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WEAKLY-INJECTIVE MODULES OVER HEREDITARY NOETHERIAN PRIME RINGS

S. K. JAIN and S. R. LÓPEZ-PERMOUTH

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Abstract

A module M is said to be weakly-injective if and only if for every finitely generated submodule N of the injective hull $E(M)$ of M there exists a submodule X of $E(M)$, isomorphic to M such that $N \subset X$. In this paper we investigate weakly-injective modules over bounded hereditary noetherian prime rings. In particular we show that torsion-free modules over bounded hnp rings are always weakly-injective, while torsion modules with finite Goldie dimension are weakly-injective only if they are injective.

As an application, we show that weakly-injective modules over bounded Dedekind prime rings have a decomposition as a direct sum of an injective module B, and a module C satisfying that if a simple module S is embeddable in C then the (external) direct sum of all proper submodules of the injective hull of S is also embeddable in C. Indeed, we show that over a bounded hereditary noetherian prime ring every uniform module has periodicity one if and only if every weakly-injective module has such a decomposition.

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1. Introduction

The study of hereditary noetherian prime (hnp) rings generalizes that of bounded Dedekind prime rings and in particular of their best known example, the ring of integers $\mathbb{Z}$. These rings and their modules have been studied extensively; see [3, 2, 4, 8], for example. McConnell and Robson's book [10] has a nice chapter on hnp and related rings. In [8], Lenagan proved that an hnp ring is either primitive or bounded. Special classes of modules over bounded hnp rings (including injective, projective, quasi-injective and quasi-projective) have been studied in [4, 9, 12, 13, 14]. In this paper we discuss weakly-injective modules over bounded hnp rings.

Given an arbitrary ring R and R-modules M and N, we say that M is *weakly N-injective* if and only if every map $\varphi : N \to E(M)$ from N into the injective hull $E(M)$ of M may be written as a composition $\sigma \circ \hat{\varphi}$ where $\hat{\varphi} : N \to M$ is a homomorphism and $\sigma : M \to E(M)$ is a monomorphism. This is equivalent to saying that for every map $\varphi : N \to E(M)$ there exists a submodule X of $E(M)$, isomorphic to M such that $\varphi(N)$ is contained in X. In particular, M is weakly R-injective if and only if for every $x \in E(M)$ there exists $X \subset E(M)$ such that $x \in X \cong M$. We say that M is *weakly-injective* if and only if it is weakly N−injective for every finitely generated module N. Cleary, M is weakly-injective if and only if for every finitely generated submodule N of $E(M)$ there exists $X \subset E(M)$ such that $N \subset X \cong M$.

Any weakly N-injective module M satisfies the closely related property that for every submodule K of N, if N/K embeds in $E(M)$ then N/K embeds in M. Following [5], we refer to any such module as being *N-tight*. If M is N-tight for every finitely generated module N, we simply say that M is *tight*.

Weakly-injective (tight) modules are closed under finite sums and under essential extensions. However, they remarkably fail to be closed under direct summands [7]. Furthermore, arbitrary sums of weakly-injective right modules over a ring R are weakly-injective if and only if R is a right q.f.d. ring (that is, all cyclic right R-modules have finite Goldie dimension) [1].

Throughout all rings have 1 and all modules are right unital modules unless otherwise stated. If N is a submodule of M, $N \subset' M$ will mean that N is essential in M.

2. Preliminaries

The exact relation between weak relative-injectivity and relative tightness is given in the following lemma from [7].

LEMMA 2.1. *Given two modules M and N, M is weakly N-injective if and only if for every submodule $K \subset N$ and for every monomorphism $\varphi : N/K \to E(M)$:*

(1) *there exists a monomorphism $\varphi' : N/K \to M$, and*
(2) *for every complement L of $\varphi'(N/K)$ in M there exists $K' \subset E(M)$ such that $K' \cap \varphi(N/K) = 0$ and $K' \cong L$.*

PROOF. See [7, Lemma 1.3].

It follows easily from the previous lemma that a uniform module U is weakly-injective if and only if it is tight. As a matter of fact, for any module M, if $E(M)$ is a

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direct sum of indecomposables, M is tight if and only if it is weakly injective. This is the subject of our next proposition.

PROPOSITION 2.2. *Let M be an R-module such that the injective hull $E(M)$ of M is a direct sum of indecomposables. Then M is tight if and only if it is weakly-injective.*

PROOF. Let M be a tight right R-module such that $E(M)$ equals a direct sum of indecomposables, say $E(M) = \bigoplus_{i\in I} E_i$. Let N be a finitely generated submodule of $E(M)$. Then there exists a finite subset $J \subset I$ such that $N \subset \bigoplus_{i\in J} E_i$. Without loss of generality we may assume that $E(N) = \bigoplus_{i\in J} E_i$. Let $\varphi : N \to M$ be an embedding of N into M as is guaranteed by the tightness of M. Then $E(M) = E(\varphi(N)) \oplus K$, for some submodule $K \subset E(M)$. It follows from the Azumaya-Krull-Schmidt theorem that $K \cong \bigoplus_{i\in I-J} E_i$. Let $A = M \cap K$. Then $A \subset' K$ and hence $\varphi(N) \oplus A$ may be embedded in $E(M)$ via a map σ such that $N = \sigma(\varphi(N))$. By the injectivity of $E(M)$ and the essentiality of the inclusion $\varphi(N) \oplus A \subset' M$, we obtain a monomorphism $\hat{\sigma} : M \to E(M)$, extending σ, such that $N \subset \hat{\sigma}(M)$, as desired.

Proposition 2.2 has the following immediate corollary.

COROLLARY 2.3. *For a right noetherian ring R, a right R-module is weakly-injective if and only if it is tight.*

PROOF. Obvious.

The following lemmas, due to Singh, are listed here without proof for easy reference.

LEMMA 2.4. *Let R be a bounded* hnp *ring and let E be an indecomposable injective torsion right R-module. Then E has a unique chain of submodules*

$$0 = x_0R \subset x_1R \subset x_2R \subset \cdots \subset x_nR \subset \dots$$

whose union is E such that

(1) *each $x_{i+1}R/x_iR$ is a simple R-module;*
(2) *the members of the chain are the only submodules of E different from E; and*
(3) *there exists a positive integer n such that for any i, j, $x_{i+1}R/x_iR \cong x_{j+1}R/x_jR$ if and only if $i \equiv j \pmod{n}$.*

PROOF. See [12, Theorem 4] and [14, Corollary 2.9].

DEFINITION 2.5. Let E be an indecomposable injective torsion right R-module over a bounded hnp ring R. The unique infinite ascending chain of submodules of R described in Lemma 2.4 is called the *composition series* of E and the positive integer n is referred to as the *periodicity* of E. Furthermore, for any uniform module U over R, the periodicity of U is defined to be the periodicity of $E(U)$.

LEMMA 2.6. *For any uniform right R-module over a bounded Dedekind prime ring R the periodicity of U is* 1.

PROOF. See [12, Corollary 1].

DEFINITION 2.7. Let R be a bounded hnp ring. Two indecomposable injective torsion right R-modules are *equivalent* if they are homomorphic images of each other. Due to the finite periodicity, this is indeed equivalent to requiring that one of them be a homomorphic image of the other. Two torsion uniform modules are equivalent if their injective hulls are equivalent. Furthermore, two uniform elements x and y in a torsion right R-module are said to be equivalent if xR and yR are equivalent uniform right R-modules. A torsion right R-module M is said to be *primary* if every pair of uniform elements of M is equivalent. Given a uniform element x in a torsion R-module M, the submodule N of M generated by all the uniform elements of M equivalent to x is primary. Such an N is called a (the) *primary component* of M (corresponding to x).

LEMMA 2.8. *Every torsion module over a bounded* hnp *ring is the direct sum of its primary components.*

PROOF. See [13, Lemma 9].

We believe that the following result must be well-known but we have not been able to find it anywhere in the literature. We include it here without a proof.

LEMMA 2.9. *Let A be a submodule of a module B, and let $n \in \mathbb{Z}^+$. Then* $\operatorname{Soc}^n A = A \cap \operatorname{Soc}^n B$ *and* $\operatorname{Soc}^n A/\operatorname{Soc}^{n-1} A$ *is embeddable in* $\operatorname{Soc}^n B/\operatorname{Soc}^{n-1} B$.

3. Weakly-injective modules over bounded HNP rings

It has been shown that any noetherian prime ring is a weakly-injective ring (i.e. it is weakly-injective as a module over itself) [7]. Indeed, more is true:

PROPOSITION 3.1. *Every torsion-free module over a noetherian prime ring is weakly-injective.*

PROOF. Over a noetherian prime ring R every torsion-free right module contains an essential submodule which is a direct sum of uniform submodules. Since weakly-injective modules over noetherian rings are closed under arbitrary direct sums and under essential extensions, it suffices to show that every uniform right R-module is weakly-injective. Let U be a uniform right R-module and let V be a finitely generated submodule of $E(U)$. Since R is prime and noetherian it follows that V is isomorphic to a right ideal of R and that therefore it embeds in U. In light of Corollary 2.3, this completes our proof.

The above proposition has the following corollary.

COROLLARY 3.2. *For any module A over a noetherian prime ring R, A is weakly-injective if and only if its singular submodule $Z(A)$ is weakly-injective.*

PROOF. The injective hull of A may be written as $E(A) = E(Z(A)) \oplus K$, where $Z(A)$ is the torsion submodule of A and K is some submodule of $E(A)$. If A is weakly injective and N is a finitely generated submodule of $E(Z(A))$ then N embeds in A. But N is itself torsion and hence N embeds in $Z(A)$. In light of Corollary 2.3 this proves our claim that $Z(A)$ is weakly-injective. On the other hand, if $Z(A)$ is weakly-injective then A must be also weakly-injective since it contains as an essential submodule the direct sum of weakly-injective modules $Z(A) \oplus (K \cap A)$.

Due to the above corollary, in order to characterize weakly-injective modules over bounded hnp rings it suffices to center our attention on torsion modules.

By Lemma 2.8, any torsion module over a bounded hnp ring can be expressed as the direct sum of its primary components. While weak-injectivity does not usually come down to summands, we have the following result.

LEMMA 3.3. *A torsion module over a bounded* hnp *ring is weakly-injective if and only if its primary components are weakly-injective.*

PROOF. Let A be a torsion module over the bounded hnp ring R. By Lemma 2.8, we may write $A = \bigoplus_{i \in I} A_i$, where the A_i's are the primary components of A. Since sums of weakly-injective modules over noetherian rings are weakly-injective we only need to show that if A is weakly-injective so is A_j for each $j \in I$. Let N be a finitely generated submodule of $E(A_j) \subset E(A) = \bigoplus_{i \in I} E(A_i)$. Clearly, for every $i \in I$, $E(A_i)$ is a primary component of $E(A)$. By the weak-injectivity of A there exists an embedding $\varphi : N \to A$. Since $\varphi(N) \cong N \subset E(A_j)$ it follows that the uniform elements in $\varphi(N)$ are equivalent to those in A_j. Hence $\varphi(N) \subset A_j$. So A_j is tight and therefore, due to Corollary 2.3, weakly-injective as claimed.

The above lemma has, as an immediate application, the following characterization of weakly-injective torsion modules with finite Goldie dimension.

LEMMA 3.4. *If a torsion module A over a bounded* hnp *ring has finite Goldie dimension, then A is weakly-injective only if it is injective.*

PROOF. Let R be a bounded hnp ring and let A be a torsion right R-module with finite Goldie dimension n. Assume that A is weakly-injective. Since $\operatorname{Soc} A \subset' A$, we may write $\operatorname{Soc} A = S_1 \oplus \cdots \oplus S_n$, where for every $i = 1, \ldots, n$, S_i is simple. For every $i = 1, \ldots, n$, let $0 \subset a_{i1}R \subset a_{i2}R \subset \cdots$ be the composition series of $E(S_i)$. Then for every $m \in \mathbb{Z}^+$, $\operatorname{Soc}^m E(A) = a_{1m}R \oplus a_{2m}R \oplus \cdots \oplus a_{nm}R$. It follows that $E(A) = \bigcup_{m=1}^{\infty} \operatorname{Soc}^m E(A)$. So, in order to prove that A is injective it suffices to prove that for every $m \in \mathbb{Z}^+$, $\operatorname{Soc}^m E(A) = \operatorname{Soc}^m A$. Since A is weakly-injective, for every $n \in \mathbb{Z}^+$ there exists an embedding $\varphi : \operatorname{Soc}^m E(A) \to A$. We will first prove by induction that for every embedding $\varphi : \operatorname{Soc}^m E(A) \to A$, $\varphi(\operatorname{Soc}^m E(A)) = \operatorname{Soc}^m A = \operatorname{Soc}^m E(A)$. The result is clear if $m = 1$. Suppose it is true for $m = j-1$ and assume that $\varphi : \operatorname{Soc}^j E(A) \to A$ is an embedding. By the inductive hypothesis, the restriction of φ to $\operatorname{Soc}^{j-1} E(A)$ is an isomorphism onto $\operatorname{Soc}^{j-1} A = \operatorname{Soc}^{j-1} E(A)$. Then

$$\frac{\operatorname{Soc}^j E(A)}{\operatorname{Soc}^{j-1} E(A)} \cong \frac{\varphi(\operatorname{Soc}^j E(A))}{\varphi(\operatorname{Soc}^{j-1} E(A))} = \frac{\varphi(\operatorname{Soc}^j E(A))}{\operatorname{Soc}^{j-1} E(A)} \subset \operatorname{Soc}\left(\frac{A}{\operatorname{Soc}^{j-1} E(A)}\right) \subset \operatorname{Soc}\left(\frac{E(A)}{\operatorname{Soc}^{j-1} E(A)}\right). \tag{1}$$

From the first inequality in (1), $\varphi(\operatorname{Soc}^j E(A)) \subset \operatorname{Soc}^j A$. Also by (1), the Goldie dimension of $\operatorname{Soc}^j A/\operatorname{Soc}^{j-1} A$ is at least n, since $\operatorname{Soc}^j E(A)/\operatorname{Soc}^{j-1} E(A) = \sum_{i=1}^{n} a_{ij}R/a_{i,j-1}R$, a direct sum of n simples. On the other hand, Lemma 2.9 implies that the Goldie dimension of $\operatorname{Soc}^j A/\operatorname{Soc}^{j-1} A$ is at most equal to the Goldie dimension of $\operatorname{Soc}^j E(A)/\operatorname{Soc}^{j-1} E(A)$, which equals n. So, using (1) once again, we obtain $\varphi(\operatorname{Soc}^j E(A))/\operatorname{Soc}^{j-1} E(A) = \operatorname{Soc}^j A/\operatorname{Soc}^{j-1} A$ and hence $\varphi(\operatorname{Soc}^j E(A)) = \operatorname{Soc}^j A = \operatorname{Soc}^j E(A)$, as desired. This concludes our induction.

Weakly-injective torsion modules with infinite Goldie dimension will be characterized in the next lemma but first we need to introduce some notation. Let S be a simple module over a bounded hnp ring R. We define N_S to be the serial module consisting of the external direct sum of all proper submodules of $E(S)$. Namely,

$$N_S = \bigoplus_{B \subsetneq E(S)} B.$$

LEMMA 3.5. *Let A be a torsion module with homogeneous socle and infinite Goldie dimension. The following statements are equivalent:*

(1) *A is weakly-injective.*
(2) *For any simple module S, if S embeds in A then N_S embeds in A.*
(3) *For every $n \in \mathbb{Z}^+$, $\mathrm{Soc}^n(A)/\mathrm{Soc}^{n-1}(A)$ is infinite dimensional.*

PROOF. Let S be a simple submodule of A. From the hypotheses, the injective hull of A is a direct sum of infinitely many copies of $E(S)$. By Lemma 2.4, $E(S)$ has a composition series $0 \subset S = x_1R \subset x_2R \subset \cdots \subset E(S)$. Clearly, any finitely generated submodule of $E(A)$ can be embedded in N_S and therefore (2) implies (1). If we assume that A is weakly-injective then for every $m, n \in \mathbb{Z}^+$, the finitely generated module $(x_mR)^n$ is embeddable in A. In light of Lemma 2.9 this implies that for every $m, n \in \mathbb{Z}^+$, the Goldie dimension of $\mathrm{Soc}^m A/\mathrm{Soc}^{m-1}A$ is larger than n and hence it must be infinite. Thus (1) implies (3). So it is only left to show that (3) implies (2). Let us assume that for every $m \in \mathbb{Z}^+$, $\mathrm{Soc}^m A/\mathrm{Soc}^{m-1}A$ is infinite dimensional. We shall proceed inductively to construct an ascending sequence of submodules of A, $0 = N_0 \subset N_1 \subset N_2 \subset \cdots$ such that, for every $i \in \mathbb{Z}^+$, $N_i = N_{i-1} \oplus y_iR$, for some $y_i \in A$ such that $y_iR \cong x_iR$. Obviously, $N = \bigcup_{i=1}^{\infty} N_i$ will then be a submodule of A isomorphic to N_S, proving our claim. For $n = 1$, since $\mathrm{Soc}(A) \neq 0$ we have a simple submodule $0 \neq y_1R$ of A. Since the socle is homogeneous, $y_1R \cong x_1R$. Thus, let $N_1 = y_1R$. Suppose N_{m-1} has been constructed, then $N_{m-1} \cong x_1R \oplus x_2R \oplus \cdots \oplus x_{m-1}R$. Since $\mathrm{Soc}^m A/\mathrm{Soc}^{m-1}A$ is infinite dimensional, it has a submodule consisting of a sum of m simple submodules, say $S_1 \oplus S_2 \oplus \cdots \oplus S_m \subset \mathrm{Soc}^m A/\mathrm{Soc}^{m-1}A$. Let us write $S_i = \bar{z}_iR$, where $\bar{z}_i = z_i + \mathrm{Soc}^{m-1}A$ (for some $z_i \in \mathrm{Soc}^m A$). The finitely generated submodule $z_1R + \cdots + z_mR$ of A, being torsion, is equal to a direct sum $t_1R \oplus \cdots \oplus t_kR$ of cyclic submodules (See [12, Lemma 1], for example). One can easily check that (i) $k \geq m$, (ii) for each $i = 1, \ldots, k$ there exists $1 \leq j \leq m$ such that $t_iR \cong x_jR$, and (iii) there exist exactly m t_i's such that $t_iR \cong x_mR$, say, $t_{i_1}, t_{i_2}, \ldots$ and t_{i_m}. Among $t_{i_1}R$, $t_{i_2}R, \ldots$ and $t_{i_m}R$ there exists at least one whose intersection with N_{m-1} is zero (otherwise the socle of N_{m-1} would contain a direct sum of m distinct simple submodules). Let $t_{i_j}R$ be one such module, then let $y_m = t_{i_j}$ and define $N_m = N_{m-1} \oplus y_mR$. This completes the proof of our lemma.

4. Bounded HNP rings whose uniform modules have periodicity one

THEOREM 4.1. *Let A be a right module over a bounded* hnp *ring. If all uniform submodules of A have periodicity one, then the following statements are equivalent:*

(1) *A is weakly-injective.*

(2) *There is a decomposition* $A = B \oplus C$ *such that* (i) B *is torsion, injective and has finite dimensional primary components,* (ii) C *satisfies that if a simple module* S *embeds in* C *then the module* N_S *embeds in* C, *and* (iii) B *and* C *have no isomorphic simple submodules.*
(3) *There is a decomposition* $A = B \oplus C$ *such that* B *is injective and* C *satisfies that if a simple module* S *embeds in* C *then the module* N_S *embeds in* C.

PROOF. Let A be a right module over a bounded hnp ring R. If A is weakly-injective, so is $Z(A)$ (Corollary 3.2), and also so are the primary components of $Z(A)$ (Lemma 3.3). Let B be the (direct) sum of all the primary components of $Z(A)$ with finite Goldie dimension. By Lemma 3.4, each such primary component is injective and therefore so is B. It follows that we may write $A = B \oplus C$, where C is chosen so that it contains the primary components of $Z(A)$ not already contained in B. If S is a simple module and a monomorphism φ embeds S in C then S actually embeds in the primary component N (say) of $Z(A)$ corresponding to $\varphi(S)$. By the weak-injectivity of N and in light of Lemma 3.5, we conclude that N_S embeds in N and consequently in C, as claimed. The decomposition $A = B \oplus C$ satisfies conditions (i), (ii) and (iii) in (2) and therefore we conclude that (1) implies (2). Obviously (2) implies (3). The conditions in (3) imply that $Z(C)$ is weakly-injective (by Lemma 3.5). Therefore, by Corollary 3.2, C is weakly-injective and hence A, being the sum of two weakly-injective modules, is weakly-injective. Thus, (3) implies (1).

COROLLARY 4.2. *The statements in Theorem* 4.1 *about a right module* A *over the ring* R *are equivalent if* R *is a bounded Dedekind prime ring.*

PROOF. Lemma 2.6 guarantees that if R is a bounded Dedekind prime ring, then A satisfies the hypotheses of the theorem.

Let R be a bounded hnp ring and let E be an indecomposable injective right R-module with periodicity ≥ 2. Let $0 \subset x_1R \subset x_2R \subset \cdots \subset E$ be the compositon series of E. Then $x_1R \ncong x_2R/x_1R$. We refer to $E\,(x_2R/x_1R) = E/x_1R$ as $\bar{E}$ and, for each $x \in E$, $\bar{x}$ denotes $x + x_1R \in \bar{E}$. For every $j \in \mathbb{Z}^+$, let M_j be the submodule of $E \oplus \bar{x}_2R \oplus \cdots \oplus \bar{x}_jR$ consisting of those elements $(a_1, \bar{a}_2, \ldots, \bar{x}_j)$ such that $\bar{a}_1 = \bar{a}_2 + \cdots + \bar{a}_j$. Also, let M be the submodule of the infinite sum $E \oplus \bar{x}_2R \oplus \bar{x}_3R \oplus \cdots$ consisting of those elements $(a_1, \bar{a}_2, \bar{a}_3, \ldots)$ such that $\bar{a}_1 = \sum_{i=2}^{\infty} \bar{a}_i$. For convenience we shall employ the usual unit vectors (sequences), $e_i = (0, 0, \ldots, 0, 1, 0, \ldots)$ where the only 1 is in the i-th place as a notational device so that we may write $(a_1, \bar{a}_2, \ldots, \bar{a}_j) = a_1e_1 + \bar{a}_2e_2 + \cdots + \bar{a}_je_j$ in M_j and also $(a_1, \bar{a}_2, \bar{a}_3, \ldots) = a_1e_1 + \sum_{i=2}^{\infty} \bar{a}_ie_i$ in M.

LEMMA 4.3. *Let* $\varphi : x_j R \to M_j$ *be a monomorphism and let* $\varphi(x_j) = b_1 e_1 + \bar{b_2} e_2 + \cdots + \bar{b_j} e_j$. *Then* $b_1 R = x_j R = b_j R$. *Moreover,* $r \cdot \operatorname{ann}(\bar{b_1}) = r \cdot \operatorname{ann}(\bar{b_j})$.

PROOF. Notice first of all that $\varphi(x_1 R) = x_1 e_1 R$. Hence $\operatorname{Soc}(\varphi(x_j R)) = x_1 e_1 R$. It follows that $\pi_1 \circ \varphi$, the composition of φ with the projection π_1 of M_j onto E, is one to one, for if $\pi_1 \circ \varphi(x) = 0$, then $\varphi(x) \in \sum_{i=2}^{j} \bar{x}_i R$. If $\varphi(x) \neq 0$ then $\operatorname{Soc}(\varphi(x)R) \subset \sum_{i=2}^{j} \bar{x}_i R$, while on the other hand $\varphi(x)R \subset \varphi(x_j R)$, and hence $\operatorname{Soc}(\varphi(x)R) = x_1 e_1 R$, a contradiction. We conclude that $\varphi(x) = 0$ and therefore, since φ is one to one, $x = 0$. Consequently, $\pi_1 \circ \varphi(x_j R) = x_j R$, which shows that indeed $b_1 R = x_j R$, as claimed. Now by definition of M_j, $\bar{b}_1 = \bar{b}_2 + \cdots + \bar{b}_j$. We conclude that $b_j \notin x_{j-1} R$. Thus $x_j R = b_j R$. Having shown that $\pi_1 \circ \varphi$ is one to one, it follows that $r \cdot \operatorname{ann}(b_1 R) \subset [x_1 R : b_j R]$. So $\bar{b}_j R$ is a homomorphic image of $b_1 R$ under the map given by $b_1 \mapsto \bar{b}_j$. Since $\bar{b}_j R$ is of length $j - 1$, the kernel of the above map must be $x_1 R$. We therefore conclude that $r \cdot \operatorname{ann}(\bar{b}_1 R) = r \cdot \operatorname{ann}(\bar{b}_j R)$, as claimed.

THEOREM 4.4. *Let* R *be a bounded* hnp *ring having an indecomposable injective right* R*-module* E *with periodicity* ≥ 2. *Then there exists a weakly-injective module* M *which does not admit a decomposition of the type described in Theorem* 4.1 *(3).*

PROOF. We shall prove that M, as defined in the remarks preceeding Lemma 4.3, is a weakly-injective module, but that E does not embed in M. Consequently, M does not have a decomposition as described in the statement of the theorem. Notice first of all that $\operatorname{Soc}(M) \cong x_1 R \oplus \bar{x}_2 R \oplus \bar{x}_3 R \oplus \cdots$, since $x_1 e_1 \in M$ and the set $\{(\bar{x}_2 e_2 - \bar{x}_2 e_3)R, (\bar{x}_2 e_2 - \bar{x}_2 e_4)R, (\bar{x}_2 e_2 - \bar{x}_2 e_5)R, \ldots\}$ of submodules of M constitutes an independent family of simple submodules of M each isomorphic to $\bar{x}_2 R$. It follows that $E(M) \cong E \oplus \bar{E} \oplus \bar{E} \oplus \cdots$. Next, we show that M is weakly-injective. Let N be a finitely generated submodule of $E(M)$. Then $N = y_1 R \oplus y_2 R \oplus \cdots \oplus y_n R$, where each $y_i R$ is uniserial. If each $y_i R$ has socle isomorphic to $\bar{x}_2 R$, for $i = 1, \ldots n$, then there exists $j_i \in \mathbb{Z}^+$ such that $y_i R \cong \bar{x}_{j_i} R \subset \bar{E}$. Let $j = \max\{j_i | i = 1, \ldots, n\}$. The submodules of M,

$$(2) \qquad (\bar{x}_{j_1} e_j - \bar{x}_{j_1} e_{j+1})R \cong y_1 R, \qquad (\bar{x}_{j_2} e_{j+2} - \bar{x}_{j_2} e_{j+3})R \cong y_2 R, \quad \ldots$$
$$\text{and} \quad (\bar{x}_{j_n} e_{j+2(n-1)} - \bar{x}_{j_n} e_{j+2n-1})R \cong y_n R,$$

are an independent family whose sum is isomorphic to N. On the other hand, if for some i, $\operatorname{Soc}(y_i R) \cong x_1 R$, then, for some $l \in \mathbb{Z}^+$, $y_i R \cong x_l R$. So, replace the corresponding submodule of M in (2) by $(x_l e_1 + \bar{x}_l e_l)R \cong x_l R$. Once again this yields an independent family of submodules whose sum is isomorphic to N. In light of Corollary 2.3, this concludes our proof of the weak-injectivity of M. Next we show that E is not embeddable in M. Assume on the contrary that $\varphi : E \to M$ is

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an embedding. We first observe that $\varphi(x_1R) = x_1e_1R$. Similarly as in Lemma 4.3, if π_1 is the projection of M onto E, $\pi_1 \circ \varphi$ is one to one. We obtain that for every $j \in \mathbb{Z}^+$, if $\varphi(x_j) = a_1e_1 + \bar{a}_2e_2 + \cdots + \bar{a}_ke_k$, with $\bar{a}_k \neq 0$, then (i) $a_1R = x_jR$, (ii) $k \geq j$, and (iii) there exists $l \in \mathbb{Z}$ such that $j \leq l \leq k$ and $a_l \notin x_{j-1}R$. Let $\varphi(x_2) = b_1e_1 + \bar{b}_2e_2 + \cdots + \bar{b}_ke_k$, with $\bar{b}_k \neq 0$ and consider then $\varphi(x_{k+1}) = c_1e_1 + \bar{c}_2e_2 + \cdots + \bar{c}_te_t$, say. As observed above, $t \geq k+1$ and there exists $l \in \mathbb{Z}^+$ such that $k+1 \leq l \leq t$ and $c_l \notin x_kR$. Define a map $\varphi' : x_{k+1}R \to M_{k+1}$ via $\varphi'(x_{k+1}r) = c_1re_1 + \bar{c}_2re_2 + \cdots + \bar{c}_kre_k + \sum_{i=k+1}^{t} \bar{c}_ire_{k+1}$. Since $\pi_1 \circ \varphi' = \pi_1 \circ \varphi$ is one to one, we conclude that φ' is also one to one. Applying Lemma 4.3, we get that $r \cdot \operatorname{ann}(\bar{d}) = r \cdot \operatorname{ann}(\bar{c}_1)$, where $\bar{d} = \sum_{i=k+1}^{t} \bar{c}_ir$. On the other hand, there exists $y \in R$ such that $x_{k+1}y = x_2$. Hence $\varphi'(x_{k+1}y) = \varphi'(x_2)$. This implies that $\bar{d}y = 0$ and therefore $c_1y \in x_1R$. However, since $x_1e_1R \subset \varphi'(x_{k+1}R)$, we would then get that $\bar{b}_2e_2 + \cdots + \bar{b}_ke_k = \bar{c}_2e_2y + \cdots + \bar{c}_ke_ky \in \varphi'(x_{k+1}R)$. But $\operatorname{Soc}(\varphi'(x_{k+1}R) = x_1e_1R$ and therefore we get $\bar{b}_2e_2 + \cdots + \bar{b}_ke_k = 0$, a contradiction to the facts that $k \geq 2$ and $\bar{b}_k \neq 0$. Thus, we conclude that E is not embeddable in M.

THEOREM 4.5. *Let R be a bounded* hnp *ring. Then the following conditions are equivalent:*

(1) *Every uniform R-module has periodicity one.*
(2) *Every weakly-injective R-module M has a decomposition $M = B \oplus C$ such that B is injective and C satisfies that if a simple module S embeds in C then the module N_S embeds in C.*

PROOF. Apply Theorems 4.1 and 4.4

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Ohio University
Athens, Ohio 45701