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Plane Stokes flows with time-dependent free boundaries in which the fluid occupies a doubly-connected region

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Consider the two-dimensional quasi-steady Stokes flow of an incompressible Newtonian fluid occupying a time-dependent region bounded by free surfaces, the motion being driven solely by a constant surface tension acting at the free boundaries. When the fluid region is simply-connected, it is known that this Stokes flow problem is closely related to a Hele-Shaw free boundary problem when the zero-surface-tension model is employed. Specifically, if the initial configuration for the Stokes flow problem can be produced by injection at N points into an empty Hele-Shaw cell, then so can all later configurations. Moreover, there are N invariants; while the N points at which injection must take place move, the amount to be injected at each of these points remains the same. In this paper, we consider the situation when the fluid region is doubly-connected and show that, provided the geometry has an appropriate rotational symmetry, the same results continue to hold and can be exploited to determine the solution of the Stokes flow problem.

1 Introduction

In an earlier paper (Richardson, 1997), a particular class of two-dimensional Stokes flows involving a time-dependent free boundary was considered, the motion being driven solely by a constant surface tension acting at the free boundary. Attention was confined to situations where the region occupied by the fluid was simply-connected, and an effective solution procedure was presented for following the evolution of initial configurations that can be produced by injecting at N points into an empty Hele-Shaw cell, the evolution of the Hele-Shaw motion being governed by the zero-surface-tension model. All subsequent configurations in the evolution of the Stokes flow problem can then also be produced by injecting at N points into a Hele-Shaw cell and, while the N injection points must move, the amount to be injected at each point is invariant. Analytically, the fluid region throughout the motion is the image of the unit disk by a conformal map produced by a rational function having just N simple poles. One of these simple poles can be taken at infinity, so there are just $2N-1$ time-dependent parameters in the map – the positions of the other $N-1$ poles and the N residues at the poles. The above invariants yield N algebraic relations to be satisfied by these parameters, and there are $N-1$ first order differential equations that control the motion of the $N-1$ finite poles. The efficacy of

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the solution procedure based on this formulation was demonstrated in the earlier paper by giving a number of examples. These all began with initial states that had the fluid occupying N touching circular disks (a configuration of much interest in the industrial context) and the link with Hele-Shaw flows proved to be vital in dealing with the extreme distortions and near-singular behaviour inevitably involved in the conformal map of the unit disk onto such a region.

In the present paper, we consider the corresponding problems when the fluid occupies a doubly-connected region. In the Hele-Shaw context, doubly-connected domains were considered by Richardson (1994); there it was shown that, if we have a doubly-connected fluid region that can be produced by injection at isolated points into an empty cell, then it is the image of a rectangle by a conformal map produced by an elliptic function. The particular example analysed (injection into a right-angled corner when air is trapped in the corner, with images invoked to deal with the barriers forming the corner) enjoyed a great deal of symmetry; in § 2 here we derive the results for a general situation when no symmetry is present. They are, in essence, the natural homologues of those in the simply-connected case, with rational functions (the most general meromorphic functions on the Riemann sphere, of genus 0) being replaced by elliptic functions (the most general meromorphic functions on a torus, of genus 1). With the doubly-connected domain produced by injection at N points, it is the image of a rectangle by a conformal map produced by an elliptic function that has just N simple poles in its period-parallelogram, which can therefore be expressed as a sum of N zeta functions.

In § 3, we consider the Stokes flow problem when the fluid region is doubly-connected. As for the simply-connected case, initial states consisting of N touching circular disks are of particular interest; these can still be produced by injection into an empty Hele-Shaw cell at N points, and one might therefore expect the Stokes flow to pass through configurations that are produced by conformal maps given by elliptic functions with N simple poles in their period-parallelogram. However, in general this does not happen and we must impose some restriction if the motion is to evolve through such a family of maps. This *does* happen, and the Stokes flow passes through states that can be produced in a Hele-Shaw cell by injection at N points, if the initial configuration is such that it is invariant under rotation through an angle $2\pi/n$ for some integer $n \geqslant 2$ about some point. In general, when the geometry does not satisfy this condition, the analysis of the Stokes flow is a good deal more complicated; this situation will not be considered here.

While the work reported here was in progress, a preprint that was subsequently published in revised form as Crowdy & Tanveer (1998) appeared. This considers the same problem of plane Stokes flows with free boundaries involving doubly-connected domains as is addressed here, but makes the *a priori* assumption that a certain constant vanishes. This assumption does indeed simplify the theory and is implied by our symmetry requirement, but it is not justified by Crowdy & Tanveer (1998), and is not valid for the particular example they use to illustrate their theory.

For the most part, we will use the standard notations associated with the Jacobian elliptic functions for which Bowman (1961) and Byrd & Friedman (1971) may serve as references. We have the modulus k, where $0 \leqslant k \leqslant 1$, the complementary modulus $k' = (1-k^2)^{1/2}$, the complete elliptic integral of the first kind $K(k) = K$, and its complement $K'(k) = K(k') = K'$. A basic function in the theory is Jacobi's zeta function $Z(\zeta, k) = Z(\zeta)$:

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just as any rational function can be expressed as a finite sum of translates of the function $1/\zeta$ and its derivatives (with appropriate interpretations for poles at infinity), so can any elliptic function be expressed as a finite sum of translates of $Z(\zeta)$ and its derivatives. Jacobi's zeta function $Z(\zeta)$ has just one simple pole within any fundamental parallelogram (and so is not itself elliptic), and is convenient for many purposes because it is real and analytic for all real values of ζ, and has $2K$ as a real period. However, for our use of these ideas in connection with Stokes flows it is preferable to employ a function that, like $1/\zeta$, has a simple pole at the origin $\zeta = 0$. Moreover, it will be convenient to have $2iK'$ rather than $2K$ as a period. For these reasons, we choose to work with a modified zeta function $Z_m(\zeta)$ defined in terms of the usual zeta function $Z(\zeta)$ by

$$Z_m(\zeta) = Z(\zeta + iK') + \frac{i\pi}{2K} + \frac{\pi}{2KK'}\zeta. \tag{1.1}$$

This function has the periodic and quasi-periodic properties

$$Z_m(\zeta + 2iK') = Z_m(\zeta) \quad \text{and} \quad Z_m(\zeta + 2K) = Z_m(\zeta) + \pi/K'. \tag{1.2}$$

Like $Z(\zeta)$, we have $Z_m(\zeta)$ as an odd function that is real when ζ is real, so

$$Z_m(-\zeta) = -Z_m(\zeta) \quad \text{and} \quad \overline{Z_m(\overline{\zeta})} = Z_m(\zeta), \tag{1.3}$$

the retention of these desirable properties being the reason that the constant has been included in (1.1).

Note that the symmetries (1.3) also imply that

$$\overline{Z_m(-\overline{\zeta})} = -Z_m(\zeta)\,; \tag{1.4}$$

thus $Z_m(\zeta)$ is purely imaginary when ζ is purely imaginary.

The only singularities of $Z_m(\zeta)$ in the complex plane are simple poles of residue 1 at the points $\zeta = 2mK + 2inK'$, where m and n are integers.

2 Hele-Shaw flows

We describe the motion in a Hele-Shaw cell in terms of the usual projection onto a Cartesian (x, y)-plane that is parallel to the solid planes forming the cell, and exploit the complex variable $z = x + iy$. The situation is illustrated in the z-plane of Figure 1, where the fluid occupies the doubly-connected region D which has the curve C_0 as its outer boundary and C_1 as its inner boundary; we assume that both C_0 and C_1 are regular analytic curves.

When we suppose a constant pressure condition to be applicable on the free boundaries (with, perhaps, a different constant pressure on each of the two free boundaries C_0 and C_1 in the present situation), Hele-Shaw flows are conveniently treated by exploiting properties of the Cauchy transform. As in Richardson (1994), this is a function $h(x, y)$ that is continuous in the entire (x, y)-plane, with

$$h(x,y) = \begin{cases} h_e(z) & \text{for } z \text{ outside } C_0, \\ h_1(z) & \text{for } z \text{ inside } C_1, \\ \overline{z} - h_i(z) & \text{for } z \text{ in } D, \end{cases} \tag{2.1}$$

where $h_e(z)$ is analytic for z outside C_0, while $h_1(z)$ is analytic for z inside C_1 and $h_i(z)$

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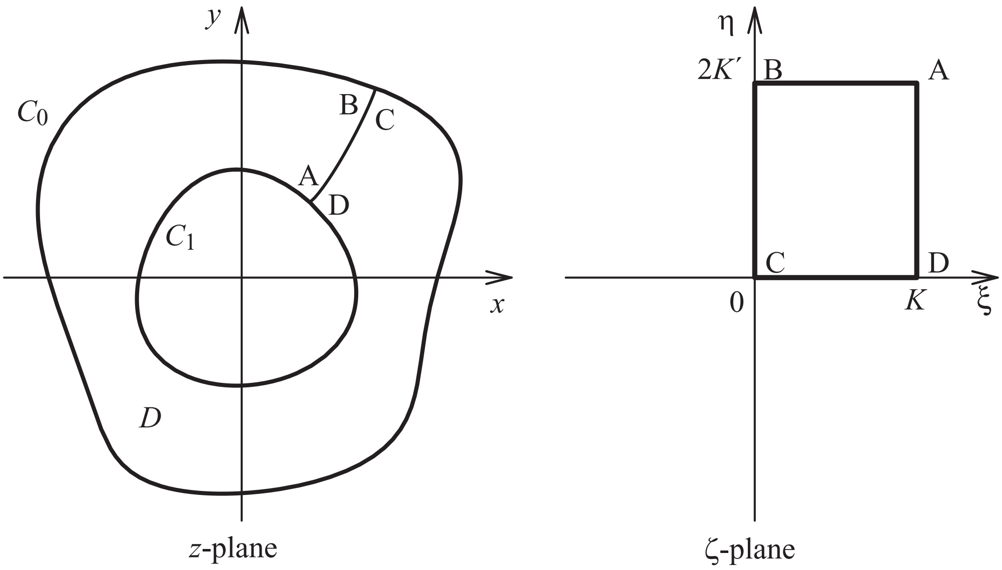

FIGURE 1. The z-plane and ζ-plane used in the analysis.

is analytic for z in D. All three of these functions may be analytically continued beyond the regions where they are known to be analytic, and we use the same notation for these analytically-continued functions. A knowledge of the functions $h_e(z)$ and $h_1(z)$, plus just one other item of information, suffices to determine D in the present circumstances. The extra information required concerns the assumptions made about the hole lying within C_1. For example, we might contemplate injection with an air vent inside C_1, so the pressure has the same value on both C_0 and C_1, or we might suppose there to be incompressible air inside C_1 and no air vent, so the area enclosed by C_1 remains constant. The mathematical forms such assumptions take are considered by Richardson (1994) and need not concern us here; we merely note that we anticipate having to impose one further constraint to obtain a determinate mathematical problem.

In general, the functions $h_e(z)$ and $h_1(z)$ are not equal. They *are* equal when the domain D can be produced by injecting fluid at a finite number of points into an initially empty cell, and it is in these circumstances that D is the image of a rectangle by an elliptic function. This is no longer true in other circumstances, and then the conformal maps involved are necessarily somewhat more complicated; examples of this are given by Richardson (1996).

If D has been produced by injection into an initially empty cell of an area πr_j^2 at the point $z = a_j$ for $j = 1, 2, \ldots, N$, then we have

$$h_e(z) = h_1(z) = \sum_{j=1}^{N} \frac{r_j^2}{z - a_j}; \tag{2.2}$$

we regard suction as negative injection, so that if we actually have suction at $z = a_j$ then $r_j^2 < 0$.

To map the doubly-connected region D onto a rectangle, we envisage a cut drawn in

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D from a point on C_1 to a point on C_0 as in Figure 1, where A and D are neighbouring points on either side of the cut on C_1, while B and C are similar points on C_0. To effect the necessary analytic continuations, this cut must have special properties. If we consider the Dirichlet problem for a harmonic function u that is equal to 1 on C_1 and 0 on C_0, then it can be any orthogonal trajectory of the level curves of u; equivalently, if we consider any conformal map of a concentric circular annulus onto D, then it can be the image of any radius of that annulus.

We choose to map this simply-connected region obtained by introducing the cut in D onto the rectangle in the plane of $\zeta = \xi + i\eta$ specified by $0 < \xi < K$ and $0 < \eta < 2K'$, with the points A, B, C and D in the z-plane mapping to the vertices marked A, B, C and D, respectively, in the ζ-plane of Figure 1. Since K/K' increases from 0 to ∞ as k increases from 0 to 1, there is just one value of k for which this is possible with D given. The orientation of the rectangle in the ζ-plane is chosen to simplify the limiting forms the map takes in the singular situations we need to consider. As $k \to 0$ we have $K \to \pi/2$, $K' \to \infty$, and the rectangle degenerates to a convenient semi-infinite strip; this corresponds to a domain D approaching one formed by a ring of touching circular disks. As $k \to 1$ we have $K \to \infty$, $K' \to \pi/2$, and the rectangle again degenerates to a convenient semi-infinite strip; this corresponds to a domain D whose central hole disappears.

Let the conformal map be effected by the function $z = w(\zeta)$. This is initially defined only for ζ within the rectangle ABCD in the ζ-plane, but our choice of cut in the z-plane allows it to be analytically continued into the entire infinite strip $0 < \xi < K$ where we have

$$w(\zeta + 2iK') = w(\zeta), \tag{2.3}$$

this periodicity reflecting the doubly-connected nature of D.

From (2.1) we have $\overline{z} = h_e(z) + h_i(z)$ on C_0. Transferring this condition to the ζ-plane, we have

$$w(\zeta) = \overline{h_e(w(-\overline{\zeta}))} + \overline{h_i(w(-\overline{\zeta}))}, \tag{2.4}$$

a condition that is initially known to hold only on the imaginary axis where $\zeta = -\overline{\zeta}$, but which then allows $w(\zeta)$ to be analytically continued into the strip $-K < \xi < 0$.

Similarly, the condition $\overline{z} = h_1(z) + h_i(z)$ on C_1 leads to

$$w(\zeta) = \overline{h_1(w(-\overline{\zeta} + 2K))} + \overline{h_i(w(-\overline{\zeta} + 2K))}, \tag{2.5}$$

which allows $w(\zeta)$ to be analytically continued into the strip $K < \xi < 3K$.

Taken together, (2.4) and (2.5) allow $w(\zeta)$ to be analytically continued into the entire ζ-plane. Moreover, if we replace ζ by $\zeta + 2K$ in (2.5), we can eliminate $\overline{h_i(w(-\overline{\zeta}))}$ between (2.4) and the resulting equation; we find that if (and only if) $h_e(z) = h_1(z)$, then

$$w(\zeta + 2K) = w(\zeta). \tag{2.6}$$

Thus $w(\zeta)$ then has both $2K$ and $2iK'$ as periods.

If $h_e(z) = h_1(z)$ is given by (2.2), then (2.4) implies that $w(\zeta)$ is meromorphic, and therefore elliptic, with just N simple poles in any fundamental parallelogram. Moreover, if the injection point $z = a_j$ is the image of the point $\zeta = \gamma_j$ under the map, so

$$w(\gamma_j) = a_j \quad \text{for } j = 1, 2, \ldots, N, \tag{2.7}$$

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then $w(\zeta)$ has simple poles at the points $\zeta = -\overline{\gamma}_j$ for $j = 1, 2, \ldots, N$. Thus

$$w(\zeta) = \sum_{j=1}^{N} \overline{\beta}_j Z_m(\zeta + \overline{\gamma}_j) + \delta \tag{2.8}$$

for some constants β_j and δ; here $Z_m(\zeta)$ is the modified zeta function defined by equation (1.1), and this is just the elliptic function equivalent of the partial fraction expansion of a rational function with only simple poles. Since the $\overline{\beta}_j$ are the residues at the poles and these must sum to zero for an elliptic function, we also have

$$\sum_{j=1}^{N} \beta_j = 0. \tag{2.9}$$

If we compare residues at the poles on both sides of (2.4), we find that we must have

$$\beta_j w'(\gamma_j) = -r_j^2 \quad \text{for } j = 1, 2, \ldots, N. \tag{2.10}$$

We now have all the equations we need to determine D, and a naïve parameter and equation count suggests that all is well. The map (2.8) involves $2N + 1$ complex parameters β_j, γ_j and δ, plus 1 real parameter k, and these are to be determined from the $2N + 1$ complex equations (2.7), (2.9) and (2.10), plus the one real equation derived from the assumption we make concerning the hole. However, it cannot be quite this straightforward because we have not yet made any assumption that fixes the location of the image in the ζ-plane in the η-direction; we could do this by taking γ_1, say, to be real, but this seems to upset the above count. In fact, with $w(\zeta)$ given by (2.8), we find that the sum of the left-hand sides of the equations in (2.10) over j from 1 to N is necessarily real, so these equations are not independent. We can, indeed, suppose γ_1 to be real, but must discard the imaginary part of one of the equations in (2.10).

The mapping obtained in Richardson (1994) looks rather different from that in (2.8); in particular, it is expressed in terms of the function $\operatorname{dn}\zeta$ rather than zeta functions. However, it can be shown that the present results reduce to the earlier ones in the special circumstances considered there. One needs to note that there the entire doubly-connected region was mapped onto the rectangle $0 < \xi < K$, $0 < \eta < 4K'$, rather than the rectangle $0 < \xi < K$, $0 < \eta < 2K'$ used here, so the elliptic functions that appear in the two approaches employ different moduli and are related by the Gauss–Landen transformation.

One may consider the limiting forms taken by the map (2.8) both as $k \to 0$ and as $k \to 1$ but, as these are not required to exploit any of the results, we will not record the details. However, it does seem to be worth noting that the standard references on elliptic functions do not contain a number of the results such an analysis requires, and they must be supplemented by formulae derived in Carlson & Todd (1983).

From equations (2.3) and (2.6), we see that the elliptic functions with which we must deal have $2K$ and $2iK'$ as periods. We refer to the rectangle $0 < \xi < 2K$, $0 < \eta < 2K'$ as *the* fundamental rectangle, while the portion in $0 < \xi < K$ will be referred to as the *left half*, denoted by L, and that in $K < \xi < 2K$ as the *right half*, denoted by R. As is familiar in the theory of elliptic functions, there are difficulties if one wishes to be precise, both here and later. (Do we include the boundary in a fundamental rectangle? What if a

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singularity lies on this boundary? Our fundamental rectangle is not the union of its left half L and right half R !) We opt for imprecision over verbosity, with apologies to those who object to this pragmatic solution to a very real dilemma.

It is the left half L of the fundamental rectangle (the rectangle ABCD in the ζ-plane of Figure 1) that maps onto D, and the N simple poles of the mapping function $w(\zeta)$ in (2.8) that lie in the fundamental rectangle are in the right half R at the points $\zeta = 2K - \overline{\gamma}_j$ for $j = 1, 2, \ldots, N$. The derivative $w'(\zeta)$ has N double poles in the fundamental rectangle, and so must also have $2N$ zeros there, and these too must lie in the right half R. These zeros will be important in later developments, and their position has an important influence on the geometry of D ; a zero of $w'(\zeta)$ that approaches the left-hand boundary of R leads to cusp formation in the inner boundary C_1 of D, while one approaching its right-hand boundary produces a cusp in the outer boundary C_0.

It is instructive to consider the symmetric case when we inject an equal area πr^2 at $N \geqslant 3$ points uniformly distributed round a circle of radius a centred on the origin, so $r_j = r$ for $j = 1, 2, \ldots, N$. We can take the injection points at

$$z = a_j = a\omega^{1-j} \quad \text{for } j = 1, 2, \ldots, N, \text{ where } \omega = e^{2\pi i/N}. \tag{2.11}$$

Choosing $\gamma_1 = \gamma$ to be real (with $0 < \gamma < K$ of course), we have

$$\gamma_j = \gamma + 2i(j-1)K'/N \text{ and } \beta_j = \beta\omega^{j-1} \quad \text{for } j = 1, 2, \ldots, N, \tag{2.12}$$

where β is real and positive. The symmetry ensures that equation (2.9) is automatically satisfied, and we have $\delta = 0$ in (2.8). Thus the mapping function $w(\zeta)$ in (2.8) now involves just three real parameters γ, β and k ; given a and r, we are to find these parameters from the two real equations $w(\gamma) = a$ and $\beta w'(\gamma) = -r^2$ to which the sets (2.7) and (2.10) reduce in this instance, plus the one further condition we know to be necessary that relates to the assumption made about the hole.

For this scenario to give us a doubly-connected domain, it is obviously necessary that $r > a\sin(\pi/N)$. With the opposite inequality, we have N disjoint circular disks, and they touch when we have equality. With r only a little larger than $a\sin(\pi/N)$, we have a doubly-connected domain that is close to one with N touching disks, and can obtain the relevant values of γ, β and k by using a path-following algorithm, as exploited by Richardson (1997) in the simply-connected case. These will serve as initial values of the parameters when we consider Stokes flows that begin from such initial states, and we can likewise find the parameters necessary to give us similar initial states that are somewhat more complicated simply by solving sets of equations.

In the symmetric situation envisaged in equations (2.11) and (2.12), the $2N$ zeros of $w'(\zeta)$ in R are $2N$ simple zeros and lie in pairs on lines parallel to the ξ-axis that bisect the line between adjacent γ_j – more precisely, they lie on the lines $\eta = (2j-1)K'/N$ for $j = 1, 2, \ldots, N$. That this must be so when r is only a little larger than $a\sin(\pi/N)$ is evident geometrically; it is these $2N$ zeros that produce the $2N$ near-cusps in this situation.

3 Stokes flows

We now consider the Stokes flow problem in the region D of the z-plane of Figure 1. We suppose a constant surface tension T to act at each free boundary and, moreover, suppose

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that we have the *same* constant pressure (which we may take to be zero) both outside C_0 and inside C_1. This difference between our assumptions for Hele-Shaw and Stokes flows is not only necessary for our theories, but is also natural from a practical standpoint: while one must take special measures, with an appropriate air vent, to ensure that these pressures are equal in a Hele-Shaw cell, it is precisely this equal-pressure scenario that arises most readily in any sintering flow for which a two-dimensional theory is relevant.

As in Richardson (1992), we can describe a Stokes flow in D in terms of two functions $\phi(z)$ and $\chi(z)$ that are analytic in D. If (u, v) is the velocity field in Cartesian form, we have

$$u + iv = \phi(z) - z\overline{\phi'(z)} - \overline{\chi'(z)}. \tag{3.1}$$

In Richardson (1992), there was just one free boundary, but the complicating feature that arises in the derivation of the relevant force balance condition when there are several free boundaries is discussed in Richardson (1968): one can exploit an arbitrariness in the definitions of $\phi(z)$ and $\chi(z)$ to ensure that a constant of integration vanishes on *one* free boundary, but no more. Thus we have

$$\left.\begin{aligned} \phi(z) + z\overline{\phi'(z)} + \overline{\chi'(z)} &= +\frac{T\,i}{2\mu}\frac{dz}{ds} && \text{on } C_0, \\ \phi(z) + z\overline{\phi'(z)} + \overline{\chi'(z)} &= -\frac{T\,i}{2\mu}\frac{dz}{ds} + \mathscr{C} && \text{on } C_1, \end{aligned}\right\} \tag{3.2}$$

where μ is the viscosity of the fluid and $\mathscr{C}$ is some constant. We have here chosen the arc-length parameter s to increase as we move anti-clockwise round both C_0 and C_1.

We also refer to Richardson (1968) for a discussion of a further complication arising with flows in doubly-connected regions. In general, though $\phi'(z)$ is single-valued, $\phi(z)$ itself need not be. However, the multi-valued nature of $\phi(z)$ is associated with a net force exerted on the flow by whatever is within C_1; in present circumstances, $\phi(z)$ *is* single-valued. This implies that $\chi'(z)$ is single-valued – but $\chi(z)$ itself need not be, and *is* not here. The function $\chi(z)$ gives rise to a potential flow, and its multi-valued nature corresponds to a source/sink flow or a potential vortex in that context. One can show that $\int \chi'(z)\,dz$, where the integral may be taken round C_0 or C_1, say, is purely imaginary in the current situation, which means that there is no potential vortex, but we *do* have (and should expect) a sink flow.

At this stage we draw attention to the quasi-steady nature of our problem. At some given time when we have a given domain D, the force balance conditions (3.2) alone determine the velocity field at that time; as in the simply-connected case, it is determined only up to a uniform translation and rotation, but this lack of uniqueness is to be expected. The kinematic condition at the boundary that we have yet to introduce serves only to tell us how D must change at that time, thereby leading to a dependence of $D, \phi(z), \chi(z)$ and $\mathscr{C}$ on time that we have thus far suppressed in our notation – and will continue to suppress for brevity.

The boundary value problem posed by (3.2) is of a kind familiar in the theory of plane elasticity and, for a given D, we have $\mathscr{C}$ determinate; an accessible proof of this result can be found in Lu (1995). An instructive special case arises when D is an eccentric circular annulus; this is *not* a geometry to which our present theory applies directly, but

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elementary methods then suffice to solve the problem. If C_0 is a circle of radius r_0 centred on $z = a_0$ and C_1 is a circle of radius r_1 centred on $z = a_1$, where we may suppose a_0 and a_1 to be real, we find that

$$\mathscr{C} = \frac{T}{2\mu}\frac{(a_0 - a_1)(r_0 - r_1)}{r_0^2 + r_1^2} \quad \text{where} \; -(r_0 - r_1) < a_0 - a_1 < r_0 - r_1; \tag{3.3}$$

the inequalities here are geometric constraints to ensure that C_1 lies inside C_0, with $r_0 > r_1$ also obviously necessary. Thus, for fixed radii, we have $\mathscr{C}$ proportional to the distance between the centres of the circles. Note that $\mathscr{C}$ tends to a finite limit as D approaches a singular state with C_1 touching C_0, corresponding to equality in one of the inequalities in (3.3).

It is of interest to determine how $\mathscr{C}$ changes when D is subjected to various transformations. A translation (i.e. replace z by $z + \alpha$ for some constant α) leaves $\mathscr{C}$ unchanged. Similarly, a simple scaling (replace z by mz for some *positive* constant m) does not change $\mathscr{C}$; this also follows from dimensional considerations, for we see from (3.1) and (3.2) that $\mathscr{C}$ has the dimensions of a velocity, and is equal to T/μ times some dimensionless function.

If we consider a rotation about the origin through an angle θ (z is replaced by $ze^{i\theta}$), we find that $\mathscr{C}$ becomes $\mathscr{C}e^{i\theta}$. Thus, if D is invariant under rotation about the origin through any angle other than an integer multiple of 2π, we must have $\mathscr{C} = 0$. This observation will be crucial to our work.

If we consider reflection in the line $y = x\tan\theta$ (replace z by $\bar{z}e^{2i\theta}$) we find that $\mathscr{C}$ becomes $\overline{\mathscr{C}}e^{2i\theta}$. As particular cases, if D is symmetric about the real axis then $\mathscr{C}$ must be real, while if D is symmetric about the imaginary axis then $\mathscr{C}$ must be purely imaginary. If D is symmetric about *both* axes we must have $\mathscr{C} = 0$, but this is a special case of the earlier observation for D must then be invariant under rotation through an angle π about the origin.

As in § 2, we now map D onto the ζ-plane in Figure 1 via the function $z = w(z)$. The doubly-connected nature of D alone implies that we have (2.3) satisfied, and we need assume no more about $w(z)$ for the moment.

Defining

$$\Phi(\zeta) = \phi(w(\zeta)) \quad \text{and} \quad X(\zeta) = \chi'(w(\zeta)), \tag{3.4}$$

the fact that $\phi(z)$ and $\chi'(z)$ are single-valued in D means that

$$\Phi(\zeta + 2iK') = \Phi(\zeta) \quad \text{and} \quad X(\zeta + 2iK') = X(\zeta), \tag{3.5}$$

and these allow $\Phi(\zeta)$ and $X(\zeta)$ to be analytically continued into the entire strip $0 < \xi < K$ as functions with period $2iK'$. Note the derivative on the right-hand side of the second equation in (3.4); without this, $X(\zeta)$ would *not* have $2iK'$ as a period but would acquire some *a priori* unknown increment when $2iK'$ was added to its argument.

Transferring conditions (3.2) to the ζ-plane, we find that we have

$$\left.\begin{aligned} \Phi(\zeta) + w(\zeta)\frac{\overline{\Phi'(\zeta)}}{\overline{w'(\zeta)}} + \overline{X(\zeta)} &= +\frac{T}{2\mu}\frac{w'(\zeta)}{|w'(\zeta)|} && \text{on } \xi = 0, \\ \Phi(\zeta) + w(\zeta)\frac{\overline{\Phi'(\zeta)}}{\overline{w'(\zeta)}} + \overline{X(\zeta)} &= -\frac{T}{2\mu}\frac{w'(\zeta)}{|w'(\zeta)|} + \mathscr{C} && \text{on } \xi = K. \end{aligned}\right\} \tag{3.6}$$

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Consider now the function

$$F(\zeta) = \frac{1}{\{w'(\zeta)\overline{w'(-\bar{\zeta})}\}^{1/2}}, \tag{3.7}$$

and add to $w(\zeta)$ the condition that it satisfy (2.6); thus $F(\zeta)$ is real on both $\xi = 0$ and $\xi = K$. With $w(\zeta)$ also supposed to be meromorphic, and therefore elliptic, with only simple poles, it follows that it can be represented as in (2.8) for some N, β_j, γ_j and δ, with the β_j so that (2.9) also holds. This does not yet imply that D is such that it could be produced by injection at N points into an initially empty Hele-Shaw cell, for this also requires that $\beta_j w'(\gamma_j)$ be real for all $j = 1, 2, \ldots, N$, as demanded by (2.10).

Because of the power of 1/2 appearing in (3.7), the function $F(\zeta)$ is defined globally on a two-sheeted Riemann surface. However, we can confine attention to just one particular sheet of this surface by introducing branch cuts, and choose both the sheet we consider and the position of the cuts at our convenience to remove the ambiguities of sign inherent in (3.7).

The function $w(\zeta)$ has N simple poles in the fundamental rectangle, all in its right half R, and $w'(\zeta)$ has $2N$ zeros there. These zeros furnish branch points of $F(\zeta)$ and we join them in pairs by branch cuts within R. Similarly, $\overline{w'(-\bar{\zeta})}$ has $2N$ zeros in the left half L of the fundamental rectangle and we join these in pairs by branch cuts within L; we can do this so that the cuts in L are the mirror images of those in R in the line $\xi = K$. Now repeat these cuts periodically throughout the ζ-plane, and we have $F(\zeta)$ single-valued and such that

$$F(\zeta + 2K) = F(\zeta), \;\; F(\zeta + 2iK') = F(\zeta) \quad \text{and} \quad \overline{F(-\bar{\zeta})} = F(\zeta) \tag{3.8}$$

throughout this cut plane.

Note that the poles of $w'(\zeta)$, being double poles, do not give rise to singularities, but are *zeros* of $F(\zeta)$. In this respect, it is useful to examine the properties of $F(\zeta)$ in the symmetric situation considered at the end of § 2. The cuts can then all be taken along lines $\eta =$ constant, and $F(\zeta)$ has a single simple zero on the portion $0 < \xi < K$ of the ξ-axis at $\zeta = \gamma$. Moreover, $F(\zeta)$ is real on the real axis. It follows that, if we take $F(\zeta)$ to be positive on $\xi = 0$ it must be negative on $\xi = K$. That is, we then have

$$F(\zeta) = \begin{cases} +\dfrac{1}{|w'(\zeta)|} & \text{on } \xi = 0, \\[2ex] -\dfrac{1}{|w'(\zeta)|} & \text{on } \xi = K. \end{cases} \tag{3.9}$$

In more general situations, the sign of $F(\zeta)$ on $\xi = K$ depends on how the branch cuts are arranged, but it will be convenient to suppose that they are always chosen so that (3.9) is true. A different choice will, of course, have no effect on the final results, but would affect our presentation. However, this convention (and the desire to have the system of cuts change in a continuous manner when we consider the evolution in time) may force us to relax the above conditions on the cuts; they may need to cross the upper and lower boundaries of the fundamental rectangle, but this causes no difficulty.

A function similar to $F(\zeta)$ arises in the analysis of the simply-connected situation. There, the simply-connected flow region is mapped onto the unit disk $|\zeta| < 1$ and the function corresponding to $F(\zeta)$ has singularities in both $|\zeta| < 1$ and $|\zeta| > 1$. A crucial

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step in the analysis presented by Richardson (1992) is the decomposition of this function into a sum of two functions, one analytic in $|\zeta| < 1$ and the other analytic in $|\zeta| > 1$. We require a similar decomposition here: $F(\zeta)$ has singularities in both the left half L and right half R of the fundamental rectangle, and we need to write it as a sum of two functions, one analytic in L and the other analytic in R. Moreover, these functions should continue to reflect the properties of $F(\zeta)$ as closely as possible. They cannot, in general, have both $2K$ and $2iK'$ as periods (an elliptic function could be thus decomposed if and only if the sum of its residues in L were zero); we will arrange to have $2iK'$ as a period, but $2K$ as a quasi-period.

We introduce a contour Γ. If the cuts introduced to make $F(\zeta)$ single-valued do not cross the boundary of L, as for the symmetric situation discussed earlier, then Γ is just this boundary traversed in the anticlockwise direction, i.e. it is the rectangle ABCDA in the ζ-plane of Figure 1. If the cuts *do* need to cross AB and CD, we deform these sides in a congruent manner to avoid the cuts in defining Γ. However, for brevity we will suppose that we can take Γ to be a rectangle in our exposition, the adjustments that are necessary if this is not possible being minor. In fact, Γ will generally appear as the contour of integration with integrands that have $2iK'$ as a period, and all that survives is the sum of integrals along the two straight lines DA and BC – indeed, we may integrate *up* any line segment of length $2K'$ on $\xi = K$ and *down* any line segment of length $2K'$ on $\xi = 0$.

Now define

$$F_L(\zeta) = \frac{1}{2\pi i}\int_\Gamma F(\tau) Z_m(\tau - \zeta)\, d\tau \quad \text{for } \zeta \text{ in } L, \tag{3.10}$$

with $F_L(\zeta)$ defined elsewhere by analytic continuation; this may be accomplished by deforming the contour of integration. $F_L(\zeta)$ is analytic in L; indeed, it is analytic in the infinite strip $0 < \xi < K$ and has $2iK'$ as a period.

We define $F_R(\zeta)$ by the same formula in (3.10), but with ζ now in R, and again invoke analytic continuation. $F_R(\zeta)$ is analytic not only in R, but also in the infinite strip $K < \xi < 2K$, and has $2iK'$ as a period.

Starting with the definition of $F_L(\zeta)$ in (3.10) with ζ in L, move ζ to the right into R. The portion of Γ along DA must be deformed, but a further deformation reduces the contour of integration to Γ plus a small circle around the point $\tau = \zeta$. Since all the poles of $Z_m(\zeta)$ have residue 1, we obtain

$$F_L(\zeta) = F_R(\zeta) + F(\zeta). \tag{3.11}$$

Now return to (3.10) and consider the value of $F_L(\zeta + 2K)$ with ζ in L, obtained by translating ζ a distance $2K$ to the right. Recalling the array of poles possessed by $Z_m(\zeta)$, this requires that both the sides DA and BC of Γ be deformed, but a further deformation reduces the contour of integration to Γ plus a small circle around each of the points $\tau = \zeta$ and $\tau = \zeta + 2K$. This yields

$$F_L(\zeta + 2K) = \frac{1}{2\pi i}\int_\Gamma F(\tau) Z_m(\tau - \zeta - 2K)\, d\tau + F(\zeta + 2K) - F(\zeta). \tag{3.12}$$

But $2K$ is a period of $F(\zeta)$ and a quasi-period of $Z_m(\zeta)$ – see equations (3.8) and (1.2) – so (3.12) reduces to a statement that it is also a quasi-period of $F_L(\zeta)$. Specifically, we

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now have

$$F_L(\zeta + 2iK') = F_L(\zeta) \quad \text{and} \quad F_L(\zeta + 2K) = F_L(\zeta) - \frac{1}{2iK'} \int_\Gamma F(\tau)\, d\tau. \tag{3.13}$$

$F_R(\zeta)$ evidently satisfies the same equations, so (3.11) furnishes the decomposition of $F(\zeta)$ that we need.

With ζ in R we have $2K - \overline{\zeta}$ in L, so

$$F_L(2K - \overline{\zeta}) = \frac{1}{2\pi i} \int_\Gamma F(\tau) Z_m(\tau + \overline{\zeta} - 2K)\, d\tau \quad \text{for } \zeta \text{ in } R.$$

Now take the conjugate of this and use the symmetries of $F(\zeta)$ and $Z_m(\zeta)$ recorded in (3.8), (1.2), (1.3) and (1.4). We obtain

$$\overline{F_L(2K - \overline{\zeta})} = -F_R(\zeta) + \frac{1}{2K'} \int_0^{2K'} F(i\eta)\, d\eta.$$

Using (3.11), we can write this as

$$F(\zeta) = F_L(\zeta) + \overline{F_L(2K - \overline{\zeta})} - \frac{1}{2K'} \int_0^{2K'} F(i\eta)\, d\eta.$$

The particular form of this equation that we will need arises when $\zeta = 2K - \overline{\zeta}$. We have

$$F(\zeta) = 2\,\mathrm{Re}\{F_L(\zeta)\} - \frac{1}{2K'} \int_0^{2K'} F(i\eta)\, d\eta \quad \text{on } \xi = K. \tag{3.14}$$

In a similar manner, we find that

$$F(\zeta) = 2\,\mathrm{Re}\{F_L(\zeta)\} - \frac{1}{2K'} \int_0^{2K'} F(i\eta + K)\, d\eta \quad \text{on } \xi = 0. \tag{3.15}$$

The integrals in (3.14) and (3.15) arise as integrals along just one side of Γ, and the range of integration can be any interval of length $2K'$. These equations are the analogue of (2.17) in Richardson (1992), and will be pivotal in our development.

After this digression to derive the necessary results concerning $F(\zeta)$, we return to conditions (3.6). The first of these can be written as

$$\Phi(\zeta) + w(\zeta) \frac{\overline{\Phi'(-\overline{\zeta})}}{\overline{w'(-\overline{\zeta})}} + \overline{X(-\overline{\zeta})} = \frac{T}{2\mu} F(\zeta) w'(\zeta), \tag{3.16}$$

while the second is

$$\Phi(\zeta) + w(\zeta) \frac{\overline{\Phi'(2K - \overline{\zeta})}}{\overline{w'(2K - \overline{\zeta})}} + \overline{X(2K - \overline{\zeta})} = \frac{T}{2\mu} F(\zeta) w'(\zeta) + \mathscr{C}. \tag{3.17}$$

Together, these allow $\Phi(\zeta)$ and $X(\zeta)$ to be analytically continued into the cut plane, though all the cuts provided for $F(\zeta)$ are not necessary to make $\Phi(\zeta)$ and $X(\zeta)$ single-valued.

If we replace ζ by $\zeta + 2K$ in (3.17), and exploit (2.6) and (3.8), we find that we can combine the result with (3.16) to obtain

$$\Phi(\zeta + 2K) = \Phi(\zeta) + \mathscr{C}. \tag{3.18}$$

Thus if (and only if) $\mathscr{C} = 0$, we have $\Phi(\zeta)$ doubly-periodic with both $2K$ and $2iK'$ as periods.

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Knowing D and the corresponding $w(\zeta)$, we could use the equations we have obtained thus far to calculate $\Phi(\zeta), X(\zeta)$ and $\mathscr{C}$. However, our primary interest is in the evolution of the geometry and the situation in the simply-connected case encourages us to believe that we can avoid an explicit calculation of these. In fact, things are not quite so simple here and we will have to return to this point, but we now proceed to consider the time evolution dictated by the kinematic condition. To emphasize the time-dependence where it seems necessary, we will sometimes write $w(\zeta\,;t)$ instead of $w(\zeta)$, but continue to suppress the time variable in the notation for the other functions – and the 'constant' $\mathscr{C}$!

A complicating feature is that, as D evolves, the proportions of the rectangle in the ζ-plane of Figure 1 change as the modulus k changes. Considerations are slightly simpler on C_0, for that always corresponds to $\xi = 0$. From (3.1) and (3.2) we see that the velocity of a fluid particle is given by

$$u + iv = 2\phi(z) - \frac{Ti}{2\mu}\frac{dz}{ds} \quad \text{on } C_0.$$

If $\zeta = i\eta$ is the corresponding point in the ζ-plane, this is

$$u + iv = 2\Phi(\zeta) - \frac{T}{2\mu}F(\zeta)w'(\zeta) \quad \text{on } \xi = 0. \tag{3.19}$$

As D evolves, a given fluid particle on C_0 corresponds to a point $\zeta(t) = i\eta(t)$ via $z = w(\zeta\,;t)$, so we also have

$$u + iv = w'(\zeta\,;t)\frac{d\zeta}{dt} + \frac{\partial w}{\partial t}(\zeta\,;t) = iw'(\zeta\,;t)\frac{d\eta}{dt} + \frac{\partial w}{\partial t}(\zeta\,;t) \quad \text{on } \xi = 0, \tag{3.20}$$

and combining (3.19) and (3.20) leads us to

$$\frac{1}{w'(\zeta\,;t)}\left\{2\Phi(\zeta) - \frac{\partial w}{\partial t}(\zeta\,;t)\right\} - \frac{T}{2\mu}F(\zeta) = i\frac{d\eta}{dt} \quad \text{on } \xi = 0. \tag{3.21}$$

The argument is similar for a point on C_1, but now the corresponding point in the ζ-plane is $\zeta(t) = K + i\eta(t)$, and K depends on k which in turn depends upon t. We obtain

$$\frac{1}{w'(\zeta\,;t)}\left\{2\Phi(\zeta) - \frac{\partial w}{\partial t}(\zeta\,;t) - \mathscr{C}\right\} - \frac{T}{2\mu}F(\zeta) = \frac{dK}{dk}\frac{dk}{dt} + i\frac{d\eta}{dt} \quad \text{on } \xi = K. \tag{3.22}$$

Now consider the function $f(\zeta)$ defined by

$$f(\zeta) = \frac{1}{w'(\zeta\,;t)}\left\{2\Phi(\zeta) - \frac{\partial w}{\partial t}(\zeta\,;t)\right\} - \frac{T}{\mu}F_L(\zeta); \tag{3.23}$$

it is analytic in L. Combining (3.15) and (3.21) we have

$$\text{Re}\{f(\zeta)\} = -\frac{T}{4\mu K'}\int_0^{2K'} F(i\eta + K)\,d\eta \quad \text{on } \xi = 0. \tag{3.24}$$

We now suppose our problem to be such that $\mathscr{C} = 0$. We know this to be so if the geometry has an appropriate rotational symmetry, and we will return to make further comments on this later. It is, of course, instructive to try to proceed *without* the assumption $\mathscr{C} = 0$, if only to convince oneself that it is indeed much simpler to proceed *with* it! If $\mathscr{C} = 0$, then combining (3.14) and (3.22) leads to

$$\text{Re}\{f(\zeta)\} = -\frac{T}{4\mu K'}\int_0^{2K'} F(i\eta)\,d\eta + \frac{dK}{dk}\frac{dk}{dt} \quad \text{on } \xi = K. \tag{3.25}$$

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At first glance, it might appear that $f(\zeta)$ defined by (3.23) has $2iK'$ as a period, because it is indeed a period of $\Phi(\zeta), w(\zeta;t)$ and $F_L(\zeta)$. However, this is not correct for the derivative of $w(\zeta;t)$ with respect to t does *not* have $2iK'$ as a period. Equation (2.3) holds for all t, so its total derivative with respect to t vanishes, but K' depends on k which varies with t. Differentiating (2.3) with respect to t, we can write the result as

$$\frac{1}{w'(\zeta+2iK';t)}\frac{\partial w}{\partial t}(\zeta+2iK';t)-\frac{1}{w'(\zeta;t)}\frac{\partial w}{\partial t}(\zeta;t)=-2i\frac{dK'}{dk}\frac{dk}{dt},$$

so for $f(\zeta)$ we have

$$f(\zeta+2iK')-f(\zeta)=2i\frac{dK'}{dk}\frac{dk}{dt}; \tag{3.26}$$

thus $2iK'$ is a quasi-period, not a period, of $f(\zeta)$.

Since $f(\zeta)$ is analytic in L, its integral round Γ must vanish. Using (3.26) we thus have

$$\int_0^{2K'}[f(i\eta+K)-f(i\eta)]\,d\eta=2K\frac{dK'}{dk}\frac{dk}{dt}.$$

If we now take the real part of this equation, and use (3.24) and (3.25), we find that

$$\left(K'\frac{dK}{dk}-K\frac{dK'}{dk}\right)\frac{dk}{dt}=\frac{T}{4\mu}\int_0^{2K'}[F(i\eta)-F(i\eta+K)]\,d\eta. \tag{3.27}$$

The left-hand side here can be simplified using standard formulae for the derivatives in conjunction with Legendre's relation; see, for example, Bowman (1961, pp. 21, 25). The right-hand side can also be expressed more explicitly in terms of $w'(\zeta)$ using (3.9). The result is

$$\frac{dk}{dt}=\frac{T}{2\mu}\frac{kk'^2}{\pi}\int_0^{2K'}\left(\frac{1}{|w'(i\eta)|}+\frac{1}{|w'(i\eta+K)|}\right)d\eta, \tag{3.28}$$

an equation controlling the change of the modulus k. Evidently, k increases with time.

With (3.27) holding, the problem for $\mathrm{Re}\{f(\zeta)\}$ posed by (3.24) and (3.25) has a unique solution (it is equivalent to a Dirichlet problem in a concentric annulus) and the corresponding $f(\zeta)$ is obvious by inspection; we have

$$f(\zeta)=-\frac{T}{4\mu K'}\int_0^{2K'}F(i\eta+K)\,d\eta+\frac{1}{K'}\frac{dK'}{dk}\frac{dk}{dt}\zeta. \tag{3.29}$$

A purely imaginary constant could be added on the right here, but it can be assumed to vanish. According to (3.23), replacing ζ by $\zeta+i\alpha(t)$ for some real function $\alpha(t)$ corresponds to adding $id\alpha/dt$ to $f(\zeta)$, so this assumption merely determines a displacement in the η-direction in the ζ-plane that has thus far remained arbitrary. If we have a geometry in the z-plane that is symmetric about the x-axis, and preserve this symmetry in the mapping to the ζ-plane with all the functions involved real on the ξ-axis, then this constant *must* vanish.

We have imposed the requirement that $2iK'$ remain a period of $w(\zeta)$ as k changes with time to obtain equation (3.28), but what of $2K$? Remarkably, that this also remains a period is implied by our equations, *provided* $\mathscr{C}=0$. Combining (3.23) and (3.29), we have

$$\frac{\partial w}{\partial t}(\zeta;t)=2\Phi(\zeta)-\frac{T}{\mu}F_L(\zeta)w'(\zeta)+\frac{T}{4\mu K'}w'(\zeta)\int_0^{2K'}F(i\eta+K)\,d\eta-\frac{1}{K'}\frac{dK'}{dk}\frac{dk}{dt}\zeta w'(\zeta). \tag{3.30}$$

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Subtract this from the equation obtained from it by replacing ζ by $\zeta+2K$, using both (2.6), and (3.18) with $\mathscr{C}=0$. The resulting equation involves the combination $F_L(\zeta+2K)-F_L(\zeta)$ which we express in terms of an integral using the second equation in (3.13). However, this integral also appears in (3.27), and we can write the result in the form

$$\frac{d}{dt}[w(\zeta+2K\,;t)-w(\zeta\,;t)]=0,$$

thus justifying our claim.

If we use the decomposition (3.11), equation (3.16) can be written as

$$\Phi(\zeta)-\frac{T}{2\mu}F_L(\zeta)w'(\zeta)=-w(\zeta)\frac{\overline{\Phi'(-\overline{\zeta})}}{\overline{w'(-\overline{\zeta})}}-\overline{X(-\overline{\zeta})}-\frac{T}{2\mu}F_R(\zeta)w'(\zeta). \tag{3.31}$$

Equations (3.30) and (3.31) are the relevant forms here of equations (4.4) and (4.3) in Richardson (1997). As there, we can eliminate $\Phi(\zeta)$ between them; this is straightforward but tedious, and the result can be written in the form

$$\begin{aligned}&\frac{\partial}{\partial t}\left[w'(\zeta\,;t)\overline{w(-\overline{\zeta}\,;t)}\right]+2X(\zeta)w'(\zeta)=\\&-\frac{\partial}{\partial\zeta}\left[w'(\zeta)\overline{w(-\overline{\zeta})}\left\{\frac{T}{\mu}F_L(\zeta)-\frac{T}{4\mu K'}\int_0^{2K'}F(i\eta+K)\,d\eta+\frac{1}{K'}\frac{dK'}{dk}\frac{dk}{dt}\zeta\right\}\right].\end{aligned} \tag{3.32}$$

The term $2X(\zeta)w'(\zeta)$ on the left-hand side here is analytic in L, the left half of the fundamental rectangle, as are the functions $w'(\zeta)$ and $F_L(\zeta)$ appearing in the other terms. With $w(\zeta)$ of the form (2.8), the only singularities in L arise from those of $\overline{w(-\overline{\zeta})}$ at $\zeta=\gamma_j$ for $j=1,2,\dots,N$. The functions comprising each side of (3.32) both have double poles at these points, but the residues of the right-hand side are zero there. Equating the residues on the left-hand side to zero, we obtain

$$\frac{d}{dt}[\beta_j w'(\gamma_j)]=0 \quad \text{for } j=1,2,\dots,N. \tag{3.33}$$

Thus the combinations $\beta_j w'(\gamma_j)$ are invariants and, if we are considering an initial state that could have been created in a Hele-Shaw cell by injecting an area πr_j^2 at the jth of N points, as in § 2, we have

$$\beta_j w'(\gamma_j)=-r_j^2 \quad \text{for } j=1,2,\dots,N. \tag{3.34}$$

Thus all later states can be created by injecting an area πr_j^2 at the jth of N points in a Hele-Shaw cell too.

If we compare the coefficients of $(\zeta-\gamma_j)^{-2}$ on both sides of (3.32), we obtain differential equations governing the motion of the γ_j in the ζ-plane. These can be written in the form

$$K'\frac{d}{dt}\left(\frac{\gamma_j}{K'}\right)=\frac{T}{\mu}\left[F_L(\gamma_j)-\frac{1}{4K'}\int_0^{2K'}F(i\eta+K)\,d\eta\right] \quad \text{for } j=1,2,\dots,N. \tag{3.35}$$

In deriving our equations, we have assumed that $\mathscr{C}=0$, and the only circumstances for which this assumption has been justified is when the geometry is invariant under a rotation about the origin through an angle $2\pi/n$ for some integer $n\geqslant 2$. With N the number of poles of $w(\zeta)$ in the fundamental rectangle, we must have $N=pn$ for some integer p, and incorporating the symmetry into the form of $w(\zeta)$ in (2.8) leaves only p

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of each of the β_j and γ_j, plus the modulus k, as parameters, for then $\delta = 0$. Moreover, condition (2.9) is automatically satisfied. Correspondingly, (3.28), (3.34) and (3.35) collapse to $2p + 1$ equations governing these parameters. However, we need to bear in mind that the motion in such a free boundary problem in Stokes flow is not determined to within a rigid body translation and rotation. Demanding that we retain the rotational invariance about the origin fixes up the translation. If we have symmetry about the real axis, say, maintaining this symmetry for all time fixes up the rotation, but with no such symmetry a mathematically convenient way of doing this is to suppose that the argument of one of the β_j remains constant. Equations (3.34) show that this is imposing a constraint that the rotation induced by the mapping at the corresponding injection point in the Hele-Shaw analogue is invariant. To compensate in the equation count, we must discard the imaginary part of one of the equations in (3.34), as already discussed in § 2.

The differential equations (3.28) and (3.35) have naturally arisen with t as the independent variable, and we expect k to increase to 1 as the hole disappears at some *a priori* unknown time; with the rotational symmetry we are assuming, it must vanish at the origin. For this reason, it is computationally convenient to recast these equations with the independent variable as k – or as $m = k^2$ if using *Mathematica* or any other computer system that employs the parameter m rather than the modulus k in its specification of the elliptic functions.

As the hole disappears and C_1 shrinks to a point, C_0 will not be circular and there will be further motion with the fluid occupying a simply-connected region that can be determined using the methods of Richardson (1997). One could effect the changeover by exploiting the Hele-Shaw connection; if the doubly-connected motion ends with the injection points in certain positions, then this gives the physical state from which the further integration must begin. However, this is not necessary, for the first phase ends with the fundamental rectangle degenerating into a semi-infinite strip, and mapping this onto the unit disk shows that, if the first phase ends with a particular γ_j having the final value c_j, then the initial value of the corresponding γ_j for the second simply-connected phase can be taken as e^{-2c_j}.

4 Examples

We present the results of some computations using equations (3.28), (3.34) and (3.35) to show the efficacy of the method we have developed. It will be convenient to introduce a new time variable, replacing $(T/\mu)t$ by t, so that values we quote for t will be purely numerical; this is as in Richardson (1997), and we note again that this has the curious consequence of making t have the dimensions of a length.

Figure 2 shows the situation when we begin with three equal circular disks that are initially just touching each other; this is the geometry discussed towards the end of § 2 with $N = 3$. With the disks of radius 1, we find that the central hole disappears at $t = 0.584$. The geometry that arises in this example has also been considered in the context of quadrature domains by Gustafsson (1988).

In Figure 3 we have the corresponding configuration with four disks. Again with the disks of radius 1, the central hole disappears at $t = 1.150$.

In both Figures 2 and 3 we have configurations that are symmetric about a line,

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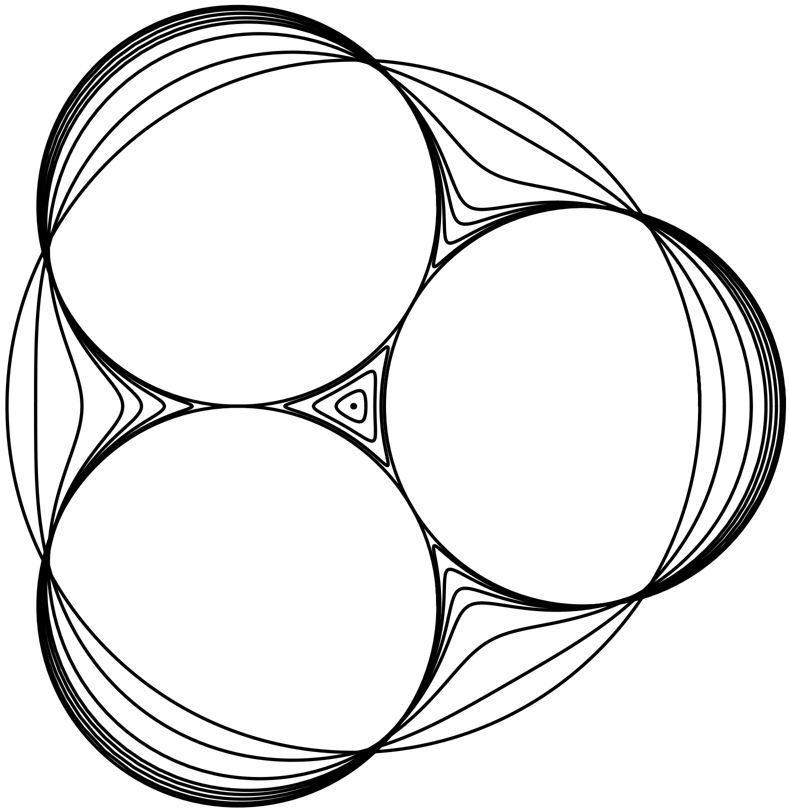

FIGURE 2. The coalescence of three circular disks of radius 1 to form a single disk of radius $\sqrt{3}$. The hole vanishes at $t = t_1 = 0.584$ and the outlines are drawn for four equal increments in t from 0 to t_1, plus $t = 1, 2$ and ∞.

and we have seen that this simplifies the computations. In Figure 4 we begin with an arrangement of six equal touching disks that does not have this feature; it is obtained from that in Figure 3 by adding a further two disks, so that we initially have part of a square lattice of touching disks. With the disks of radius 1 again, the hole now vanishes at $t = 1.156$; adding the extra disks has slightly *increased* the time it takes to reach the simply-connected state. In Figure 4, the rotation has been fixed by requiring that the map during the doubly-connected motion keeps the rotation at the point corresponding to the second initial disk from the right invariant, while during the simply-connected motion the rotation at the centre of symmetry is invariant.

Solutions for the two situations in Figures 2 and 3 using a purely numerical method have been given by van de Vorst (1993). When comparing the values we give for t, one must remember that it has the dimensions of a length and note that his initial disks are of radius 0.5. Thus the values here should be twice his. In addition, he does not start the integration as close to the initial 'touching disks' configuration as we do, nor does he continue up to the vanishing of the hole. It is, of course, true that a direct numerical integration of our equations cannot literally continue up to the instant when the hole vanishes either, for they are singular as $k \to 1$. However, with the default options,

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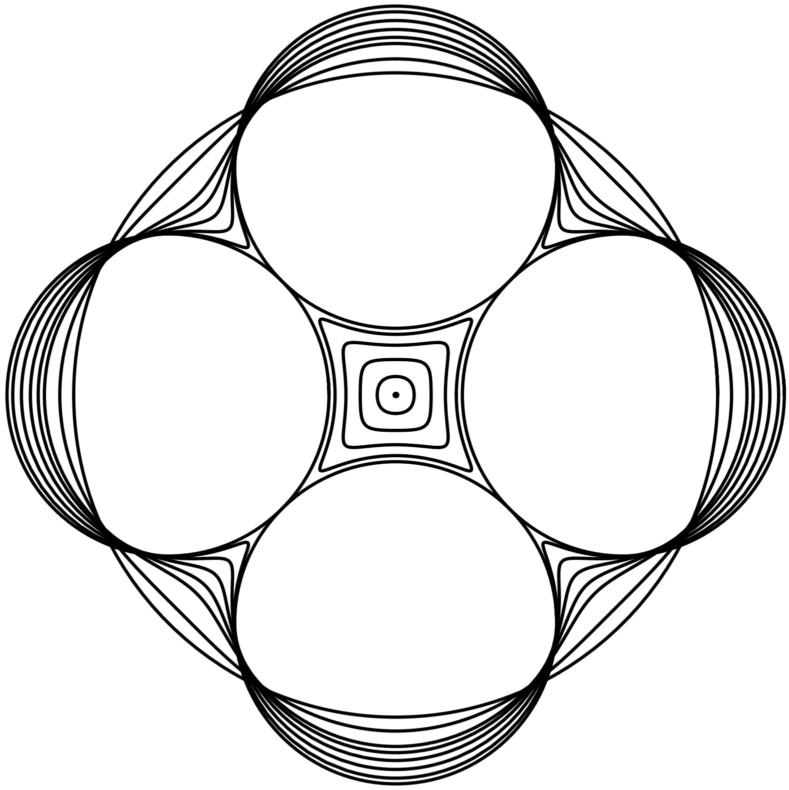

FIGURE 3. The coalescence of four circular disks of radius 1 to form a single disk of radius 2; the geometry is invariant under a rotation through an angle $\pi/2$. The hole vanishes at $t = t_2 = 1.150$ and the outlines are drawn for five equal increments in t from 0 to t_2, plus $t = 2$ and ∞.

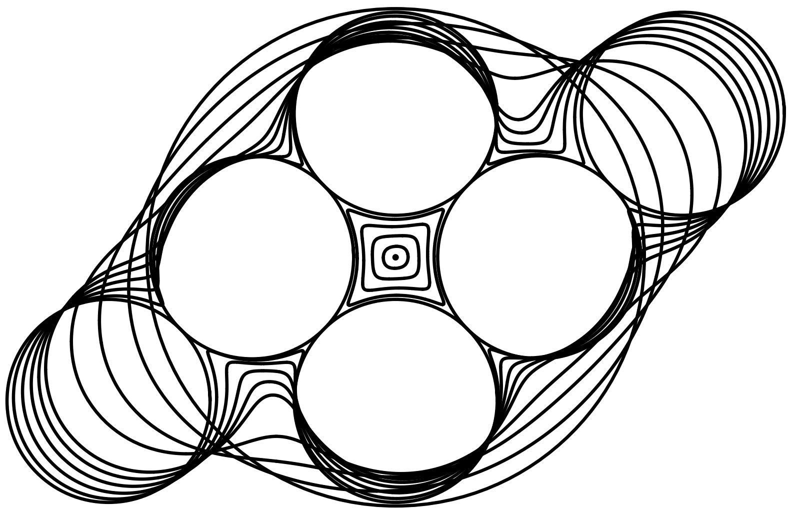

FIGURE 4. The coalescence of six circular disks of radius 1 to form a single disk of radius $\sqrt{6}$. The hole vanishes at $t = t_3 = 1.156$ and the outlines are drawn for five equal increments in t from 0 to t_3, plus $t = 2, 3, 5$ and ∞.

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NDSolve in *Mathematica* was able to integrate until the diameter of C_1 was less than 10^{-10} in all three cases we have illustrated, and this is sufficiently close to vanishing for most purposes.

The criteria adopted for the starting configurations in the figures were similar to those used in Richardson (1997). The initial values of the parameters were determined via the Hele-Shaw analogy with the injection points forming the touching disks a distance apart precisely 10^{-3} less than the distance between their centres, but with the angular relationship between the injection points the same as that between the corresponding centres. For doubly-connected examples, this still leaves some freedom, corresponding to the constraint we must impose in Hele-Shaw flows relating to the assumed properties of the hole. The extra condition used required that the geometry in the neck regions between the disks be examined, locating the two points on the inner and outer boundaries that are closest together, and finding their midpoint. In Figures 2 and 3, these neck regions are all congruent, and these midpoints were placed on the line joining the centres of the adjacent disks when these are just touching. In Figure 4, there are six neck regions that are congruent in pairs; here the sum of the squares of the distances of these midpoints from the corresponding line of centres was chosen to be a minimum.

5 Concluding remarks

We return to consider the condition $\mathscr{C} = 0$ that we have justified when the geometry is invariant under a rotation through any angle that is not an integer multiple of 2π, and which has been a crucial factor in obtaining our results. The approach of Crowdy & Tanveer (1998) is rather different from our own, but one can easily identify the source of their error in our development. If we were to take $\mathscr{C} = 0$ as an *assumption*, then we could still eliminate $\Phi(\zeta)$ between (3.30) and (3.31) to deduce our basic equations (3.28), (3.34) and (3.35); we just ignore the fact that, if the problem demands $\mathscr{C} \neq 0$, then no such function $\Phi(\zeta)$ exists.

To explore this matter further, we return to (3.16) and (3.17) to show how $\mathscr{C}$ may be computed using these equations when the map is given by a function $w(\zeta)$ that is an elliptic function of the form (2.8). Briefly, we need to write the function $F(\zeta)w'(\zeta)$ appearing in (3.16) and (3.17) as a sum of two functions, each analytic in one half of the fundamental rectangle, much as we did with $F(\zeta)$ itself in (3.11). Define

$$G_L(\zeta) = \frac{1}{2\pi i}\int_\Gamma F(\tau)w'(\tau)Z_m(\tau-\zeta)\,d\tau \quad \text{for } \zeta \text{ in } L, \tag{5.1}$$

and define $G_R(\zeta)$ by the same formula with ζ in R, so that $G_L(\zeta)$ is analytic in L and $G_R(\zeta)$ is analytic in R. Both $G_L(\zeta)$ and $G_R(\zeta)$ have $2iK'$ as a period and $2K$ as a quasi-period when analytically continued, while

$$G_L(\zeta) = G_R(\zeta) + F(\zeta)w'(\zeta). \tag{5.2}$$

Using this decomposition in (3.16) and (3.17), we find that the function

$$M(\zeta) = \Phi(\zeta) - \frac{T}{2\mu}G_L(\zeta), \tag{5.3}$$

which also has $2iK'$ as a period and $2K$ as a quasi-period, is actually *meromorphic* in the

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entire ζ-plane, with simple poles at the same points as $w(\zeta)$. With $w(\zeta)$ given by (2.8), we therefore have

$$M(\zeta) = \sum_{j=1}^{N} \overline{q}_j Z_m(\zeta + \overline{\gamma}_j) + \epsilon \tag{5.4}$$

for some constants q_j and ϵ. Equating residues at the poles on both sides of (3.16) we find that

$$q_j w'(\gamma_j) + \beta_j N'(\gamma_j) = -\frac{T}{2\mu}\beta_j G_L'(\gamma_j) \quad \text{for } j = 1, 2, \ldots, N. \tag{5.5}$$

Given the β_j and γ_j, these constitute $2N$ real, linear equations for the real and imaginary parts of the q_j.

If we equate the quasi-periodic behaviours on both sides of (5.3) when $2K$ is added to ζ, with $M(\zeta)$ given by (5.4), we find that

$$\mathscr{C} = \frac{\pi}{K'} \sum_{j=1}^{N} \overline{q}_j - \frac{T}{4i\mu K'} \int_{\Gamma} F(\tau) w'(\tau)\, d\tau. \tag{5.6}$$

Thus, given any elliptic $w(\zeta)$ of the form (2.8), we have a computational scheme to find the relevant value of $\mathscr{C}$. (A complicating feature has been glossed over, for the $2N$ real equations in (5.5) are not independent and, correspondingly, they do not completely determine the q_j, but they do uniquely determine $\mathscr{C}$.)

We can, of course, use these equations to confirm analytically what we already know: if the geometry has an appropriate rotational symmetry, then $\mathscr{C} = 0$. We can also use them computationally to show that, in the absence of such symmetry, we generally have $\mathscr{C} \neq 0$. Then our present theory is not applicable and the region occupied by fluid in such a Stokes flow evolves via maps that are not given by elliptic functions; the map is given by an elliptic function of the form (2.8) initially, but at later times the function has a more complicated structure.

We have shown that an appropriate rotational symmetry is *sufficient* to imply $\mathscr{C} = 0$, but computations show that such symmetry is not *necessary*. There *do* exist maps given by elliptic functions of the form (2.8) onto regions with no such symmetry, but for which $\mathscr{C} = 0$. However, in all the cases examined thus far with such a region as the initial state for a Stokes flow, it evolves to a region with $\mathscr{C} \neq 0$ at later times.

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