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OPTION PRICING FOR PROCESSES DRIVEN BY MIXED FRACTIONAL BROWNIAN MOTION WITH SUPERIMPOSED JUMPS

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We propose a geometric mixed fractional Brownian motion model for the stock price process with possible jumps superimposed by an independent Poisson process. Option price of the European call option is computed for such a model. Some special cases are studied in detail.

1. INTRODUCTION

Suppose we are interested in the price of a stock as it evolves over time. Let $S(t)$ denote the price of the stock at time t. The process $\{S(t), t \geq 0\}$ is said to be a geometric Brownian motion with drift parameter μ and volatility parameter σ if for all $y \geq 0$ and $t \geq 0$, the random variable

$$S(t+y)/S(y)$$

is independent of all prices up to time y and if the random variable

$$\log(S(t+y)/S(y))$$

is a Gaussian random variable with mean μt and variance $t\sigma^2$. Suppose the stock price process is a geometric Brownian motion. It is known that, given the initial price $S(0)$, the expected value of the price at time t depends on the parameters μ and σ^2 and

$$E[S(t)] = S(0)e^{\mu t+(1/2)\sigma^2 t}.$$

It is also known that the sample paths of a geometric Brownian motion are continuous almost surely and hence the geometric Brownian motion is not suitable for modeling stock price process if there are likely to be jumps. Consider a call option having the strike price K and expiration time t. Under the assumption that the stock price process follows the geometric Brownian motion and the interest rate r does not change over time, Black–Scholes formula gives the unique no arbitrage cost of the European call option (cf. Ross [8]). For a detailed explanation of "arbitrage" opportunity, see Ross [8]. It is now known that certain time series

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are long-range dependent and it was thought that a process driven by a fractional Brownian motion, in particular, a geometric fractional Brownian motion may be used as a model for modeling stock price process. A fractional Brownian motion process with Hurst parameter $H \in (0, 1]$ is an almost surely continuous centered Gaussian process with

$$\mathrm{Cov}(B_t^H, B_s^H) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), t, s \in R.$$

For properties of a fractional Brownian motion, see Prakasa Rao [7]. It is easy to see that this process reduces to the Wiener process or the Brownian motion if $H = \frac{1}{2}$. Attempts to model stock price process using the fractional Brownian motion as the driving force were not successful as such a modeling leads to an arbitrage opportunity under the model which violates the fundamental assumption of mathematical finance modeling of no arbitrage opportunity or no free lunch (cf. Kuznetsov [4]). It is known that the fractional Brownian motion $\{B_t^H, t \geq 0\}$ with Hurst index $H \in (0, 1)$ is neither a Markov process nor a semi-martingale except when $H = \frac{1}{2}$. Cheridito [1,2] introduced the concept of mixed fractional Brownian motion for modeling stock price process. He showed that the sum of a Brownian motion and a non-trivial multiple of an independent fractional Brownian motion with Hurst index $H \in (0, 1]$ is not a semimartingale for $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4})$. However, if $H \in (\frac{3}{4}, 1]$, then the mixed fractional Brownian motion is equivalent to a multiple of Brownian motion and hence is a semi-martingale. Hence, for $H \in (\frac{3}{4}, 1)$, the arbitrage opportunities can be excluded by modeling the stock price process as the geometric mixed fractional Brownian motion given by

$$X_t = X_0 \exp\{g(t) + \sigma B_t^H + \epsilon W_t\}, t \geq 0, \tag{1.1}$$

where $g(t)$ is a non-random function, $(\sigma, \epsilon) \neq (0, 0)$, and the processes $\{B_t^H, t \geq 0\}$ and $\{W_t, t \geq 0\}$ are *independent* fractional Brownian motion and Brownian motion, respectively. Cheridito [1] showed that this model is arbitrage-free (cf. Mishura and Valkeila [6]). It is known that the sample paths of the mixed fractional Brownian motion or the geometric mixed fractional Brownian motion are almost surely continuous (cf. Prakasa Rao [7], Zili [13]). In order to take into account the long-memory property as well as to model the fluctuations in the stock prices in a financial market, one can use the mixed fractional Brownian motion as the driving force to model the stock price process. Sun [9] discussed pricing currency options using the mixed fractional Brownian motion model. Sun and Yan [10] discussed application of the mixed-fractional models to credit risk pricing. Yu and Yan [12] studied the European call option pricing under the mixed fractional Brownian environment. Since the sample paths of the geometric mixed fractional Brownian motion are almost surely continuous, it is not suitable for modeling stock price process with possible jumps. Mishura [5] discussed sufficient conditions for the existence and uniqueness of solutions of stochastic differential equations driven by a mixed fractional Brownian motion. Yu and Yan [12] derived the analog of the Black–Scholes formula for the European call option price when the stock price process is the geometric mixed fractional Brownian motion with interest rate r is being constantly compounded continuously, the strike time is t and the strike price is K. Suppose the initial price of the stock at time 0 is s. They showed that the European call option price is given by the formula

$$C(s, t, K, \sigma, r, \epsilon) = s\ \Phi(d_1) - Ke^{-rt}\ \Phi(d_2)$$

where

$$d_1 = \frac{\log(s/K) + rt + \frac{1}{2}\sigma^2 t^{2H} + \frac{1}{2}\epsilon^2 t}{\sqrt{\sigma^2 t^{2H} + \epsilon^2 t}},$$

$$d_2 = \frac{\log(s/K) + rt - \frac{1}{2}\sigma^2 t^{2H} - \frac{1}{2}\epsilon^2 t}{\sqrt{\sigma^2 t^{2H} + \epsilon^2 t}}.$$

and $\Phi(.)$ is the standard Gaussian distribution function.

2. ADDING JUMPS TO GEOMETRIC MIXED FRACTIONAL BROWNIAN MOTION

It is now known that modeling of the stock price process using the geometric Brownian motion is not useful as it does now allow possibility of a discontinuous price jump either in the upward or downward direction and to model long-range dependence. Under the assumption of the geometric Brownian motion, the probability of having a jump is zero. Since such jumps do occur in practice for various reasons, it is important to consider a model for the stock price process that allows possibility of jumps in the process.

We assume that there are no transaction costs, trading is continuous and the interest rate is constant and compounded continuously. We have indicated that there are no arbitrage opportunities under the mixed fractional Brownian motion model whenever the Hurst index $H \in (\frac{3}{4}, 1]$.

We now propose a jump mixed fractional Brownian motion model to capture jumps or discontinuities, fluctuations in the stock price process and to take into account the long range dependence of the stock price process and obtain the European call option price for such models. We assume that the basic stock price follows the geometric mixed fractional Brwonian motion with Hurst index $H \in (\frac{3}{4}, 1]$.

Mixed fractional Brownian motion with superimposed jumps can be used for pricing currency options (cf. Xiao et al. [11]). It is based on the assumption that the exchange rate returns are generated by a two-part stochastic process, the first part dealing with small continuous price movements generated by a mixed fractional Brownian motion and the second part by large infrequent price jumps generated by a Poisson process. As has been pointed out by Foad and Adem [3], modeling by this two-part process is in tune with the market in which major information arrives infrequently and randomly. In addition, this process provides a model through heavy tailed distributions for modeling empirically observed distributions of exchange rate changes.

We now introduce the Poisson process as a model for the jump times in the stock price process. Let $N(0) = 0$ and let $N(t)$ denote the number of jumps in the process that occur by time t for $t > 0$. Suppose that the process $\{N(t), t \geq 0\}$ is a Poisson process with stationary independent increments. Under such a process, the probability that there is a jump in a time interval of length h is approximately λh for h small and the probability of more than one jump in a time interval of length h is almost zero for h sufficiently small. Furthermore the probability that there is a jump in an interval does not depend on the information about the earlier jumps. Suppose that when the ith jump occurs, the price of the stock is multiplied by an amount J_i and the random sequence $\{J_i, i \geq 1\}$ forms an independent and identically distributed (i.i.d.) sequence of random variables. In addition, suppose that the random sequence $\{J_i, i \geq 1\}$ is independent of the times at which the jumps occur. Let $S(t)$ denote the stock price at time t for $t \geq 0$. Then

$$S(t) = S^*(t)\Pi_{i=1}^{N(t)} J_i, \quad t \geq 0$$

where $\{S^*(t), t \geq 0\}$ is the geometric mixed fractional Brownian motion modeled according to equation (1.1) specified earlier. Note that, if there is a jump in the price process at time t, then the jump is of size J_i at the ith jump. Let

$$J(t) = \Pi_{i=1}^{N(t)} J_i, t \geq 0$$

and we define $\Pi_{i=1}^{N(t)} J_i = 1$ if $N(t) = 0$. Note that the random variable $\log((S^*(t))/(S^*(0)))$ has the Gaussian distribution with mean g(t) and variance $\epsilon^2 t + \sigma^2 t^{2H}$. Note that $S(0) = S^*(0)$ is the initial stock price and we assume that it is non-random. Observe that

$$E[S(t)] = E[S^*(t)J(t)] = E[S^*(t)]E[J(t)]$$

by the independence of the random variables $S^*(t)$ and $J(t)$. Furthermore,

$$\begin{aligned} E[S^*(t)] &= S^*(0)E[\exp\{g(t) + \sigma B_t^H + \epsilon W_t\}] \\ &= S^*(0)\exp\left\{g(t) + \frac{1}{2}\sigma^2 t^{2H} + \frac{1}{2}\epsilon^2 t\right\} \end{aligned} \tag{2.1}$$

by the independence of the processes $\{B_t^H, t \geq 0\}$ and $\{W_t, t \geq 0\}$ and the properties of Gaussian random variables. It is easy to check that

$$E[J(t)] = e^{-\lambda t(1-E[J_1])}$$

and

$$Var[J(t)] = e^{-\lambda t(1-E[J_1^2])} - e^{-2\lambda t(1-E[J_1])}.$$

In particular, the equations given above show that

$$E[S(t)] = S^*(0)\, e^{g(t)+(1/2)\sigma^2 t^{2H}+(1/2)\epsilon^2 t} e^{-\lambda t(1-E[J_1])}.$$

Suppose the interest rate r is compounded continuously. Then the future value of the stock price $S(0)$, after time t, should be $S(0)e^{rt}$ under any risk-neutral probability measure. Under the no arbitrage assumption, it follows that

$$S^{(*)}(0)\exp\{g(t) + \frac{1}{2}\sigma^2 t^{2H} + \frac{1}{2}\epsilon^2 t - \lambda t(1 - E[J_1])\} = S(0)e^{rt}.$$

Since $S(0) = S^*(0)$, it follows that the stock price process should satisfy the relation

$$g(t) + \frac{1}{2}\sigma^2 t^{2H} + \frac{1}{2}\epsilon^2 t - \lambda t(1 - E[J_1]) = rt,$$

which implies that

$$g(t) = rt - \frac{1}{2}\sigma^2 t^{2H} - \frac{1}{2}\epsilon^2 t + \lambda t(1 - E[J_1])$$

under the no arbitrage assumption. The price for an European call option with strike price K, strike time t, and interest rate r compounded continuously is equal to

$$E[e^{-rt}(S(t) - K)_+],$$

where the expectation is computed with respect to the Gaussian distribution with mean

$$rt - \frac{1}{2}\sigma^2 t^{2H} - \frac{1}{2}\epsilon^2 t + \lambda t(1 - E[J_1])$$

and variance

$$\sigma^2 t^{2H} + \epsilon^2 t.$$

Here $a_+ = a$ if $a \geq 0$ and $a_+ = 0$ if $a < 0$. Let R_t be a Gaussian random variable with mean $rt - \frac{1}{2}\sigma^2 t^{2H} - \frac{1}{2}\epsilon^2 t + \lambda t(1 - E[J_1])$ and variance $\sigma^2 t^{2H} + \epsilon^2 t$. Note that the option price for an European call option under this model is

$$
\begin{aligned}
E[e^{-rt}(S(t) - K)_+] &= e^{-rt}E[(J(t)S^*(t) - K)_+] \\
&= e^{-rt}E[(J(t)S^*(0)e^{R_t} - K)_+],
\end{aligned}
\tag{2.2}
$$

where $S^*(0) = S(0)$ is the initial price of the stock.

3. SPECIAL CASE

Let us consider a special case of the model for the stock price discussed earlier. Suppose that the jumps $\{J_i, i \geq 1\}$ are i.i.d. log-normally distributed with parameters μ_1 and σ_1^2. It is easy to see that

$$E[J_1] = e^{\mu_1 + (1/2)\sigma_1^2}.$$

Let $X_i = \log J_i, i \geq 1$. Then the random variables $X_i, i \geq 1$ are i.i.d. with Gaussian distribution with mean μ_1 and variance σ_1^2. Observe that

$$J(t) = e^{\sum_{i=1}^{N(t)} X_i}$$

in this special case. Hence, the option price of the European call option, with strike price K, interest rate r compounded continuously, and the strike time t, under the no arbitrage assumption, is equal to

$$e^{-rt}E[(S^*(0)e^{R_t + \sum_{i=1}^{N(t)} X_i} - K)_+].$$

Under the condition $N(t) = n$, the random variable $R_t + \sum_{i=1}^{N(t)} X_i$ has the Gaussian distribution with mean $rt - \frac{1}{2}\sigma^2 t^{2H} - \frac{1}{2}\epsilon^2 t + \lambda t(1 - E[J_1]) + n\mu_1$ and variance

$$\sigma^2 t^{2H} + \epsilon^2 t + n\sigma_1^2.$$

Let

$$t\epsilon^2(n) = \epsilon^2 t + n\sigma_1^2,$$

and

$$
\begin{aligned}
r(n) &= \frac{1}{t}\left[rt - \frac{1}{2}\sigma^2 t^{2H} - \frac{1}{2}\epsilon^2 t + \lambda t(1 - E[J_1]) + n\mu_1\right] + \frac{1}{2}\epsilon^2(n) + \frac{1}{2}\sigma^2 t^{2H-1} \\
&= \left(r + \lambda - \lambda E[J_1] - \frac{1}{2}\sigma^2 t^{2H-1} - \frac{1}{2}\epsilon^2\right) + \frac{n}{t}\mu_1 + \frac{1}{2}\epsilon^2(n) + \frac{1}{2}\sigma^2 t^{2H-1} \\
&= \left(r + \lambda - \lambda E[J_1] - \frac{1}{2}\sigma^2 t^{2H-1}\right) + \frac{n}{t}\left(\mu_1 + \frac{1}{2}\sigma_1^2\right) + \frac{1}{2}\epsilon^2 - \frac{1}{2}\epsilon^2 + \frac{1}{2}\sigma^2 t^{2H-1} \\
&= (r + \lambda - \lambda E[J_1]) + \frac{n}{t}\log E[J_1].
\end{aligned}
\tag{3.1}
$$

Hence, given the event $N(t) = n$, the random variable $R_t + \sum_{i=1}^{N(t)} X_i$ has the Gaussian distribution with mean $(r(n) - \frac{1}{2}\epsilon^2(n) - \frac{1}{2}\sigma^2 t^{2H-1})t$ and variance $\epsilon^2(n)t + \sigma^2 t^{2H}$. Let $S(0) = S^*(0) = s$. Under the condition $N(t) = n$, we can interpret $r(n)$ as the interest rate and compute the European call option with strike price K and strike time t when the volatility of the Brownian motion is $\epsilon(n)$ and the volatility of the fractional Brownian motion is σ in the mixed fractional Brownian motion. Let $C(s,t,K,\sigma,\epsilon(n),r(n))$ denote the European call option price under the mixed fractional Brownian motion when the volatility of the Brownian motion is $\epsilon(n)$, volatility of the fractional Brownian motion is σ, interest rate is r compounded continuously, strike price is K and strike time is t. Note that

$$C(s,t,K,\sigma,\epsilon(n),r(n)) = e^{-r(n)t}E[(se^{R_t+\sum_{i=1}^{N(t)}X_i} - K)_+|N(t)=n].$$

Therefore

$$e^{-rt}E[(se^{R_t+\sum_{i=1}^{N(t)}X_i} - K)_+|N(t)=n] = e^{(r(n)-r)t}C(s,t,K,\sigma,\epsilon(n),r(n)).$$

Hence, the European call option price under the model described above is

$$\begin{aligned}&\sum_{n=0}^{\infty} e^{(r(n)-r)t}C(s,t,K,\sigma,\epsilon(n),r(n))P(N(t)=n) \qquad \textbf{(3.2)}\\ &= \sum_{n=0}^{\infty} e^{(r(n)-r)t}C(s,t,K,\sigma,\epsilon(n),r(n))e^{-\lambda t}\frac{(\lambda t)^n}{n!}\\ &= \sum_{n=0}^{\infty} e^{-\lambda t E[J_1]}\frac{(\lambda t E[J_1])^n}{n!}C(s,t,K,\sigma,\epsilon(n),r(n)).\end{aligned}$$

4. GENERAL CASE

As an application of the Jensen's inequality, we will now show that the option price for the European call option in the jump model in the general case is not less than it is in the model without jumps.

Suppose the distribution of the jumps is some general distribution. Let the stock price at time 0 be equal to s. The the European call option price with the strike price K and the expiration time t under the no arbitrage condition is

$$C(s,t,K,\sigma,\epsilon,r) = e^{-rt}E[(J(t)se^{R_t} - K)_+]$$

where the random variable R_t has the Gaussian distribution with the mean

$$rt - \frac{1}{2}\sigma^2 t^{2H} - \frac{1}{2}\epsilon^2 t + \lambda t(1 - E[J_1])$$

and the variance

$$\sigma^2 t^{2H} + \epsilon^2 t.$$

Let $R_t^* = R_t - \lambda t(1 - E[J_1])$ and $s_t = se^{\lambda t(1-E[J_1])} = (s/(E[J(t)])$. Then the price of a European call option under the general model described earlier is

$$E[e^{-rt}(s_t J(t)e^{R_t^*} - K)_+],$$

where the random variable R_t^* has the Gaussian distribution with mean

$$rt - \frac{1}{2}\epsilon^2 t - \frac{1}{2}\sigma^2 t^{2H}$$

and the variance

$$\sigma^2 t^{2H} + \epsilon^2 t.$$

Therefore, the option price of the European call option under this model is given by

$$E[C(s_t J(t), t, K, \sigma, \epsilon, r)].$$

The option price $C(s, t, K, \sigma, \epsilon, r)$ is a convex function in s. This follows from the fact that, for any positive constant a, the function

$$e^{-rt}(sa - K)_+$$

is an increasing and convex function of s. Since the probability distribution of $(R_t, J(t))$ does not depend on s, the quantity

$$e^{-rt}(J(t)se^{R_t} - K)_+,$$

is, for all values of R_t and J_t, increasing and convex in s and hence the $E[(J(t)se^{R_t} - K)_+]$ is a convex function of the initial price s. Hence, the function $C(s, t, K, \sigma, \epsilon, r)$ is a convex function of s. Applying the Jensen's inequality, it follows that

$$E[C(s_t J(t), t, K, \sigma, \epsilon, r)] \geq C(E[s_t J(t)], t, K, \sigma, \epsilon, r).$$

This implies that the European call option price when there are jumps in the stock price process is at least as large as the European call option price when there are no jumps in the stock price process, that is, when the stock price process is continuous.

Remarks: After the original version of this paper was prepared, the author came to know of the work of Foad and Adem [3] where similar results were obtained using slightly different techniques. They discuss pricing the currency option when the spot exchange rate follows the mixed fractional Brownian motion with the jumps following a Poisson process and jump size is log-normal. They derive the price of a currency option as the solution of a partial differential equation and discuss the properties of the jump mixed fractional partial differential equation. Our method of approach is similar to that in Chapter 8, Section 4, pp. 129–135 of Ross [8] and we have derived the general formula for the European call option price when the stock price is driven by a mixed fractional Brownian motion with superimposed jumps following the Poisson process and an arbitrary jump size distribution. We have obtained a closed form for the European call option price when the jump size distribution is log normal. It does not seem to be possible to derive a closed form when the jump size distribution is of any other type as the calculations involve sum of a Gaussian random variable and independent finite sums of i.i.d. random variables distributed possibly non-Gaussian.

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