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CONSISTENCY OF MOMENT SYSTEMS

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ABSTRACT. An important question in the study of moment problems is to determine when a fixed point in $\mathbb{R}^n$ lies in the moment cone of vectors $(\int a_i\,d\mu)_1^n$, with μ a nonnegative measure. In associated optimization problems it is also important to be able to distinguish between the interior and boundary of the moment cone. Recent work of Dachuna-Castelle, Gamboa and Gassiat derived elegant computational characterizations for these problems, and for related questions with an upper bound on μ. Their technique involves a probabilistic interpretation and large deviations theory. In this paper a purely convex analytic approach is used, giving a more direct understanding of the underlying duality, and allowing the relaxation of their assumptions.

1. **Introduction.** The existence of a nonnegative Borel measure μ with given Fourier coefficients, $\int_0^1 e^{2\pi rs\sqrt{-1}}\,d\mu(s)$ $(r = 0, 1, \ldots, m)$, is a classical question which can be determined via consideration of an associated Toeplitz matrix. A similar technique can be applied when the algebraic moments, $\int_0^1 s^r\,d\mu(s)$ $(r = 0, 1, \ldots, m)$ are given (see for example [10] or [11]). More generally, when the given moments are $\int_0^1 a_i(s)\,d\mu(s)$ $(i = 1, 2, \ldots, n)$ and the functions $\{a_1, a_2, \ldots, a_n\}$ form a Tchebycheff system, there are classical computational techniques for determining the existence of the required measure μ.

For more general systems of functions $\{a_1, a_2, \ldots, a_n\}$, as observed in recent work of Dachuna-Castelle, Gamboa and Gassiat, the classical criteria are typically not helpful in practice. Motivated by this, they derive a computational procedure for checking whether a point b lies in the interior, the boundary or the complement of the moment cone

$$\Big\{\Big(\int a_i(s)\,d\mu(s)\Big)_{i=1}^{n} \;\Big|\; \mu \geq 0\Big\},$$

and for the analogous question where μ is bounded above by a given measure ρ (see [5], [4], [6] and [8]). Their technique involves an elegant probabilistic interpretation of the underlying moment problem, termed the Maximum Entropy Method on the Mean, followed by some rather technical analysis involving large deviations theory. A particularly important application is to optimization problems involving the moment conditions as constraints: the standard regularity condition or constraint qualification requires the right-hand-side vector to lie in the interior of the moment cone (see for example [2]).

A rather more direct way of understanding these main results is through convex programming duality. Dachuna-Castelle, Gamboa and Gassiat's computational criteria

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amount to the solution of natural dual problems. Motivated by this, we here give an entirely distinct approach to these results, centered more around the underlying optimization and using only convex analysis. By avoiding the probabilistic interpretation we are able to relax some of the assumptions needed in the original results. We will make free use of terminology and results from [14].

In summary, to study the moment problems described above, Dachuna-Castelle, Gamboa and Gassiat study the behaviour of the dual problem

$$\sup_{\lambda\in\mathbb{R}^n}\Big\{b^T\lambda-\int\psi\big(\lambda^Ta(s)\big)\,d\rho(s)\Big\},$$

for certain convex functions ψ. Their assumptions require that ψ is in fact Legendre-type. The duality approach which we employ here allows the use of more general closed convex functions ψ.

2. **Pseudo-Haar functions.** The key feature of the moment functions a_i in most of what follows will be the 'pseudo-Haar' property. In the next two sections, S will be a compact Hausdorff space with a fixed associated nonnegative regular Borel measure ρ which, without loss of generality, we will assume has full support, and we shall assume the moment functions a_i are real-valued and continuous on S, for each $i=1,2,\ldots,n$. We denote the $\mathbb{R}^n$-valued function with components a_i by a, and the moment map $F\colon M(S)\longrightarrow\mathbb{R}^n$ (where $M(S)$ is the space of regular Borel measures on S) is defined by $F\mu=\int a(s)\,d\mu(s)$.

We say that the system $\{a_1,a_2,\ldots,a_n\}$ is *pseudo-Haar* if it is linearly independent on every subset of S with nonzero measure. The terminology is derived from the idea of a Haar system, where we require linear independence on every subset with cardinality at least n. In particular, any linearly independent set of real-analytic functions on a compact interval of $\mathbb{R}$ (with Lebesgue measure) will be pseudo-Haar (see [1]). The reference [12] contains a proof of the same fact for compact subsets of $\mathbb{R}^k$.

The *moment cone* that we shall study is

$$K:=\{F\mu\mid 0\leq\mu\in M(S)\}.$$

We will denote the interior of a subset C of $\mathbb{R}^n$ by $\operatorname{int}C$, the relative interior by $\operatorname{ri}C$, and the boundary by $\operatorname{bd}C$. We write $\mu\perp\rho$ to mean the measures μ and ρ are mutually singular.

THEOREM 2.1. *Let the system $\{a_1,a_2,\ldots,a_n\}$ be pseudo-Haar. Then the following relations hold:*

$$\operatorname{int}K=\{Ax\mid 0\leq x\in L_1(S,\rho),x\neq 0\}. \tag{2.2}$$

$$K\setminus\operatorname{int}K\subset\{F\mu\mid 0\leq\mu\in M(S),\mu\perp\rho\}. \tag{2.3}$$

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PROOF. Let C denote the right-hand side of (2.2). Clearly C is a convex set, and we claim C is open. To see this, suppose that $A\bar{x}$ is a boundary point of C, with $0 \leq \bar{x} \in L_1(S, \rho)$ and $\bar{x} \not\equiv 0$. Taking a supporting hyperplane implies the existence of a nonzero λ in $\mathbb{R}^n$ with $\lambda^T(Ax - A\bar{x}) \geq 0$ whenever $0 \leq x \in L_1(S, \rho)$ with $x \not\equiv 0$. Since the a_i's are pseudo-Haar, $\lambda^T a(s) \not\equiv 0$ a.e., so defining

$$x(s) = \begin{cases} 2\bar{x}(s), & \text{if } \lambda^T a(s) < 0, \\ \frac{1}{2}\bar{x}(s), & \text{otherwise,} \end{cases}$$

gives $\lambda^T(Ax - A\bar{x}) = \int (x - \bar{x}) \sum \lambda_i a_i < 0$, which is a contradiction. Thus C is an open convex cone in $\mathbb{R}^n$.

Clearly $C \subset K$. On the other hand, a standard separation argument shows that $K \subset \operatorname{cl} C$. Hence since C and K are convex sets in $\mathbb{R}^n$ with $C \subset K \subset \operatorname{cl} C$, it follows that $\operatorname{int} K = \operatorname{int} C = C$, which is exactly (2.2).

To see (2.3), suppose that $0 \leq \mu \in M(S)$ and that the measure μ is not singular with respect to ρ. Taking a Lebesgue decomposition and applying the Radon-Nikodym theorem allows us to write $d\mu = x\,d\rho + d\mu_\sigma$, where $0 \leq x \in L_1(S, \rho)$ with $x \not\equiv 0$, and $\mu_\sigma \geq 0$. Hence $F\mu = Ax + F\mu_\sigma \in \operatorname{int} K + K = \operatorname{int} K$, by (2.2) and using the fact that K is a convex cone. Now (2.3) follows. ∎

We end this section with a well-known result about the moment cone K.

THEOREM 2.4. *The moment cone K is the convex cone generated by the set $\{a(s) \mid s \in S\}$. If there exists a $\hat{\lambda}$ in $\mathbb{R}^n$ with $\hat{\lambda}^T a(s) < 0$ for all s in S then K is closed.*

PROOF. The first part is an easy separation argument. The second part may be found for example in [9]. ∎

3. **Duality.** The computational criteria of Gamboa and Gassiat for testing whether a given vector b in $\mathbb{R}^n$ lies in the interior, the boundary or the complement of the moment cone K involve calculating the supremum of an unconstrained concave function on $\mathbb{R}^n$. This problem is in fact the dual problem for a convex program naturally associated with the original moment problem.

We begin by summarizing the relevant duality theory. For details, see [3]. We fix a closed, proper, convex function $\phi\colon \mathbb{R} \longrightarrow (-\infty, +\infty]$ throughout this paper, and define constants $p := \lim_{u \to -\infty} \phi(u)/u \in [-\infty, +\infty)$, and $q := \lim_{u \to +\infty} \phi(u)/u \in (-\infty, +\infty]$. We can now define a convex function $I\colon M(S) \longrightarrow (-\infty, +\infty]$ by

$$I(\mu) := \int \phi\Big(\frac{d\mu_\alpha}{d\rho}(s)\Big)\, d\rho(s) + q\mu_\sigma^+(S) - p\mu_\sigma^-(S),$$

where $\mu = \mu_\alpha + \mu_\sigma$ is the Lebesgue decomposition of μ with respect to ρ (so μ_α is absolutely continuous with respect to ρ and $\mu_\sigma \perp \rho$), and $\mu_\sigma = \mu_\sigma^+ - \mu_\sigma^-$ is the Jordan decomposition (so $\mu_\sigma^+, \mu_\sigma^- \geq 0$). We now consider the following primal convex program, a 'maximum entropy problem',

$$\alpha := \inf\{I(\mu) \mid F\mu = b, \mu \in M(S)\}, \tag{3.1}$$

and its associated dual problem,

$$\beta := \sup\Big\{b^T\lambda - \int \phi^*\big(\lambda^T a(s)\big)\, d\rho(s) \;\Big|\; \lambda \in \mathbb{R}^n\Big\}, \tag{3.2}$$

where $\phi^*\colon \mathbb{R} \longrightarrow (-\infty, +\infty]$ denotes the conjugate: $\phi^*(v) = \sup_u\{uv - \phi(u)\}$. The following theorem summarizes the duality relationships that we shall use. The *domain* of a convex function g is $\operatorname{dom} g := \{y \mid g(y) < +\infty\}$.

THEOREM 3.3. *The primal and dual values satisfy the 'weak duality' inequality* $-\infty \leq \beta \leq \alpha \leq +\infty$. *Suppose furthermore that 'dual regularity' holds:*

$$\text{there exists a } \hat\lambda \text{ in } \mathbb{R}^n \text{ with } p < \hat\lambda^T a(s) < q \text{ for all } s \text{ in } S. \tag{3.4}$$

Then the primal and dual values are equal, $\alpha = \beta > -\infty$, *and if the primal value* α *is finite then it is attained in (3.1).*

Suppose on the other hand that 'primal regularity' holds:

$$b \in \operatorname{ri}(F \operatorname{dom} I). \tag{3.5}$$

Then the primal and dual values are equal, $\alpha = \beta < +\infty$, *and if the dual value* β *is finite then it is attained in (3.2).*

All of these ideas may be found in [3]. For the primal regularity condition (3.5), see for example [13].

We say that ϕ is *Legendre-type* if it is strictly convex and essentially smooth (see [14]). Suppose that we actually wish to compute the dual value β, defined in (3.2). If we define a convex function $\Psi\colon \mathbb{R}^n \longrightarrow (-\infty, +\infty]$ by $\Psi(\lambda) = \int \phi^*\big(\lambda^T a(s)\big)\, d\rho(s)$, then

$$-\beta = \inf\{\Psi(\lambda) - b^T\lambda \mid \lambda \in \mathbb{R}^n\}. \tag{3.6}$$

Problem (3.6) is unconstrained, at least formally, but if we wish to use an unconstrained minimization routine (with a safeguarded steplength) to solve it iteratively we want any optimal solution $\bar\lambda$ to lie in $\operatorname{int}(\operatorname{dom}\Psi)$.

Now $\operatorname{int}(\operatorname{dom}\phi^*) = (p, q)$ (see [3]), from which it follows immediately that if (3.4) holds then $\operatorname{int}(\operatorname{dom}\Psi) = \{\lambda \mid p < \lambda^T a(s) < q,\ \forall s \in S\}$. A key condition for ensuring that the dual can only have interior solutions is the following (see [3] for a discussion of this condition).

INTEGRABILITY CONDITION. The function ϕ is strictly convex on its domain, and if $\lambda^T a(s) \in (p, q)$ a.e. with $(\phi^*)'\big(\lambda^T a(\cdot)\big)$ integrable then $\lambda^T a(s) \in (p, q)$ for all s in S.

THEOREM 3.7. *Suppose that dual regularity (3.4) holds, and that the Integrability Condition holds. Then if* $\bar\lambda \in \mathbb{R}^n$ *is any optimal solution of the dual problem (3.6) then* $\bar\lambda \in \operatorname{int}(\operatorname{dom}\Psi)$, *and furthermore the function*

$$\bar x(s) := (\phi^*)'\big(\bar\lambda^T a(s)\big) \in \operatorname{dom}\phi \text{ a.e.}$$

is integrable and satisfies $\int a(s)\bar x(s)\, d\rho(s) = b$. *If furthermore* ϕ *is Legendre-type then* $\bar x(s) \in \operatorname{int}(\operatorname{dom}\phi)$ *a.e.*

PROOF. Dual regularity implies that the primal and dual values are equal, and since $\bar{\lambda}$ is dual optimal, both are finite and attained, by Theorem 3.3. If $\bar{\mu}$ is a primal optimal solution then by complementary slackness (Theorem 4.10 in [3]), the integrable function

$$\bar{x}(s) = \frac{d\bar{\mu}_\alpha}{d\rho}(s) = (\phi^*)'\big(\bar{\lambda}^T a(s)\big) \in \operatorname{dom}\phi \text{ a.e.}$$

and $\bar{\mu}_\sigma$ is supported on $\{s \in S \mid \bar{\lambda}^T a(s) = p \text{ or } q\}$. The Integrability Condition then guarantees that $\bar{\lambda}^T a(s) \in (p,q)$ for all s in S, and so $\bar{\mu}_\sigma = 0$ and the result follows. The final remark is a consequence of the fact that range$(\phi^*)' = \operatorname{int}(\operatorname{dom}\phi)$ for Legendre-type ϕ. ∎

4. **The moment cone.** In this section we will use the Duality Theorem (3.3) to diagnose whether a fixed vector b in $\mathbb{R}^n$ lies in the interior, the boundary, or the complement of the moment cone K. The following easy lemma is our starting point.

LEMMA 4.1. *Suppose that* $0 \in \operatorname{dom}\phi$ *and* q *is finite. Then*

$$\phi(u) \leq \phi(0) + qu \quad \textit{for all } u > 0, \tag{4.2}$$

with strict inequality unless ϕ *is affine on* $[0, +\infty)$.

The next result, a straightforward consequence of the previous lemma, gives conditions on the function ϕ which ensure that the primal feasible region is exactly the moment cone K. For such problems the primal regularity condition (3.5) becomes $b \in \operatorname{ri} K$. This explains one of our main motivations for seeking a computational criterion for characterizing the interior and boundary of the moment cone: an *a priori* check that $b \in K$ ensures primal consistency, and an *a priori* check that $b \in \operatorname{int} K$ ensures the solvability of the dual problem (3.2) when it is consistent. We write $M(S)_+$ for the nonnegative cone in $M(S)$.

LEMMA 4.3. *Suppose that* $\operatorname{dom}\phi = [0, +\infty)$ *and that* q *is finite. Then for any* μ *in* $M(S)_+$, *we have that*

$$I(\mu) \leq \phi(0)\rho(S) + q\mu(S), \tag{4.4}$$

so in particular, $\operatorname{dom} I = M(S)_+$. *Strict inequality holds in (4.4) unless either* ϕ *is affine on* $[0, +\infty)$, *or* μ *is singular with respect to* ρ.

We are now ready for our main results, which give a computational criterion for checking whether a vector lies in the interior, the boundary, or the complement of the moment cone, via solving the dual problem (3.2).

THEOREM 4.5. *Suppose that* $\operatorname{dom}\phi = [0, +\infty)$, *that* q *is finite, and that there exists a* $\hat{\lambda}$ *in* $\mathbb{R}^n$ *with* $\hat{\lambda}^T a(s) < q$ *for all* s *in* S. *Then* $b \in K$ *if and only if* $\beta < +\infty$. *Suppose furthermore that the Integrability Condition holds. If* $\beta < +\infty$ *and is attained at* $\bar{\lambda} \in \mathbb{R}^n$ *then* $\bar{x}(s) := (\phi^*)'\big(\bar{\lambda}^T a(s)\big)$ *is a solution of the moment problem:*

$$\int a(s)\bar{x}(s)\, d\rho = b \quad \textit{with } 0 \leq \bar{x} \in L_1(S, \rho).$$

If in fact ϕ *is Legendre-type then* $\bar{x}(s) > 0$ *a.e.*

PROOF. By Lemma 4.3, $b \in K$ if and only if the primal value $\alpha < +\infty$, and by Theorem 3.3, $\alpha = \beta$. The remainder follows by Theorem 3.7. ■

More generally, if the dual value β is attained but the Integrability Condition fails then semi-infinite linear programming techniques may be needed to find a solution μ for the moment problem. See [3] for a further discussion.

THEOREM 4.6. *Suppose that* $\operatorname{dom}\phi = [0, +\infty)$, *that the function* ϕ *is not affine on* $[0, +\infty)$, *and that the system* $\{a_1, a_2, \ldots, a_n\}$ *is pseudo-Haar. Suppose further that either* q *is finite with* $a_1 \equiv 1$, *or that* $q = 0$ *and there exists a* $\hat{\lambda}$ *in* $\mathbb{R}^n$ *with* $\hat{\lambda}^T a(s) < 0$ *for all* s *in* S. *Then the moment cone* K *is closed and convex with:*

(i) $b \in \operatorname{int} K$ *if and only if* $\beta < \phi(0)\rho(S) + qb_1$, *(in which case* β *is attained in (3.2)).*

(ii) $b \in \operatorname{bd} K$ *if and only if* $\beta = \phi(0)\rho(S) + qb_1$, *(in which case, if the Integrability Condition holds and* $b \neq 0$, β *is not attained).*

(iii) $b \notin K$ *if and only if* $\beta = +\infty$.

PROOF. Note that K is closed, by Theorem 2.4, and dual regularity (3.4) holds, so the primal and dual values are equal by Theorem 3.3: $\alpha = \beta$. Part (iii) follows from Theorem 4.5. Now if $b \in \operatorname{bd} K$ then the primal problem (3.1) has at least one feasible solution μ (by Lemma 4.3), but any such μ must be singular with respect to ρ, by Theorem 2.1, and hence $I(\mu) = \phi(0)\rho(S) + qb_1$. Thus the primal value $\alpha = \phi(0)\rho(S) + qb_1$.

If on the other hand $b \in \operatorname{int} K$ then by Theorem 2.1 there is a (nonzero) feasible solution μ which is absolutely continuous with respect to ρ, and by Lemma 4.3, $I(\mu) < \phi(0)\rho(S) + qb_1$. Thus the primal value $\alpha < \phi(0)\rho(S) + qb_1$. In this case the dual value β is attained, again by Theorem 3.3. If the Integrability Condition holds then β will never be attained in the boundary case (ii) since if it was, Theorem 4.5 shows the existence of a nonzero absolutely continuous solution of the moment problem, contradicting Theorem 2.1. ■

In practice we would apply a standard finite-dimensional concave maximization algorithm to the dual problem (3.2), which for suitable choices of ϕ will be smooth and unconstrained (see [1]). As soon as a point λ is detected with dual value larger than $\phi(0)\rho(S) + qb_1$ we know that $b \notin K$.

The following example illustrates how this theorem can be used to rederive an example of Gamboa and Gassiat.

COROLLARY 4.7. *Suppose that the system* $\{a_1, a_2, \ldots, a_n\}$ *is pseudo-Haar and that there exists a* $\hat{\lambda}$ *in* $\mathbb{R}^n$ *with* $\hat{\lambda}^T a(s) < 0$ *for all* s *in* S. *Then the moment cone* K *is closed, and if we define*

$$\beta_1 := \sup\Big\{ b^T\lambda + \int \log(1 - e^{\lambda^T a(s)})\, d\rho(s) \mid \lambda \in \mathbb{R}^n \Big\}, \tag{4.8}$$

then we have that

(i) $b \in \operatorname{int} K$ *if and only if* $\beta_1 < 0$ *(in which case* β_1 *is attained in (4.8)).*

(ii) $b \in \operatorname{bd} K$ *if and only if* $\beta_1 = 0$.

(iii) $b \notin K$ *if and only if* $\beta_1 = +\infty$.

PROOF. We apply Theorem 4.6 with

$$\phi(u) = \begin{cases} u \log u - (u+1)\log(u+1), & \text{if } u > 0, \\ 0, & \text{if } u = 0, \\ +\infty, & \text{if } u < 0, \end{cases}$$

and the result is immediate. ∎

Suppose in the above Corollary, that S is a compact interval in $\mathbb{R}$ with Lebesgue measure and that the a_i's are Lipschitz. It follows that the Integrability Condition holds, by Theorem 6.14 in [3], so β_1 is attained at an interior point of the dual domain in case (i), and is not attained in case (ii) (the case $b = 0$ is clear). In case (i), if $\bar{\lambda}$ achieves the supremum then, by Theorem 4.5, a solution of the moment problem is

$$d\mu = \Big(\exp\big(-\bar{\lambda}^T a(s)\big) - 1\Big)^{-1} d\rho. \tag{4.9}$$

The original motivation of Dachuna-Castelle, Gamboa and Gassiat in studying this problem was in large part the search for tests amenable to computation. It is therefore highly instructive to test Corollary 4.7 computationally. The following example is illustrative of the difficulties encountered in the boundary case (due to recession effects): we choose $S = [0, 1]$ with ρ Lebesgue measure, $a_1(s) = 1$, $a_2(s) = s$ and $a_3(s) = s^2$, $b_1 = 1$, $b_2 = \frac{1}{2}$, and then we consider the three cases $b_3 = \frac{1}{3}, \frac{1}{4}$ and $\frac{1}{5}$. In each case we attempt to evaluate β_1 in (4.8) by Newton's method.

(i) $b_3 = \frac{1}{3}$. After 6 interations we obtain that $\beta_1 = -1.38629$ (to 5 decimal places), and hence $b \in \operatorname{int} K$. Formula (4.9) gives simply $\mu = \rho$.

(ii) $b_3 = \frac{1}{4}$. In this case the unique solution of the moment problem is $\mu = \delta_{\{\frac{1}{2}\}}$, so by Theorem 2.1, $b \in \operatorname{bd} K$, and by Corollary 4.7, $\beta_1 = 0$. We obtain the following sequence of iterates:

iteration	dual variables			dual value
	λ_1	λ_2	λ_3	
1	−1.9	6.8	−7.7	−.997
2	−4.1	16.7	−17.5	−.794
3	−7.0	28.3	−29.2	−.638
4	−12.1	48.6	−49.4	−.518
5	−20.2	81.4	−82.3	−.427
6	−33.0	132.7	−133.7	−.360
7	−52.6	211.3	−212.5	−.313
8	−81.6	327.7	−329.2	−.283
9	−122.1	490.0	−491.7	−.265
10	−171.6	688.2	−690.1	−.256

Evidently the dual value is converging to 0 extremely slowly, as the dual variables tend to infinity with the recession direction $(-1, 4, -4)$ as asymptote.

(iii) $b_3 = \frac{1}{5}$. After 3 iterations we obtain a dual value of $1.37636\cdots > 0$. Hence we can conclude that $b \notin K$.

5. The bounded moment set. In this section we will apply similar techniques to the easier problem of the bounded moment set. We now simply assume that (S, ρ) is a finite measure space with $a_1, a_2, \ldots, a_n$ in $L_1(S)$.

We define subsets of $L_\infty(S)$ by

$$[0,1]_\infty = \{x \mid 0 \le x(s) \le 1 \text{ a.e.}\}, \text{ and}$$
$$[0,1]_e = \{x \mid x(s) \in \{0,1\} \text{ a.e.}\}.$$

It is easily seen ([16], p. 65) that the set of extreme points of $[0,1]_\infty$ is exactly $[0,1]_e$. We also define a weak-star continuous linear map $A: L_\infty(S) \longrightarrow \mathbb{R}^n$ by $Ax = \int ax$. We say a convex subset of $\mathbb{R}^n$ is *rotund* if every boundary point is an extreme point. The moment set that we wish to study is $K_1 = A[0,1]_\infty$.

THEOREM 5.1. *The moment set K_1 is a compact, convex subset of $\mathbb{R}^n$. If the system $\{a_1, a_2, \ldots, a_n\}$ is pseudo-Haar then K_1 is rotund, with*

$$\operatorname{int} K_1 = A([0,1]_\infty \setminus [0,1]_e). \tag{5.2}$$

PROOF. The set $[0,1]_\infty$ is weak-star compact and convex, so $A[0,1]_\infty$ is compact and convex. Let $D = [0,1]_\infty \setminus [0,1]_e$. Clearly the set D is convex with $AD \subset K_1 \subset \operatorname{cl} AD$. Equation (5.2) will follow if we show AD is open.

Suppose that $A\bar{x}$ is a boundary point of AD, with $\bar{x}$ in D. Taking a supporting hyperplane implies the existence of a nonzero λ in $\mathbb{R}^n$ with

$$0 \le \lambda^T(Ax - A\bar{x}) = \int (\lambda^T a)(x - \bar{x})$$

whenever x lies in D. Since the a_i's are pseudo-Haar, $\lambda^T a \not\equiv 0$ a.e., so defining

$$x(s) = \begin{cases} \frac{1}{2}\big(1 + \bar{x}(s)\big), & \text{if } (\lambda^T a)(s) < 0, \\ \frac{1}{2}\bar{x}(s), & \text{otherwise,} \end{cases}$$

gives a contradiction.

Finally, if $A\bar{x}$ is a boundary point of K_1 with $\bar{x}$ in $[0,1]_\infty$, then (5.2) and our characterization of extreme points of $[0,1]_\infty$, imply $A\bar{x}$ is extreme in $A[0,1]_\infty$. ∎

We now fix a closed, convex function $\phi: \mathbb{R} \longrightarrow (-\infty, +\infty]$ with $\operatorname{dom} \phi = [0,1]$, and we define a convex function $I: L_\infty(S) \longrightarrow (-\infty, +\infty]$ by

$$I(x) = \int_S \phi\big(x(s)\big)\, d\rho(s).$$

This time we consider the dual pair of problems

$$\alpha := \inf\{I(x) \mid Ax = b, x \in L_\infty(S)\}, \tag{5.3}$$

$$\beta := \sup\Big\{b^T\lambda - \int \phi^*\big(\lambda^T a(s)\big)\, d\rho(s) \mid \lambda \in \mathbb{R}^n\Big\}. \tag{5.4}$$

It is straightforward to check that the objective function in (5.4) is everywhere finite. The following result is an easy consequence of standard duality results (see for example [15], [2, Theorem 4.2], or [12] for details).

THEOREM 5.5. *The primal and dual values are equal,* $\alpha = \beta > -\infty$, *and if the primal value* α *is finite then it is attained in (5.3). If* b *lies in the interior of the moment set* K_1 *then* $\alpha = \beta$ *and both are finite and attained in (5.3) and (5.4) respectively.*

As a consequence we obtain the following approach to solving the bounded moment problem.

THEOREM 5.6. *The point* b *lies in the moment set* K_1 *if and only if* $\beta < +\infty$ *in (5.4). If* ϕ *is strictly convex on* $[0, 1]$ *then when* β *is finite and attained at some* $\bar{\lambda}$ *in* $\mathbb{R}^n$, *a solution of the bounded moment problem is given by* $\bar{x}(s) = (\phi^*)'\big(\bar{\lambda}^T a(s)\big)$:

$$\int a(s)\bar{x}(s)\,d\rho(s) = b \quad \text{with } 0 \le \bar{x}(s) \le 1 \text{ a.e.} \tag{5.7}$$

If in fact ϕ *is Legendre-type then* $0 < \bar{x}(s) < 1$ *a.e. in (5.7).*

PROOF. The first part follows from Theorem 5.5, and when b lies in K_1 the primal value α is attained by some $\bar{x}$ in (5.3). Now suppose that $\bar{\lambda}$ attains β in (5.4). Note that

$$\phi\big(\bar{x}(s)\big) + \phi^*\big(\bar{\lambda}^T a(s)\big) \ge \bar{\lambda}^T a(s) x(s) \text{ a.e.}, \tag{5.8}$$

and when ϕ is strictly convex, equality holds if and only if

$$\bar{x}(s) \in \partial\phi^*\big(\bar{\lambda}^T a(s)\big) = \big\{(\phi^*)'\big(\bar{\lambda}^T a(s)\big)\big\}. \tag{5.9}$$

Now integrating (5.8) and using $\alpha = \beta$ proves (5.9), and (5.7) follows as $\bar{x}$ is feasible. The last statement follows from the fact that range$(\phi^*)' = \operatorname{int}(\operatorname{dom}\phi)$ if ϕ is Legendre-type. ∎

Notice that, if a unified approach is desired, when S is compact with ρ in $M(S)_+$ and $a_1, a_2, \ldots, a_n$ continuous, Theorems 5.5 and 5.6 are actually special cases of Theorems 3.3 and 3.7.

THEOREM 5.10. *Suppose that* $\phi(0) = \phi(1)$, *and that* ϕ *is not constant on* $[0, 1]$. *Suppose further that the system* $\{a_1, a_2, \ldots, a_n\}$ *is pseudo-Haar. Then the moment set* K_1 *is compact, convex and rotund with*

(i) $b \in \operatorname{int} K_1$ *if and only if* $\beta < \phi(0)\rho(S)$ *(in which case* β *is attained in (5.4)).*
(ii) $b \in \operatorname{bd} K$, *if and only if* $\beta = \phi(0)\rho(S)$ *(in which case, if* ϕ *is Legendre-type,* β *is not attained in (5.4)).*
(iii) $b \notin K_1$ *if and only if* $\beta = +\infty$.

PROOF. Part (iii) follows from Theorem 5.5. Now $b \in \operatorname{bd} K_1$ implies that there exists an $\bar{x}$ satisfying (5.7), but by Theorem 5.1 any such $\bar{x}(s) \in \{0, 1\}$ a.e. Hence the primal value $\alpha = \phi(0)\rho(S)$, and $\alpha = \beta$ by Theorem 5.5. If ϕ is Legendre-type, β cannot be attained, by Theorem 5.6.

On the other hand, if $b \in \operatorname{int} K_1$ then Theorem 5.1 shows the existence of an $\bar{x}$ satisfying (5.2) but with $\bar{x}(s) \in (0, 1)$ on a set of positive measure. The assumptions on ϕ guarantee that $\phi(u) < \phi(0)$ for all u in $(0, 1)$, so the primal value $\alpha < \phi(0)\rho(S)$, and $\alpha = \beta$, with β attained by Theorem 5.5. ∎

Elegant illustrations of Theorems 5.6 and 5.10 may be obtained, for example, by making the choice

$$\phi(u) = \begin{cases} u \log u + (1-u)\log(1-u), & \text{if } u \in (0,1), \\ 0, & \text{if } u = 0, 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

6. **Comparison with previous results.** In this concluding section the results obtained here will be compared with recent results of Dachuna-Castelle, Gamboa and Gassiat. Theorems analogous to the two main results, Theorems 4.6 and 5.5, may be found in [4], [6] and [8] (see also [5] and [7]). The main observation is that their corresponding results can be subsumed by this direct convex-analytic approach, but that some of the restrictions on the function ϕ imposed by their probabilistic approach are not in fact necessary.

The common framework for this previous work assumes that the underlying measure space (S, ρ) is compact and completely metrizable, with ρ a Borel probability measure. However the main difference occurs in their choice of ϕ, which is effectively restricted to be the Legendre-Fenchel conjugate of the log-Laplace transform of a probability measure F on $\mathbb{R}_+ : \phi = \psi^*$, where

$$\psi(v) = \log \int_0^\infty e^{vy}\, dF(y). \tag{6.1}$$

We shall see that the additional conditions imposed on the function ψ ensure that the conjugate ϕ satisfies the assumptions for Theorems 4.6 and 5.5 respectively, but also that the resulting ϕ must be Legendre-type.

When the probability measure F is concentrated on a single point the function ψ is linear: we exclude this case.

LEMMA 6.2. *Suppose that the probability measure F is not a point mass. Then the function* $\psi\colon \mathbb{R} \to (-\infty, +\infty]$ *defined by (6.1) is proper, strictly increasing and lower semicontinuous, and is strictly convex and differentiable on the interior of its domain, with*

$$-\psi^*(0) = \lim_{v \downarrow -\infty} \psi(v) = \begin{cases} \log(F\{0\}), & \text{if } F\{0\} > 0, \\ -\infty, & \text{if } F\{0\} = 0. \end{cases} \tag{6.3}$$

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PROOF. Clearly $\psi(0) = 0$, so ψ is proper, and obviously strictly increasing. Hence we can write $\operatorname{int}(\operatorname{dom}\psi) = (-\infty, d)$ for some d in $[0, +\infty]$. Strict convexity of ψ on $(-\infty, d)$ is easily seen by applying Cauchy-Schwartz to the inner product of the functions $e^{v_1 y/2}$ and $e^{v_2 y/2}$, for any $v_1, v_2 < d$.

Equation (6.3) is a consequence of monotone convergence. Since ψ is convex, it is continuous on $(-\infty, d)$, so to show lower semicontinuity we need to show that $\lim_{v\uparrow d}\psi(v) = \psi(d)$, which again is a consequence of monotone convergence. It is also straightforward to check by monotone convergence that $\frac{d}{dv}\int_0^\infty e^{vy}\,dF(y) = \int_0^\infty ye^{vy}\,dF(y) > 0$, for v in $(-\infty, d)$, so ψ is differentiable on $(-\infty, d)$. ∎

The next result details the conditions imposed on ϕ by this framework.

THEOREM 6.4. *Suppose that* $F\{0\} > 0$ *and that* $d := \sup(\operatorname{dom}\psi) \in (0, +\infty)$ *for the function* ψ *defined by (6.1). Then the conjugate function* $\phi = \psi^*$ *is closed, convex and essentially smooth, with* $\operatorname{dom}\phi = [0, +\infty)$*, and furthermore* $\lim_{u\uparrow+\infty}\phi(u)/u = d$ *is finite. If in fact* $\psi(d) = +\infty$ *then* ϕ *is Legendre-type.*

PROOF. Since ϕ is a conjugate function, it is closed and convex, and since $d < +\infty$, F is not a point mass, so ψ is strictly convex by Lemma 6.2, and thus ϕ is essentially smooth. If $\psi(d) = +\infty$ then since ψ is differentiable on $(-\infty, d)$, by Lemma 6.2 it must be essentially smooth, and hence Legendre-type. Thus its conjugate ϕ is also Legendre-type. An easy computation shows $\operatorname{int}(\operatorname{dom}\phi) = (0, +\infty)$, and since $\phi(0)$ is finite by Lemma 6.2, we have that $\operatorname{dom}\phi = [0, +\infty)$. Finally $\lim_{u\uparrow+\infty}\phi(u)/u = d$ also follows by Lemma 6.2. ∎

Versions of Theorems 4.5 and 4.6 in the work of Dachuna-Castelle, Gamboa and Gassiat use the dual problem (3.2) with the function $\phi = \psi^*$, where ψ is defined by (6.1). Furthermore, the probability measure F satisfies $F\{0\} > 0$ and $d = \sup(\operatorname{dom}\psi) \in (0, +\infty)$. Hence Theorem 6.4 demonstrates that the resulting function ϕ satisfies the conditions of Theorems 4.5 and 4.6, and thus we rederive their results. However, Theorem 6.4 shows that functions ϕ obtained in this way are more restricted than those allowed in Theorems 4.5 and 4.6: they are essentially smooth, and in fact, since it is also assumed that $\psi(d) = +\infty$, they are Legendre-type.

Returning to the case of the bounded moment set K_1, suppose that the probability measure F is supported on $[0, 1]$, with $F\{0\} = F\{1\} > 0$. Then similar arguments to those for Theorem 6.4 show that the conjugate function ϕ of the function ψ defined by (6.1) is a Legendre-type closed convex function with domain $[0, 1]$ and $\phi(0) = \phi(1)$ (see [12]). The versions of Theorems 5.6 and 5.10 in the work of Dachuna-Castelle, Gamboa and Gassiat work with precisely these assumptions, and hence can be obtained from our theorems. Once again however we see that the functions ϕ allowed in their results are restricted to be Legendre-type.

It should be noted that, although functions ϕ which are not Legendre-type could be used in Theorems 4.5, 4.6, 5.6 and 5.10, the most elegant and perhaps the most computationally useful criteria will result from using Legendre-type ϕ: Corollary 4.7

and the example after Theorem 5.10 are typical. Hence the techniques presented here should be viewed as primarily of theoretical interest, allowing the results of Dachuna-Castelle, Gamboa and Gassiat to be interpreted and derived as duality theorems.

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