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PARITY OF THE LANGLANDS PARAMETERS OF CONJUGATE SELF-DUAL REPRESENTATIONS OF GL(n) AND THE LOCAL JACQUET–LANGLANDS CORRESPONDENCE

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Abstract We determine the parity of the Langlands parameter of a conjugate self-dual supercuspidal representation of GL(n) over a non-archimedean local field by means of the local Jacquet–Langlands correspondence. It gives a partial generalization of a previous result on the self-dual case by Prasad and Ramakrishnan.

Keywords: local Langlands correspondence; local Jacquet–Langlands correspondence; Lubin–Tate space

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1. Introduction

Let F be a p-adic field. By the local Langlands correspondence, irreducible smooth representations of $\mathrm{GL}_n(F)$ are known to be parameterized by n-dimensional representations of $W_F \times \mathrm{SL}_2(\mathbb{C})$, where W_F denotes the Weil group of F. For an irreducible smooth representation π of $\mathrm{GL}_n(F)$, we write $\mathrm{rec}_F(\pi)$ for the attached parameter, which is called the Langlands parameter of π.

Let us assume that π is self-dual, namely, π is isomorphic to its contragredient $\pi^\vee$. Since rec_F is compatible with dual, $\mathrm{rec}_F(\pi)$ is again self-dual. Therefore, we can consider the problem whether $\mathrm{rec}_F(\pi)$ is symplectic or orthogonal, under the condition that $\mathrm{rec}_F(\pi)$ is irreducible; in other words, π is a discrete series representation. In [19], Prasad and Ramakrishnan answered this question by means of the local Jacquet–Langlands correspondence. Let D be a central division algebra of rank n over F. Recall that the local Jacquet–Langlands correspondence [4, 22] gives a bijection between isomorphism classes of irreducible discrete series representations of $\mathrm{GL}_n(F)$ and those of irreducible smooth representations of $D^\times$. We write $\mathrm{JL}(\pi)$ for the representation of $D^\times$ attached to π by this correspondence. The theorem of Prasad and Ramakrishnan is as follows:

Theorem 1.1 [19, Theorem B]. *Assume that π is self-dual. If n is odd, $\mathrm{rec}_F(\pi)$ is always orthogonal (this part is clear). If n is even, then $\mathrm{rec}_F(\pi)$ is symplectic (respectively orthogonal) if and only if $\mathrm{JL}(\pi)$ is orthogonal (respectively symplectic).*

The purpose of this paper is to extend this theorem to the conjugate self-dual setting. Let F/F^+ be a quadratic extension of p-adic fields and τ the generator of $\mathrm{Gal}(F/F^+)$. A smooth representation (π, V) of $\mathrm{GL}_n(F)$ is said to be conjugate self-dual if $\pi^\tau \cong \pi^\vee$, where π^τ denotes the representation $\mathrm{GL}_n(F) \xrightarrow{\tau} \mathrm{GL}_n(F) \xrightarrow{\pi} \mathrm{GL}(V)$. If π is conjugate self-dual, its Langlands parameter $\mathrm{rec}_F(\pi)$ is also conjugate self-dual in the following sense. Take $c \in W_{F^+} \setminus W_F$. For a representation ϕ of $W_F \times \mathrm{SL}_2(\mathbb{C})$, define a new representation ϕ^c by $\phi^c(w) = \phi(cwc^{-1})$; it is independent of the choice of c up to isomorphism. A representation ϕ is said to be conjugate self-dual if $\phi^c \cong \phi^\vee$ holds. For an irreducible conjugate self-dual representation ϕ of $W_F \times \mathrm{SL}_2(\mathbb{C})$, we can define its parity $C_\phi \in \{\pm 1\}$ in the similar way as in the self-dual case (for the detail, see [8, §3], [16, §2.2] and § 2 of this paper). If $C_\phi = 1$, ϕ is said to be conjugate orthogonal, otherwise conjugate symplectic. For an irreducible conjugate self-dual discrete series representation π, the parity of $\mathrm{rec}_F(\pi)$ knows whether π comes from the standard base change lifting or the twisted base change lifting from the quasi-split unitary group $U_{F/F^+}(n)$ (see [16, §2]).

In this paper, we determine the parity of $\mathrm{rec}_F(\pi)$ by means of $\mathrm{JL}(\pi)$, under the conditions that

- F/F^+ is at worst tamely ramified;
- the invariant of D is $1/n$;
- and π is supercuspidal (in other words, $\mathrm{rec}_F(\pi)$ is trivial on the $\mathrm{SL}_2(\mathbb{C})$-factor).

Under the first two assumptions, we construct explicitly an automorphism $\tau\colon D^\times \to D^\times$ such that $\tau|_{F^\times}$ coincides with $\tau \in \mathrm{Gal}(F/F^+)$, and $t \in D^\times$ such that $\tau^2(d) = tdt^{-1}$ for $d \in D^\times$ (Definition 2.10). For such a pair (τ, t), we can define the conjugate self-duality and the parity of an irreducible smooth representation of $D^\times$ (see § 2). Our main theorem is summarized as follows:

Theorem 1.2 (Main theorem, Theorem 2.12). *Assume that F/F^+ is at worst tamely ramified and the invariant of D is $1/n$. Let π be an irreducible conjugate self-dual supercuspidal representation of $\mathrm{GL}_n(F)$. Then, $\mathrm{JL}(\pi)$ is conjugate self-dual with respect to (τ, t), and its parity $C_{\mathrm{JL}(\pi)}$ satisfies*

$$C_{\mathrm{rec}_F(\pi)} = (-1)^{n-1} C_{\mathrm{JL}(\pi)}.$$

Theorems 1.1 and 1.2 are useful in the study of $\mathrm{rec}_F(\pi)$, because the determination of $\mathrm{JL}(\pi)$ is usually much easier than that of $\mathrm{rec}_F(\pi)$. In § 4, we apply Theorems 1.1 and 1.2 to compute the parity of $\mathrm{rec}_F(\pi)$ for conjugate (or usual) self-dual simple supercuspidal representations of $\mathrm{GL}_n(F)$ (for simple supercuspidal representations, see [9, 13, 21]). For example, we prove that the Langlands parameter of a self-dual simple supercuspidal representation of $\mathrm{GL}_{2n}(F)$ is symplectic if and only if its central character is trivial. This result plays a crucial role in the recent study of Oi [17] on the endoscopic lifting of simple supercuspidal representations of $\mathrm{SO}_{2n+1}(F)$ to $\mathrm{GL}_{2n}(F)$.

Let us explain the strategy of our proof of Theorem 1.2. We use a geometric method. The non-abelian Lubin–Tate theory [2, 3, 10] tells us that the correspondences rec_F and JL for supercuspidal representations appear in the ℓ-adic étale cohomology of the Lubin–Tate tower, which is a projective system of universal deformation spaces of a one-dimensional formal $\mathcal{O}_F$-module $\mathbb{X}$ of height n with suitable level structures. By using the cup product of the cohomology and a result in [15], we can construct a perfect pairing

$$(\mathrm{JL}(\pi)\boxtimes \mathrm{rec}_F(\pi))\times(\mathrm{JL}(\pi^\vee)\boxtimes \mathrm{rec}_F(\pi^\vee))\to\mathbb{C}$$

for an irreducible supercuspidal representation π of $\mathrm{GL}_n(F)$. It enables us to compare the parity of $\mathrm{rec}_F(\pi)$ and that of $\mathrm{JL}(\pi)$, provided that π is self-dual. As in the introduction of [19], this method had already been found by Fargues; he announced the supercuspidal case of Theorem 1.1 in [6, §5] without proof. The new point of this paper is to adapt the argument above to the conjugate self-dual case. In the conjugate self-dual case, we need to make the pairing 'Hermitian'. For this purpose, we introduce a new operator on the Lubin–Tate tower, which we call the twisting operator. In the definition of it, we need to fix an additional structure on the fixed formal $\mathcal{O}_F$-module $\mathbb{X}$. This extra structure naturally induces the pair (τ, t) in Theorem 1.2, as $D^\times$ can be identified with the group of self-$\mathcal{O}_F$-isogenies of $\mathbb{X}$.

Since our method is geometric, our theorem is also valid in the equal characteristic case. On the other hand, we need to assume that the invariant of D is $1/n$ and π is supercuspidal, because this is the only case in which $\mathrm{rec}_F(\pi)$ and $\mathrm{JL}(\pi)$ have nice geometric descriptions. The author expects that Theorem 1.2 is true for any conjugate self-dual discrete series representation π; in fact, we can easily verify it for a character twist of the Steinberg representation (see Remark 2.13). It seems also an interesting question to extend Theorem 1.2 to general division algebras. These problems will be considered in our future works.

The outline of this paper is as follows. In § 2, we give some basic definitions on conjugate self-dual representations and their parity. We need a slightly general framework than usual, in order to formulate Theorem 1.2. § 3 is devoted to a proof of the main theorem. After a brief review of the non-abelian Lubin–Tate theory, we introduce and study the twisting operator, which is a key of our proof. To describe the pair (τ, t) explicitly, we also need some explicit computations of Dieudonné modules. In § 4, we apply the main theorem to determine the parity of conjugate self-dual simple supercuspidal representations of $\mathrm{GL}_n(F)$.

Notation For a field L and an integer $m \geqslant 1$, we write $\mu_m(L)$ for the set of mth roots of unity in L. If L is a discrete valuation field, we denote the ring of integers of L by $\mathcal{O}_L$, and the maximal ideal of $\mathcal{O}_L$ by $\mathfrak{p}_L$. Every representation is considered over $\mathbb{C}$.

2. Parity of conjugate self-dual representations

2.1. Basic definitions and properties

Let G be a totally disconnected locally compact topological group. We fix a continuous automorphism $\tau\colon G\to G$ and an element $t\in G$ satisfying

$$\tau^2=\mathrm{Int}(t),\quad \tau(t)=t,$$

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where $\mathrm{Int}(t)\colon G \to G$ denotes the isomorphism $g \mapsto tgt^{-1}$. For a smooth representation (π, V) of G, we write (π^τ, V) for the smooth representation defined by $\pi^\tau(g) = \pi(\tau(g))$. We say that π is conjugate self-dual with respect to τ if π^τ is isomorphic to the contragredient representation $\pi^\vee$. If π is conjugate self-dual with respect to τ, we have $\pi^{\vee\vee} \cong (\pi^\tau)^\vee = (\pi^\vee)^\tau \cong (\pi^\tau)^\tau = \pi^t \cong \pi$ (the last isomorphism is given by $\pi(t)^{-1}$). Hence π is admissible.

Let π be a smooth representation of G which is conjugate self-dual with respect to τ. Then, there exists a non-degenerate bilinear pairing $\langle\,,\rangle\colon V \times V \to \mathbb{C}$ satisfying $\langle \pi(\tau(g))x, \pi(g)y\rangle = \langle x, y\rangle$ for every $g \in G$ and $x, y \in V$. If π is irreducible, such a pairing is unique up to scalar by Schur's lemma (recall that π is admissible).

Lemma 2.1. *There exists $C_\pi \in \{\pm 1\}$ such that $\langle \pi(t)y, x\rangle = C_\pi \langle x, y\rangle$ for every $x, y \in V$.*

Proof. Put $\langle x, y\rangle' = \langle \pi(t)y, x\rangle$. Let $g \in G$ and $x, y \in V$ be arbitrary elements, and we put $g' = \tau^{-1}(g)$. Then we have

$$\begin{aligned}\langle \pi(\tau(g))x, \pi(g)y\rangle' &= \langle \pi(t)\pi(g)y, \pi(\tau(g))x\rangle = \langle \pi(\tau(tg'))y, \pi(tg't^{-1})x\rangle\\ &= \langle y, \pi(t)^{-1}x\rangle = \langle \pi(t)y, x\rangle = \langle x, y\rangle'.\end{aligned}$$

Therefore, there exists $C_\pi \in \mathbb{C}^\times$ such that $\langle x, y\rangle' = C_\pi \langle x, y\rangle$ for every $x, y \in V$.

For $x, y \in V$, we have

$$\langle x, y\rangle = \langle \pi(\tau(t))x, \pi(t)y\rangle = \langle \pi(t)x, \pi(t)y\rangle = C_\pi \langle \pi(t)y, x\rangle = C_\pi^2 \langle x, y\rangle.$$

Hence we have $C_\pi^2 = 1$. This concludes the proof. □

Remark 2.2. The sign C_π depends not only on τ but also on t. Let $t' \in G$ be another element satisfying $\tau^2 = \mathrm{Int}(t')$. Then $z = t't^{-1}$ lies in the center of G and fixed by τ. It is immediate to see that C_π for t' equals $\omega_\pi(z)C_\pi$, where ω_π denotes the central character of π.

We call C_π the parity of π (with respect to (τ, t)). If $C_\pi = 1$ (respectively $C_\pi = -1$), we say that π is conjugate orthogonal (respectively conjugate symplectic). If $\tau = \mathrm{id}$ and $t = 1$, this notion coincides with the standard one.

Remark 2.3. Consider the case where (π, V) is finite-dimensional, and put $m = \dim_{\mathbb{C}} V$.

(i) Assume that $m = 1$, and identify V with $\mathbb{C}$. Then, $\langle\ ,\ \rangle\colon \mathbb{C} \times \mathbb{C} \to \mathbb{C};\ (x, y) \mapsto xy$ gives a non-degenerate bilinear pairing satisfying $\langle \pi(\tau(g))x, \pi(g)y\rangle = \langle x, y\rangle$. From this pairing we can deduce $C_\pi = \pi(t)$.

(ii) Let $\langle\,,\rangle\colon V \times V \to \mathbb{C}$ be a non-degenerate bilinear pairing as in the definition of the parity. Put $(\det\pi, \det V) = (\bigwedge^m \pi, \bigwedge^m V)$. Then, $\langle\ ,\ \rangle$ induces a pairing $\det V \times \det V \to \mathbb{C}$ by

$$\langle x_1 \wedge \cdots \wedge x_m, y_1 \wedge \cdots \wedge y_m\rangle = \sum_{\sigma \in \mathfrak{S}_m} \mathrm{sgn}(\sigma)\langle x_1, y_{\sigma(1)}\rangle \cdots \langle x_m, y_{\sigma(m)}\rangle.$$

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It is non-degenerate and satisfies

$$\langle (\det\pi)(\tau(g))x, (\det\pi)(g)y\rangle = \langle x, y\rangle, \quad \langle (\det\pi)(t)y, x\rangle = C_\pi^m\langle x, y\rangle$$

for $x, y \in \det V$ and $g \in G$. Hence we have $C_{\det\pi} = C_\pi^m$.

In particular, if m is odd, the parity C_π can be computed as follows:

$$C_\pi = C_\pi^m = C_{\det\pi} = \det\pi(t).$$

In contrast, if m is even, the parity is a more subtle invariant.

We give two elementary lemmas.

Lemma 2.4. *Assume that (G, τ, t) is decomposed into $(G_1 \times G_2, \tau_1 \times \tau_2, (t_1, t_2))$, where G_i is a totally disconnected locally compact topological group, $\tau_i : G_i \to G_i$ a continuous automorphism and $t_i \in G_i$ satisfying $\tau_i^2 = \mathrm{Int}(t_i)$. For each $i = 1, 2$, let (π_i, V_i) be an irreducible smooth representation of G_i conjugate self-dual with respect to τ_i. Then, $(\pi_1 \boxtimes \pi_2, V_1 \otimes V_2)$ is an irreducible smooth representation of G conjugate self-dual with respect to τ, and $C_{\pi_1\boxtimes\pi_2}$ is equal to $C_{\pi_1}C_{\pi_2}$.*

Proof. It is well known that the exterior tensor product of irreducible admissible representations is irreducible. The parity can be computed by using the pairing $\langle x_1 \otimes x_2, y_1 \otimes y_2\rangle = \langle x_1, y_1\rangle_1\langle x_2, y_2\rangle_2$, where $\langle\, ,\rangle_i : V_i \times V_i \to \mathbb{C}$ is an appropriate pairing attached to π_i. □

Lemma 2.5. *Take an element $h \in G$ and put $\tau' = \mathrm{Int}(h) \circ \tau$, $t' = h\tau(h)t$. Then we have $\tau'^2 = \mathrm{Int}(t')$. For an irreducible smooth representation π of G, π is conjugate self-dual with respect to τ if and only if it is conjugate self-dual with respect to τ'. If π is conjugate self-dual with respect to τ and τ', its parity with respect to (τ, t) coincides with that with respect to (τ', t').*

Proof. The claim $\tau'^2 = \mathrm{Int}(t')$ is immediate. We write V for the representation space of π. Assume that π is conjugate self-dual with respect to τ, and take a non-degenerate pairing $\langle\ ,\ \rangle : V \times V \to \mathbb{C}$ satisfying $\langle\pi(\tau(g))x, \pi(g)y\rangle = \langle x, y\rangle$. Let $\langle\ ,\ \rangle_h : V \times V \to \mathbb{C}$ be the pairing defined by $\langle x, y\rangle_h = \langle\pi(h)^{-1}x, y\rangle$. It is a non-degenerate pairing and satisfies

$$\begin{aligned}\langle\pi(\tau'(g))x, \pi(g)y\rangle_h &= \langle\pi(h)^{-1}\pi(h\tau(g)h^{-1})x, \pi(g)y\rangle = \langle\pi(\tau(g))\pi(h)^{-1}x, \pi(g)y\rangle\\ &= \langle\pi(h)^{-1}x, y\rangle = \langle x, y\rangle_h.\end{aligned}$$

Therefore, $\pi^{\tau'} \cong \pi^\vee$, that is, π is conjugate self-dual with respect to τ'. Since $\tau = \mathrm{Int}(h^{-1}) \circ \tau'$, the converse is also the case.

Let us denote by C (respectively C') the parity of π, which is assumed to be conjugate self-dual, with respect to (τ, t) (respectively (τ', t')). We use the pairing $\langle\, ,\rangle_h$ to compute C'. For $x, y \in V$, we have

$$\begin{aligned}C'\langle x, y\rangle_h &= \langle\pi(t')y, x\rangle_h = \langle\pi(h^{-1}t')y, x\rangle = \langle\pi(\tau(h)t)y, x\rangle = \langle\pi(t)y, \pi(h)^{-1}x\rangle\\ &= C\langle\pi(h)^{-1}x, y\rangle = C\langle x, y\rangle_h.\end{aligned}$$

Hence we conclude that $C = C'$. □

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Let H be an open subgroup of G. Take a smooth character $\chi\colon H\to\mathbb{C}^\times$ such that $\pi=\text{c-Ind}_H^G\chi$ is irreducible and admissible. Note that $\text{Ind}_H^G\chi^{-1}=\text{c-Ind}_H^G\chi^{-1}$ in this case. Indeed, since π is irreducible and admissible, so is $\pi^\vee\cong\text{Ind}_H^G\chi^{-1}$. As $\text{c-Ind}_H^G\chi^{-1}$ is a non-zero G-invariant subspace of $\text{Ind}_H^G\chi^{-1}$, it equals $\text{Ind}_H^G\chi^{-1}$.

We consider when π is conjugate self-dual with respect to τ, and how to compute the parity of π.

Proposition 2.6. *Put $H^\tau=\tau^{-1}(H)$, and write χ^τ for the character $H^\tau\to\mathbb{C}^\times$; $h\mapsto\chi(\tau(h))$. Assume that there exists $a\in G$ which intertwines (H,χ^{-1}) and (H^τ,χ^τ); namely, satisfies the following conditions:*

$$aHa^{-1}=H^\tau,\quad \chi(h)^{-1}=\chi^\tau(aha^{-1})\quad \textit{for every } h\in H.$$

Then, the representation $\pi=\text{c-Ind}_H^G\chi$ is conjugate self-dual with respect to τ. Furthermore, an element $z=\tau(a)ta$ lies in H, and the parity C_π of π is given by $\chi(z)$.

Proof. For $f\in\text{c-Ind}_H^G\chi$, let $f^\tau\colon G\to\mathbb{C}$ be the function $g\mapsto f(\tau(g))$. Then, it is easy to see that f^τ belongs to $\text{c-Ind}_{H^\tau}^G\chi^\tau$, and $f\mapsto f^\tau$ gives an isomorphism $(\text{c-Ind}_H^G\chi)^\tau\xrightarrow{\cong}\text{c-Ind}_{H^\tau}^G\chi^\tau$ of G-representations. On the other hand, for $f\in\text{c-Ind}_{H^\tau}^G\chi^\tau$, let $f^a\colon G\to\mathbb{C}$ be the function $g\mapsto f(ag)$. We can check that f^a belongs to $\text{c-Ind}_H^G\chi^{-1}$ and $f\mapsto f^a$ gives an isomorphism $\text{c-Ind}_{H^\tau}^G\chi^\tau\xrightarrow{\cong}\text{c-Ind}_H^G\chi^{-1}$. Hence we have $\pi^\tau=(\text{c-Ind}_H^G\chi)^\tau\cong\text{c-Ind}_{H^\tau}^G\chi^\tau\cong\text{c-Ind}_H^G\chi^{-1}=\text{Ind}_H^G\chi^{-1}\cong\pi^\vee$. In other words, π is conjugate self-dual with respect to τ.

Next we prove $z\in H$. First we see that z normalizes (H,χ). Since $H^\tau=aHa^{-1}$, we have $H=\tau(a)\tau^2(H^\tau)\tau(a)^{-1}=\tau(a)tH^\tau t^{-1}\tau(a)^{-1}=zHz^{-1}$. Therefore, z normalizes H. Moreover, for $h\in H$ we have

$$\begin{aligned}\chi(z^{-1}hz)&=\chi^\tau(az^{-1}hza^{-1})^{-1}=\chi^\tau(t^{-1}\tau(a)^{-1}h\tau(a)t)^{-1}=\chi(a^{-1}t^{-1}\tau(h)ta)^{-1}\\&=\chi^\tau(t^{-1}\tau(h)t)=\chi(h).\end{aligned}$$

Thus z fixes χ.

Recall that we are assuming that $\pi=\text{c-Ind}_H^G\chi$ is irreducible. Therefore,

$$\text{Hom}_G(\pi,\pi)=\text{Hom}_H(\chi,(\text{c-Ind}_H^G\chi)|_H)\cong\text{Hom}_H\Bigl(\chi,\bigoplus_{g\in H\backslash G/H}\text{c-Ind}_{H\cap g^{-1}Hg}^H\chi^g\Bigr)$$

is one-dimensional (here χ^g denotes the character $h'\mapsto\chi(gh'g^{-1})$ on $H\cap g^{-1}Hg$). Since $\text{c-Ind}_{H\cap z^{-1}Hz}^H\chi^z=\chi$, z must lie in H; otherwise the direct sum above contains $\chi\oplus\chi$.

Finally we compute the parity of π. Define a pairing $\langle\ ,\ \rangle\colon\text{c-Ind}_H^G\chi\times\text{c-Ind}_H^G\chi\to\mathbb{C}$ by

$$\langle f_1,f_2\rangle=\sum_{g\in H\backslash G}(f_1^\tau)^a(g)f_2(g),$$

where the sum is essentially finite since the support of f_2 is compact modulo H. This pairing is the composite of $(\text{c-Ind}_H^G\chi)^\tau\times\text{c-Ind}_H^G\chi\underset{\cong}{\xrightarrow{(1)}}\text{c-Ind}_H^G\chi^{-1}\times\text{c-Ind}_H^G\chi=\text{Ind}_H^G\chi^{-1}\times\text{c-Ind}_H^G\chi\xrightarrow{(2)}\mathbb{C}$, where (1) denotes the isomorphism $(f_1,f_2)\mapsto((f_1^\tau)^a,f_2)$

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and (2) the canonical pairing. Hence $\langle\,,\rangle$ is a non-degenerate pairing satisfying $\langle\pi(\tau(g))f_1, \pi(g)f_2\rangle = \langle f_1, f_2\rangle$ for every $g \in G$ and $f_1, f_2 \in \text{c-Ind}_H^G \chi$.

By definition we can compute as follows:

$$\begin{aligned}\langle\pi(t)f_2, f_1\rangle &= \sum_{g\in H\backslash G}(f_2^\tau)^a(gt)f_1(g) = \sum_{g\in H\backslash G} f_2(\tau(agt))f_1(g)\\ &\overset{(*)}{=} \sum_{g'\in H\backslash G} f_2(g')f_1(a^{-1}t^{-1}\tau(g')) = \sum_{g'\in H\backslash G} f_1(z^{-1}\tau(ag'))f_2(g')\\ &= \sum_{g'\in H\backslash G}\chi(z)^{-1}f_1(\tau(ag'))f_2(g') = \chi(z)^{-1}\langle f_1, f_2\rangle.\end{aligned}$$

At the equality $(*)$, we put $g' = \tau(agt)$. As $\tau(aHgt) = \tau(aHa^{-1})\tau(agt) = H\tau(agt)$, this replacement is well defined. Hence the parity $C_\pi = C_\pi^{-1}$ of π equals $\chi(z)$. This completes the proof. □

Proposition 2.6 will be used in § 4, in which case G is a p-adic reductive group. Suppose that G is a p-adic reductive group. Then, every irreducible smooth representation of G is known to be admissible. Therefore, to apply Proposition 2.6, we have only to check the irreducibility of π.

2.2. Division algebra setting

Let F^+ be a non-archimedean local field and F a separable extension of F^+ such that $[F : F^+] \leqslant 2$. Denote by τ the generator of $\mathrm{Gal}(F/F^+)$. Let q (respectively q') denote the cardinality of the residue field of $\mathcal{O}_F$ (respectively $\mathcal{O}_{F^+}$). We denote the characteristic of $\mathbb{F}_q$ by p.

The extension F/F^+ provides two well-known examples of (G, τ, t) in the previous subsection.

Example 2.7. For an integer $n \geqslant 1$, put $G = \mathrm{GL}_n(F)$. Let $\tau : G \to G$ be an automorphism induced by $\tau \in \mathrm{Gal}(F/F^+)$. Then we have $\tau^2 = \mathrm{id}$, and we can set $t = 1$.

Example 2.8. Let G be the Weil group W_F of F. Fix an element $c \in W_{F^+}$ whose image in W_{F^+}/W_F is a generator, and let $\tau : G \to G$ be $\mathrm{Int}(c)$. Then c^2 lies in W_F, and we can set $t = c^2$. The conjugate self-duality and the parity are independent of the choice of c. Indeed, another choice of c is of the form wc with $w \in W_F$. Use Lemma 2.5 to $\tau' = \mathrm{Int}(wc) = \mathrm{Int}(w) \circ \tau$ and $t' = (wc)^2 = w(cwc^{-1})c^2 = w\tau(w)t$.

The conjugate self-duality and the parity in this case coincide with those in [8, §3] and [16, §2.2].

The parity under the setting in Example 2.8 is interesting because of the following theorem:

Theorem 2.9. *Let π be an irreducible supercuspidal representation of $\mathrm{GL}_n(F)$ and $\mathrm{rec}_F(\pi)$ the corresponding n-dimensional irreducible smooth representation of W_F under the local Langlands correspondence.*

(i) *The representation* π *is conjugate self-dual under the setting in Example 2.7 if and only if* $\operatorname{rec}_F(\pi)$ *is conjugate self-dual under the setting in Example 2.8.*

(ii) *Assume that* $F \neq F^+$ *and the characteristic of* F *is* 0*. The representation* π *belongs to the image of the standard (respectively twisted) base change lift from the quasi-split unitary group* $U_{F/F^+}(n)$ *if and only if the parity* $C_{\operatorname{rec}_F(\pi)}$ *is equal to* $(-1)^{n-1}$ *(respectively* $(-1)^n$*). For the notion of the base change lift, see [16, § 2].*

In the following, we introduce another setting. Fix a separable closure $\overline{F}$ and a uniformizer ϖ of F. For an integer $n \geqslant 1$, we denote by F_n (respectively F_n^+) the unramified extension of degree n of F (respectively F^+) contained in $\overline{F}$, and by $\sigma \in \operatorname{Gal}(F_n/F)$ the arithmetic Frobenius lift. Let D be the central division algebra over F with invariant $1/n$. Recall that D can be written as $F_n[\Pi]$, where Π is an element satisfying $\Pi^n = \varpi$ and $\Pi a = \sigma(a)\Pi$ for every $a \in F_n$. Assuming the tameness of F/F^+, we explicitly construct an isomorphism $\tau\colon D \to D$ whose restriction to the center F coincides with $\tau \in \operatorname{Gal}(F/F^+)$.

Definition 2.10. Assume that F/F^+ is at worst tamely ramified.

(i) Suppose that F/F^+ is an unramified quadratic extension. Then, $\tau \in \operatorname{Gal}(F/F^+)$ is canonically extended to the arithmetic Frobenius lift in $\operatorname{Gal}(F_n/F^+)$, that is also denoted by τ. It satisfies $\sigma = \tau^2$. In this case, we take ϖ in F^+ and define $\tau\colon D \to D$ by $a \mapsto \tau(a)$ $(a \in F_n)$ and $\Pi \mapsto \Pi$. We put $t = \Pi$.

(ii) Suppose that F/F^+ is a ramified quadratic extension (thus $p \neq 2$). Then, the restriction map $\operatorname{Gal}(F_n/F_n^+) \to \operatorname{Gal}(F/F^+)$ is an isomorphism. We also write τ for the generator of $\operatorname{Gal}(F_n/F_n^+)$. It commutes with $\sigma \in \operatorname{Gal}(F_n/F)$.
In this case, we can (and do) take ϖ so that $\tau(\varpi) = -\varpi$. Fix an element $\beta \in \overline{F}$ such that $\beta^{q^n-1} = -1$ and put $\alpha = \beta^{q-1}$. Since $\alpha^{q^n-1} = (-1)^{q-1} = 1$, α belongs to $\mu_{q^n-1}(\overline{F}) = \mu_{q^n-1}(F_n^+)$ and $\operatorname{Nr}_{F_n/F}(\alpha) = \alpha^{1+q+\cdots+q^{n-1}} = \beta^{q^n-1} = -1$. We define $\tau\colon D \to D$ by $a \mapsto \tau(a)$ $(a \in F_n)$ and $\Pi \mapsto \alpha\Pi$. Note that $(\alpha\Pi)^n = \operatorname{Nr}_{F_n/F}(\alpha)\Pi^n = -\varpi = \tau(\varpi)$ and $(\alpha\Pi)\tau(a) = \alpha\sigma(\tau(a))\Pi = \tau(\sigma(a))(\alpha\Pi)$, which ensure the well-definedness of τ. We put $t = \beta^{-2} \in \mu_{q^n-1}(\overline{F}) = \mu_{q^n-1}(F_n^+)$.

(iii) If $F = F^+$, then we define $\tau\colon D \to D$ to be the identity map. We put $t = 1$.

In each case we can check that $\tau^2(d) = tdt^{-1}$ holds for every $d \in D$. Therefore, the triple $(D^\times, \tau, t)$ gives an example of the setting in § 2.1.

Remark 2.11. (i) In the second case, the conjugate self-duality and the parity are independent of the choice of β. Indeed, let $\beta' \in \overline{F}$ be another element such that $\beta'^{q^n-1} = -1$. Then $\gamma = \beta/\beta'$ lies in $\mu_{q^n-1}(\overline{F}) = \mu_{q^n-1}(F_n^+)$. We put $\alpha' = \beta'^{q-1}$ and write τ', t' for τ, t attached to β', respectively. Since $\alpha'\Pi = \gamma(\alpha\Pi)\gamma^{-1}$, we have $\tau' = \operatorname{Int}(\gamma) \circ \tau$ and $t' = \beta'^{-2} = \gamma^2 t = \gamma\tau(\gamma)t$. Hence the independence follows from Lemma 2.5.

(ii) In the second case, assume that n is odd. Take $\varepsilon \in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^2$ and $\eta \in \mathbb{F}_{q^2}$ such that $\eta^2 = \varepsilon^{-1}$. We have $\eta^{q-1} = -1$.

Then, the unique lifting $\beta \in \mu_{q^2-1}(\mathcal{O}_{F_2})$ of η satisfies $\beta^{q^n-1} = (-1)^{1+q+\cdots+q^{n-1}} = -1$. Under this choice of β, we have $\alpha = \beta^{q-1} = -1$. Moreover, the element $t = \beta^{-2}$ is the unique element of $\mu_{q-1}(\mathcal{O}_F)$ lifting ε.

Our main theorem is as follows:

Theorem 2.12. *Assume that F/F^+ is at worst tamely ramified. Let π be an irreducible supercuspidal representation of $\mathrm{GL}_n(F)$ which is conjugate self-dual under the setting in Example 2.7. We write $\mathrm{JL}(\pi)$ for the irreducible smooth representation of $D^\times$ attached to π under the local Jacquet–Langlands correspondence.*

Then, $\mathrm{JL}(\pi)$ is conjugate self-dual with respect to $\tau\colon D^\times \to D^\times$ introduced in Definition 2.10. Moreover, we have

$$C_{\mathrm{rec}_F(\pi)} = (-1)^{n-1} C_{\mathrm{JL}(\pi)}.$$

Remark 2.13. (i) The case where $F = F^+$ and the characteristic of F is 0 has been obtained in [19], in which a discrete series representation π is also treated. The same statement for the case $F = F^+$ is also announced in [6, §5] without proof.

(ii) It is natural to expect that Theorem 2.12 remains true for conjugate self-dual discrete series representations of $\mathrm{GL}_n(F)$. For example, let us consider a twist of the Steinberg representation $\pi = \mathbf{St} \otimes (\chi \circ \det)$, where $\chi\colon F^\times \to \mathbb{C}^\times$ is a smooth character. Since $\mathbf{St}^\tau \cong \mathbf{St} \cong \mathbf{St}^\vee$, the representation π is conjugate self-dual if and only if $\chi^\tau = \chi^{-1}$. The Langlands parameter $\mathrm{rec}_F(\pi)$ is given by $(\chi \circ \mathrm{Art}_F^{-1}) \boxtimes \mathrm{Sym}^{n-1}\mathbf{Std}\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$, where $\mathrm{Art}_F\colon F^\times \xrightarrow{\cong} W_F^{\mathrm{ab}}$ denotes the isomorphism of the local class field theory, and $\mathbf{Std}$ the standard representation of $\mathrm{SL}_2(\mathbb{C})$. The parity of $\mathrm{Sym}^{n-1}\mathbf{Std}$ equals $(-1)^{n-1}$. By Remark 2.3(i), the parity of $\chi \circ \mathrm{Art}_F^{-1}$ is given by $\chi(\mathrm{Art}_F^{-1}(c^2)) = \chi(\mathrm{Art}_{F^+}^{-1}(c))$ (recall that the image of c under the transfer map $W_{F^+}^{\mathrm{ab}} \to W_F^{\mathrm{ab}}$ is c^2). Hence we obtain $C_{\mathrm{rec}_F(\pi)} = (-1)^{n-1}\chi(\mathrm{Art}_{F^+}^{-1}(c))$. On the other hand, we have $\mathrm{JL}(\pi) = \chi \circ \mathrm{Nrd}$, where Nrd denotes the reduced norm of D. Its parity $C_{\chi\circ\mathrm{Nrd}}$ equals $\chi(\mathrm{Nrd}(t))$. By definition, both $\mathrm{Art}_{F^+}^{-1}(c)$ and $\mathrm{Nrd}(t)$ lie in $(F^+)^\times \setminus \mathrm{Nr}_{F/F^+}(F^\times)$. Since $\chi|_{(F^+)^\times}$ factors through $(F^+)^\times/\mathrm{Nr}_{F/F^+}(F^\times)$, we conclude that $\chi(\mathrm{Art}_{F^+}^{-1}(c)) = \chi(\mathrm{Nrd}(t))$ and $C_{\mathrm{rec}_F(\pi)} = (-1)^{n-1} C_{\mathrm{JL}(\pi)}$.

3. Proof of the main theorem

3.1. Review of the non-abelian Lubin–Tate theory

To prove Theorem 2.12, we use the non-abelian Lubin–Tate theory, which is a geometric realization of the local Langlands correspondence for GL_n. Here we recall it briefly. Let F be a non-archimedean local field and ϖ its uniformizer. Take an integer $n \geqslant 1$. We write F^{ur} for the maximal unramified extension of F inside the fixed separable closure $\overline{F}$, and $\breve{F}$ for the completion of F^{ur}.

Let **Nilp** be the category of schemes over $\mathcal{O}_{\breve{F}}$ on which ϖ is locally nilpotent. For an object S of **Nilp**, we denote the structure morphism $S \to \operatorname{Spec} \mathcal{O}_{\breve{F}}$ by ϕ_S. Put $\overline{S} =$

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$S \otimes_{\mathcal{O}_{\breve{F}}} \mathcal{O}_{\breve{F}}/\mathfrak{p}_{\breve{F}}$. Recall that a formal $\mathcal{O}_F$-module over S is a formal group X over S endowed with an $\mathcal{O}_F$-action $\iota\colon \mathcal{O}_F \to \mathrm{End}(X)$ such that the following two actions of $\mathcal{O}_F$ on the Lie algebra $\mathrm{Lie}(X)$ coincide:

– the action induced by ι; and

– that induced by the $\mathcal{O}_S$-module structure of $\mathrm{Lie}(X)$ and the structure homomorphism $\mathcal{O}_F \to \mathcal{O}_{\breve{F}} \to \mathcal{O}_S$.

Fix a one-dimensional formal $\mathcal{O}_F$-module $\mathbb{X}$ of $\mathcal{O}_F$-height n over $\overline{\mathbb{F}}_q = \mathcal{O}_{\breve{F}}/\mathfrak{p}_{\breve{F}}$. Such $\mathbb{X}$ is unique up to isomorphism. Put $D = \mathrm{End}_{\mathcal{O}_F}(\mathbb{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$, which is known to be a central division algebra over F with invariant $1/n$.

Let $\mathcal{M}\colon \mathbf{Nilp} \to \mathbf{Set}$ be the functor that sends S to the set of isomorphism classes of pairs (X, ρ), where X is a formal $\mathcal{O}_F$-module over S and $\rho\colon \phi_{\overline{S}}^{*}\mathbb{X} \to X \times_S \overline{S}$ is an $\mathcal{O}_F$-quasi-isogeny. It is known that $\mathcal{M}$ is represented by a formal scheme over $\mathcal{O}_{\breve{F}}$, which is non-canonically isomorphic to the disjoint union of countable copies of $\mathrm{Spf}\, \mathcal{O}_{\breve{F}}[[T_1, \ldots, T_{n-1}]]$ (see [5, 14, 20]). The group of self-isogenies $\mathbf{QIsog}_{\mathcal{O}_F}(\mathbb{X}) = D^\times$ naturally acts on $\mathcal{M}$ on the right; $h \in D^\times$ sends (X, ρ) to $(X, \rho \circ \phi_{\overline{S}}^{*}h)$. The formal scheme $\mathcal{M}$ is endowed with another structure, called a Weil descent datum. It is an isomorphism $\alpha\colon \mathcal{M} \to \mathcal{M}$ that makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\ \alpha\ } & \mathcal{M} \\ \downarrow & & \downarrow \\ \mathrm{Spf}\, \mathcal{O}_{\breve{F}} & \xrightarrow{\ \sigma^*\ } & \mathrm{Spf}\, \mathcal{O}_{\breve{F}}. \end{array}$$

Here $\sigma\colon \mathcal{O}_{\breve{F}} \to \mathcal{O}_{\breve{F}}$ is induced from the unique element $\sigma \in \mathrm{Gal}(F^{\mathrm{ur}}/F)$ lifting the arithmetic Frobenius automorphism $\overline{\sigma}\colon \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q$, as in § 2.2. In order to describe this isomorphism, it suffices to construct a bijection $\alpha\colon \mathcal{M}(S) \to \mathcal{M}(S^\sigma)$ for each $S \in \mathbf{Nilp}$ compatibly, where S^σ denotes the object $S \xrightarrow{\phi_S} \mathrm{Spec}\, \mathcal{O}_{\breve{F}} \xrightarrow{\sigma^*} \mathrm{Spec}\, \mathcal{O}_{\breve{F}}$ of $\mathbf{Nilp}$. For $(X, \rho) \in \mathcal{M}(S)$, we define $\alpha(X, \rho) = (X, \rho \circ \phi_{\overline{S}}^{*}\mathrm{Frob}_{\mathbb{X}}^{-1})$, where $\mathrm{Frob}_{\mathbb{X}}\colon \mathbb{X} \to (\overline{\sigma}^*)^*\mathbb{X}$ denotes the qth power Frobenius morphism, which is an $\mathcal{O}_F$-isogeny of $\mathcal{O}_F$-height 1.

Next we consider level structures. For $m \geqslant 0$, let $\mathcal{M}_m\colon \mathbf{Nilp} \to \mathbf{Set}$ be the functor that sends S to the set of isomorphism classes of triples (X, ρ, η), where $(X, \rho) \in \mathcal{M}(S)$ and η is a Drinfeld m-level structure on X (for its definition, see [5, §4] and [10, §II.2]). It is represented by a formal scheme finite and flat over $\mathcal{M}$, and $\{\mathcal{M}_m\}_{m \geqslant 0}$ form a projective system called the Lubin–Tate tower. The action of $D^\times$ and the Weil descent datum on $\mathcal{M}$ naturally extend to $\mathcal{M}_m$, and they are compatible with the transition morphisms of the tower. Further, the group $\mathrm{GL}_n(F)$ acts on $\{\mathcal{M}_m\}_{m \geqslant 0}$ on the right as a pro-object (see [23, §2.2] for the definition). This action is called the Hecke action. The principal congruence subgroup $K_m = \mathrm{Ker}(\mathrm{GL}_n(\mathcal{O}_F) \to \mathrm{GL}_n(\mathcal{O}_F/\mathfrak{p}_F^m))$ of $\mathrm{GL}_n(F)$ acts trivially on $\mathcal{M}_m$.

By taking the rigid generic fiber, we obtain a projective system $\{M_m\}_{m \geqslant 0}$ of rigid spaces, whose transition maps are finite and étale. Each M_m is an $n-1$-dimensional smooth rigid space over $\breve{F}$. For a compact open subgroup K of $\mathrm{GL}_n(\mathcal{O}_F)$, we can define the rigid space M_K as the quotient of M_m by K/K_m, where $m \geqslant 0$ is an integer satisfying $K_m \subset K$. It is independent of the choice of m, and M_{K_m} coincides with M_m. These rigid spaces form a

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projective system $\{M_K\}_{K\subset \mathrm{GL}_n(\mathcal{O}_F)}$ with finite étale transition maps. The actions of $D^\times$ and $\mathrm{GL}_n(F)$, and the Weil descent datum naturally extend to it.

For a discrete torsion-free cocompact subgroup Γ of $F^\times$ (e.g., $\varpi^{d\mathbb{Z}}$ for an integer $d \geqslant 1$), we may consider the quotient towers $\{\mathcal{M}_m/\Gamma\}_m$ and $\{M_K/\Gamma\}_K$, where Γ is regarded as a discrete subgroup of $D^\times$ by $F^\times \subset D^\times$. It is known that the actions of $\mathrm{GL}_n(F)$ on these towers are trivial on $\Gamma \subset F^\times \subset \mathrm{GL}_n(F)$ (see [20, Lemma 5.36]).

Now we take a prime number $\ell \neq p$ and consider the ℓ-adic étale cohomology of the Lubin–Tate tower

$$H^i_{\mathrm{LT}/\Gamma,c} = \varinjlim_K H^i_c\big((M_K/\Gamma)\otimes_{\breve{F}} \widehat{\overline{F}}, \overline{\mathbb{Q}}_\ell\big), \qquad H^i_{\mathrm{LT}/\Gamma} = \varinjlim_K H^i\big((M_K/\Gamma)\otimes_{\breve{F}} \widehat{\overline{F}}, \overline{\mathbb{Q}}_\ell\big),$$

where $\widehat{\overline{F}}$ denotes the completion of $\overline{F}$. The groups $\mathrm{GL}_n(F)$ and $D^\times$ act on $H^i_{\mathrm{LT}/\Gamma,c}$ and $H^i_{\mathrm{LT}/\Gamma}$. The actions of $\mathrm{GL}_n(F)$ on both spaces are obviously smooth, and moreover admissible. The action of $D^\times$ on $H^i_{\mathrm{LT}/\Gamma,c}$ is also known to be smooth (see [23, Lemma 2.5.1]). Furthermore, by using the Weil descent datum α, we can define the actions of W_F on $H^i_{\mathrm{LT}/\Gamma,c}$ and $H^i_{\mathrm{LT}/\Gamma}$ as follows. For $w \in W_F$, let $\nu(w)$ denote the integer satisfying $w|_{F^{\mathrm{ur}}} = \sigma^{\nu(w)}$. By taking the fiber product of diagrams

$$\begin{array}{ccc} \mathrm{Spa}(\widehat{\overline{F}}, \mathcal{O}_{\widehat{\overline{F}}}) & \xrightarrow{w^*} & \mathrm{Spa}(\widehat{\overline{F}}, \mathcal{O}_{\widehat{\overline{F}}}) \\ \downarrow & & \downarrow \\ \mathrm{Spa}(\breve{F}, \mathcal{O}_{\breve{F}}) & \xrightarrow{(\sigma^*)^{\nu(w)}} & \mathrm{Spa}(\breve{F}, \mathcal{O}_{\breve{F}}), \end{array} \qquad \begin{array}{ccc} M_K/\Gamma & \xrightarrow{\alpha^{\nu(w)}} & M_K/\Gamma \\ \downarrow & & \downarrow \\ \mathrm{Spa}(\breve{F}, \mathcal{O}_{\breve{F}}) & \xrightarrow{(\sigma^*)^{\nu(w)}} & \mathrm{Spa}(\breve{F}, \mathcal{O}_{\breve{F}}), \end{array}$$

we obtain an isomorphism $\alpha_w\colon (M_K/\Gamma)\otimes_{\breve{F}} \widehat{\overline{F}} \to (M_K/\Gamma)\otimes_{\breve{F}} \widehat{\overline{F}}$ of adic spaces. The action of w is defined to be α_w^*. By these constructions, we obtain two representations $H^i_{\mathrm{LT}/\Gamma,c}$ and $H^i_{\mathrm{LT}/\Gamma}$ of $\mathrm{GL}_n(F)\times D^\times \times W_F$.

Recall that any admissible representation V of $\mathrm{GL}_n(F)/\Gamma$ is decomposed canonically into $V = (\bigoplus_\pi V_\pi)\oplus V_{\text{non-cusp}}$, where

- π runs through irreducible supercuspidal representations of $\mathrm{GL}_n(F)$ whose central characters are trivial on Γ;
- V_π is a direct sum of finitely many copies of π;
- and $V_{\text{non-cusp}}$ has no supercuspidal subquotient

(see [1, 1.11, Variantes c)]). We call V_π the π-isotypic component of V. By definition we have $(V_\pi)^\vee = (V^\vee)_{\pi^\vee}$ and $(V_{\text{non-cusp}})^\vee = (V^\vee)_{\text{non-cusp}}$.

We fix an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ and identify them. Here is a form of the non-abelian Lubin–Tate theory.

Theorem 3.1 [2, 10, 15]. *For an irreducible supercuspidal representation π of $\mathrm{GL}_n(F)$ whose central character is trivial on Γ, we have*

$$H^{n-1}_{\mathrm{LT}/\Gamma,c,\pi}\left(\frac{n-1}{2}\right) = \pi \boxtimes \mathrm{JL}(\pi)^\vee \boxtimes \mathrm{rec}_F(\pi)^\vee$$

as representations of $\mathrm{GL}_n(F)\times D^\times\times W_F$. *Here* $(\frac{n-1}{2})$ *denotes the twist by the character* $W_F\to\mathbb{C}^\times;\, w\mapsto q^{\frac{n-1}{2}\nu(w)}$, *and* $\mathrm{JL}(\pi)$ *denotes the irreducible smooth representation of* $D^\times$ *attached to* π *under the local Jacquet–Langlands correspondence. Unless* $i=n-1$, *we have* $H^i_{\mathrm{LT}/\Gamma,c,\pi}=0$.

The following theorem was obtained in [15], in the course of the proof of the latter part of Theorem 3.1.

Theorem 3.2. *For every integer* i, *the kernel and cokernel of the natural map* $H^i_{\mathrm{LT}/\Gamma,c}\to H^i_{\mathrm{LT}/\Gamma}$ *have no supercuspidal subquotient as representations of* $\mathrm{GL}_n(F)$. *In particular, for every irreducible supercuspidal representation* π *of* $\mathrm{GL}_n(F)$ *whose central character is trivial on* Γ, *the induced map* $H^i_{\mathrm{LT}/\Gamma,c,\pi}\to H^i_{\mathrm{LT}/\Gamma,\pi}$ *is an isomorphism.*

Definition 3.3. For a compact open subgroup K of $\mathrm{GL}_n(\mathcal{O}_F)$, put $\mathrm{Tr}_K=(\mathrm{GL}_n(\mathcal{O}_F):K)^{-1}\,\mathrm{Tr}_{M_K}$, where Tr_{M_K} denotes the trace map $H^{2(n-1)}_c((M_K/\Gamma)\otimes_{\breve F}\widehat{\overline{F}},\overline{\mathbb{Q}}_\ell)(n-1)\to\overline{\mathbb{Q}}_\ell$. It is easy to see that Tr_K is compatible with the change of K. We write Tr for the homomorphism $H^{2(n-1)}_{\mathrm{LT}/\Gamma,c}(n-1)\to\overline{\mathbb{Q}}_\ell$ induced from $\{\mathrm{Tr}_K\}_K$.

Proposition 3.4. *Let* π *be an irreducible supercuspidal representation of* $\mathrm{GL}_n(F)$ *whose central character is trivial on* Γ. *Then, the cup product pairing*

$$\mathrm{Tr}(-\cup-)\colon H^{n-1}_{\mathrm{LT}/\Gamma,c}\left(\frac{n-1}{2}\right)\times H^{n-1}_{\mathrm{LT}/\Gamma,c}\left(\frac{n-1}{2}\right)\to\overline{\mathbb{Q}}_\ell$$

induces a $D^\times\times W_F$*-invariant pairing* $H^{n-1}_{\mathrm{LT}/\Gamma,c,\pi^\vee}(\frac{n-1}{2})\times H^{n-1}_{\mathrm{LT}/\Gamma,c,\pi}(\frac{n-1}{2})\to\overline{\mathbb{Q}}_\ell$ *satisfying the following condition:*

> *for every compact open subgroup* K *of* $\mathrm{GL}_n(F)$, *the restriction of it to* $(H^{n-1}_{\mathrm{LT}/\Gamma,c,\pi^\vee})^K(\frac{n-1}{2})\times(H^{n-1}_{\mathrm{LT}/\Gamma,c,\pi})^K(\frac{n-1}{2})$ *is a perfect pairing.*

Proof. First, by the Poincaré duality for M_K/Γ, we know that the cup product pairing $(H^{n-1}_{\mathrm{LT}/\Gamma,c})^K(\frac{n-1}{2})\times(H^{n-1}_{\mathrm{LT}/\Gamma})^K(\frac{n-1}{2})\to\overline{\mathbb{Q}}_\ell$ is perfect for every compact open subgroup K of $\mathrm{GL}_n(\mathcal{O}_F)$. This tells us that the induced map

$$H^{n-1}_{\mathrm{LT}/\Gamma}\left(\frac{n-1}{2}\right)\to\left(H^{n-1}_{\mathrm{LT}/\Gamma,c}\left(\frac{n-1}{2}\right)\right)^\vee$$

is an isomorphism. By taking π-isotypic parts and composing with the isomorphism in Theorem 3.2, we obtain an isomorphism

$$H^{n-1}_{\mathrm{LT}/\Gamma,c,\pi}\left(\frac{n-1}{2}\right)\xrightarrow{\cong}H^{n-1}_{\mathrm{LT}/\Gamma,\pi}\left(\frac{n-1}{2}\right)\xrightarrow{\cong}\left(H^{n-1}_{\mathrm{LT}/\Gamma,c,\pi^\vee}\left(\frac{n-1}{2}\right)\right)^\vee.$$

Therefore, for every compact open subgroup K of $\mathrm{GL}_n(F)$, we have an isomorphism

$$(H^{n-1}_{\mathrm{LT}/\Gamma,c,\pi})^K\left(\frac{n-1}{2}\right)\xrightarrow{\cong}\left((H^{n-1}_{\mathrm{LT}/\Gamma,c,\pi^\vee})^K\left(\frac{n-1}{2}\right)\right)^\vee.$$

It is easy to see that this isomorphism is induced from the restriction of the cup product pairing to $(H^{n-1}_{\mathrm{LT}/\Gamma,c,\pi^\vee})^K(\frac{n-1}{2})\times(H^{n-1}_{\mathrm{LT}/\Gamma,c,\pi})^K(\frac{n-1}{2})$. This concludes the proof. □

3.2. Twisting operator

Here we use the notation introduced in the beginning of § 2.2. We construct the twisting operator $\theta\colon \mathcal{M}_m \to \mathcal{M}_m$.

First we consider the case where F/F^+ is an unramified quadratic extension. In this case we have $F^{\mathrm{ur}} = (F^+)^{\mathrm{ur}}$. We write τ for the unique element of $\mathrm{Gal}(F^{\mathrm{ur}}/F^+)$ lifting the q'th power Frobenius automorphism $\overline{\tau}$ on $\overline{\mathbb{F}}_q = \overline{\mathbb{F}}_{q'}$. It extends $\tau \in \mathrm{Gal}(F/F^+)$, and satisfies $\tau^2 = \sigma$. For an object S of **Nilp**, we write S^τ for the object $S \to \mathrm{Spec}\,\mathcal{O}_{\breve{F}} \xrightarrow{\tau^*} \mathrm{Spec}\,\mathcal{O}_{\breve{F}}$ of **Nilp**.

We write $\overline{\tau}_*\mathbb{X}$ for the pull-back of the formal $\mathcal{O}_F$-module $\mathbb{X}$ by $\overline{\tau}^*\colon \mathrm{Spec}\,\overline{\mathbb{F}}_q \to \mathrm{Spec}\,\overline{\mathbb{F}}_q$. On the other hand, we denote by $\mathbb{X}^\tau$ the formal group $\mathbb{X}$ endowed with the $\mathcal{O}_F$-action twisted by τ (that is, $\mathcal{O}_F \xrightarrow{\tau} \mathcal{O}_F \to \mathrm{End}(\mathbb{X})$). It is easy to see that $\overline{\tau}_*\mathbb{X}$ and $\mathbb{X}^\tau$ are one-dimensional formal $\mathcal{O}_F$-modules of $\mathcal{O}_F$-height n over $(\mathrm{Spec}\,\overline{\mathbb{F}}_q)^\tau \in$ **Nilp**. Hence these are isomorphic as formal $\mathcal{O}_F$-modules. We fix an isomorphism $\iota\colon \overline{\tau}_*\mathbb{X} \xrightarrow{\cong} \mathbb{X}^\tau$ between them. This isomorphism induces an automorphism on D:

Definition 3.5. (i) An element $h \in D = \mathrm{End}_{\mathcal{O}_F}(\mathbb{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ determines an element $\tau(h) = \iota \circ \overline{\tau}_* h \circ \iota^{-1} \in \mathrm{End}_{\mathcal{O}_F}(\mathbb{X}^\tau) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{End}_{\mathcal{O}_F}(\mathbb{X}) \otimes_{\mathbb{Z}} \mathbb{Q} = D$. This gives an isomorphism $\tau\colon D \to D$ such that $\tau|_F = \tau \in \mathrm{Gal}(F/F^+)$.

(ii) We denote the composite of $\mathcal{O}_F$-isogenies

$$\mathbb{X} \xrightarrow{\mathrm{Frob}_{\mathbb{X}}} \overline{\sigma}_*\mathbb{X} = \overline{\tau}_*^2\mathbb{X} \xrightarrow{\overline{\tau}_*\iota} \overline{\tau}_*\mathbb{X}^\tau \xrightarrow{\iota} \mathbb{X}$$

by t. It is an element of $D^\times$.

Lemma 3.6. *The element $t \in D^\times$ satisfies $\tau^2 = \mathrm{Int}(t)$ and $\tau(t) = t$.*

Proof. For $h \in D$, we have

$$\begin{aligned}\tau^2(h) &= \iota \circ \overline{\tau}_*(\iota \circ \overline{\tau}_* h \circ \iota^{-1}) \circ \iota^{-1} = (\iota \circ \overline{\tau}_*\iota) \circ \overline{\sigma}_* h \circ (\iota \circ \overline{\tau}_*\iota)^{-1}\\ &= (t \circ \mathrm{Frob}_{\mathbb{X}}^{-1}) \circ \overline{\sigma}_* h \circ (t \circ \mathrm{Frob}_{\mathbb{X}}^{-1})^{-1}\\ &= t \circ (\mathrm{Frob}_{\mathbb{X}}^{-1} \circ \overline{\sigma}_* h \circ \mathrm{Frob}_{\mathbb{X}}) \circ t^{-1}.\end{aligned}$$

By the functoriality of the relative Frobenius morphisms, the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\mathrm{Frob}_{\mathbb{X}}} & \overline{\sigma}_*\mathbb{X} \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \overline{\sigma}_* h} \\ \mathbb{X} & \xrightarrow{\mathrm{Frob}_{\mathbb{X}}} & \overline{\sigma}_*\mathbb{X}. \end{array}$$

Hence we have $\tau^2(h) = t \circ h \circ t^{-1}$, as desired.

Next consider $\tau(t)$. We have

$$\begin{aligned}\tau(t) &= \iota \circ \overline{\tau}_*(\iota \circ \overline{\tau}_*\iota \circ \mathrm{Frob}_{\mathbb{X}}) \circ \iota^{-1} = \iota \circ \overline{\tau}_*\iota \circ \overline{\sigma}_*\iota \circ \overline{\tau}_*\,\mathrm{Frob}_{\mathbb{X}} \circ \iota^{-1}\\ &= \iota \circ \overline{\tau}_*\iota \circ \overline{\sigma}_*\iota \circ \mathrm{Frob}_{\overline{\tau}_*\mathbb{X}} \circ \iota^{-1} \overset{(*)}{=} \iota \circ \overline{\tau}_*\iota \circ \mathrm{Frob}_{\mathbb{X}} = t.\end{aligned}$$

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The equality $(*)$ follows from the functoriality of the relative Frobenius morphisms with respect to $\iota\colon \overline{\tau}_*\mathbb{X} \to \mathbb{X}$. This completes the proof. $\square$

Now we construct an isomorphism $\theta\colon \mathcal{M}_m \to \mathcal{M}_m$ that makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{M}_m & \xrightarrow{\theta} & \mathcal{M}_m \\ \downarrow & & \downarrow \\ \operatorname{Spf} \mathcal{O}_{\breve{F}} & \xrightarrow{\tau^*} & \operatorname{Spf} \mathcal{O}_{\breve{F}}. \end{array}$$

Definition 3.7. Let S be an object of **Nilp**. For $(X, \rho, \eta) \in \mathcal{M}_m(S)$, we put $\theta(X, \rho, \eta) = (X^\tau, \rho \circ \phi_{\overline{S}}^* \iota, \eta^\tau) \in \mathcal{M}_m(S^\tau)$, where

- X^τ is the formal group X over S endowed with the $\mathcal{O}_F$-action twisted by τ;
- $\rho \circ \phi_{\overline{S}}^* \iota$ is the $\mathcal{O}_F$-quasi-isogeny $\phi_{\overline{S^\tau}}^* \mathbb{X} = \phi_{\overline{S}}^*(\overline{\tau}_* \mathbb{X}) \xrightarrow{\phi_{\overline{S}}^* \iota} \phi_{\overline{S}}^* \mathbb{X}^\tau \xrightarrow{\rho} X^\tau \times_S \overline{S}$;
- and η^τ is the composite of $(\mathcal{O}_F/\mathfrak{p}_F^m)^n \xrightarrow{\tau} (\mathcal{O}_F/\mathfrak{p}_F^m)^n$ and η.

This gives a bijection $\theta\colon \mathcal{M}_m(S) \xrightarrow{\cong} \mathcal{M}_m(S^\tau)$, and an isomorphism $\theta\colon \mathcal{M}_m \xrightarrow{\cong} \mathcal{M}_m$ which covers $\tau^*\colon \operatorname{Spf} \mathcal{O}_{\breve{F}} \to \operatorname{Spf} \mathcal{O}_{\breve{F}}$.

The isomorphism θ is compatible with the transition maps of the tower $\{\mathcal{M}_m\}$. Hence it induces automorphisms of the towers $\{\mathcal{M}_m\}$ and $\{M_m\}$.

Lemma 3.8. (i) *For $g \in \mathrm{GL}_n(F)$, we have $g \circ \theta = \theta \circ \tau(g)$, where $\tau\colon \mathrm{GL}_n(F) \to \mathrm{GL}_n(F)$ is the isomorphism in Example 2.7.*

(ii) *For $h \in D^\times$, we have $h \circ \theta = \theta \circ \tau(h)$, where $\tau\colon D^\times \to D^\times$ is the isomorphism in Definition 3.5(i).*

(iii) *We have $\theta^2 = \alpha \circ t$ and $\alpha \circ \theta = \theta \circ \alpha$.*

Proof. The claim (i) is clear from the definition of θ.

As for (ii), take $(X, \rho, \eta) \in \mathcal{M}_m(S)$. Then we have

$$(h \circ \theta)(X, \rho, \eta) = (X^\tau, \rho \circ \phi_{\overline{S}}^* \iota \circ \phi_{\overline{S^\tau}}^* h, \eta^\tau).$$

Since $\phi_{\overline{S}}^* \iota \circ \phi_{\overline{S^\tau}}^* h = \phi_{\overline{S}}^*(\iota \circ \overline{\tau}_* h) = \phi_{\overline{S}}^*(\tau(h) \circ \iota) = \phi_{\overline{S}}^*(\tau(h)) \circ \phi_{\overline{S}}^* \iota$, we have

$$(h \circ \theta)(X, \rho, \eta) = (X^\tau, \rho \circ \phi_{\overline{S}}^*(\tau(h)) \circ \phi_{\overline{S}}^* \iota, \eta^\tau) = (\theta \circ \tau(h))(X, \rho, \eta).$$

Thus $h \circ \theta = \theta \circ \tau(h)$, as desired.

We prove (iii). For $(X, \rho, \eta) \in \mathcal{M}_m(S)$, $\theta^2(X, \rho, \eta)$ equals $(X, \rho \circ \phi_{\overline{S}}^* \iota \circ \phi_{\overline{S^\tau}}^* \iota, \eta)$. Since $\phi_{\overline{S}}^* \iota \circ \phi_{\overline{S^\tau}}^* \iota = \phi_{\overline{S}}^*(\iota \circ \overline{\tau}_* \iota) = \phi_{\overline{S}}^*(t \circ \mathrm{Frob}_{\mathbb{X}}^{-1}) = \phi_{\overline{S}}^*(t) \circ \phi_{\overline{S}}^* \mathrm{Frob}_{\mathbb{X}}^{-1}$, we have $\theta^2(X, \rho, \eta) = \alpha(t(X, \rho, \eta))$. Hence $\theta^2 = \alpha \circ t$. Finally, by (ii) and Lemma 3.6 we conclude that

$$\alpha \circ \theta = \theta^2 \circ t^{-1} \circ \theta = \theta^3 \circ \tau(t)^{-1} = \theta^3 \circ t^{-1} = \theta \circ \alpha. \qquad \square$$

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We fix $c \in W_{F^+}$ such that $c|_{F^{\mathrm{ur}}} = \tau$. Assume that the subgroup $\Gamma \subset F^\times$ is stable under τ. Then, $\theta \colon M_m/\Gamma \to M_m/\Gamma$ is induced. By taking the fiber product of diagrams

$$\begin{array}{ccc} \mathrm{Spa}(\widehat{\overline{F}}, \mathcal{O}_{\widehat{\overline{F}}}) & \xrightarrow{c^*} & \mathrm{Spa}(\widehat{\overline{F}}, \mathcal{O}_{\widehat{\overline{F}}}) \\ \downarrow & & \downarrow \\ \mathrm{Spa}(\breve{F}, \mathcal{O}_{\breve{F}}) & \xrightarrow{\tau^*} & \mathrm{Spa}(\breve{F}, \mathcal{O}_{\breve{F}}), \end{array} \qquad \begin{array}{ccc} M_m/\Gamma & \xrightarrow{\theta} & M_m/\Gamma \\ \downarrow & & \downarrow \\ \mathrm{Spa}(\breve{F}, \mathcal{O}_{\breve{F}}) & \xrightarrow{\tau^*} & \mathrm{Spa}(\breve{F}, \mathcal{O}_{\breve{F}}), \end{array}$$

we obtain an isomorphism $\theta_c \colon (M_m/\Gamma) \otimes_{\breve{F}} \widehat{\overline{F}} \to (M_m/\Gamma) \otimes_{\breve{F}} \widehat{\overline{F}}$ of adic spaces. It induces an automorphism θ_c^* on the cohomology $H^i_{\mathrm{LT}/\Gamma,c}$, for which we simply write θ.

Corollary 3.9. *The following equalities of automorphisms on $H^i_{\mathrm{LT}/\Gamma,c}$ hold.*

(i) *For $g \in \mathrm{GL}_n(F)$, we have $\theta \circ g = \tau(g) \circ \theta$, where $\tau \colon \mathrm{GL}_n(F) \to \mathrm{GL}_n(F)$ is the isomorphism in Example 2.7.*

(ii) *For $h \in D^\times$, we have $\theta \circ h = \tau(h) \circ \theta$, where $\tau \colon D^\times \to D^\times$ is the isomorphism in Definition 3.5(i).*

(iii) *We have $\theta^2 = t \circ c^2$ and $\theta \circ w = cwc^{-1} \circ \theta$ for every $w \in W_F$.*

Proof. The claims (i) and (ii) follow from Lemma 3.8(i), (ii), respectively. For (iii), it suffices to show $\theta_c^2 = \alpha_{c^2} \circ t$ and $\alpha_w \circ \theta_c = \theta_c \circ \alpha_{cwc^{-1}}$. These are consequences of Lemma 3.8(iii), the definitions of α_w and θ_c, and the equality $\nu(cwc^{-1}) = \nu(w)$. □

Next we consider the case where F/F^+ is a ramified quadratic extension (here we do not need the tameness assumption). We also write τ for the unique non-trivial element of $\mathrm{Gal}(F^{\mathrm{ur}}/(F^+)^{\mathrm{ur}})$. It gives an extension of the original $\tau \in \mathrm{Gal}(F/F^+)$. Note that $\tau^2 = 1$ and $\overline{\tau} = 1$, where $\overline{\tau}$ denotes the automorphism of the residue field $\overline{\mathbb{F}}_q$ of $\mathcal{O}_{F^{\mathrm{ur}}}$ induced by τ. Further, we have $\sigma \circ \tau = \tau \circ \sigma$ as automorphisms of F^{ur}. For an object S of **Nilp**, we write S^τ for the object $S \to \mathrm{Spec}\, \mathcal{O}_{\breve{F}} \xrightarrow{\tau^*} \mathrm{Spec}\, \mathcal{O}_{\breve{F}}$ of **Nilp**.

As in the unramified case, we fix an isomorphism $\iota \colon \mathbb{X} \xrightarrow{\cong} \mathbb{X}^\tau$ between formal $\mathcal{O}_F$-modules over $(\mathrm{Spec}\, \overline{\mathbb{F}}_q)^\tau = \mathrm{Spec}\, \overline{\mathbb{F}}_q \in$ **Nilp**.

Definition 3.10. (i) An element $h \in D = \mathrm{End}_{\mathcal{O}_F}(\mathbb{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ determines an element $\tau(h) = \iota \circ h \circ \iota^{-1} \in \mathrm{End}_{\mathcal{O}_F}(\mathbb{X}^\tau) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{End}_{\mathcal{O}_F}(\mathbb{X}) \otimes_{\mathbb{Z}} \mathbb{Q} = D$. This gives an isomorphism $\tau \colon D \to D$ such that $\tau|_F = \tau \in \mathrm{Gal}(F/F^+)$.

(ii) We denote the composite $\mathbb{X} \xrightarrow{\iota} \mathbb{X}^\tau \xrightarrow{\iota} \mathbb{X}$ by t. It is an element of $D^\times$.

Lemma 3.11. *The element $t \in D^\times$ satisfies $\tau^2 = \mathrm{Int}(t)$ and $\tau(t) = t$.*

Proof. Clear from definition. □

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Exactly in the same way, we can construct an isomorphism $\theta\colon \mathcal{M}_m \to \mathcal{M}_m$ that makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{M}_m & \xrightarrow{\theta} & \mathcal{M}_m \\ \downarrow & & \downarrow \\ \operatorname{Spf} \mathcal{O}_{\breve{F}} & \xrightarrow{\tau^*} & \operatorname{Spf} \mathcal{O}_{\breve{F}}. \end{array}$$

It induces automorphisms of the towers $\{\mathcal{M}_m\}$ and $\{M_m\}$.

Lemma 3.12. (i) *For $g \in \mathrm{GL}_n(F)$, we have $g \circ \theta = \theta \circ \tau(g)$, where $\tau\colon \mathrm{GL}_n(F) \to \mathrm{GL}_n(F)$ is the isomorphism in Example 2.7.*

(ii) *For $h \in D^\times$, we have $h \circ \theta = \theta \circ \tau(h)$, where $\tau\colon D^\times \to D^\times$ is the isomorphism in Definition 3.10(i).*

(iii) *We have $\theta^2 = t$ and $\alpha \circ \theta = \theta \circ \alpha$.*

Proof. As in the proof of Lemma 3.8, it suffices to show $\theta^2 = t$. For an object S of **Nilp** and $(X, \rho, \eta) \in \mathcal{M}_m(S)$, we have

$$\theta^2(X, \rho, \eta) = (X, \rho \circ \phi_{\overline{S}}^{*}\iota \circ \phi_{\overline{S}}^{*}\iota, \eta) = (X, \rho \circ \phi_{\overline{S}}^{*} t, \eta) = t(X, \rho, \eta),$$

as desired (note that $\overline{S^\tau} = \overline{S}$). □

We fix $c \in W_{F^+}$ such that $c|_{F^{\mathrm{ur}}} = \tau$. Assume that $\Gamma \subset F^\times$ is stable under τ. As in the unramified case, we obtain an isomorphism $\theta_c\colon (M_m/\Gamma) \otimes_{\breve{F}} \widehat{\overline{F}} \to (M_m/\Gamma) \otimes_{\breve{F}} \widehat{\overline{F}}$ of adic spaces. It induces an automorphism θ_c^* on the cohomology $H^i_{\mathrm{LT}/\Gamma,c}$, for which we simply write θ.

Corollary 3.13. *The following equalities of automorphisms on $H^i_{\mathrm{LT}/\Gamma,c}$ hold.*

(i) *For $g \in \mathrm{GL}_n(F)$, we have $\theta \circ g = \tau(g) \circ \theta$, where $\tau\colon \mathrm{GL}_n(F) \to \mathrm{GL}_n(F)$ is the isomorphism in Example 2.7.*

(ii) *For $h \in D^\times$, we have $\theta \circ h = \tau(h) \circ \theta$, where $\tau\colon D^\times \to D^\times$ is the isomorphism in Definition 3.10(i).*

(iii) *We have $\theta^2 = t \circ c^2$ and $\theta \circ w = cwc^{-1} \circ \theta$ for every $w \in W_F$.*

Proof. Similar as Corollary 3.9. □

Finally, consider the case $F = F^+$.

Definition 3.14. We put $\tau = \mathrm{id}_{D^\times}$, $t = 1 \in D^\times$, $c = 1 \in W_{F^+}$ and $\theta = \mathrm{id}$ on $H^i_{\mathrm{LT}/\Gamma,c}$. Then the same statements as in Corollaries 3.9, 3.13 obviously hold.

Now we return to a general separable extension F/F^+ with $[F : F^+] \leqslant 2$.

Lemma 3.15. *Assume that* $\Gamma \subset F^\times$ *is stable under* τ*. The cup product pairing*

$$\mathrm{Tr}(- \cup -)\colon H_{\mathrm{LT}/\Gamma,c}\left(\frac{n-1}{2}\right) \times H_{\mathrm{LT}/\Gamma,c}\left(\frac{n-1}{2}\right) \to \overline{\mathbb{Q}}_\ell$$

in Proposition 3.4 satisfies $\mathrm{Tr}(\theta x, \theta y) = q^{-\frac{n-1}{2}\nu(c^2)} \mathrm{Tr}(x \cup y)$.

Proof. Recall that the isomorphism $\theta_c\colon (M_m/\Gamma) \otimes_{\breve{F}} \widehat{\overline{F}} \to (M_m/\Gamma) \otimes_{\breve{F}} \widehat{\overline{F}}$ covers $c^*\colon \mathrm{Spa}(\widehat{\overline{F}}, \mathcal{O}_{\widehat{\overline{F}}}) \to \mathrm{Spa}(\widehat{\overline{F}}, \mathcal{O}_{\widehat{\overline{F}}})$.

If F/F^+ is an unramified quadratic extension, c induces the q'th power map on $\mu_{\ell^k}(\widehat{\overline{F}}) = \mu_{\ell^k}(\breve{F}^+)$. Therefore, we have $\mathrm{Tr}(\theta x, \theta y) = q'^{-(n-1)} \mathrm{Tr}(x \cup y)$. Since $q = q'^2$ and $\nu(c^2) = 1$, this equals $q^{-\frac{n-1}{2}\nu(c^2)} \mathrm{Tr}(x \cup y)$.

Otherwise c acts trivially on $\mu_{\ell^k}(\widehat{\overline{F}}) = \mu_{\ell^k}(\breve{F}^+)$, and $\nu(c^2) = 0$. Hence we have $\mathrm{Tr}(\theta x, \theta y) = \mathrm{Tr}(x \cup y) = q^{-\frac{n-1}{2}\nu(c^2)} \mathrm{Tr}(x \cup y)$. □

Theorem 3.16. *Here we consider* (τ, t) *as in Definitions 3.5, 3.10, 3.14. Let* π *be an irreducible supercuspidal representation of* $\mathrm{GL}_n(F)$ *which is conjugate self-dual under the setting in Example 2.7. Then,* $\mathrm{JL}(\pi)$ *is conjugate self-dual with respect to* τ*. Moreover, we have*

$$C_{\mathrm{rec}_F(\pi)} = (-1)^{n-1} C_{\mathrm{JL}(\pi)},$$

where $C_{\mathrm{JL}(\pi)}$ *denotes the parity of* $\mathrm{JL}(\pi)$ *with respect to* (τ, t).

Proof. Since π is conjugate self-dual, its central character ω_π satisfies $\omega_\pi(\tau(z)) = \omega_\pi(z)^{-1}$ for every $z \in F^\times \subset \mathrm{GL}_n(F)$. Hence, for a uniformizer ϖ' of F^+, we have $\omega_\pi(\varpi'^2) = 1$. Put $\Gamma = \varpi'^{2\mathbb{Z}} \subset (F^+)^\times \subset F^\times$. It is a τ-stable discrete cocompact subgroup of $F^\times$ on which ω_π is trivial.

Let $\tau\colon \mathrm{GL}_n(F) \to \mathrm{GL}_n(F)$ be as in Example 2.7, $\tau = \mathrm{Int}(c)\colon W_F \to W_F$ as in Example 2.8, and $\tau = (\tau, \tau, \tau)\colon \mathrm{GL}_n(F) \times D^\times \times W_F \to \mathrm{GL}_n(F) \times D^\times \times W_F$. Then, Corollaries 3.9, 3.13 tell us that θ gives an isomorphism $H^{n-1}_{\mathrm{LT}/\Gamma,c} \xrightarrow{\cong} (H^{n-1}_{\mathrm{LT}/\Gamma,c})^\tau$. Since the character $W_F \to \mathbb{C}^\times$; $w \mapsto q^{\frac{n-1}{2}\nu(w)}$ is τ-invariant, we have $H^{n-1}_{\mathrm{LT}/\Gamma,c}(\frac{n-1}{2}) \xrightarrow{\cong} (H^{n-1}_{\mathrm{LT}/\Gamma,c}(\frac{n-1}{2}))^\tau$ by twisting. By taking $\pi^\vee$-isotypic parts and using $\pi^\tau = \pi^\vee$, we obtain an isomorphism $\theta\colon H^{n-1}_{\mathrm{LT}/\Gamma,c,\pi^\vee}(\frac{n-1}{2}) \xrightarrow{\cong} (H^{n-1}_{\mathrm{LT}/\Gamma,c,\pi}(\frac{n-1}{2}))^\tau$ of representations of $\mathrm{GL}_n(F) \times D^\times \times W_F$.

Take a τ-stable compact open subgroup K of $\mathrm{GL}_n(F)$. Then, θ induces an isomorphism $(H^{n-1}_{\mathrm{LT}/\Gamma,c,\pi^\vee}(\frac{n-1}{2}))^K \xrightarrow{\cong} ((H^{n-1}_{\mathrm{LT}/\Gamma,c,\pi}(\frac{n-1}{2}))^K)^\tau$ of representations of $D^\times \times W_F$. Consider the pairing

$$\begin{aligned}\langle\, ,\rangle\colon &\left(H^{n-1}_{\mathrm{LT}/\Gamma,c,\pi}\left(\frac{n-1}{2}\right)\right)^K \times \left(H^{n-1}_{\mathrm{LT}/\Gamma,c,\pi}\left(\frac{n-1}{2}\right)\right)^K \\ &\xrightarrow[\cong]{\theta^{-1}\times\mathrm{id}} \left(H^{n-1}_{\mathrm{LT}/\Gamma,c,\pi^\vee}\left(\frac{n-1}{2}\right)\right)^K \times \left(H^{n-1}_{\mathrm{LT}/\Gamma,c,\pi}\left(\frac{n-1}{2}\right)\right)^K \xrightarrow{\mathrm{Tr}(-\cup-)} \overline{\mathbb{Q}}_\ell.\end{aligned}$$

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It satisfies $\langle(\tau(h),\tau(w))x,(h,w)y\rangle=\langle x,y\rangle$ for every $h\in D^\times$ and $w\in W_F$. Moreover, Proposition 3.4 tells us that it is a perfect pairing. We have

$$\begin{aligned}\langle y,x\rangle&=\operatorname{Tr}(\theta^{-1}(y)\cup x)=(-1)^{n-1}\operatorname{Tr}(x\cup\theta^{-1}(y))\overset{(1)}{=}(-1)^{n-1}q^{\frac{n-1}{2}\nu(c^2)}\operatorname{Tr}(\theta(x)\cup y)\\&=(-1)^{n-1}q^{\frac{n-1}{2}\nu(c^2)}\langle\theta^2(x),y\rangle\overset{(2)}{=}(-1)^{n-1}q^{\frac{n-1}{2}\nu(c^2)}\langle q^{-\frac{n-1}{2}\nu(c^2)}(t,c^2)(x),y\rangle\\&=(-1)^{n-1}\langle(t,c^2)x,y\rangle.\end{aligned}$$

Here (1) follows from Lemma 3.15, and (2) from the identity $\theta^2=t\circ c^2$ on $H^{n-1}_{\mathrm{LT}/\Gamma,c,\pi}$ (Corollary 3.9(iii) and Corollary 3.13(iii)); the factor $q^{-\frac{n-1}{2}\nu(c^2)}$ arises from the twist $(\frac{n-1}{2})$.

Now we specify K. Since π is supercuspidal, it is generic. Hence by [12, §5, Théorème], there exists an integer $m\geqslant 0$ such that $\dim\pi^{K_1(m)}=1$. Here $K_1(m)$ is the subgroup of $\mathrm{GL}_n(\mathcal{O}_F)$ consisting of matrices (g_{ij}) with $g_{n,1},\dots,g_{n,n-1}\in\mathfrak{p}_F^m$ and $g_{n,n}\in 1+\mathfrak{p}_F^m$. Clearly $K_1(m)$ is τ-stable. We take K as $K_1(m)$. Then, Theorem 3.1 tells us that $(H^{n-1}_{\mathrm{LT}/\Gamma,c,\pi}(\frac{n-1}{2}))^K\cong\mathrm{JL}(\pi)^\vee\boxtimes\mathrm{rec}_F(\pi)^\vee$ as representations of $D^\times\times W_F$. Since $(\pi^\vee)^K=(\pi^K)^\vee$ is also one-dimensional, the existence of $\theta\colon(H^{n-1}_{\mathrm{LT}/\Gamma,c,\pi^\vee}(\frac{n-1}{2}))^K\xrightarrow{\cong}((H^{n-1}_{\mathrm{LT}/\Gamma,c,\pi}(\frac{n-1}{2}))^K)^\tau$ tells us that

$$\mathrm{JL}(\pi)\boxtimes\mathrm{rec}_F(\pi)=\mathrm{JL}(\pi^\vee)^\vee\boxtimes\mathrm{rec}_F(\pi^\vee)^\vee\cong\mathrm{JL}(\pi)^{\vee\tau}\boxtimes\mathrm{rec}_F(\pi)^{\vee\tau}.$$

Thus $\mathrm{JL}(\pi)$ is conjugate self-dual with respect to τ. Finally, by the existence of the pairing $\langle\,,\rangle$, we conclude that the parity of the irreducible representation $\mathrm{JL}(\pi)^\vee\boxtimes\mathrm{rec}_F(\pi)^\vee$ of $D^\times\times W_F$ with respect to $(\tau\times\tau,(t,c^2))$ is equal to $(-1)^{n-1}$. Replacing π by $\pi^\vee$, we get the same result for $\mathrm{JL}(\pi)\boxtimes\mathrm{rec}_F(\pi)$. Therefore, by Lemma 2.4 we have $C_{\mathrm{JL}(\pi)}C_{\mathrm{rec}_F(\pi)}=(-1)^{n-1}$, and $C_{\mathrm{rec}_F(\pi)}=(-1)^{n-1}C_{\mathrm{JL}(\pi)}$. This completes the proof. □

3.3. Formal $\mathcal{O}_F$-module over $\overline{\mathbb{F}}_q$ and division algebra

Our remaining task for proving Theorem 2.12 is to describe (τ,t) in Definitions 3.5 and 3.10 explicitly, under the assumption that F/F^+ is at worst tamely ramified and quadratic.

First we consider the easier case where F has equal characteristic. In this case, we have $F=\mathbb{F}_q((\varpi))$. We can take a one-dimensional formal $\mathcal{O}_F$-module $\mathbb{X}$ over $\overline{\mathbb{F}}_q$ as follows:

$$\mathbb{X}=\widehat{\mathbb{G}}_a\text{ as a formal group},\quad[a]_{\mathbb{X}}(X)=aX\ (a\in\mathbb{F}_q),\quad[\varpi]_{\mathbb{X}}(X)=X^{q^n}.$$

Any element $a\in\mathbb{F}_{q^n}$ gives an endomorphism $X\mapsto aX$ of $\mathbb{X}$. On the other hand, we write Π for the endomorphism $X\mapsto X^q$ of $\mathbb{X}$. Note that $\Pi a=a^q\Pi$ for $a\in\mathbb{F}_{q^n}$ and $\Pi^n=\varpi$ in $\operatorname{End}_{\mathcal{O}_F}(\mathbb{X})$. These elements are known to generate $\operatorname{End}_{\mathcal{O}_F}(\mathbb{X})$, and we have $\operatorname{End}_{\mathcal{O}_F}(\mathbb{X})=\mathbb{F}_{q^n}[\Pi]=\mathcal{O}_{F_n}[\Pi]$, which is a maximal order of the central division algebra over F with invariant $1/n$.

Assume that F/F^+ is an unramified quadratic extension. We may assume that $F^+=\mathbb{F}_{q'}((\varpi))$. Then, $\overline{\tau}_*\mathbb{X}$ and $\mathbb{X}^\tau$ are described explicitly as follows:

$$[a]_{\overline{\tau}_*\mathbb{X}}(X)=\overline{\tau}(aX)=a^{q'}X\ (a\in\mathbb{F}_q),\quad[\varpi]_{\overline{\tau}_*\mathbb{X}}(X)=\overline{\tau}(X^{q^n})=X^{q^n},$$

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$$[a]_{\mathbb{X}^\tau}(X) = [\overline{\tau}(a)]_{\mathbb{X}}(X) = a^{q'}X \ (a \in \mathbb{F}_q), \quad [\varpi]_{\mathbb{X}^\tau}(X) = X^{q^n}.$$

Hence we may take $\iota = \mathrm{id}_{\mathbb{X}} \colon \overline{\tau}_*\mathbb{X} \xrightarrow{\cong} \mathbb{X}^\tau$. The following lemma is immediate.

Proposition 3.17. *The pair (τ, t) constructed from $\iota = \mathrm{id}_{\mathbb{X}}$ as in Definition 3.5 coincides with that in Definition 2.10(i).*

Next assume that $p \neq 2$ and F/F^+ is a ramified quadratic extension. We may assume that $F^+ = \mathbb{F}_q((\varpi^2))$. Then $\mathbb{X}^\tau$ is described as follows:

$$[a]_{\mathbb{X}^\tau}(X) = aX \ (a \in \mathbb{F}_q), \quad [\varpi]_{\mathbb{X}^\tau}(X) = [-\varpi]_{\mathbb{X}}(X) = -X^{q^n}.$$

Take $\beta \in \overline{\mathbb{F}}_q$ such that $\beta^{q^n-1} = -1$, and put $\alpha = \beta^{q-1}$. Then, we may take an isomorphism $\iota \colon \mathbb{X} \xrightarrow{\cong} \mathbb{X}^\tau$; $X \mapsto \beta^{-1}X$.

Proposition 3.18. *The pair (τ, t) constructed from $\iota \colon X \mapsto \beta^{-1}X$ as in Definition 3.10 coincides with that in Definition 2.10(ii).*

Proof. For $a \in \mathbb{F}_{q^n} \subset \mathcal{O}_{F_n}[\Pi]$, we have $\tau(a) \colon X \mapsto \beta^{-1}a\beta X = aX$; that is, $\tau(a) = a$. On the other hand, we have $\tau(\Pi) \colon X \mapsto \beta^{-1}(\beta X)^q = \alpha X^q$, and thus $\tau(\Pi) = \alpha\Pi$. Clearly we have $t = \beta^{-2}$. Hence the pair (τ, t) coincides with that in Definition 2.10(ii). □

Now we consider the case where F is a p-adic field. We regard formal $\mathcal{O}_F$-modules over $\overline{\mathbb{F}}_q$ as ϖ-divisible $\mathcal{O}_F$-modules. We use the Dieudonné theory for ϖ-divisible $\mathcal{O}_F$-modules over $\overline{\mathbb{F}}_q$ developed in [7, Chapitre I, §B.8]. Here we identify $\mathcal{O}_{\breve{F}}$ with $W_{\mathcal{O}_F}(\overline{\mathbb{F}}_q) = \mathcal{O}_F \otimes_{W(\mathbb{F}_q)} W(\overline{\mathbb{F}}_q)$. Let $\mathbb{D} = \mathcal{O}_{\breve{F}}^n$ be a free $\mathcal{O}_{\breve{F}}$-module of rank n. We define a σ-linear map $F \colon \mathbb{D} \to \mathbb{D}$ and a σ^{-1}-linear map $V \colon \mathbb{D} \to \mathbb{D}$ by

$$F(e_i) = \begin{cases} \varpi e_{i+1} & i \neq n, \\ e_1 & i = n, \end{cases} \qquad V(e_i) = \begin{cases} e_{i-1} & i \neq 1, \\ \varpi e_n & i = 1, \end{cases}$$

where $(e_1, \dots, e_n)$ denotes the standard basis of $\mathbb{D}$. Then, by [7, Chapitre I, Proposition B.8.2], we can find a ϖ-divisible $\mathcal{O}_F$-module $\mathbb{X}$ of $\mathcal{O}_F$-height n over $\overline{\mathbb{F}}_q$ satisfying $\mathbb{D}_{\mathcal{O}_F}(\mathbb{X}) \cong (\mathbb{D}, F, V)$. Since V is topologically nilpotent and $\dim_{\overline{\mathbb{F}}_q} \mathbb{D}/V\mathbb{D} = 1$, $\mathbb{X}$ is a one-dimensional formal $\mathcal{O}_F$-module.

Let $D = F_n[\Pi]$ be the central division algebra over F with invariant $1/n$ as in § 2.2, and $\mathcal{O}_D = \mathcal{O}_{F_n}[\Pi]$ its maximal order. We construct a homomorphism $\mathcal{O}_D \to \mathrm{End}_{\mathcal{O}_F}(\mathbb{X})$. First, any $a \in \mathcal{O}_{F_n}$ defines an $\mathcal{O}_{\breve{F}}$-linear endomorphism on $\mathbb{D}$ by $e_i \mapsto \sigma^i(a)e_i$. Since it commutes with F and V, it gives an element of $\mathrm{End}_{\mathcal{O}_F}(\mathbb{X})$. Let Π be the $\mathcal{O}_{\breve{F}}$-linear endomorphism on $\mathbb{D}$ such that

$$\Pi(e_i) = \begin{cases} e_{i-1} & i \neq 1, \\ \varpi e_n & i = 1. \end{cases}$$

It also commutes with F and V, and gives an element of $\mathrm{End}_{\mathcal{O}_F}(\mathbb{X})$. It is immediate to observe that $\Pi a = \sigma(a)\Pi$ for $a \in \mathcal{O}_{F_n}$ and $\Pi^n = \varpi$ as endomorphisms of $\mathbb{D}$. Therefore, we obtain a homomorphism $\mathcal{O}_D = \mathcal{O}_{F_n}[\Pi] \to \mathrm{End}_{\mathcal{O}_F}(\mathbb{X})$, which is in fact an isomorphism. In the following, we identify $\mathcal{O}_D$ and $\mathrm{End}_{\mathcal{O}_F}(\mathbb{X})$ by this isomorphism.

We assume that F/F^+ is an unramified quadratic extension, and take ϖ in F^+. Recall that in this case τ also denotes the unique element of $\mathrm{Gal}(F^{\mathrm{ur}}/F^+)$ lifting the q'th power Frobenius automorphism $\overline{\tau}$ on $\overline{\mathbb{F}}_q = \overline{\mathbb{F}}_{q'}$. We describe $\overline{\tau}_*\mathbb{X}$ and $\mathbb{X}^\tau$ by means of the Dieudonné module as follows.

Proposition 3.19. *Let $\tau_W \colon W(\overline{\mathbb{F}}_q) \to W(\overline{\mathbb{F}}_q)$ denote the homomorphism induced from $\overline{\mathbb{F}}_q \xrightarrow{\overline{\tau}} \overline{\mathbb{F}}_q$. We also write τ_W for the composite $W(\mathbb{F}_q) \to W(\overline{\mathbb{F}}_q) \xrightarrow{\tau_W} W(\overline{\mathbb{F}}_q)$. Note that the Dieudonné module of a formal $\mathcal{O}_F$-module over $(\operatorname{Spec}\overline{\mathbb{F}}_q)^\tau \in$ **Nilp** is a free $\mathcal{O}_F \otimes_{W(\mathbb{F}_q),\tau_W} W(\overline{\mathbb{F}}_q)$-module endowed with F and V. We identify $\mathcal{O}_F \otimes_{W(\mathbb{F}_q),\tau_W} W(\overline{\mathbb{F}}_q)$ with $\mathcal{O}_{\breve{F}}$ by the isomorphism $\mathcal{O}_F \otimes_{W(\mathbb{F}_q),\tau_W} W(\overline{\mathbb{F}}_q) \xrightarrow[\cong]{\tau\otimes\mathrm{id}} \mathcal{O}_F \otimes_{W(\mathbb{F}_q)} W(\overline{\mathbb{F}}_q) = \mathcal{O}_{\breve{F}}$.*

- (i) *For a formal $\mathcal{O}_F$-module $\mathbb{Y}$ over $\overline{\mathbb{F}}_q$, we have $\mathbb{D}_{\mathcal{O}_F}(\overline{\tau}_*\mathbb{Y}) = \tau_*\mathbb{D}_{\mathcal{O}_F}(\mathbb{Y})$ and $\mathbb{D}_{\mathcal{O}_F}(\mathbb{Y}^\tau) = \mathbb{D}_{\mathcal{O}_F}(\mathbb{Y})$, where τ_* denotes the base change by $\tau \colon \mathcal{O}_{\breve{F}} \to \mathcal{O}_{\breve{F}}$.*
- (ii) *For $\mathbb{X}$ introduced above, we have $\mathbb{D}_{\mathcal{O}_F}(\overline{\tau}_*\mathbb{X}) \cong \mathbb{D}_{\mathcal{O}_F}(\mathbb{X})$.*

Proof. We prove (i). By functoriality we have $\mathbb{D}_{\mathcal{O}_F}(\overline{\tau}_*\mathbb{Y}) = (\mathrm{id}\otimes\tau_W)_*\mathbb{D}_{\mathcal{O}_F}(\mathbb{Y})$. Under the identification $\mathcal{O}_F \otimes_{W(\mathbb{F}_q),\tau_W} W(\overline{\mathbb{F}}_q) = \mathcal{O}_{\breve{F}}$, this equals

$$(\tau\otimes\mathrm{id})_*(\mathrm{id}\otimes\tau_W)_*\mathbb{D}_{\mathcal{O}_F}(\mathbb{Y}) = \tau_*\mathbb{D}_{\mathcal{O}_F}(\mathbb{Y}).$$

On the other hand, we have $\mathbb{D}_{\mathcal{O}_F}(\mathbb{Y}^\tau) = (\tau^{-1}\otimes\mathrm{id})_*\mathbb{D}_{\mathcal{O}_F}(\mathbb{Y})$. Under the identification, this clearly corresponds to the $\mathcal{O}_{\breve{F}}$-module $\mathbb{D}_{\mathcal{O}_F}(\mathbb{Y})$.

The assertion (ii) is clear from the definition of $\mathbb{X}$ and the identification

$$\tau_*\mathbb{D} \cong \mathbb{D};\ (x_1, \ldots, x_n) \mapsto (\tau(x_1), \ldots, \tau(x_n)),$$

as $\tau(\varpi) = \varpi$. □

Proposition 3.20. *Let $\iota \colon \overline{\tau}_*\mathbb{X} \xrightarrow{\cong} \mathbb{X}^\tau$ be the isomorphism that induces the isomorphism in Proposition 3.19(ii) on the Dieudonné modules. The pair (τ, t) constructed from this ι as in Definition 3.10 coincides with that in Definition 2.10(i).*

Proof. The claim on τ is clear from the definition. We prove $\iota\circ\overline{\tau}_*\iota\circ\mathrm{Frob}_{\mathbb{X}} = \Pi$. Recall that $\mathrm{Frob}_{\mathbb{X}} \colon \mathbb{X} \to \overline{\sigma}_*\mathbb{X}$ induces $V \colon \mathbb{D}_{\mathcal{O}_F}(\mathbb{X}) \to \sigma_*\mathbb{D}_{\mathcal{O}_F}(\mathbb{X}) = \mathbb{D}_{\mathcal{O}_F}(\overline{\sigma}_*\mathbb{X})$. On the other hand, the composite $\sigma_*\mathbb{D} = \mathbb{D}_{\mathcal{O}_F}(\overline{\sigma}_*\mathbb{X}) \xrightarrow{\mathbb{D}(\overline{\tau}_*\iota)} \mathbb{D}_{\mathcal{O}_F}(\overline{\tau}_*\mathbb{X}^\tau) \xrightarrow{\mathbb{D}(\iota)} \mathbb{D}_{\mathcal{O}_F}(\mathbb{X}) = \mathbb{D}$ is equal to $(x_1, \ldots, x_n) \mapsto (\sigma(x_1), \ldots, \sigma(x_n))$. Since $\Pi(e_i) = V(e_i)$ for every i, we conclude that $\iota\circ\overline{\tau}_*\iota\circ\mathrm{Frob}_{\mathbb{X}} = \Pi$. □

Next we assume that $p \neq 2$ and F/F^+ is a ramified quadratic extension, and take ϖ so that $\tau(\varpi) = -\varpi$. Recall that in this case τ also denotes the unique non-trivial element of $\mathrm{Gal}(F^{\mathrm{ur}}/(F^+)^{\mathrm{ur}})$.

Proposition 3.21. (i) *For a formal $\mathcal{O}_F$-module $\mathbb{Y}$ over $\overline{\mathbb{F}}_q$, we have $\mathbb{D}_{\mathcal{O}_F}(\mathbb{Y}^\tau) = \tau_*^{-1}\mathbb{D}_{\mathcal{O}_F}(\mathbb{Y})$, where τ_*^{-1} denotes the base change by $\tau^{-1} \colon \mathcal{O}_{\breve{F}} \to \mathcal{O}_{\breve{F}}$. For*

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every $\mathcal{O}_F$-homomorphism $h\colon \mathbb{Y} \to \mathbb{Y}'$ between formal $\mathcal{O}_F$-modules over $\overline{\mathbb{F}}_q$, the homomorphism $\mathbb{D}_{\mathcal{O}_F}(\mathbb{Y}^\tau) \to \mathbb{D}_{\mathcal{O}_F}(\mathbb{Y}'^\tau)$ induced by $h\colon \mathbb{Y}^\tau \to \mathbb{Y}'^\tau$ coincides with $\tau_*^{-1}\mathbb{D}(h)$.

(ii) *For $\mathbb{X}$ introduced above, we have $\mathbb{D}_{\mathcal{O}_F}(\mathbb{X}^\tau) = (\mathbb{D}, F', V')$, where F' and V' are determined by*

$$F'(e_i) = \begin{cases} -\varpi e_{i+1} & i \neq n, \\ e_1 & i = n, \end{cases} \qquad V'(e_i) = \begin{cases} e_{i-1} & i \neq 1, \\ -\varpi e_n & i = 1. \end{cases}$$

(iii) *For an element $h \in \mathrm{End}_{\mathcal{O}_F}(\mathbb{X})$, regard $\mathbb{D}(h)\colon \mathbb{D} \to \mathbb{D}$ as a matrix $(h_{ij}) \in M_n(\mathcal{O}_{\breve{F}})$. Then, the homomorphism $\mathbb{D} = \mathbb{D}_{\mathcal{O}_F}(\mathbb{X}^\tau) \to \mathbb{D}_{\mathcal{O}_F}(\mathbb{X}^\tau) = \mathbb{D}$ induced by $h\colon \mathbb{X}^\tau \to \mathbb{X}^\tau$ is given by the matrix $(\tau^{-1}(h_{ij}))$.*

Proof. The first assertion is clear from functoriality. The second is obvious from the definition of $\mathbb{X}$ and the identification

$$\mathbb{D} \cong \tau_*^{-1}\mathbb{D};\ (x_1, \dots, x_n) \mapsto (\tau(x_1), \dots, \tau(x_n)),$$

as $\tau(\varpi) = -\varpi$. Let h and (h_{ij}) be as in (iii). Under the identification $\mathbb{D} \cong \tau_*^{-1}\mathbb{D}$ above, $\tau_*^{-1}\mathbb{D}(h)$ corresponds to $(\tau^{-1}(h_{ij}))$. The third assertion immediately follows from this. □

Proposition 3.22. *As in Definition 2.10(ii), we take $\beta \in \mathcal{O}_{\breve{F}}$ such that $\beta^{q^n-1} = -1$ and put $\alpha = \beta^{q-1}$. Let $\iota\colon \mathbb{X} \xrightarrow{\cong} \mathbb{X}^\tau$ be the isomorphism such that the induced homomorphism $\mathbb{D} = \mathbb{D}_{\mathcal{O}_F}(\mathbb{X}) \to \mathbb{D}_{\mathcal{O}_F}(\mathbb{X}^\tau) = \mathbb{D}$ is given by $e_i \mapsto \sigma^i(\beta)^{-1}e_i$.*

Then, the pair (τ, t) constructed from this ι as in Definition 3.10 coincides with that in Definition 2.10(ii).

Proof. For $a \in \mathcal{O}_{F_n} \subset \mathcal{O}_D$, the composite $\mathbb{D} = \mathbb{D}_{\mathcal{O}_F}(\mathbb{X}^\tau) \xrightarrow{\mathbb{D}(\iota\circ a\circ\iota^{-1})} \mathbb{D}_{\mathcal{O}_F}(\mathbb{X}^\tau) = \mathbb{D}$ maps e_i to $\sigma^i(a)e_i$. Hence, by Proposition 3.21(iii), $\iota\circ a\circ\iota^{-1} \in \mathrm{End}_{\mathcal{O}_F}(\mathbb{X}^\tau)$ corresponds to $\tau(a) \in \mathcal{O}_{F_n} \subset \mathrm{End}_{\mathcal{O}_F}(\mathbb{X})$ under the identification $\mathrm{End}_{\mathcal{O}_F}(\mathbb{X}) = \mathrm{End}_{\mathcal{O}_F}(\mathbb{X}^\tau)$. Similarly, the composite $\mathbb{D} = \mathbb{D}_{\mathcal{O}_F}(\mathbb{X}^\tau) \xrightarrow{\mathbb{D}(\iota\circ\Pi\circ\iota^{-1})} \mathbb{D}_{\mathcal{O}_F}(\mathbb{X}^\tau) = \mathbb{D}$ maps e_i to

$$\begin{cases} \dfrac{\sigma^i(\beta)}{\sigma^{i-1}(\beta)} e_{i-1} & i \neq 1, \\ \dfrac{\sigma(\beta)}{\sigma^n(\beta)} \varpi e_n & i = 1. \end{cases}$$

Since $\beta \in \mu_{2(q^n-1)}(\mathcal{O}_{\breve{F}})$, we have $\sigma(\beta)/\beta = \beta^{q-1} = \alpha$. Hence $\sigma^i(\beta)/\sigma^{i-1}(\beta)$ equals $\sigma^{i-1}(\alpha)$. Similarly, we have $\sigma(\beta)/\sigma^n(\beta) = \beta^q/\beta^{q^n} = \beta^q/(-\beta) = -\beta^{q-1} = -\alpha$. Noting that $\alpha \in \mu_{q^n-1}(\mathcal{O}_{\breve{F}}) \subset \mathcal{O}_{F_n^+}$ is fixed by τ, we can conclude that $\iota\circ\Pi\circ\iota^{-1} \in \mathrm{End}_{\mathcal{O}_F}(\mathbb{X}^\tau)$ corresponds to $\alpha\Pi \in \mathrm{End}_{\mathcal{O}_F}(\mathbb{X})$ under the identification $\mathrm{End}_{\mathcal{O}_F}(\mathbb{X}) = \mathrm{End}_{\mathcal{O}_F}(\mathbb{X}^\tau)$.

We can observe that the composite $\mathbb{X} \xrightarrow{\iota} \mathbb{X}^\tau \xrightarrow{\iota} \mathbb{X}$ equals β^{-2} in the same way, by using the fact that $\beta \in \mu_{2(q^n-1)}(\mathcal{O}_{\breve{F}}) \subset \mathcal{O}_{(F^+)^{\mathrm{ur}}}$ is fixed by τ. □

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By Theorem 3.16 and Propositions 3.17, 3.18, 3.20, 3.22, we complete the proof of Theorem 2.12.

4. The case of simple supercuspidal representations

4.1. Conjugate self-dual simple supercuspidal representations

Here we apply our main theorem to simple supercuspidal representations. Let the notation be as in § 2.2. We briefly recall the notion of simple supercuspidal representations of $\mathrm{GL}_n(F)$ and $D^\times$. See [9, 11, 13, 21] for detail. Throughout this section, we fix a non-trivial additive character $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$ which factors through $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}\colon \mathbb{F}_q \to \mathbb{F}_p$.

First consider the case of $\mathrm{GL}_n(F)$. Let us denote by Iw the standard Iwahori subgroup of $\mathrm{GL}_n(F)$, namely, the subgroup of $\mathrm{GL}_n(\mathcal{O}_F)$ consisting of matrices whose image in $\mathrm{GL}_n(\mathbb{F}_q)$ is upper triangular. We write Iw_+ for the pro-p unipotent radical of Iw; it consists of matrices in Iw whose diagonal entries lie in $1+\mathfrak{p}_F$. Each element $\zeta \in \mathbb{F}_q^\times$ gives rise to a character

$$\psi_\zeta\colon \mathrm{Iw}_+ \to \mathbb{C}^\times; \quad (a_{ij}) \mapsto \psi(\overline{a_{12}} + \overline{a_{23}} + \dots + \overline{a_{n-1,n}} + \zeta^{-1}\overline{\varpi^{-1}a_{n1}}).$$

Here we denote the image of $a \in \mathcal{O}_F$ in $\overline{\mathbb{F}}_q$ by $\overline{a}$.

Let φ_ζ denote the matrix

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \widetilde{\zeta}\varpi & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $\widetilde{\zeta}$ denotes the unique element of $\mu_{q-1}(F)$ lifting ζ. It normalizes Iw_+. Put $H_\zeta = \mathcal{O}_F^\times \varphi_\zeta^{\mathbb{Z}} \mathrm{Iw}_+$. It is an open compact-mod-center subgroup of $\mathrm{GL}_n(F)$ (note that it contains the center $F^\times$, since $\varphi_\zeta^n = \widetilde{\zeta}\varpi$). We write $(\mathbb{F}_q^\times)^\vee$ for the set of characters $\mathbb{F}_q^\times \to \mathbb{C}^\times$. For a triple $(\zeta,\chi,c) \in \mathbb{F}_q^\times \times (\mathbb{F}_q^\times)^\vee \times \mathbb{C}^\times$, define the character $\Lambda_{\zeta,\chi,c}\colon H_\zeta \to \mathbb{C}^\times$ by

$$\Lambda_{\zeta,\chi,c}(x) = \chi(\overline{x}) \ (x \in \mathcal{O}_F^\times), \quad \Lambda_{\zeta,\chi,c}(\varphi_\zeta) = c, \quad \Lambda_{\zeta,\chi,c}|_{\mathrm{Iw}_+} = \psi_\zeta.$$

We put $\pi_{\zeta,\chi,c} = \text{c-Ind}_{H_\zeta}^{\mathrm{GL}_n(F)} \Lambda_{\zeta,\chi,c}$, which turns out to be an irreducible supercuspidal representation of $\mathrm{GL}_n(F)$. A representation obtained in this way is called a simple supercuspidal representation of $\mathrm{GL}_n(F)$. For another triple $(\zeta',\chi',c') \in \mathbb{F}_q^\times \times (\mathbb{F}_q^\times)^\vee \times \mathbb{C}^\times$, one can prove that $\pi_{\zeta,\chi,c} \cong \pi_{\zeta',\chi',c'}$ if and only if $(\zeta,\chi,c) = (\zeta',\chi',c')$ (see [11, Proposition 1.3]). Thus simple supercuspidal representations of $\mathrm{GL}_n(F)$ are parameterized by the set $\mathbb{F}_q^\times \times (\mathbb{F}_q^\times)^\vee \times \mathbb{C}^\times$.

Remark 4.1. Note that $\pi_{\zeta,\chi,c}$ implicitly depends on the choice of the uniformizer ϖ of F. Later we take it as in Definition 2.10.

The contragredient of $\pi_{\zeta,\chi,c}$ can be computed as follows:

Proposition 4.2. *For $(\zeta,\chi,c) \in \mathbb{F}_q^\times \times (\mathbb{F}_q^\times)^\vee \times \mathbb{C}^\times$, we have $\pi_{\zeta,\chi,c}^\vee \cong \pi_{(-1)^n\zeta,\chi^{-1},\chi(-1)c^{-1}}$.*

Proof. For $a = \mathrm{diag}(1, -1, \ldots, (-1)^{n-1})$, we have $a\varphi_\zeta a^{-1} = -\varphi_{(-1)^n\zeta}$. As a normalizes $\mathcal{O}_F^\times \mathrm{Iw}_+$, we obtain $aH_\zeta a^{-1} = H_{(-1)^n\zeta}$. Moreover, we can directly check that $\Lambda_{\zeta,\chi,c}(h)^{-1} = \Lambda_{(-1)^n\zeta,\chi^{-1},\chi(-1)c^{-1}}(aha^{-1})$ for $h \in H_\zeta$. Therefore, a intertwines $(H_\zeta, \Lambda_{\zeta,\chi,c}^{-1})$ and $(H_{(-1)^n\zeta}, \Lambda_{(-1)^n\zeta,\chi^{-1},\chi(-1)c^{-1}})$. By the same way as in the proof of Proposition 2.6, we conclude that

$$\pi_{(-1)^n\zeta,\chi^{-1},\chi(-1)c^{-1}} = \text{c-Ind}_{H_{(-1)^n\zeta}}^{\mathrm{GL}_n(F)} \Lambda_{(-1)^n\zeta,\chi^{-1},\chi(-1)c^{-1}} \cong \text{c-Ind}_{H_\zeta}^{\mathrm{GL}_n(F)} \Lambda_{\zeta,\chi,c}^{-1} \cong \pi_{\zeta,\chi,c}^\vee. \qquad \square$$

Corollary 4.3. *Let (ζ, χ, c) be an element of $\mathbb{F}_q^\times \times (\mathbb{F}_q^\times)^\vee \times \mathbb{C}^\times$.*

(i) *If F/F^+ is an unramified quadratic extension and $\varpi \in F^+$, then $\pi_{\zeta,\chi,c}$ is conjugate self-dual with respect to τ if and only if $\tau(\zeta) = (-1)^n\zeta$, $\chi^\tau = \chi^{-1}$ and $c^2 = \chi(-1)$, where τ denotes the q'th power Frobenius automorphism on $\mathbb{F}_q$.*

(ii) *If $p \neq 2$, F/F^+ is a ramified quadratic extension and ϖ satisfies $\tau(\varpi) = -\varpi$, then $\pi_{\zeta,\chi,c}$ is conjugate self-dual with respect to τ if and only if n is odd, $\chi^2 = 1$ and $c^2 = \chi(-1)$.*

(iii) *If $F = F^+$, then $\pi_{\zeta,\chi,c}$ is conjugate self-dual with respect to $\tau = \mathrm{id}$ (that is, self-dual) if and only if n is even, $\chi^2 = 1$ and $c^2 = \chi(-1)$.*

Proof. In the proof of Proposition 2.6, we obtained an isomorphism $(\text{c-Ind}_H^G \chi)^\tau \cong \text{c-Ind}_{H^\tau}^G \chi^\tau$. We can use it to determine $(\pi_{\zeta,\chi,c})^\tau$ in each case as follows:

(i) $(\pi_{\zeta,\chi,c})^\tau \cong \pi_{\tau^{-1}(\zeta),\chi^\tau,c} = \pi_{\tau(\zeta),\chi^\tau,c}$ (note that $\psi(\tau(x)) = \psi(x)$ for $x \in \mathbb{F}_q$).

(ii) $(\pi_{\zeta,\chi,c})^\tau \cong \pi_{-\zeta,\chi,c}$.

(iii) $(\pi_{\zeta,\chi,c})^\tau = \pi_{\zeta,\chi,c}$.

Together with Proposition 4.2, we conclude the proof. $\square$

Next we consider the case of $D^\times$. Let (ζ, χ, c) be an element of $\mathbb{F}_q^\times \times (\mathbb{F}_q^\times)^\vee \times \mathbb{C}^\times$. Take $\xi \in \mathbb{F}_{q^n}^\times$ such that $\mathrm{Nr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(\xi) = \zeta$, and write b for the unique element of $\mu_{q^n-1}(\mathcal{O}_{F_n})$ lifting ζ. Note that $(b\Pi)^n = \mathrm{Nr}_{F_n/F}(b)\Pi^n = \tilde{\zeta}\varpi$.

Put $H_\xi^D = \mathcal{O}_F^\times (b\Pi)^{\mathbb{Z}}(1 + \Pi\mathcal{O}_D)$. It is an open compact-mod-center subgroup of $D^\times$. We define the character $\Lambda_{\xi,\chi,c}^D \colon H_\xi^D \to \mathbb{C}^\times$ by

$$\Lambda_{\xi,\chi,c}^D(x) = \chi(\overline{x})(x \in \mathcal{O}_F^\times), \quad \Lambda_{\xi,\chi,c}^D(b\Pi) = c,$$
$$\Lambda_{\xi,\chi,c}^D(1 + b\Pi d) = \psi(\mathrm{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(\overline{d}))\ (d \in \mathcal{O}_D).$$

Here, $\overline{d}$ denotes the image of d under $\mathcal{O}_D \twoheadrightarrow \mathcal{O}_D/\Pi\mathcal{O}_D \overset{\cong}{\leftarrow} \mathcal{O}_{F_n}/\mathfrak{p}_{F_n} = \mathbb{F}_{q^n}$. We put $\pi_{\zeta,\chi,c}^D = \text{c-Ind}_{H_\xi^D}^{D^\times} \Lambda_{\xi,\chi,c}^D$, which turns out to be an irreducible smooth representation of $D^\times$ whose isomorphism class depends only on (ζ, χ, c). A representation of $D^\times$, which is automatically supercuspidal, obtained in this way is called a simple supercuspidal representation of $D^\times$.

The following theorem is proved in [11, Theorem 3.5].

Theorem 4.4. *For $(\zeta, \chi, c) \in \mathbb{F}_q^\times \times (\mathbb{F}_q^\times)^\vee \times \mathbb{C}^\times$, we have $\mathrm{JL}(\pi_{\zeta,\chi,c}) = \pi_{\zeta,\chi,(-1)^{n-1}c}^D$.*

4.2. Computation of parity

Here we compute the parity of $\mathrm{rec}_F(\pi_{\zeta,\chi,c})$ for a conjugate self-dual simple supercuspidal representation $\pi_{\zeta,\chi,c}$. We use Proposition 2.6 to compute the parity of $\pi^D_{\zeta,\chi,(-1)^{n-1}c}$.

Proposition 4.5. *Let (ζ,χ,c) be an element of $\mathbb{F}_q^\times\times(\mathbb{F}_q^\times)^\vee\times\mathbb{C}^\times$ such that $\pi_{\zeta,\chi,c}$ is conjugate self-dual with respect to τ under the setting in Example 2.7.*

(i) *Suppose that F/F^+ is an unramified quadratic extension and $\varpi\in F^+$. Let ε be an element of $\mathbb{F}_q^\times$ satisfying $\varepsilon^{q'-1}=-1$. Then the parity of $\pi^D_{\zeta,\chi,(-1)^{n-1}c}$ is equal to $(-1)^{n-1}\chi(\varepsilon)c$.*

(ii) *Suppose that $p\neq 2$, F/F^+ is a ramified quadratic extension and $\tau(\varpi)=-\varpi$. Then the parity of $\pi^D_{\zeta,\chi,(-1)^{n-1}c}$ is equal to*

$$\begin{cases}1 & \text{if } \chi \text{ is trivial,}\\ -1 & \text{if } \chi \text{ is non-trivial.}\end{cases}$$

(iii) *Suppose that $F=F^+$. Then, the parity of $\pi^D_{\zeta,\chi,(-1)^{n-1}c}$ is equal to*

$$\begin{cases}1 & \text{if } \chi \text{ is trivial,}\\ -1 & \text{if } \chi \text{ is non-trivial.}\end{cases}$$

Proof. For simplicity, we put $c'=(-1)^{n-1}c$ and $\Psi=\Lambda^D_{\zeta,\chi,(-1)^{n-1}c}$. In each case we find $a\in\mu_{q^n-1}(\mathcal{O}_{F_n})\subset D^\times$ which intertwines (H^D_ξ,Ψ^{-1}) and $((H^D_\xi)^\tau,\Psi^\tau)$.

Consider the case (i). Corollary 4.3 tells us that $\zeta^{q'-1}=(-1)^n$, $\chi^\tau=\chi^{-1}$ and $c^2=\chi(-1)$. Therefore, we have $(\varepsilon\xi)^{(1+q+\cdots+q^{n-1})(q'-1)}=(-1)^{1+q+\cdots+q^{n-1}}\zeta^{q'-1}=(-1)^n\cdot(-1)^n=1$. Hence there exists $\eta\in\mathbb{F}_{q^n}^\times$ satisfying $\eta^{q'+1}=\varepsilon\xi$. Let a_0 be the unique element of $\mu_{q^n-1}(\mathcal{O}_{F_n})$ lifting η and put $a=\tau^{-1}(a_0)$. Since $\eta^{1-q}\xi^{q'}=(\varepsilon\xi)^{1-q'}\xi^{q'}=-\xi$, we have $a_0^{1-q}b^{q'}=-b$. Thus $a_0\tau(b\Pi)a_0^{-1}=a_0b^{q'}a_0^{-q}\Pi=-b\Pi$ and $a(b\Pi)a^{-1}=-\tau^{-1}(b\Pi)$. In particular we have $aH^D_\xi a^{-1}=(H^D_\xi)^\tau$.

Let us prove that $\Psi(h)^{-1}=\Psi^\tau(aha^{-1})$ for every $h\in H^D_\xi$. If $h\in\mathcal{O}_F^\times$, we have $\Psi(h)^{-1}=\chi(\overline{h})^{-1}=\chi^\tau(\overline{h})=\Psi^\tau(aha^{-1})$, as $\chi^\tau=\chi^{-1}$. If $h=b\Pi$, we have $\Psi(h)^{-1}=c'^{-1}$ and $\Psi^\tau(aha^{-1})=\Psi(-b\Pi)=\chi(-1)c'$. These are equal since $c'^2=c^2=\chi(-1)$. If $h=1+b\Pi d\in 1+\Pi\mathcal{O}_D$, we have $\Psi(h)^{-1}=\psi(\mathrm{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(\overline{d}))^{-1}$ and $\Psi^\tau(aha^{-1})=\Psi^\tau(1+a(b\Pi)a^{-1}\cdot ada^{-1})=\Psi(1-b\Pi a_0\tau(d)a_0^{-1})=\psi(\mathrm{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(\overline{d}^{q'}))^{-1}$. Since ψ factors through $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_{q'}}$, they are equal.

Therefore, a intertwines (H^D_ξ,Ψ^{-1}) and $((H^D_\xi)^\tau,\Psi^\tau)$. In this case, the element z in Proposition 2.6 becomes $a_0\Pi\tau^{-1}(a_0)=(a_0^{q'+1}b^{-1})\cdot(b\Pi)$. Note that the reduction of $a_0^{q'+1}b^{-1}\in\mu_{q^n-1}(\mathcal{O}_{F_n})$ is equal to $\eta^{q'+1}\xi^{-1}=\varepsilon\in\mathbb{F}_q^\times$, and thus $a_0^{q'+1}b^{-1}$ lies in $\mathcal{O}_F^\times$.

Therefore, by Proposition 2.6 the parity of $\pi^D_{\zeta,\chi,c'}$ is equal to

$$\Psi(a_0\Pi\tau^{-1}(a_0)) = \Psi(a_0^{q'+1}b^{-1})\Psi(b\Pi) = \chi(\varepsilon)c' = (-1)^{n-1}\chi(\varepsilon)c,$$

as desired.

Consider the case (ii). Corollary 4.3 tells us that n is odd, $\chi^2 = 1$ and $c^2 = \chi(-1)$. Fix $\varepsilon \in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^2$. As in Remark 2.11(ii), we can take $\tau\colon D \to D$ so that $\tau(\Pi) = -\Pi$, and t as the unique element of $\mu_{q-1}(\mathcal{O}_F)$ lifting ε. Since $b \in \mu_{q^n-1}(\mathcal{O}_{F_n}) \subset F_n^+$, we have $\tau(b\Pi) = -b\Pi$, and thus $H^D_\xi = (H^D_\xi)^\tau$. By the similar computation as in (i), we can observe that $\Psi(h)^{-1} = \Psi^\tau(h)$ for every $h \in H^D_\xi$. Therefore, 1 intertwines (H^D_ξ, Ψ^{-1}) and $((H^D_\xi)^\tau, \Psi^\tau)$, and $z = t$. By Proposition 2.6 the parity of $\pi^D_{\zeta,\chi,c'}$ is equal to $\Psi(t) = \chi(\varepsilon) \in \{\pm 1\}$. Since $\chi^2 = 1$, $\chi(\varepsilon) = 1$ if and only if χ is trivial.

Finally consider the case (iii). Corollary 4.3 tells us that n is even, $\chi^2 = 1$ and $c^2 = \chi(-1)$. Take $\varepsilon \in \mathbb{F}_{q^2}^\times$ such that $\varepsilon^{q-1} = -1$, and let a be the unique element of $\mu_{q^2-1}(\mathcal{O}_{F_2})$ lifting ε. Since n is even, a belongs to $\mu_{q^n-1}(\mathcal{O}_{F_n})$. We have $a(b\Pi)a^{-1} = a^{1-q}b\Pi = -b\Pi$. Therefore, a normalizes H^D_ξ. By the similar computation as in (i), we can observe that $\Psi(h)^{-1} = \Psi(aha^{-1})$ for every $h \in H^D_\xi$. Therefore, a intertwines (H^D_ξ, Ψ^{-1}) and (H^D_ξ, Ψ), and $z = a^2 \in \mu_{q^n-1}(\mathcal{O}_{F_n})$. Since $(\varepsilon^2)^{q-1} = 1$, the reduction ε^2 of z lies in $\mathbb{F}_q^\times$, and thus z lies in $\mu_{q-1}(\mathcal{O}_F)$. Hence, by Proposition 2.6 the parity of $\pi^D_{\zeta,\chi,c'}$ is equal to $\Psi(z) = \chi(\varepsilon^2)$. As $\varepsilon^2 \in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^2$ and $\chi^2 = 1$, $\chi(\varepsilon^2) = 1$ if and only if χ is trivial. This completes the proof. □

Corollary 4.6. *Let (ζ, χ, c) be as in Proposition 4.5.*

(i) *Suppose that F/F^+ is an unramified quadratic extension and $\varpi \in F^+$. Let ε be an element of $\mathbb{F}_q^\times$ satisfying $\varepsilon^{q'-1} = -1$. Then the parity of $\mathrm{rec}_F(\pi_{\zeta,\chi,c})$ is equal to $\chi(\varepsilon)c$.*

(ii) *Suppose that $p \neq 2$, F/F^+ is a ramified quadratic extension and $\tau(\varpi) = -\varpi$. Then the parity of $\mathrm{rec}_F(\pi_{\zeta,\chi,c})$ is equal to*

$$\begin{cases} 1 & \text{if } \chi \text{ is trivial,} \\ -1 & \text{if } \chi \text{ is non-trivial.} \end{cases}$$

(iii) *Suppose that $F = F^+$. Then the parity of $\mathrm{rec}_F(\pi_{\zeta,\chi,c})$ is equal to*

$$\begin{cases} -1 & \text{if } \chi \text{ is trivial,} \\ 1 & \text{if } \chi \text{ is non-trivial.} \end{cases}$$

Proof. Clear from Theorems 2.12, 4.4 and Proposition 4.5. Recall that in the case (ii) (respectively (iii)), n is odd (respectively even). □

Remark 4.7. By Corollary 4.6(iii), if a simple supercuspidal representation π of $\mathrm{GL}_{2n}(F)$ is self-dual and has trivial central character, $\mathrm{rec}_F(\pi)$ is symplectic and π comes from $\mathrm{SO}(2n+1)$ by the endoscopic lifting. It is a starting point of a recent work of Oi [17].

On the other hand, if F has characteristic 0 and $p \neq 2$, Corollary 4.6(i) has been obtained in [18] by using the endoscopic character relation.

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