otimes \mathcal{A}$ is in a natural way an $\mathcal{A}$-bimodule), and $\pi(d) = 1$, the identity, where π is the natural product map from $\mathcal{A} \otimes \mathcal{A}$ to $\mathcal{A}$. As a consequence, every derivation from $\mathcal{A}$ into an $\mathcal{A}$-bimodule E is inner - the cohomology $H^1(\mathcal{A}, E)$ is trivial.

A Banach algebra is said to be amenable if the continuous cohomology $\mathcal{H}^1(\mathcal{A}, E)$ is trivial for every dual Banach $\mathcal{A}$-bimodule E; by a theorem of B. E. Johnson this happens if and only if $\mathcal{A}$ has an 'approximate diagonal', that is, there is a bounded net (d_α) in the projective tensor product $\mathcal{A} \hat{\otimes} \mathcal{A}$ such that $\|a \cdot d_\alpha - d_\alpha \cdot a\| \to 0$ for every $a \in \mathcal{A}$, and $\pi(d_\alpha)$ is a bounded approximate identity for $\mathcal{A}$. This notion of 'approximate diagonal' seems to be the correct generalisation of the notion of a diagonal to the Banach algebra setting; certainly the question of which well known Banach algebras are amenable has an extensive literature, see Dales [3], especially Section 2.8 and Section 5.6, for a survey. However, the question has hitherto remained largely unsolved when the Banach algebra is $B(E)$, the algebra of all bounded operators on a Banach space E. This is somewhat embarrassing because the archetypal algebra with a diagonal is $M_n(\mathbb{C})$, that is, $B(\mathbb{C}^n)$, so if 'approximate diagonal' is the correct generalisation for the infinite dimensional setting, we really ought to know whether $B(E)$ has one, for a fair variety of infinite dimensional Banach spaces E. However, the question has proved quite difficult. There is still no infinite dimensional Banach space E for which $B(E)$ is known to be amenable; and until the ideas in this paper became available, the only Banach spaces for which $B(E)$ was known not to be amenable were the infinite dimensional Hilbert spaces.

The result that $B(\mathcal{H})$ is not amenable (for an infinite dimensional Hilbert space $\mathcal{H}$) is a corollary of the result of Connes [2] that a C^*-algebra is amenable only if it is nuclear. The converse result, that a C^*-algebra is amenable if it is nuclear, was eventually proved by Haagerup [4], and the full theorem is considered one of the deeper results in modern analysis.

This paper begins with a fairly short, direct proof that the algebra $B(l^1)$ of bounded operators on the Banach space l^1 is not amenable. Then, we prove that certain l^∞ direct sums of finite-dimensional matrix algebras are not amenable either; see Theorem 1.3 for the exact statement. As a corollary, we find that the l^∞ direct sum $\mathcal{A} = \bigoplus_{n=1}^\infty B(l_n^p)$ is not amenable for any $1 \le p \le \infty$, $p \ne 2$.

Using ideas from this paper, Pisier [6] has produced a variant proof that is simpler but not quite so self contained, see our concluding section for a discussion. Ozawa [5] has generalised the result somewhat, though not to the extent of answering the all-too-obvious question of whether $B(l^p)$ is amenable for general p. He does, however, give a version of the proof which shows that for certain other Banach algebras $\mathcal{A}$, $\mathcal{A}$ is not amenable.

1. Statement of our main results

As we have indicated, our first result is as follows.

THEOREM 1.1. *$B(l^1)$ is not amenable.*

Our second result involves l^∞ direct sums; the finite dimensional algebras $B(X_i)$ that we are summing need to be as in the following definition.

DEFINITION 1.2. Let $(n_i)_{i=1}^{\infty}$ be a sequence of natural numbers, and for each i, let X_i be a finite dimensional Banach space having normalised, 1-symmetric, 1-unconditional basis $(e_j^{(i)})_{j=1}^{n_i}$. Let $\mathcal{A}_i$ denote the finite dimensional matrix algebra $B(X_i)$, with its operator norm, and let $\mathcal{A}$ be the l^∞ direct sum

$$\mathcal{A} = \left(\bigoplus_{i=1}^{\infty} \mathcal{A}_i\right)_{\infty},$$

consisting of all norm-bounded sequences $(a_i)_{i=1}^{\infty}$, $a_i \in \mathcal{A}_i$.

Our second theorem is then as follows.

THEOREM 1.3. *With the notation of Definition* 1.2, *let*

$$M_i = \frac{1}{\sqrt{n_i}} \left\| \sum_{j=1}^{n_i} e_j^{(i)} \right\|_{X_i}. \tag{1.1}$$

Then if either $\limsup M_i = \infty$, *or* $\liminf M_i = 0$, *the algebra* $\mathcal{A}$ *is not amenable.*

Informally, then, as long as the spaces X_i 'cannot be mistaken for $l_{n_i}^2$', $\mathcal{A}$ is not amenable.

2. Ideas involved in the proofs

There are two main ideas involved in our proofs: permutation operators and random hypergraphs. Of these, permutation operators will be familiar to most readers already, so we will be brief when introducing them. We will spend a little more time in introducing the random hypergraphs and giving a preliminary lemma. The connection between random hypergraphs and permutation operators will then be made in the following section, and after that, the main proofs can be given.

Very broadly, the outline of the proof that $B(l^1)$ is not amenable is that if it were, we can show that the free group F_2 on two generators g_1 and g_2 would be an amenable group; this we know is not the case, so $B(l^1)$ is not amenable. Now the Banach space l^1 is implicitly $l^1(\mathbb{N})$—one indexes the unit vector basis (e_i) with integers $i \in \mathbb{N}$. However, as a Banach space this is isometrically isomorphic to the space $X = l^1(F_2)$ whose unit vector basis $(e_g)_{g\in F_2}$ is indexed by the free group F_2. It is convenient to us to prove that $B(X)$ is not amenable, where $X = l^1(F_2)$ (plainly that is exactly the same statement as saying $B(l^1(\mathbb{N}))$ is not amenable). This is because there are two very convenient operators $T_i \in B(X)$, namely the isometries T_i $(i = 1, 2)$ which send each unit vector e_g to $e_{g_i \cdot g}$. The T_i implement left multiplication by the two generators.

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More generally, if $\pi \in S(F)$ is any permutation on a set F, and Y is a Banach space with normalised 1-symmetric, 1-unconditional basis $(e_g)_{g\in F}$, we define the permutation operator T_π to be the isometry such that $T_\pi(e_g) = e_{\pi(g)}$ for all $g \in F$. And it is at this point that we have the need for random hypergraphs; we shall use them to select some specially useful permutations $\pi \in S(F_2)$.

Now, the reader will be familiar with the notion that a graph G is a pair (V, E), where V is its vertex set, and $E \subset V^{(2)}$ is its set of edges, $V^{(\lambda)}$ denoting the collection of all (unordered) subsets of V of size λ. In a graph, every edge involves exactly two (unordered) vertices, so if V has size n, the number of possible edges is the combinatorial function $\binom{n}{2}$.

Now, a hypergraph is a pair (V, E) where the edge set E is an arbitrary subset of the power set $\mathcal{P}(V)$. The edges can involve arbitrary numbers of vertices.

A λ-regular hypergraph (the kind of hypergraph we are interested in for the present proof) is a pair (V, E) with $E \subset V^{(\lambda)}$, so each edge involves exactly λ vertices. If V has size n, the number of possible edges is $\binom{n}{\lambda}$, and for fixed $v \in V$, the number of edges that contain v is $\binom{n-1}{\lambda-1}$.

The lemma we need is as follows.

LEMMA 2.1. *For all $d, \lambda, \rho \in \mathbb{N}$ with $\lambda \geq 2$ and $\rho \geq 9$, there is a $C > 0$ and a $N_0 \in \mathbb{N}$ with the following property. For all $n \geq N_0$, one can find a λ-regular hypergraph G on n vertices, having at least dn edges, such that* (a) *for all $1 \leq r \leq n/C$ the union of any r edges of G contains at least* $(\lambda - 1)r/2$ *vertices, and* (b) *no vertex of G is contained in as many as $\rho d\lambda$ edges of G.*

Let us remark that, while it is likely that the exact result given in Lemma 2.1 has never appeared in print before, the general methods for proving such results are well known, and many similar results will be found in the standard reference [1].

Let us bring this introductory section to a close by proving Lemma 2.1

PROOF OF LEMMA 2.1. Fix n and choose an arbitrary vertex set V of size n. Take a random λ-regular hypergraph G on vertex set V, random in the following sense: the probability of the event that a given edge $e \in V^{(\lambda)}$ is in the edge set of G is $p = 3dn/\binom{n}{\lambda}$, and all these events are independent.

The expected number of edges in G is $3dn$, and the value $2dn$ is (for large n) many standard deviations below the mean, so for large n, with probability $1 - o(1)$ we have at least $2dn$ edges in G. Let us consider how many edges we would have to delete from G in order to have condition (b) satisfied.

The number of possible edges $e \in V^{(\lambda)}$ incident at a given vertex $v \in V$ is $\binom{n-1}{\lambda-1}$; so the number of 'clusters' of $\rho d\lambda$ edges all incident at the same vertex v is

$$\binom{\binom{n-1}{\lambda-1}}{\rho d\lambda}.$$

There are n vertices v and the probability of all the edges of a given cluster being picked is $p^{\rho d\lambda}$. So the expected number of 'bad' clusters of $\rho d\lambda$ edges with a common vertex all in G is no more than

$$M = np^{\rho d\lambda}\binom{\binom{n-1}{\lambda-1}}{\rho d\lambda} = n\left(\frac{3dn}{\binom{n}{\lambda}}\right)^{\rho d\lambda}\binom{\binom{n-1}{\lambda-1}}{\rho d\lambda}$$
$$\leq n\frac{(3dn)^{\rho d\lambda}}{(\rho d\lambda)!}\frac{\binom{n-1}{\lambda-1}^{\rho d\lambda}}{\binom{n}{\lambda}^{\rho d\lambda}} = n\frac{(3d\lambda)^{\rho d\lambda}}{(\rho d\lambda)!},$$

because $\binom{n-1}{\lambda-1}/\binom{n}{\lambda} = \lambda/n$. By Stirling's formula, $m! \geq \sqrt{2\pi m}(m/e)^m$ so for $\rho \geq 9 > 3e$, when $m = \rho d\lambda$ we have $m! \geq \sqrt{2\pi m}(3d\lambda)^m$. Hence

$$M \leq \frac{n}{\sqrt{2\pi\rho d\lambda}} \leq \frac{n}{10}.$$

Markov's inequality (that for a nonnegative random variable X of mean μ, and for each $a > 0$, the probability $\mathbb{P}(X \geq a) \leq \mu/a$) tells us that the probability of getting n or more bad clusters is at most $1/10$. So with probability at least $9/10$, one may obtain a graph satisfying (b) by deleting at most n edges of G—one from each bad cluster in G.

Let us finally consider the probability that (a) is satisfied. Let us define

$$C = C(d, \lambda) = (22(\lambda - 1)ed)^3.$$

Let $2 \leq r \leq n/C$ (note that (a) is always satisfied when $r = 1$), and let σ be the integer part of $(\lambda - 1)r/2$ (so certainly $\sigma \leq n$). The expectation of the number of sets of σ vertices of G that contain at least r edges of G is no more than

$$\mu_r = \binom{n}{\sigma}\binom{\binom{\sigma}{\lambda}}{r}p^r \leq \frac{n^\sigma}{\sigma!}\frac{\binom{\sigma}{\lambda}^r}{r!}\left(\frac{4dn}{\binom{n}{\lambda}}\right)^r \leq \left(\frac{ne}{\sigma}\right)^\sigma \frac{(\sigma/n)^{\lambda r}}{(r/e)^r}(4dn)^r \tag{2.1}$$

since $\binom{\sigma}{\lambda}/\binom{n}{\lambda} \leq (\sigma/n)^\lambda$, and $\sigma! \geq (\sigma/e)^\sigma$, and $r! \geq (r/e)^r$ by Stirling's formula. For $1 \leq \sigma \leq \min(n, \lambda r)$, the right-hand side of (2.1) is an increasing function of σ, and we have $\sigma \leq r(\lambda - 1)/2$, so

$$\mu_r \leq \left(\frac{2ne}{r(\lambda-1)}\right)^{r(\lambda-1)/2}\left(\frac{r(\lambda-1)}{2n}\right)^{\lambda r}\left(\frac{4dne}{r}\right)^r$$
$$= (2d)^r\left(\frac{r}{2n}\right)^{r(\lambda-1)/2}((\lambda-1)e)^{r(\lambda+1)/2}$$
$$\leq (2d)^r(2C)^{-r(\lambda-1)/2}((\lambda-1)e)^{r(\lambda+1)/2},$$

because $r/n \le C^{-1}$. Now $(\lambda+1)/(\lambda-1) \le 3$, so

$$\mu_r \le (2d(\lambda-1)e \cdot C^{-1/3})^{r(\lambda+1)/2} \le 11^{-r(\lambda+1)/2}$$

because of the choice of C. In particular, $\mu_r \le 11^{-r}$ so $\sum_{1\le r\le n/C} \mu_r \le 1/10$. By Markov's inequality, the probability that condition (a) fails for any such r is no more than $1/10$.

So, with probability at least $8/10 - o(1)$ as $n \to \infty$ the following hold: the graph G has at least $2dn$ edges to begin with; G satisfies condition (a); we may delete at most n edges from G and obtain a graph that also satisfies (b). The final graph has at least dn edges left, and satisfies all the conditions of the lemma. Given d, λ, and ρ it remains to choose an N_0 so large that the probability estimated as $8/10 - o(1)$ as $n \to \infty$ really is strictly positive for all $n \ge N_0$. We then know that a suitable graph G exists on a vertex set of size n, for any $n \ge N_0$. $\square$

3. Using our graph-theoretic lemma

We now use Lemma 2.1 to define permutation operators on $X = l^1(F_2)$ in the following slightly peculiar manner. We begin by defining some notation to use when handling the group F_2.

Recall that the length $l(w)$ of a word w in the free group F_2 is the least n such that w is equal to a product of n elements $\gamma_1\gamma_2\cdots\gamma_n$, $\gamma_i \in \{g_1, g_2, g_1^{-1}, g_2^{-1}\}$ (and $l(1) = 0$). For $g \in F_2$ and $n \ge 0$, we define the ball $B(g,n) = \{hg : h \in F_2, l(h) \le n\}$ and the sphere $\partial B(g,n) = \{hg : l(h) = n\}$. Let us choose, once and for all, a disjoint collection of balls $B_n = B(\gamma_n, n)$, each B_n having radius n. Let ∂B_n be the corresponding spheres, and let $\operatorname{int} B_n = B_n \setminus \partial B_n$. Let us choose them in such a way that even the slightly larger balls $B(\gamma_n, n+1)$ are all disjoint.

Note that the size of B_n is $2\cdot 3^n - 1$ and so the sizes of $\operatorname{int} B_n$ and ∂B_n are $2\cdot 3^{n-1} - 1$ and $2\cdot(3^n - 3^{n-1})$ respectively. In particular, for $n > 1$ we have $|\partial B_n|/|\operatorname{int} B_n| \le 3$. Let us write $\beta_n = 2\cdot 3^{n-1} - 1 = |\operatorname{int} B_n|$.

Next, we choose specific values of d, ρ, and λ to use in Lemma 2.1.

DEFINITION 3.1. We define $d = 3$ and $\rho = 9$. Let us then choose, once and for all, an $\varepsilon > 0$ which is, in the following sense, a witness to the fact that F_2 is not an amenable group: there is no $\phi \in l^\infty(F_2)^*$ with $\phi(1) = 1$ and $\|\phi \circ T_i^* - \phi\| < \epsilon$ $(i = 1, 2)$. Then, choose $\lambda \in \mathbb{N}$ large enough that $192/(\lambda - 1) < \varepsilon/4$, and choose C and N_0 as in Lemma 2.1, for these values of ρ, d, and λ.

Note that non-amenability plainly implies that such an epsilon exists, for a weak-$*$ limit of such functionals ϕ as $\varepsilon \to 0$ would be a translation invariant mean. In fact, $\varepsilon = 1/2$ will do.

DEFINITION 3.2. For each n large enough that $\beta_n \geq N_0$, let us choose a λ-regular hypergraph G_n on vertex set int B_n, having $|\partial B_n|$ edges and such that (a) no vertex is contained in more than 27λ edges and (b) for all $r \leq \beta_n/C$, the union of any r edges contains at least $(\lambda - 1)r/2$ vertices. Let us use the elements of ∂B_n $(n \geq N_0(\lambda))$ to index the edges of G_n. Let us say the edges of G_n are λ-element sets $\gamma_g = (\gamma_{g,1}, \gamma_{g,2}, \dots, \gamma_{g,\lambda})$ for each $g \in \partial B_n$.

Note that the hypergraphs G_n exist because of Lemma 2.1. Having got this far, we now seek a finite sequence of 'extra' permutation operators T_j, $j = 3, \dots, l$, such that for every $n \geq N_0$, $i = 1, \dots, \lambda$, and $g \in \partial B_n$ there is a j such that

$$T_j e_g = e_{\gamma_{g,i}}. \tag{3.1}$$

In fact the number of extra permutations needed is not too large.

LEMMA 3.3. *A suitable collection of permutation operators* $T_i = T_{\pi_i}$ $(\pi_i \in S(F_2)$, $i = 3, \dots, l)$ *can be found, satisfying* (3.1), *with* $l \leq 2 + 27\lambda^2$.

PROOF. Let us totally order F_2 in some arbitrary way. We know that for each $\gamma \in \operatorname{int} B_n$ there may be up to 27λ edges of G_n that are incident at γ; and of course permutations must be injective. So let us choose $27\lambda^2$ permutations $\pi_{i,j}$ $(i = 1, \dots, \lambda$, $j = 1, \dots, 27\lambda)$ with the property that for every n with $\beta_n \geq N_0$ and every $g \in \partial B_n$, one has $\pi_{i,j}(g) = \gamma$ provided $\gamma = \gamma_{g,i}$, and g is (in our total ordering) the jth highest of the up to 27λ elements $h \in \partial B_n$ whose edges γ_h involve γ. It is plain that such permutations exist; if we relabel them as π_i, $i = 3, \dots, l$, we have $l = 2 + 27\lambda^2$ and the condition (3.1) is satisfied. The main thing is, the number of maps needed is finite. Let us choose, once and for all, a set of permutation operators $(T_i)_{i=3}^{l}$, and let π_i denote the permutations on F_2 such that $T_i = T_{\pi_i}$, $i = 1, \dots, l$. □

We shall show that the finite collection of operators $(T_i)_{i=1}^{l}$, which we have now defined, generate a 'relatively non-amenable' subalgebra of $B(X)$, in the following sense.

DEFINITION 3.4. Let $\mathcal{B}$ be a subalgebra of the unital Banach algebra $\mathcal{A}$. We say $\mathcal{B}$ is *relatively amenable in* $\mathcal{A}$ (with constant D) if there is a net $(d_\lambda)_{\lambda \in \Lambda}$ of elements of $\mathcal{A} \hat{\otimes} \mathcal{A}$ such that $\|d_\lambda\| \leq D$, $\pi(d_\lambda) = 1$, and $x \cdot d_\lambda - d_\lambda \cdot x \to 0$ for all $x \in \mathcal{B}$. Otherwise, we say $\mathcal{B}$ is *relatively non-amenable in* $\mathcal{A}$.

Obviously this implies that $B(X)$ cannot have an approximate diagonal, and so is not amenable. We cannot find any candidate for an approximate diagonal in $B(X)$ that even 'works' for the subalgebra finitely generated by the permutation operators $(T_i)_{i=1}^{l}$. And next we give the proof.

4. Proof of Theorem 1.1

To prove the theorem, we begin by making some definitions which help us to probe the nature of a (hypothetical) approximate diagonal in $B(X)$.

DEFINITION 4.1. Let $\mathcal{A}$ denote the algebra $B(Y)$, where in immediate applications in this section Y will be $X = l^1(F_2)$ with its unit vector basis $(e_g)_{g\in F_2}$, but more generally Y can be any Banach space with given normalised 1-symmetric 1-unconditional basis $(e_g)_{g\in F}$ for some set F. Let $(e_g^*)_{g\in F}$ be the coefficient functionals in Y^* with $\langle e_g, e_h^*\rangle = \delta_{g,h}$, and let $d = \sum_m A_m \otimes B_m \in \mathcal{A} \hat{\otimes} \mathcal{A}$. We define coefficients $d_{g,h,k}$ $(g, h, k \in F)$ by $d_{g,h,k} = \sum_m \langle B_m e_g, e_h^*\rangle \cdot \langle A_m e_h, e_k^*\rangle$. We define the formal sum $d_{g,k} = \sum_h d_{g,h,k} e_h$.

Note that in terms of the familiar method of multiplying matrices together (to get the (k, g)th coefficient of AB you multiply corresponding entries of row k of A and column g of B, and add up the results) the numbers $d_{g,h,k}$ are obtained by multiplying entries of the kth 'row' of A_m and the gth 'column' of B_m, and then *forgetting* to add up the results over the index h. The following is therefore no surprise.

LEMMA 4.2. *For all g and k, we have $d_{g,k} \in l^1(F)$ and $\|d_{g,k}\|_1 \le \|d\|$ (where $\|\cdot\|_1$ is the l_1 norm, and $\|d\|$ the projective tensor norm of d).*

PROOF. For

$$\|d_{g,k}\|_1 = \sum_h |d_{g,h,k}| \le \sum_{h,m} |\langle B_m e_g, e_h^*\rangle|\,|\langle e_h, A_m^* e_k^*\rangle| \le \sum_m \|B_m e_g\|\,\|A_m^* e_k^*\|$$

(since the basis is 1-unconditional)

$$\le \sum_m \|A_m\|\,\|B_m\|. \qquad \square$$

DEFINITION 4.3. Let $s : l^1(F) \to \mathbb{C}$ be the sum functional, $s\left(\sum_g \lambda_g e_g\right) = \sum_g \lambda_g$.

LEMMA 4.4. *If $\pi(d) = I$, then $s(d_{g,k}) = \delta_{gk}$ for all g and k. In particular, $\|d_{g,g}\| \ge 1$ for all g.*

PROOF. For $s(d_{g,k}) = \sum_{h,m} \langle B_m e_g, e_h^*\rangle \cdot \langle A_m e_h, e_k^*\rangle = \langle \pi(d) e_g, e_k^*\rangle$. $\square$

LEMMA 4.5. *Let $d \in \mathcal{A} \hat{\otimes} \mathcal{A}$, and let $T_i = T_{\pi_i}$, $i = 1, \dots, l$, be permutation operators on Y. If $\|T_i \cdot d - d \cdot T_i\| \le \delta$, $i = 1, \dots, l$, then we have*

$$\|d_{g,g} - d_{\pi_i(g),\pi_i(g)}\|_1 \le \delta$$

for all $g \in F$ and $i = 1, \dots, l$.

PROOF. For any $g, h \in F_2$ and $T = \sum_m A_m \otimes B_m \in B(X)$, we have

$$(4.1)\qquad (T \cdot d)_{g,k} = \sum_{m,h} \langle B_m e_g, e_h^* \rangle \langle T A_m e_h, e_k^* \rangle e_h$$
$$= \sum_{m,h} \langle B_m e_g, e_h^* \rangle \langle e_h, A_m^* T^* e_k^* \rangle e_h \in l^1.$$

If $T^* e_k^* = e_l^*$ for some l, this gives us $(T \cdot d)_{g,k} = d_{g,l}$; and likewise one may check that if $T e_g = e_p$, then $(d \cdot T)_{g,k} = d_{p,k}$. For a permutation operator T_π, we have $T_\pi e_g = e_{\pi(g)}$ and $T_\pi^* e_k^* = e_{\pi^{-1}(k)}^*$ for all g, k. Accordingly for all g, k, we have

$$(4.2)\qquad (T_\pi \cdot d - d \cdot T_\pi)_{g,k} = d_{g,\pi^{-1}(k)} - d_{\pi(g),k},$$

and $\|T_\pi \cdot d - d \cdot T_\pi\| \le \delta$ implies $\|d_{g,\pi^{-1}(k)} - d_{\pi(g),k}\|_1 \le \delta$ for all g and k; in particular, $\|d_{g,g} - d_{\pi(g),\pi(g)}\|_1 \le \delta$ for every $g \in F$. Thus the lemma is proved. □

So far the paper has lacked a lemma that specifically works only for $Y = X = l^1(F_2)$, rather than some other space with symmetric basis. Here is one.

LEMMA 4.6. *Let $\delta > 0$ and let $d \in B(X) \hat{\otimes} B(X)$ be a finite sum $\sum_{m+1}^M A_m \otimes B_m$ with $\sum_{m=1}^M \|A_m\| \|B_m\| \le D$. Let vectors $d_{g,k}$ $(g, k \in F_2)$ be as in Definition 4.1. Then there is a sequence of vectors $(d'_{g,k})_{g,k \in F_2}$, $d'_{g,k} = \sum_h d'_{g,h,k} e_h \in l^1$ with the following properties*:

(i) $\|d_{g,k} - d'_{g,k}\|_1 \le \delta$ *for all* $g, k \in F_2$.

(ii) *For each $h \in F_2$ the number of k with $d'_{g,h,k} \neq 0$ for any g is at most $R = DM/\delta$.*

In particular, the number of g with $\langle d'_{g,g}, e_h^* \rangle \neq 0$ is at most R.

PROOF. Define $d'_{g,h,k} = d_{g,h,k}$ if for any $m = 1, \ldots, M$ we have $|\langle A_m e_h, e_k^* \rangle| > \|A_m\| \delta / D$; otherwise, define $d'_{g,h,k} = 0$. Now $\|A_m e_h\|_1 \le \|A_m\|$, so for fixed h, m the number of k with $|\langle A_m e_h, e_k^* \rangle| > \|A_m\| \delta / D$ is at most D/δ; for fixed h, the number of k such that this happens for any $m = 1, \ldots, M$ is accordingly at most R. Furthermore, if $S = S_k = \{h : |\langle A_m e_h, e_k^* \rangle| \le \|A_m\| \delta / D,\ m = 1, \ldots, M\}$ then

$$\|d_{g,k} - d'_{g,k}\|_1 = \sum_{h \in S_k} \left| \sum_{m=1}^M \langle B_m e_g, e_h^* \rangle \cdot \langle A_m e_h, e_k^* \rangle \right|$$
$$\le \sum_{m=1}^M \|B_m e_g\|_1 \cdot \|A_m\| \frac{\delta}{D} \le \sum_{m=1}^M \|B_m\| \cdot \|A_m\| \frac{\delta}{D} \le \delta,$$

because $X = l^1$. Thus the lemma is proved. □

We continue our examination of the special case when F_2 is involved.

LEMMA 4.7. *Let $\delta > 0$ and let $E \subset F_2$. Let $(\pi_i)_{i=1}^l$ be the special permutations defined in Section 3. Let vectors $d_g \in l^1(E)$ be given, for each $g \in E$, with $s(d_g) \geq 1 - \delta$, and $\|d_g - d_{\pi_i(g)}\|_1 \leq 3\delta$ for all $i = 1, \ldots, l$ and all g such that $g, \pi_i(g) \in E$. Then for each nonempty finite subset $S \subset E$ there is an $h \in E$ such that the linear functional $\phi_h \in l^\infty(F_2)^*$,*

$$\phi_h(x) = \frac{\sum_{g\in S} x_g |\langle d_g, e_h^*\rangle|}{\sum_{g\in S} |\langle d_g, e_h^*\rangle|} \tag{4.3}$$

satisfies $\|\phi_h\|_1 = 1$, and writing $S_i^- = S \cap \pi_i^{-1}(S)$, we have

$$\sum_{i=1}^{l} \sum_{g\in S_i^-} |\phi_h(e_g) - \phi_h(e_{\pi_i(g)})| \leq \frac{3l\delta}{1-\delta}. \tag{4.4}$$

PROOF. On the one hand,

$$\sum_{g\in S, h\in E} |\langle d_g, e_h^*\rangle| = \sum_{g\in S} \|d_g\|_1 \geq (1-\delta)|S|,$$

since $\|d_g\|_1 \geq |s(d_g)| \geq 1 - \delta$ for all $g \in E$. On the other hand,

$$\sum_{i=1}^{l} \sum_{g\in S, h\in E} |\langle d_g - d_{\pi_i(g)}, e_h^*\rangle| = \sum_{i=1}^{l} \sum_{g\in S} \|d_g - d_{\pi_i(g)}\|_1 \leq 3l\delta|S|.$$

Choose, then, an $h \in E$ such that

$$0 \neq \sum_{g\in S} |\langle d_g, e_h^*\rangle| \geq \frac{1-\delta}{3l\delta} \sum_{i=1}^{l} \sum_{g\in S} |\langle d_g - d_{\pi_i(g)}, e_h^*\rangle|.$$

Since for $g \in S_i^-$ we have

$$\phi_h(e_g) - \phi_h(e_{\pi_i(g)}) = \frac{|\langle d_g, e_h^*\rangle| - |\langle d_{\pi_i(g)}, e_h^*\rangle|}{\sum_{g\in S} |\langle d_g, e_h^*\rangle|},$$

it is easily seen that (4.4) is satisfied. □

LEMMA 4.8. *Let us assume the hypotheses of Lemma 4.7, and assume $E \supset B_n$ for some n with $\beta_n \geq N_0$. Suppose that for every $h \in F_2$, the number of $g \in F_2$ such that $\langle d_g, e_h^*\rangle \neq 0$ is at most $R \leq |\operatorname{int} B_n|/C$. Let $\phi = \phi_h \in l^\infty(F_2)^*$ be the linear functional given by Lemma 4.7 when the finite subset S involved is B_n. Then*

$$\sum_{g\in\partial B_n} |\phi(e_g)| \leq \frac{48}{\lambda - 1}\left(1 + \frac{l\delta}{1-\delta}\right). \tag{4.5}$$

PROOF. For each $i > 0$, let $R_i = \{g \in \partial B_n : \phi(e_g) \in (2^{-i}, 2^{1-i}]\}$, and let $r_i = |R_i|$ so that $\sum r_i \le R$ (after all, we have $\phi(e_g) = 0$ unless $\langle d_g, e_h^* \rangle \ne 0$, where h is the element of F_2 involved in the definition of ϕ in Lemma 4.7; so there can be no more than R such values altogether).

Next, let Γ_i be the set of $\gamma \in \operatorname{int} B_n$ such that $\phi(e_\gamma) \in (2^{-i-1}, 2^{2-i}]$, and let A be the set of i such that $|\Gamma_i| \ge r_i(\lambda - 1)/4$. Now the intervals $(2^{-i-1}, 2^{2-i}]$ overlap, but no $x \in (0, 1]$ lies in more than 3 of them, hence

$$\sum_{\gamma \in \operatorname{int} B_n} \phi(e_\gamma) \ge \frac{1}{3} \sum_{i=1}^{\infty} \sum_{\gamma \in \Gamma_i} \phi(e_\gamma) \ge \frac{\lambda - 1}{12} \sum_{i \in A} 2^{-i-1} r_i. \tag{4.6}$$

On the other hand, the number of $\gamma \in \operatorname{int} B_n$ with $\gamma = \gamma_{g,j}$ for some g, j with $g \in R_i$ is equal to the total number of vertices covered by $r_i \le R$ edges of the graph G_n in Definition 3.2. Since $R \le |\operatorname{int} B_n|/C$, we know that any r_i edges of G_n contain at least $(\lambda - 1)r_i/2$ vertices. So the number of such γ is at least $(\lambda - 1)r_i/2$. If $i \notin A$, then at least $(\lambda - 1)r_i/4$ of these do not have $\phi(e_\gamma) \in (2^{-i-1}, 2^{2-i}]$, thus the distance from $\phi(e_\gamma)$ to the interval $(2^{-i}, 2^{1-i}]$ is at least 2^{-i-1}, so with $\gamma = \gamma_{g,j}$ the 'error' $|\phi(e_\gamma) - \phi(e_g)| \ge 2^{-i-1}$. So, the sum

$$\sum_{i \notin A} \sum_{g \in R_i} \sum_{i=1}^{l} |\phi(e_g) - \phi(e_{\gamma_{g,j}})| \ge \sum_{i \notin A} \frac{(\lambda - 1) r_i}{4} \cdot 2^{-i-1}. \tag{4.7}$$

For each $g \in \partial B_n$, we have $\hat{\pi}_i(g) \in B_n$ for $i = 3, \ldots, l$, so in the notation of Lemma 4.7, we have $g \in S_i^- = B_n \cap \pi_i^{-1}(B_n)$. Accordingly (4.4) tells us that the left-hand side of (4.7) is at most $3l\delta/(1 - \delta)$. Therefore,

$$\sum_{i \notin A} (\lambda - 1) r_i 2^{-i-3} \le 3l\delta/(1 - \delta). \tag{4.8}$$

But $\sum_{i=1}^{\infty} 2^{-i} r_i \ge (1/2) \sum_{g \in \partial B_n} \phi(e_g)$, so

$$\begin{aligned} \sum_{g \in \partial B_n} \phi(e_g) &\le \sum_{i \in A} 2^{1-i} r_i + \sum_{i \notin A} 2^{1-i} r_i \\ &\le \frac{48}{\lambda - 1} \sum_{\gamma \in \operatorname{int} B_n} \phi(e_g) + \frac{48 l\delta}{(\lambda - 1)(1 - \delta)} \le \frac{48}{\lambda - 1} \left(1 + \frac{l\delta}{1 - \delta}\right) \end{aligned}$$

by (4.6) and (4.8), and because $\sum_g \phi(e_g) = 1$. Thus the lemma is proved. □

COROLLARY 4.9. *Given the hypotheses of Lemma* 4.8, *we have*

$$\sum_{i=1}^{2} \|\phi - \phi \circ T_i\|_1 \le \frac{3l\delta}{1 - \delta} + \frac{192}{\lambda - 1} \left(1 + \frac{l\delta}{1 - \delta}\right).$$

PROOF. By (4.4) we know that for $n \geq N_0$, the sum

$$\sum_{i=1}^{2} \sum_{g\in B_n\cap g_i^{-1}B_n} |\phi(e_g) - \phi(e_{g_ig})| \leq \frac{3l\delta}{1-\delta},$$

because the left-hand side is the first two terms $i = 1, 2$ of the full sum (4.4) for the functional ϕ. Now if $g \notin B_n$ then $\phi(e_g) = 0$; hence the full sum

$$\begin{aligned}\sum_{i=1}^{2} \|\phi - \phi\circ T_i\| &= \sum_{i=1}^{2}\sum_{g\in F_2} |\phi(e_g) - \phi(e_{g_ig})| \\ &= \sum_{i=1}^{2}\left(\sum_{g\in B_n\cap g_i^{-1}B_n} |\phi(e_g)-\phi(e_{g_ig})| + \sum_{g\in B_n\setminus g_i^{-1}B_n} |\phi(e_g)| + \sum_{g\in B_n\setminus g_iB_n} |\phi(e_g)|\right) \\ &\leq \frac{3l\delta}{1-\delta} + 4\sum_{g\in\partial B_n} |\phi(e_g)| \leq \frac{3l\delta}{1-\delta} + \frac{192}{\lambda-1}\left(1+\frac{l\delta}{1-\delta}\right)\end{aligned}$$

by (4.5), as required. □

PROOF OF THEOREM 1.1. Suppose towards a contradiction that $B(X)$ is amenable. We may find a $D > 0$ and an approximate diagonal $(d_\alpha)_{\alpha\in A}$ with $\pi(d_\alpha) = I$ and $\|d_\alpha\| < D$ for all α. We may further assume that each d_α is equal to a finite sum of tensors

$$d = \sum_{m=1}^{M} A_m \otimes B_m \tag{4.9}$$

with $\sum_{m=1}^{M} \|A_m\|\|B_m\| < D$.

Pick $\delta > 0$ so small that $l\delta/(1-\delta) < \min(1, \varepsilon/6)$, and choose d from the approximate diagonal in such a way that $\|d\cdot T_i - T_i\cdot d\| \leq \delta$ for $i = 1, \dots, l$. Let us write $d = \sum_{m=1}^{M} A_m\otimes B_m$ as in (4.9), with $\sum_{m=1}^{M}\|A_m\|\|B_m\| < D$.

By Lemma 4.5, the vectors $d_{g,g}$ satisfy $\|d_{g,g} - d_{\pi_i(g),\pi_i(g)}\|_1 \leq \delta$. By Lemma 4.6, we can pick vectors $d'_{g,g} = \sum_h d'_{g,h,g}e_h$ with $\|d_{g,g} - d'_{g,g}\|_1 \leq \delta$, (so certainly $\|d'_{g,g} - d'_{\pi_i(g),\pi_i(g)}\|_1 \leq 3\delta$) yet for all $h \in F_2$ the number of g with $d'_{g,h,g} \neq 0$ is at most $R = DM/\delta$.

Writing $d_g = d'_{g,g}$ and $E = F_2$, we find that the conditions of Lemma 4.7 are satisfied. If we choose n so large that $\beta_n \geq N_0$ and $R/C \leq \beta_n$, then so also are the further conditions of Lemma 4.8. By Corollary 4.9, the sum

$$\sum_{i=1}^{2} \|\phi - \phi\circ T_i\| \leq \frac{3l\delta}{1-\delta} + \frac{192}{\lambda-1}\left(1+\frac{l\delta}{1-\delta}\right) < \varepsilon,$$

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because λ was chosen such that $192/(\lambda-1) < \varepsilon/4$, and $l\delta/(1-\delta) < \min(1, \varepsilon/6)$. This contradicts the definition of ε (Definition 3.1) so no such d can be found, hence $B(l^1)$ is not amenable. □

Note that non-amenability is witnessed to by testing $\|d \cdot T_i - T_i \cdot d\|$ for the specific sequence $(T_i)_{i=1}^l$ that we defined; hence the subalgebra $\mathcal{B}$ generated by $(T_i)_{i=1}^l$ is indeed relatively non-amenable in $\mathcal{A}$.

5. Proof of Theorem 1.3

In this section, we vary the arguments of the previous section so as to obtain our second result, Theorem 1.3. As claimed in the Abstract, this implies the corollary.

COROLLARY 5.1. *Let $1 \le p \le \infty$, $p \ne 2$. Then the l^∞ direct sum of matrix algebras $\mathcal{A} = \bigoplus_{n=1}^\infty B(l_n^p)$ is not amenable*

This result is interesting and suggestive because the C^*-algebras proof that $B(l^2)$ is not amenable 'goes via' a proof that the direct sum of matrix algebras $\bigoplus_{n=1}^\infty B(l_n^2)$ is not amenable. So this corollary hints strongly that $B(l^p)$ is probably not amenable.

PROOF OF THEOREM 1.3. Let us begin with a minor simplification. Since $B(X_i)$ is isometrically anti-isomorphic to $B(X_i^*)$, the algebra $\mathcal{A}$ is amenable if and only if $\mathcal{B} = \bigoplus_{n=1}^\infty B(X_n^*)$ is amenable; hence we may assume (with the notation of (1.1)) that $\limsup M_i = \infty$, rather than $\liminf M_i = 0$. Further, since amenability passes to quotient algebras such as $\bigoplus_{i\in E} B(X_i)$ for a subset $E \subset \mathbb{N}$, it is enough to show that $\mathcal{A}$ cannot be amenable in the case when M_i actually tends to infinity.

In order to prove Theorem 1.3, let us take the sequence of dimensions (n_i), and relate them to the sizes of the balls B_μ of radius μ in F_2. Since (as we noted after Definition 3.1) the size of such a ball is $2 \cdot 3^\mu - 1$, it makes sense to pick integers μ_i such that $n_i \in [2 \cdot 3^{\mu_i} - 1, 2 \cdot 3^{1+\mu_i} - 1)$. Quotienting out unneeded X_i as required, we can also assume that the sequence μ_i is strictly increasing, and all $\mu_i > N_0$.

DEFINITION 5.2. For each i, let us choose an injective map $\varepsilon_i : \{1, \ldots, n_i\} \to F_2$ such that $B(\gamma_{\mu_i}, \mu_i) \subset \operatorname{Im}\varepsilon_i \subset B(\gamma_{\mu_i}, 1+\mu_i)$ for each i, where $\gamma_i \in F_2$ is the sequence chosen in Definition 3.1 (so that the balls $B(\gamma_k, k+1)$ are disjoint).

DEFINITION 5.3. Let (π_j) be the permutations of Section 3. We define some operators $T_j' \in \mathcal{A}$, closely related to the permutation operators $T_j \in B(X)$, as follows: T_j' is an element of $\mathcal{A} = \left(\bigoplus_{i=1}^\infty \mathcal{A}_i\right)_\infty$, whose ith element is a permutation operator $T_{j,i} : X_i \to X_i$ such that $T_{j,i}(e_m^{(i)}) = e_n^{(i)}$ whenever $\pi_j(\varepsilon_i(m)) = \varepsilon_i(n)$.

Note that the permutation operators $T_{j,i}$ are not unique because the original permutations π_j on F_2 do not map the finite sets $\operatorname{Im}\varepsilon_i$ to themselves (for example, π_1 is multiplication by a generator of F_2, and as such it does not map *any* finite set to itself). So, we pick our elements T_j' in a slightly arbitrary way. But we pick them now, once and for all, for each $\lambda > 0$.

DEFINITION 5.4. We let P_K be the natural map $\mathcal{A} = \bigoplus_{j=1}^{\infty} \mathcal{A}_j \to \mathcal{A}_K$ and $\hat{P}_K = P_K \otimes P_K$ the natural norm 1 linear map $\mathcal{A} \hat{\otimes} \mathcal{A} \to \mathcal{A}_K \hat{\otimes} \mathcal{A}_K$ such that $\hat{P}_K(a \otimes b) = P_K(a) \otimes P_K(b)$ for all a and b in $\mathcal{A}$.

Now suppose that, contrary to Theorem 1.3, the algebra $\mathcal{A}$ is amenable. Let $(d_\alpha)_{\alpha \in A}$ be an approximate diagonal for $\mathcal{A}$ with $d_\alpha \in \mathcal{A} \otimes \mathcal{A}$, $\pi(d_\alpha) = I$ and, say $\|d_\alpha\| < D$ for every α. We may assume that each d_α is a finite sum of tensors $\sum_{m=1}^{M} A_m \otimes B_m$ with $\sum_{m=1}^{M} \|A_m\| \cdot \|B_m\| < D$.

As before, pick $\delta > 0$ so $l\delta/(1-\delta) < \min(1, \varepsilon/6)$ and choose d from the diagonal, $d = \sum_{m=1}^{M} A_m \otimes B_m$, so that $\|d \cdot T_j' - T_j' \cdot d\| \le \delta$ for $j = 1, \ldots, l$. Then for all K,

$$\left\| \hat{P}_K(d) \cdot T_{j,K} - T_{j,K} \cdot \hat{P}_K(d) \right\|_{\mathcal{A}_K} \le \delta.$$

Now each $\mathcal{A}_K = B(X_K)$, $Y = X_K$ a Banach space with given symmetric basis as in Definition 4.1. So we may write

$$d_{i,j,k}^{(K)} = \sum_{m=1}^{M} \langle P_K B_m e_i, e_j^* \rangle \cdot \langle P_K A_m e_j, e_k^* \rangle$$

as in Definition 4.1, where $\hat{P}_K(d) = \sum_{m=1}^{M} P_K A_m \otimes P_K B_m$. By Lemma 4.5, we have $\|d_{j,j}^{(K)} - d_{k,k}^{(K)}\|_1 \le \delta$ whenever $T_{i,K} e_j = e_k$, in particular, whenever $\varepsilon_K(j) = g$, $\varepsilon_K(k) = \pi_i(g)$ for some $i \in [1, l]$.

At this point we need a finite dimensional version of Lemma 4.6 that works for Banach spaces X other than $X = l^1$. It is as follows:

LEMMA 5.5. *Let X be a finite dimensional normed space of dimension n, let $\delta > 0$ and let $d \in B(X) \hat{\otimes} B(X)$ be a finite sum $\sum_{m=1}^{M} A_m \otimes B_m$ with*

$$\sum_{m=1}^{M} \|A_m\| \cdot \|B_m\| \le D.$$

Let X have normalised 1-symmetric 1-unconditional basis $(e_i)_{i=1}^{n}$, let

$$d_{ijk} = \sum_{m=1}^{M} \langle B_m e_i, e_j^* \rangle \cdot \langle A_m e_j, e_k^* \rangle,$$

and let $d_{ik} = \sum_{j=1}^{n} d_{ijk} e_j \in l_n^1$. Then for every i, k the vector d_{ik} can be approximated by d'_{ik}, $\|d_{ik} - d'_{ik}\|_1 \le \delta$, in such a way that for each $j = 1, \ldots, n$ the number of k with $d'_{i,j,k} \ne 0$ for any i, is at most $R = M \cdot \mathcal{F}\big(D\big\| \sum_{h=1}^{n} e_h^*\big\|_{X^*}/\delta\big)$, where $\mathcal{F}(\alpha)$ is the greatest integer m such that $\big\| \sum_{h=1}^{m} e_h \big\|_X \le \alpha$, or n if $\big\| \sum_{h=1}^{n} e_h \big\| \le \alpha$. In particular, the number of k with $\langle d_{kk}, e_j^* \rangle \ne 0$ is at most R.

PROOF. Define $d'_{ijk} = d_{ijk}$ if for any $m = 1, \ldots, M$ we have

$$|\langle A_m e_i, e_k^* \rangle| > \frac{\|A_m\|\delta}{D\big\| \sum_{h=1}^{n} e_h^* \big\|_{X^*}}.$$

Otherwise, define $d'_{ijk} = 0$. Now $\|A_m e_j\|_X \le \|A_m\|$ so for fixed j, m the number of k with $|\langle A_m e_j, e_k^* \rangle| > \|A_m\|\delta / \big(D\big\| \sum_{h=1}^{n} e_h^* \big\|_{X^*}\big)$ is at most $\mathcal{F}\big(D\big\| \sum_{h=1}^{n} e_h^* \big\|_{X^*}/\delta\big)$. So the number of k such that this happens for any $m = 1, 2, \ldots, M$ is at most $M \cdot \mathcal{F}\big(D\big\| \sum_{h=1}^{n} e_h^* \big\|_{X^*}/\delta\big)$. And if

$$S_k = \left\{ j : |\langle A_m e_j, e_k^* \rangle| \le \frac{\|A_m\|\delta}{D\big\| \sum_{h=1}^{n} e_h^* \big\|_{X^*}} \right\},$$

then

$$\begin{aligned} \|d_{ik} - d'_{ik}\|_1 &= \sum_{j \in S_k} \left| \sum_{m=1}^{M} \langle B_m e_i, e_j^* \rangle \cdot \langle A_m e_j, e_k^* \rangle \right| \\ &\le \sum_{m=1}^{M} \|B_m e_i\|_X \cdot \|A_m\| \frac{\delta}{D} \le \sum_{m=1}^{M} \|A_m\| \cdot \|B_m\| \frac{\delta}{D} \le \delta. \end{aligned}$$ □

We now apply Lemma 5.5 once for each $K > 0$, with $X = X_K$, $n = n_K$, and $P_K(d)$, $P_K B_m$, $P_K A_m$ substituted for d, B_m, A_m respectively. We find there are vectors $d'^{(K)}_{ik}$, $\|d'^{(K)}_{ik} - d^{(K)}_{ik}\|_1 \le \delta$, $d'^{(K)}_{ik} = \sum_{j=1}^{n_K} d'^{(K)}_{ijk} e_j$, such that for each $j = 1, \ldots, n_K$ the number of k with $d'^{(K)}_{ijk} \ne 0$ for any i is at most $R_K = M \cdot \mathcal{F}_K(D\|e_1^* + \cdots + e_{n_K}^*\|_{X_K^*}/\delta)$, where $\mathcal{F}_K(\alpha)$ is the greatest integer m such that $\|e_1 + \cdots + e_m\|_{X_K} \le \alpha$.

The vitally important fact is that we know $\|e_1 + \cdots + e_{n_K}\|_{X_K}/\sqrt{n_K} \to \infty$ as $K \to \infty$, so $R_K = o(n_K)$ as $K \to \infty$.

Let us therefore pick a large enough K that $\mu_K \ge N_0$ and $R_K/n_K \le 1/17C$. Define the set $E = \operatorname{Im} \varepsilon_K \subset B(\gamma_{\mu_K}, 1 + \mu_K)$, and for $g \in E$ define the vector $d_g \in l^1(E)$ by $\langle d_g, e_h \rangle = \langle d'^{(K)}_{ii}, e_j \rangle$, where $i = \varepsilon_K^{-1}(g)$ and $j = \varepsilon_K^{-1}(h)$. Now $n_K \in [|B_{\mu_K}|, |B_{1+\mu_K}|)$, where $|B_\mu|$ is the size of a ball of radius μ in F_2. Since for $\mu > 1$ we have $|B_{\mu+1}|/|B_{\mu-1}| \le |B_3|/|B_1| = 17$, we find

$$\frac{R_K}{|B_{\mu_K - 1}|} = \frac{R_K}{|\operatorname{int} B_{\mu_K}|} \le C. \tag{5.1}$$

The conditions of Lemma 4.7 are satisfied with δ, E, $d_g \in l^1(E)$ as above, and $(\pi_i)_{i=1}^{l}$ the usual permutations π_i. The further conditions of Lemma 4.8 are satisfied with $n = \mu_K$ and $R = R_K$. That lemma tells us that

$$\sum_{g \in \partial B_{\mu_K}} |\phi(e_g)| \leq \frac{48}{\lambda - 1}\left(1 + \frac{l\delta}{1-\delta}\right),$$

where ϕ is the linear functional obtained from Lemma 4.7 when $S = B_{\mu_K}$. Corollary 4.9 then tells us that

$$\sum_{i=1}^{2} \|\phi - \phi \circ T_i\|_1 \leq \frac{3l\delta}{1-\delta} + \frac{192}{\lambda - 1}\left(1 + \frac{l\delta}{1-\delta}\right) < \varepsilon.$$

So as with the algebra $B(X)$, we conclude that if our algebra $\mathcal{A} = \left(\bigoplus_{i=1}^{\infty} X_i\right)_{\infty}$ were amenable, so also the free group F_2 would be amenable (or at least, our particular constant $\varepsilon > 0$ could not be a 'witness' to its nonamenability as required by Definition 3.1). This contradiction shows that $\mathcal{A} = \left(\bigoplus_{i=1}^{\infty} B(X_i)\right)_{\infty}$ is not an amenable Banach algebra, and once again, we have a finitely generated subalgebra of $\mathcal{A}$ (the one generated by the operators T_j') that is not relatively amenable in $\mathcal{A}$. □

6. Conclusion

It is remarkable how difficult it is to resolve the question of whether certain well-known Banach algebras are amenable. When the Banach algebra is $B(X)$ (for an infinite dimensional Banach space X), intuition suggests that it is most unlikely to be amenable; the counterexample, if any, likely to be a really weird Banach space X with relatively few operators on it. Here we show $B(X)$ is non-amenable when $X = l^1$. Recently Ozawa [5] has generalised our methods to prove non-amenability results for some further Banach algebras. Pisier [6] has also provided a variant of the present proof, in which he shortens the graph theory involved by making reference to the work of Lubotzky *et al.* on 'expanding graphs', work which was unknown to the present author. So this construction, and its generalisations, provides slow but genuine progress in a difficult area.

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