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A note on a positive solution of a null mass nonlinear field equation in exterior domains

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We consider the Null Mass nonlinear field equation

$$\begin{cases} -\Delta u = f(u) \quad \text{in} \quad \Omega \\ u > 0 \\ u|_{\partial\Omega} = 0 \end{cases} \qquad (\mathcal{P})$$

where $\mathbb{R}^N \setminus \Omega$ is a bounded regular domain. The existence of a bound state solution is established in situations where this problem does not have a ground state.

Keywords: Nonlinear null mass equation, exterior domain, variational methods.

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1. Introduction

In this work, we look for a positive bound state solution for problem $(\mathcal{P})$ where a ground state cannot be obtained. Here we study a general nonhomogeneous nonlinearity with double-power growth condition on f, which behaves as a subcritical power u^p at infinity and a supercritical power u^q near the origin, where $p < 2^* < q$, in any exterior domain. Using the ideas introduced in [**14**, **15**, **20**], we extend the results of V. Benci and A. Micheletti [**7**] by removing any assumption on the size of hole $\mathbb{R}^N \setminus \Omega$.

The method used in this note, of finding a solution of $(\mathcal{P})$ as a critical point of the functional associated with the equation, constrained to the Nehari manifold of the functional, is rather natural because of the geometry of this functional due to the super-quadratic growth of the nonlinear terms. However, the novelty in our approach is found mostly in some clever technical results such as the sharp estimates on the decay of the positive ground state solution of the problem in $\mathbb{R}^N$ and its implications in the interaction of two distinct and distant copies of these solitons, and on the other hand, a new compactness result which allows us to circumvent the difficulties created by an unbounded nonsymmetric domain and embrace a very general problem.

Problems like $(\mathcal{P})$ with $f'(0) = 0$, the so-called zero mass case, appear in the study of Yang-Mills equations and have attracted the interest of researchers mostly in the case $\Omega = \mathbb{R}^N$(see [**8**, **18**, **22**]). Also, the electrostatic problem of capacitors that is modelled by exterior boundary-value problems (see [**16**],Volume 1, Chapter II, for instance).

When $\Omega = \mathbb{R}^N$, we distinguish three different cases $f'(0) < 0$, $f'(0) > 0$ and $f'(0) = 0$. In the first case, there is quite a large literature, where the first results on this subject can be seen in [**17**] and [**21**]. Also, H. Berestycki and P. L. Lions analysed this problem in [**8**, **9**]. In the second case, there are no finite energy solutions in general. Finally, when $f'(0) = 0$, the so-called zero mass case has seen a growing interest in recent mathematical literature as the zero mass limit case of noncritical elliptic problems of the form

$$-\Delta u + V(x)u = g(u),$$

for $g'(0) = 0$, and potentials satisfying $\liminf_{x\to\infty} V(x) = 0$. The existence of solutions for a null potential $V = 0$ was obtained by H. Berestycki and P. L. Lions in [**8**], where they used double-power growth condition on g and showed there is solution u in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Further, many authors resumed the study of this kind of equation under the double power growth condition, after it was successfully exploited in [**5**, **6**].

The main purpose of the present note is to solve problem $(\mathcal{P})$, in the null mass case, where Ω is an exterior regular domain with no restriction on its size. In order to do so, we make use of the ground state soluton in the whole $\mathbb{R}^N$, namely w, and show there exists $u \in \mathcal{D}^{1,2}(\Omega)$ which is solution of $(\mathcal{P})$, but not a ground state solution. In fact, there is no solution of $(\mathcal{P})$ which minimizes the energy function on the Nehari manifold. We extend the results in V. Benci and A. Micheletti [**7**], that worked with Ω such that $\mathbb{R}^N \setminus \Omega \subset B_\epsilon$ when ϵ is sufficiently small. This assumption on the size of Ω is removed in our work.

An important feature when Ω is an unbounded exterior domain is that $\mathcal{D}^{1,2}(\Omega)$ is not necessarily contained in any Lebesgue space $L^q(\Omega)$ with $q \neq 2^*$ and thus, there are no standard Sobolev embeddings like those of $H_0^1(\Omega)$. For this reason, we study the Orlicz spaces related to the right-hand side term f and require that it satisfies a double power growth condition and obtain the regularity required in the energy functional. These Lebesgue spaces have several important and essential properties that play the same role for the Hilbert space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ that the usual Lebesgue spaces play for $H_0^1(\Omega)$. In an exterior domain, the main difficulty is the lack of compactness. Here we use a splitting lemma that is an important key to overcome the lack of compactness. This lemma is a variant of a well-known result of M. Struwe (see [**23**]) related to the space $\mathcal{D}^{1,2}(\Omega)$ and also V. Benci and G. Cerami [**4**] with a clever description of what happens when a Palais-Smale sequence does not converge in norm to its weak limit. Note that since the space $\mathcal{D}^{1,2}(\Omega)$ is not necessarily contained in $H_0^1(\Omega)$, we cannot use Lions lemma as in [**19**], so we need another version of that and a splitting lemma in Orlicz spaces which we show in lemmas 3.4 and 3.5.

Finally, according to the method that we apply in this paper, we need to compare energy functionals associated with the equation in $(\mathcal{P})$ and that associated with the equation in $\mathbb{R}^N$. Suitable decay estimates for w, the positive radial solution of limit problem and ∇w will be crucial in order to compare all the terms in the energy functional and the ground state level. Thanks to J. Vétois [**24**], we find very fine and exact decay estimates for w and ∇w, that play essential roles in this work.

We consider the problem

$$\begin{cases} -\Delta u = f(u) \quad \text{in} \ \ \Omega, \\ u > 0, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{$\mathcal{P}$}$$

where $N \geqslant 3$, $\mathbb{R}^N \setminus \Omega \subseteq B_K(0)$ is a regular domain and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is odd and of class $C^1(\mathbb{R}, \mathbb{R})$, satisfying the conditions:

(f_1) Let $F(s) := \displaystyle\int_0^s f(t)\mathrm{d}t$, then $0 < \mu F(s) \leqslant f(s)s < f'(s)s^2$ for any $s \neq 0$ and for some $\mu > 2$;

(f_2) $F(0) = f(0) = f'(0) = 0$. There exist $C_1 > 0$ and $2 < p < 2^* < q$ such that

$$\begin{cases} |f^{(k)}(s)| \leqslant C|s|^{p-(k+1)} \quad \text{for} \ \ |s| \geqslant 1 \\ |f^{(k)}(s)| \leqslant C|s|^{q-(k+1)} \quad \text{for} \ \ |s| \leqslant 1 \end{cases}$$

for $k \in \{0, 1\}$, $s \in \mathbb{R}$.

Remark 1.1. It is straightforward from (f_1) that

$$F(s) \geqslant C|s|^{\mu}, \quad \text{for all} \ \ |s| \geqslant 1, \tag{1.1}$$

and by (f_2) we can write

$$|f^{(k)}(s)| \leqslant C|s|^{2^*-(k+1)}, \quad \text{for all} \ \ s \in \mathbb{R}. \tag{1.2}$$

Moreover, since $\mu F(s) \leqslant f(s)s$, then $C_1|s|^{\mu} \leqslant C_2|s|^p$ and so $\mu \leqslant p$.

A model nonlinear term which satisfies all assumptions is

$$f(u) = \begin{cases} u^q \quad \text{if } u \leqslant 1 \\ a + bu + cu^p \quad \text{if } u \geqslant 1 \end{cases}$$

with an appropriate choice of constants a, b and c for which f belongs to C^1. The energy functional associated with problem $(\mathcal{P})$ is

$$I_\Omega(u) = \frac{1}{2}\|u\|_\Omega^2 - \int_\Omega F(u)\mathrm{d}x, \quad \text{with } u \in \mathcal{D}^{1,2}(\Omega).$$

The main result of this paper is the following theorem.

THEOREM 1.2. *Assume that the positive solution in the whole* $\mathbb{R}^N$ *is unique, up to translations. Then, under assumptions* $(f_1)-(f_2)$*, problem* $(\mathcal{P})$ *has a positive classical solution* $u \in \mathcal{D}^{1,2}(\Omega)$.

REMARK 1.3. Note that the assumption of uniqueness of a positive solution in the whole $\mathbb{R}^N$:

$$\begin{cases} -\Delta u = f(u) \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^{\mathbb{N}}) \end{cases} \tag{$\mathcal{P}_{\mathbb{R}^N}$}$$

is a natural one. For instance, L. A. Caffarelli, B. Gidas and J. Spruck [**12**] proved that the functions

$$u_{\gamma,x_0}(x) = (N(N-2)\gamma)^{(N-2)/4}(\gamma + |x + x_0|)^{(2-N)/2}$$

are the only positive solutions of $(\mathcal{P}_{\mathbb{R}^N})$ with $f(u) = u^{2^*-1}$ for some real number $\gamma > 0$ and point $x_0 \in \mathbb{R}^N$.
For other nonlinearities $f(u)$ for which the uniqueness of positive solution holds see [**15**].

REMARK 1.4. We may assume in theorem 1.2 that the critical ground level c of the functional $I_{\mathbb{R}^N}$ is isolated with radius $r \geqslant c$, rather than assuming the uniqueness of positive solution of $(\mathcal{P}_{\mathbb{R}^N})$.

2. Preliminary results

We will use the following notation,

$$\langle u, v\rangle_\Omega = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x, \quad \|u\|_\Omega^2 = \int_\Omega |\nabla u|^2 \, \mathrm{d}x,$$

and we denote by $\mathcal{D}^{1,2}(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_\Omega$ or $\|\cdot\|_{\mathcal{D}^{1,2}(\Omega)}$.

Likewise, we write

$$\langle u, v\rangle_{\mathbb{R}^N} = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, \mathrm{d}x, \quad \|u\|_{\mathbb{R}^N}^2 = \int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x$$

and also denote by $\mathcal{D}^{1,2}(\mathbb{R}^{\mathbb{N}})$ the completion of $C_0^\infty(\mathbb{R}^{\mathbb{N}})$ with respect to the norm $\|\cdot\|_{\mathbb{R}^N}$ or $\|\cdot\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}$.

Set

$$J_\Omega(u) = I'_\Omega(u)u = \|u\|_\Omega^2 - \int_\Omega f(u)u \, \mathrm{d}x,$$
$$\mathcal{N}_\Omega := \{u \in \mathcal{D}^{1,2}(\Omega)\backslash\{0\} : J_\Omega(u) = 0\},$$

and

$$c_\Omega := \inf_{u \in \mathcal{N}_\Omega} I(u).$$

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The variational approach to solve this problem requires the study of the problem $(\mathcal{P}_{\mathbb{R}^N})$ in the whole $\mathbb{R}^{\mathbb{N}}$ associated with the functional

$$I_{\mathbb{R}^N}(u) = \frac{1}{2}\|u\|^2_{\mathbb{R}^N} - \int_{\mathbb{R}^N} F(u)\,\mathrm{d}x, \quad \text{with } u \in \mathcal{D}^{1,2}(\mathbb{R}^{\mathbb{N}}),$$

and in the same way

$$J_{\mathbb{R}^N}(u) = I'_{\mathbb{R}^N}(u)u = \|u\|^2_{\mathbb{R}^N} - \int_{\mathbb{R}^N} f(u)u\,\mathrm{d}x,$$
$$\mathcal{N}_{\mathbb{R}^N} := \{u \in \mathcal{D}^{1,2}(\mathbb{R}^{\mathbb{N}})\backslash\{0\} : J_{\mathbb{R}^N}(u) = 0\},$$

and

$$c := \inf_{u \in \mathcal{N}_{\mathbb{R}^N}} I_{\mathbb{R}^N}(u).$$

Let w be a positive radial solution of $(\mathcal{P}_{\mathbb{R}^N})$ which is well known to exist by [**8**] and $c = I_{\mathbb{R}^N}(w)$. Moreover, by [**24**] that there are positive constants C_1, C_2 and C_3 such that

$$C_1(1+|x|)^{-(N-2)} \leqslant w(x) \leqslant C_2(1+|x|)^{-(N-2)}, \quad \forall x \in \mathbb{R}^{\mathbb{N}}, \tag{2.1}$$

and

$$|\nabla w(x)| \leqslant C_3(1+|x|)^{-(N-1)}, \quad \forall x \in \mathbb{R}^{\mathbb{N}}. \tag{2.2}$$

Given $1 \leqslant p < q$, now we consider the space $L^p + L^q$ of functions $v : \Omega \to \mathbb{R}$ such that

$$v = v_1 + v_2 \quad \text{with} \;\; v_1 \in L^p(\Omega), v_2 \in L^q(\Omega).$$

$L^p + L^q$ is a Banach space with the norm (see [**2**, **6**, **10**])

$$\|v\|_{L^p+L^q} = \inf\{\|v_1\|_{L^p} + \|v_2\|_{L^q} : v = v_1 + v_2\}.$$

Remark 2.1. V. Benci and D. Fortunato in [**6**] showed that $L^{2^*} \subset L^p + L^q$ when $2 < p < 2^* < q$. Then, by the Sobolev inequality, we get the continuous embedding $\mathcal{D}^{1,2}(\Omega) \subset L^p + L^q$.

Now we present a fundamental lemma which may be found in [**7**] (lemma 2.6) and which will be systematically used in the forthcoming arguments.

Lemma 2.2. *The functional $\mathcal{F} : L^p + L^q \to \mathbb{R}$ defined by*

$$\mathcal{F}(u) := \int_\Omega F(u)\mathrm{d}x,$$

is of class C^2 and we have

$$\mathcal{F}'(u)v = \int_\Omega f(u)v\mathrm{d}x,$$
$$\mathcal{F}''(u)vw := \int_\Omega f'(u)vw\mathrm{d}x.$$

Remark 2.3. Lemma 2.2 ensures that the functional

$$I_\Omega(u) = \frac{1}{2}\|u\|_\Omega^2 - \int_\Omega F(u)\mathrm{d}x, \quad \text{with } u \in \mathcal{D}^{1,2}(\Omega)$$

is well defined, of class C^2 and any critical point of I_Ω is a weak solution of $(\mathcal{P})$.

Lemma 2.4.

(a) *$\mathcal{N}_{\mathbb{R}^N}$ is a closed C^1 manifold;*

(b) *given $u \neq 0$; there exists a unique number $t = t(u) > 0$ such that $ut(u) \in \mathcal{N}_{\mathbb{R}^N}$ and $I_{\mathbb{R}^N}(t(u)u)$ is the maximum for $I_{\mathbb{R}^N}(tu)$ when $t \geqslant 0$;*

(c) *the dependence of $t(u)$ on u is of class C^1;*

(d) $\inf_{u\in\mathcal{N}_{\mathbb{R}^N}} \|u\|_{\mathbb{R}^N} = \rho > 0$.

Proof. Item (a) follows from (f_1) and lemma 2.2. For $u \in \mathcal{N}_{\mathbb{R}^N}$

$$I'_{\mathbb{R}^N}(u)u = \int_{\mathbb{R}^N} 2|\nabla u|^2 - f(u)u - f'(u)u^2\mathrm{d}x = \int_{\mathbb{R}^N} f(u)u - f'(u)u^2\mathrm{d}x < 0$$

and $\mathcal{N}_{\mathbb{R}^N} = J_{\mathbb{R}^N}^{-1}(\{0\})$ is a closed subset of $\mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}$.
(b) Given $u \neq 0$, if we set

$$G_u(t) := \int_{\mathbb{R}^N} \frac{1}{2}t^2|\nabla u|^2 - F(tu)\mathrm{d}x \quad \text{for } t \geqslant 0,$$

then

$$G'_u(t) = \int_{\mathbb{R}^N} t|\nabla u|^2 - f(tu)u\mathrm{d}x, \quad G''_u(t) = \int_{\mathbb{R}^N} |\nabla u|^2 - f'(tu)u^2\mathrm{d}x.$$

By (f_1), if $\bar{t} > 0$ is a critical point of G_u, then $G''_u(\bar{t}) < 0$ so $\bar{t}$ is a point of maximum for G. Furthermore, $0 = G_u(0) = G'_u(0)$ and $G''_u(0) > 0$, and hence 0 is a point of minimum for G_u. By (1.1) and $F(u) > 0$ in (f_1), we obtain

$$\begin{aligned} G_u(t) &\leqslant \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\mathrm{d}x - C\int_{t|u|<1} F(tu)\mathrm{d}x - Ct^\mu \int_{t|u|>1} |u|^\mu \mathrm{d}x \\ &\leqslant \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\mathrm{d}x - Ct^\mu \int_{t|u|>1} |u|^\mu \mathrm{d}x. \end{aligned}$$

Since $u \neq 0$, then there exists $\Lambda \subset \mathbb{R}^N$ with Lebesgue positive measure such that $|u\,|_\Lambda\,| > 0$. By the monotone convergence theorem, $G_u(t) \to -\infty$ as $t \to \infty$ and this proves b)

(c) We define the operator $g : \mathbb{R}^+ \times \mathcal{D}^{1,2}(\Omega) \to \mathbb{R}$ by

$$g(t, u) = t \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x - \int_{\mathbb{R}^N} f(tu)u \, \mathrm{d}x.$$

By lemma 2.2, g is of class C^1 and if (t_0, u_0) is such that $g(t_0, u_0) = 0$ and $t_0 \neq 0$, then by (f_1)

$$g_t'(t_0, u_0) = \int_{\mathbb{R}^N} |\nabla u_0|^2 - f'(t_0 u_0)u_0^2 \mathrm{d}x = \int_{\mathbb{R}^N} \frac{f(t_0 u_0)u_0}{t_0} - f'(t_0 u_0)u_0^2 \mathrm{d}x < 0.$$

By the Implicit Function theorem, we get that $u \to t(u)$ is of class C^1 and

$$t'(u_0)[\varphi] = \frac{t_0^2 \displaystyle\int_{\mathbb{R}^N} 2t_0 \nabla u_0 \nabla \varphi - f(t_0 u_0)\varphi - f'(t_0 u_0)t_0 u_0 \varphi \mathrm{d}x}{\displaystyle\int_{\mathbb{R}^N} f'(t_0 u_0)(t_0 u_0)^2 - f(t_0 u_0)t_0 u_0 \mathrm{d}x}$$

where $t_0 = t(u_0)$.
(d) By contradiction, suppose that the minimizing sequence (u_n) converges to 0. We set $u_n = t_n v_n$ with $\|v_n\|_{\mathbb{R}^N} = 1$. Since $u_n \in \mathcal{N}_{\mathbb{R}^N}$ and (t_n) converges to 0, we have

$$t_n = \int_{\mathbb{R}^N} f(t_n v_n)v_n \leqslant C t_n^{2^*-1} \int_{\mathbb{R}^N} |v_n|^{2^*}.$$

Hence, it holds that

$$1 \leqslant C t_n^{2^*-2} \int_{\mathbb{R}^N} |v_n|^{2^*},$$

which yields a contradiction if $t_n \to 0$. □

Remark 2.5. Similarly, by substituting $\mathbb{R}^N$ with Ω, lemma 2.4 holds also for $\mathcal{N}_\Omega$.

Remark 2.6. If $u \neq 0$ is a critical point of the functional I_Ω on $\mathcal{N}_\Omega$, then u is a critical point of I_Ω. Indeed, consider $u \in \mathcal{N}_\Omega$ and use (f_1) to obtain

$$\langle J_\Omega' u, u\rangle = 2\|u\|_\Omega^2 - \int_\Omega f'(u)u^2 + f(u)u \leqslant \int_\Omega \left(\frac{f(u)}{u} - f'(u)\right) u^2 < 0.$$

Now, suppose that $u \in \mathcal{N}_\Omega$ is a constrained critical point of I_Ω, then there exists a real number ϑ such that $I_\Omega'(u) - \vartheta J_\Omega'(u) = 0$; taking u as test function one gets $\vartheta \langle J_\Omega' u, u\rangle = 0$, which yields $\vartheta = 0$, that is, u is a free critical point.

Lemma 2.7. *$c_\Omega = c > 0$.*

Proof. We have $c \leqslant c_\Omega$, because we can consider $\mathcal{N}_\Omega \subset \mathcal{N}_{\mathbb{R}^N}$ by extending u in $\mathcal{D}^{1,2}(\Omega)$ as zero outside Ω. On the other hand, by lemma 4.5 in § 4, we have $c_\Omega \leqslant c$ and so $c_\Omega = c$.

Now we show that $c > 0$. Let $(u_n) \subset \mathcal{N}_{\mathbb{R}^N}$ be a minimizing sequence of c, then

$$\left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}|\nabla u_n|^2 \mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u_n|^2 - \frac{1}{\mu}\int_{\mathbb{R}^N} f(u_n)u_n$$
$$\leqslant \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u_n|^2 - \int_{\mathbb{R}^N} F(u_n)$$
$$= I_{\mathbb{R}^N}(u_n). \tag{2.3}$$

Now suppose by contradiction that $c = 0$. Then the minimizing sequence (u_n) is such that $(I_{\mathbb{R}^N}(u_n))$ goes to zero, hence by (2.3) (u_n) converges to zero in $\mathcal{D}^{1,2}(\Omega)$ which is a contradiction with d) in lemma 2.4. □

Lemma 2.8. *Problem $(\mathcal{P})$ has no ground state, in other words, c_Ω is not attained.*

Proof. We proved in the previous lemma that $c_\Omega = c > 0$. At this point, we suppose by contradiction, that there exists $\overline{u} \in \mathcal{N}_\Omega$ such that $I_\Omega(\overline{u}) = c_\Omega$. Setting $\overline{u} = 0$ in $\mathbb{R}^N \setminus \Omega$, $\overline{u}$ can be regarded as an element of $\mathcal{N}_{\mathbb{R}^N}$. We can assume $\overline{u} \geqslant 0$ since if $\overline{u} \in \mathcal{N}_{\mathbb{R}^N}$ then $|\overline{u}| \in \mathcal{N}_{\mathbb{R}^N}$ and $I_{\mathbb{R}^N}(|\overline{u}|) = I_{\mathbb{R}^N}(\overline{u}^+ + \overline{u}^-) = I_{\mathbb{R}^N}(\overline{u}^+) + I_{\mathbb{R}^N}(\overline{u}^-) = I_{\mathbb{R}^N}(\overline{u}) = c$. Hence, $\overline{u}$ is a minimizer of $I_{\mathbb{R}^N}$ on $\mathcal{N}_{\mathbb{R}^N}$ and a solution of $(\mathcal{P}_{\mathbb{R}^N})$ in $\mathbb{R}^{\mathbb{N}}$. Now by Brezis-Kato theorem we see that $\overline{u} \in C^2(\mathbb{R}^N)$ (details are in the end of this paper, by bootstrap procedure). Then, by the strong maximum principle, $\overline{u}$ is strictly positive in $\mathbb{R}^N$ and so we have a contradiction. □

Lemma 2.9. *For every $0 < \nu < q-2$ and $\rho > 0$ there exists $C_\rho > 0$ such that for all $0 \leqslant u, v \leqslant \sigma$ it holds*

$$F(u+v) - F(u) - F(v) - f(u)v - f(v)u \geqslant -C_\sigma (uv)^{1+\nu/2} \tag{2.4}$$

Proof. The inequality (2.4) is obviously satisfied if $u = 0$ or $v = 0$. By (f_1) the function $f(s)$ is increasing in $s > 0$, which yields for $u, v > 0$

$$F(u+v) - F(u) = \int_u^{u+v} f(w)\mathrm{d}w \geqslant f(u)v.$$

Moreover, by (f_2) for every $0 < \nu < q-2$ it follows

$$f(u) = o(|u|^{1+\nu}) \quad \text{as} \quad |u| \to 0,$$

and so $\tilde{C}_\sigma := \sup_{0<u\leqslant\sigma} f(u)/(u^{1+\nu}) < \infty$. Now if $0 < v \leqslant u$, we deduce

$$F(u+v) - F(u) - F(v) - f(u)v - f(v)u \geqslant -F(v) - f(v)u$$
$$= \int_0^v -\frac{f(w)}{w^{1+\nu}} w^{1+\nu}\mathrm{d}w - \frac{f(v)}{v^{1+\nu}} uv^{1+\nu} \geqslant -\tilde{C}_\sigma \frac{v^{2+\nu}}{2+\nu} - \tilde{C}_\sigma uv^{1+\nu}$$
$$\geqslant \left[-(\frac{v}{u})^{\nu/2}\left(\left(\frac{v}{u}\right)^{\nu/2} + \frac{1}{2}\left(\frac{v}{u}\right)^{1+\nu/2}\right)\right]\tilde{C}_\sigma (uv)^{1+\nu/2} \geqslant -\frac{3}{2}\tilde{C}_\sigma (uv)^{1+\nu/2}.$$

Using the symmetry of the expressions with respect to u and v, the same estimate holds for $0 < u \leqslant v$, and the proof is complete. □

Now let us fix $y_0 \in \mathbb{R}^N$ with $|y_0| = 1$ and consider $B_2(y_0) := \{x \in \mathbb{R}^N : |x - y_0| \leqslant 2\}$. We write for each $y \in \partial B_2(y_0)$

$$w_0^R := w(\cdot - Ry_0), \quad w_y^R := w(\cdot - Ry).$$

LEMMA 2.10. *Let $R > 0$ be large enough and $r > 1$, then*

$$a) \int_{B_{2K}(0)} |w_0^R|^r \leqslant CR^{-r(N-2)} \quad \textit{and} \quad \int_{B_{2K}(0)} |w_y^R|^r \leqslant CR^{-r(N-2)}; \tag{2.5}$$

$$b) \int_{B_{2K}(0)} |\nabla w_0^R|^r \leqslant CR^{-r(N-2)} \quad \textit{and} \quad \int_{B_{2K}(0)} |\nabla w_y^R|^r \leqslant CR^{-r(N-2)}. \tag{2.6}$$

Proof. In order to prove the first estimate, note that for $2K < 1/2R$ and $x \in B_{2K}(0)$,

$$\frac{1}{2}R = R - \frac{1}{2}R < |Ry_0| - |x| < |x - Ry_0| < 1 + |x - Ry_0|. \tag{2.7}$$

Now by (2.1) and $r > 1$, we have

$$\int_{B_{2K}(0)} |w(x - Ry_0)|^r \mathrm{d}x \leqslant C \int_{B_{2K}(0)} (1 + |x - Ry_0|)^{-r(N-2)} \mathrm{d}x \leqslant CR^{-r(N-2)}.$$

The proofs of the other estimates in (2.5) and (2.6) are similar. □

Now we are going to obtain some crucial estimates of the integrals in the whole $\mathbb{R}^N$, inspired in the work of M. Clapp and L. Maia [**15**].

LEMMA 2.11. *Let $r > 2^*/2$ and $s \geqslant 1$ then*

$$\int_{\mathbb{R}^N} (w_0^R)^r (w_y^R)^s \leqslant CR^{-s(N-2)}, \tag{2.8}$$

and

$$\int_{\mathbb{R}^N} (w_y^R)^r (w_0^R)^s \leqslant CR^{-s(N-2)}. \tag{2.9}$$

Proof. In order to prove the first estimate

$$\int_{\mathbb{R}^N} (w_0^R)^r (w_y^R)^s = \int_{\mathbb{R}^N} (w(x - Ry_0))^r (w(x - Ry))^s \mathrm{d}x,$$

we consider a change of variables $x = z + (Ry_0 + Ry)/2$, thus

$$\begin{aligned}
&\int_{\mathbb{R}^N} (w(x - Ry_0))^r (w(x - Ry))^s \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^N} \left(w \left(z - \frac{Ry_0 - Ry}{2} \right) \right)^r \left(w \left(z + \frac{Ry_0 - Ry}{2} \right) \right)^s \mathrm{d}z, \\
&\quad = \int_{\mathbb{R}^N} (w(z - P_R))^r (w(z + P_R))^s \mathrm{d}z = 2 \int_{\mathbb{Q}^+} (w(z - P_R))^r (w(z + P_R))^s \mathrm{d}z \\
&\quad = 2 \int_{B_1(P_R)} (w(z - P_R))^r (w(z + P_R))^s \mathrm{d}z \\
&\qquad + 2 \int_{\mathbb{Q}^+ \setminus B_1(P_R)} (w(z + P_R))^r (w(z - P_R))^s \mathrm{d}z,
\end{aligned}$$

by denoting $P_R = (Ry_0 - Ry)/2$, using the symmetry of the integrals and denoting $\mathbb{Q}^+ = \{z \in \mathbb{R}^N : \langle z - P_R, P_R \rangle \geqslant 0\}$. Note that for $\xi \in \mathbb{Q}^+$ and R sufficiently large

$$\begin{cases} \text{if} \quad |\xi| > 1 \quad \text{then} \quad R < 1 + |\xi + 2P_R|, \\ \text{if} \quad |\xi| < 1 \quad \text{then} \quad 2R < 1 + |\xi + 2P_R|. \end{cases} \tag{2.10}$$

Now, by another change of variables $\xi = z - P_R$, (2.10) and (2.1) in the previous statement we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} (w_0^R)^r (w_y^R)^s &= 2 \int_{B_1(0)} (w(\xi))^r (w(\xi + 2P_R))^s \mathrm{d}\xi \\
&\quad + 2 \int_{\{\mathbb{Q}^+ - P_R\} \setminus B_1(0)} (w(\xi))^r (w(\xi + 2P_R))^s \mathrm{d}\xi \\
&\leqslant C \int_{B_1(0)} (1 + |\xi + 2P_R|)^{-s(N-2)} \mathrm{d}\xi \\
&\quad + C \int_{\{\mathbb{Q}^+ - P_R\} \setminus B_1(0)} (1 + |\xi|)^{-r(N-2)} (1 + |\xi + 2P_R|)^{-s(N-2)} \mathrm{d}\xi \\
&\leqslant CR^{-s(N-2)} \int_{B_1(0)} \mathrm{d}\xi + CR^{-s(N-2)} \int_{\{\mathbb{Q}^+ - P_R\} \setminus B_1(0)} |\xi|^{-r(N-2)} \mathrm{d}\xi \\
&\leqslant CR^{-s(N-2)},
\end{aligned}$$

since

$$\int_{\{\mathbb{Q}^+ - P_R\} \setminus B_1(0)} |\xi|^{-r(N-2)} \mathrm{d}\xi < \int_1^\infty y^{-r(N-2)} y^{N-1} \mathrm{d}y$$

and for $r > 2^*/2$ it holds $-r(N-2) + N - 1 < -1$. The proof of estimative (2.9) is similar and this completes the proof of this lemma. □

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Define for $\lambda \in [0,1]$

$$Z^R_{\lambda,y} := \lambda w^R_0 + (1-\lambda) w^R_y$$

and

$$U^R_{\lambda,y} := Z^R_{\lambda,y}\psi \tag{2.11}$$

where $\psi \in C^\infty(\mathbb{R}^N)$ is continuous radially symmetric and increasing cutoff function given by

$$\psi(x) = \begin{cases} 0 & |x| \leqslant K, \\ 0 < \psi < 1 & K < |x| < 2K, \\ 1 & |x| \geqslant 2K\,, \end{cases}$$

with K the radius of the smallest sphere $B_K(0)$ that contains $\mathbb{R}^N \setminus \Omega$. We can consider $U^R_{\lambda,y} \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ by extending $U^R_{\lambda,y} = 0$ outside Ω.

Lemma 2.12. *$U^R_{\lambda,y} - Z^R_{\lambda,y} \to 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, as $R \to \infty$.*

Proof. First of all, if $R > 0$ is sufficiently large we claim that

$$\|\nabla w^R_0 - \nabla(\psi w^R_0)\|_{L^2(B_{2K}(0))} \leqslant CR^{-(N-2)} \tag{2.12}$$

and

$$\|\nabla w^R_y - \nabla(\psi w^R_y)\|_{L^2(B_{2K}(0))} \leqslant CR^{-(N-2)}. \tag{2.13}$$

By the claim we have

$$\begin{aligned}\|U^R_{\lambda,y} - Z^R_{\lambda,y}\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} &\leqslant \lambda\|w^R_0 - \psi w^R_0\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} + (1-\lambda)\|w^R_y - \psi w^R_y\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \\ &= \lambda\|\nabla w^R_0 - \nabla(\psi w^R_0)\|_{L^2(B_{2K}(0))} \\ &\quad + (1-\lambda)\|w^R_y - \psi w^R_y\|_{L^2(B_{2K}(0))} \leqslant CR^{-(N-2)}\end{aligned}$$

and this shows that $U^R_{\lambda,y} - Z^R_{\lambda,y} \to 0$ if $R \to \infty$.

Now, in order to complete the proof we have to show the claim. Since $\psi \in C^\infty$, then there exist positive constants C_1 and C_2 such that

$$|\nabla(\psi w^R_0)| = |(\nabla\psi) w^R_0 + (\nabla w^R_0)\psi| \leqslant C_1|w^R_0| + C_2|\nabla w^R_0| \quad \text{in } B_{2K}(0) \tag{2.14}$$

and so by lemma 2.10 with $r = 2$ and (2.14),

$$\begin{aligned}\|\nabla w^R_0 - \nabla(\psi w^R_0)\|^2_{L^2(B_{2K}(0))} &\leqslant \int_{B_{2K}(0)} (C_1|w^R_0| + (C_2+1)|\nabla w^R_0|)^2 \mathrm{d}x \\ &\leqslant CR^{-2(N-2)}\end{aligned}$$

as claimed. □

Lemma 2.13. *If $t > 0$, then $J_{\mathbb{R}^N}(tU^R_{\lambda,y}) - J_{\mathbb{R}^N}(tZ^R_{\lambda,y}) \to 0$, as $R \to \infty$.*

Proof. By the definition of $J_{\mathbb{R}^N}$, we have

$$\begin{aligned}|J_{\mathbb{R}^N}(tU^R_{\lambda,y}) - J_{\mathbb{R}^N}(tZ^R_{\lambda,y})| \\ = \left| \|tU^R_{\lambda,y}\|^2 - \int_{\mathbb{R}^N} f(tU^R_{\lambda,y})(tU^R_{\lambda,y}) - \|tZ^R_{\lambda,y}\|^2 + \int_{\mathbb{R}^N} f(tZ^R_{\lambda,y})(tZ^R_{\lambda,y}) \right| \\ \leqslant \|tU^R_{\lambda,y} - tZ^R_{\lambda,y}\|^2 + \left| \int_{\mathbb{R}^N} f(tZ^R_{\lambda,y})tZ^R_{\lambda,y} - f(tU^R_{\lambda,y})tU^R_{\lambda,y} \right|. \qquad (2.15)\end{aligned}$$

By lemma 2.12 the first parcel of (2.15) is equal to $o_R(1)$ where $o_R(1) \to 0$ as $R \to 0$. So it's enough to show that

$$\left| \int_{\mathbb{R}^N} f(tZ^R_{\lambda,y})(tZ^R_{\lambda,y}) - f(tU^R_{\lambda,y})(tU^R_{\lambda,y}) \right| = \left| \int_{B_{2K}(0)} f(tZ^R_{\lambda,y})(tZ^R_{\lambda,y}) - f(tU^R_{\lambda,y})(tU^R_{\lambda,y}) \right| \\ = o_R(1).$$

For this purpose, (1.2), lemma 2.10 and the inequality $(a+b)^p \leqslant 2^p(a^p + b^p)$, for a, $b \geqslant 0$, yield

$$\begin{aligned}&\left| \int_{B_{2K}(0)} f(tZ^R_{\lambda,y})(tZ^R_{\lambda,y}) - f(tU^R_{\lambda,y})(tU^R_{\lambda,y}) \right| \leqslant \int_{B_{2K}(0)} |tZ^R_{\lambda,y}|^{2^*} + |tU^R_{\lambda,y}|^{2^*} \\ &\leqslant \int_{B_{2K}(0)} |1+\psi^{2^*}||tZ^R_{\lambda,y}|^{2^*} \leqslant C \int_{B_{2K}(0)} |Z^R_{\lambda,y}|^{2^*} \leqslant C \int_{B_{2K}(0)} |\lambda w^R_0 + (1-\lambda) w^R_y|^{2^*} \\ &\leqslant C \int_{B_{2K}(0)} |w^R_0|^{2^*} + |w^R_y|^{2^*} \leqslant C R^{-2^*(N-2)} = o_R(1).\end{aligned}$$

□

LEMMA 2.14.

(a) *There exist* $R_0 > 0$, $T_0 > 2$ *and for each* $R \geqslant R_0$, $y \in \partial B_2(y_0)$ *and* $\lambda \in [0, 1]$, *a unique* $T^R_{\lambda,y}$ *such that*

$$T^R_{\lambda,y} U^R_{\lambda,y} \in \mathcal{N}_\Omega,$$

$T^R_{\lambda,y} \in [0, T_0]$ *and* $T^R_{\lambda,y}$ *is a continuous function of the variables* λ, y *and* R.

(b) *for* $\lambda = 1/2$ *it holds that* $T^R_{1/2,y} \to 2$ *as* $R \to \infty$ *uniformly in* $y \in \partial B_2(y_0)$.

Proof. By lemma 2.4 for each $R > 0$, $y \in \partial B_2(y_0)$ and $\lambda \in [0, 1]$ there exists $T^R_{\lambda,y} = t(U^R_{\lambda,y})$. Now for such fixed $R > 0$, the function $(\lambda, y) \to U^R_{\lambda,y}$ is continuous and $t(U^R_{\lambda,y})$ is in C^1. Since $[0, 1] \times \partial B_2(y_0)$ is a compact set in $\mathbb{R}^2$, there is $T_0(R) = \max_{(\lambda,y)\in[0,1]\times\partial B_2(y_0)} T^R_{\lambda,y}$ such that $T^R_{\lambda,y} U^R_{\lambda,y} \in \mathcal{N}_\Omega$ and $T^R_{\lambda,y} \in [0, T_0(R)]$. Suppose by contradiction that $T_0(R_j) \longrightarrow \infty$ as $R_j \longrightarrow \infty$. Since $T_0(R_j) = \max_{(\lambda,y)\in[0,1]\times\partial B_2(y_0)} T^{R_j}_{\lambda,y}$, then $T_0(R_j) = T^{R_j}_{\lambda,y}$ for some (λ, y). Let $u, v > 0$, and $r \in$

$(0,\infty)$, then using that $\frac{f(s)}{s}$ is increasing by assumption (f_1), it holds

$$\begin{aligned} J_{\mathbb{R}^N}(ru+rv) &= r^2(\|u\|^2+\|v\|^2+2\langle u,v\rangle) - \int_{\mathbb{R}^N} \frac{f(ru+rv)}{ru+rv}(ru+rv)^2 \\ &\leqslant r^2\left(\|u\|^2 - \int_{\mathbb{R}^N}\frac{f(ru)}{ru}u^2 + \|v\|^2 - \int_{\mathbb{R}^N}\frac{f(rv)}{rv}v^2 + 2\langle u,v\rangle\right). \end{aligned} \tag{2.16}$$

Now for $\lambda\in[0,1]$ and $y\in\partial B_2(y_0)$, setting $u:=\lambda w_0^{R_j},\ v:=(1-\lambda)w_y^{R_j},\ r=T_{\lambda,y}^{R_j}$ and (2.16), we have

$$\begin{aligned} 0 &= J_{\mathbb{R}^N}(T_{\lambda,y}^{R_j}U_{\lambda,y}^{R_j}) \\ &\leqslant (T_{\lambda,y}^{R_j})^2(\|\lambda w_0^{R_j}\|_{\mathbb{R}^N}^2 - \int_{\mathbb{R}^N}\frac{f(T_{\lambda,y}^{R_j}\lambda w_0^{R_j})}{T_{\lambda,y}^{R_j}\lambda w_0^{R_j}}(\lambda w_0^{R_j})^2 \\ &\quad + \|(1-\lambda)w_y^{R_j}\|_{\mathbb{R}^N}^2 - \int_{\mathbb{R}^N}\frac{f(T_{\lambda,y}^{R_j}(1-\lambda)w_y^{R_j})}{T_{\lambda,y}^{R_j}(1-\lambda)w_y^{R_j}}((1-\lambda)w_y^{R_j})^2 \\ &\quad + 2\langle\lambda w_0^{R_j},(1-\lambda)w_y^{R_j}\rangle_{\mathbb{R}^N}) \\ &\leqslant (T_{\lambda,y}^{R_j})^2\left\{\int_{\mathbb{R}^N}\left(\frac{f(w_0^{R_j})}{w_0^{R_j}} - \frac{f(T_{\lambda,y}^{R_j}\lambda w_0^{R_j})}{T_{\lambda,y}^{R_j}\lambda w_0^{R_j}}\right)(\lambda w_0^{R_j})^2\right. \\ &\quad \left. + \int_{\mathbb{R}^N}\left(\frac{f(w_y^{R_j})}{w_y^{R_j}} - \frac{f(T_{\lambda,y}^{R_j}(1-\lambda)w_y^{R_j})}{T_{\lambda,y}^{R_j}(1-\lambda)w_y^{R_j}}\right)((1-\lambda)w_y^{R_j})^2 + o_R(1)\right\}. \end{aligned}$$

As we are assuming that $T_{\lambda,y}^{R_j}\to\infty$ as $R_j\to\infty$, then we get a contradiction since by (f_1) and the Monotone Convergence theorem

$$\int_{\mathbb{R}^N}\left(\frac{f(w_0^R)}{w_0^R} - \frac{f(T_{\lambda,y}^R\lambda w_0^R)}{T_{\lambda,y}^R\lambda w_0^R}\right)(\lambda w_0^R)^2 < S_0 < 0$$

and

$$\int_{\mathbb{R}^N}\left(\frac{f(w_y^R)}{w_y^R} - \frac{f(T_{\lambda,y}^R(1-\lambda)w_y^R)}{T_{\lambda,y}^R(1-\lambda)w_y^R}\right)((1-\lambda)w_y^R)^2 < S_0 < 0,$$

for R_j sufficiently large, $\lambda\in[0,1]$ and $y\in\partial B_2(y_0)$, where S_0 may be taken, for instance, as $S_0 := (f(w_0^R))/(w_0^R) - (f(2w_0^R))/(2w_0^R)$.

In order to prove part (b) let $\varphi(u,v)=f(u+v)-f(u)-f(v)$, then from lemma 2.11 we have

$$\int_{\mathbb{R}^N}|\varphi(w_0^R,w_y^R)(w_0^R+w_y^R)| \leqslant \int_{\mathbb{R}^N}(w_0^Rw_y^R)^q(w_0^R+w_y^R) = o_R(1).$$

Thus

$$
\begin{aligned}
J_{\mathbb{R}^N}(w_0^R + w_y^R) &= \|w_0^R + w_y^R\|^2 - \int_{\mathbb{R}^N} f(w_0^R + w_y^R)(w_0^R + w_y^R) \\
&= \|w_0^R\|^2 + \|w_y^R\|^2 + 2\langle w_0^R, w_y^R\rangle - \int_{\mathbb{R}^N} f(w_0^R)(w_0^R) - \int_{\mathbb{R}^N} f(w_y^R)(w_y^R) \\
&\quad - \int_{\mathbb{R}^N} f(w_0^R)(w_y^R) - \int_{\mathbb{R}^N} f(w_y^R)(w_0^R) + \int_{\mathbb{R}^N} \varphi(w_0^R, w_y^R)(w_0^R + w_y^R) \\
&= J_{\mathbb{R}^N}(w_0^R) + J_{\mathbb{R}^N}(w_y^R) + o_R(1) = o_R(1),
\end{aligned}
$$

recalling that w is a solution of $(P_{\mathbb{R}^N})$. Together with lemma 2.13, this gives

$$
J_{\mathbb{R}^N}((w_0^R + w_y^R)\psi) = J_{\mathbb{R}^N}(w_0^R + w_y^R) + o_R(1) = o_R(1) \quad \text{as} \quad R \to \infty. \tag{2.17}
$$

Therefore, (2.17) and $\mathcal{D}^{1,2}(\Omega) \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ yield

$$
\begin{aligned}
J_\Omega(2U^R_{1/2,y}) &= J_\Omega((w_0^R + w_y^R)\psi)) \\
&= J_{\mathbb{R}^N}((w_0^R + w_y^R)\psi) = o_R(1)
\end{aligned}
$$

and so $T^R_{1/2,y} \to 2$. Indeed, without loss of generality, suppose by contradiction that $T^R_{1/2,y} \to T > 2$. Given $\delta > 1$ such that $2 < 2\delta < T$, there exists $R_0 > 0$ such that $T^R_{1/2,y} > 2\delta$ for all $R > R_0$, $y \in \partial B_2(y_0)$. Then applying the previous argument, $f(s)/s$ increasing and the translation invariance of integrals,

$$
\begin{aligned}
0 = J_{\mathbb{R}^N}\left(\frac{T^R_{1/2,y}}{2}w_0^R + \frac{T^R_{1/2,y}}{2}w_y^R\right) &\leqslant \left\|\frac{T^R_{1/2,y}}{2}w_0^R\right\|^2_{\mathbb{R}^N} \\
&\quad - \int_{\mathbb{R}^N}\left(\frac{f((T^R_{1/2,y})/2w_0^R)}{(T^R_{1/2,y})/2w_0^R}\right)\left(\frac{T^R_{1/2,y}}{2}w_0^R\right)^2 \\
&\quad + \left\|\frac{T^R_{1/2,y}}{2}w_y^R\right\|^2_{\mathbb{R}^N} - \int_{\mathbb{R}^N}\left(\frac{f((T^R_{1/2,y})/2w_y^R)}{(T^R_{1/2,y})/2w_y^R}\right)\left(\frac{T^R_{1/2,y}}{2}w_y^R\right)^2 + o_R(1) \\
&\leqslant 2\int_{\mathbb{R}^N}\left(\frac{f(w)}{w} - \frac{f(\delta w)}{\delta w}\right)(\delta w)^2 + o_R(1) < 0
\end{aligned}
$$

which yields a contradiction. Likewise, if $T^R_{1/2,y} \to T < 2$ then $J_{\mathbb{R}^N}((T^R_{1/2,y})/2w_0^R + (T^R_{1/2,y})/2w_y^R) > 0$, and this completes the proof of the lemma. □

3. Compactness condition

First, we present two crucial lemmas which will be used in the proof of the splitting lemma.

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LEMMA 3.1.

(a) *If v and u are in a bounded subset of $L^p + L^q$, then $f'(v)u$ is in a bounded subset of $L^{p'} + L^{q'}$;*

(b) *f' is a bounded map from $L^p + L^q$ into $L^{p/(p-2)} + L^{q/(q-2)}$.*

Proof. Lemma 2.3 in [7]. □

LEMMA 3.2. *Assume that the sequence (u_k) converges to u_0 weakly in $\mathcal{D}^{1,2}(\Omega)$. Set $u_k^1 = u_k - u_0$ then it holds:*

(a) $\|u_k^1\|^2_{\mathcal{D}^{1,2}(\Omega)} = \|u_k\|^2_{\mathcal{D}^{1,2}(\Omega)} - \|u_0\|^2_{\mathcal{D}^{1,2}(\Omega)} + o(1)$;

(b) $\displaystyle\int_\Omega f(u_k^1)u_k^1 = \int_\Omega f(u_k)u_k - \int_\Omega f(u_0)u_0 + o(1)$;

(c) $\displaystyle\int_\Omega F(u_k^1) = \int_\Omega F(u_k) - \int_\Omega F(u_0) + o(1)$.

Proof. Lemma 2.8 in [7] and lemma 3.6 in [15]. □

Note that $I'_{\mathcal{N}_V} I(u)$ is orthogonal projection of $I'_\Omega(u)$ onto the tangent space of $\mathcal{N}_\Omega$ at u, that is defined by $T_u(\mathcal{N}_\Omega) := \{v \in \mathcal{D}^{1,2}(\Omega);\quad J'_\Omega(u)v = 0\}$. Recall that a sequence (u_k) in $\mathcal{D}^{1,2}(\Omega)$ is said to be a $(PS)_d$-sequence for I_Ω restricted to $\mathcal{N}_\Omega$ if $I_\Omega(u_k) \to d$ and $\|I'_{\mathcal{N}_\Omega}(u_k)\| \to 0$. The functional I_Ω satisfies the Palais-Smale condition on $\mathcal{N}_\Omega$ at the level d if every $(PS)_d$-sequence for I_Ω restricted to $\mathcal{N}_\Omega$ contains a convergent subsequence.

Now we proceed with the study of Palais Smale sequences of I_Ω. Usually, the compactness results depend on P-L. Lion's lemma [19]. However that lemma does not apply directly if (u_k) is bounded in $\mathcal{D}^{1,2}(\Omega)$. We present the following result in the lines of [1], lemma 2, (see also [15]).

LEMMA 3.3. *Suppose (u_k) is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and there exists $R > 0$ such that*

$$\lim_{k\to\infty}\left(\sup_{y\in\mathbb{R}^N}\int_{B(y,R)} |u_k|^2\right) = 0,$$

then $\displaystyle\int_{\mathbb{R}^N} f(u_k)u_k \to 0.$

Proof. Fix $\varepsilon \in (0,1)$ and for every k consider the new sequence of functions

$$w_k := \begin{cases} |u_k| & |u_k| \geqslant \varepsilon, \\ |u_k|^{2^*/2}\varepsilon^{-(2^*/2-1)} & |u_k| \leqslant \varepsilon. \end{cases}$$

It is easy to verify that

$$|w_k|^2 \leqslant |u_k|^2, \quad |w_k|^2 \leqslant |u_k|^{2^*}\varepsilon^{-(2^*-2)}, \quad |\nabla w_k|^2 \leqslant \left(\frac{2^*}{2}\right)^2 |\nabla u_k|^2,$$

since

$$|w_k|^2 = \begin{cases} |u_k|^2 & |u_k| \geqslant \varepsilon, \\ |u_k|^{2^*}\varepsilon^{-(2^*-2)} \leqslant |u_k|^2 \dfrac{|u_k|^{2^*-2}}{\varepsilon^{(2^*-2)}} \leqslant |u_k|^2 & |u_k| \leqslant \varepsilon, \end{cases}$$

$$|w_k|^2 = \begin{cases} |u_k|^2 = \dfrac{|u_k|^{2^*}}{|u_k|^{2^*-2}} \leqslant \dfrac{|u_k|^{2^*}}{\varepsilon^{2^*-2}} = |u_k|^{2^*}\varepsilon^{-(2^*-2)} & |u_k| \geqslant \varepsilon, \\ |u_k|^{2^*}\varepsilon^{-(2^*-2)} & |u_k| \leqslant \varepsilon, \end{cases}$$

$$\nabla w_k = \begin{cases} \nabla |u_k| & |u_k| \geqslant \varepsilon, \\ \nabla(|u_k|^{2^*/2}\varepsilon^{-(2^*/2-1)}) & \\ = \dfrac{2^*}{2}\varepsilon^{-(2^*/2-1)}|u_k|^{2^*/2-1}\nabla|u_k| \leqslant \dfrac{2^*}{2}\nabla|u_k| & |u_k| \leqslant \varepsilon. \end{cases}$$

So

$$\begin{aligned} \|w_k\|^2_{H^1(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} |w_k|^2 + |\nabla w_k|^2 \\ &\leqslant \int_{\mathbb{R}^N} |u_k|^{2^*}\varepsilon^{-(2^*-2)} + \int_{\mathbb{R}^N} \left(\frac{2^*}{2}\right)^2 |\nabla u_k|^2 \leqslant C\varepsilon^{-(2^*-2)}, \end{aligned}$$

in particular, $w_k \in H^1(\mathbb{R}^N)$. We claim that $w_k \to 0$ in $L^s(\mathbb{R}^N)$ for each $2 < s < 2^*$. Indeed, for any $y \in \mathbb{R}^N$ and $s \in (2, 2^*)$, using the Sobolev continuous embedding $H^1(B(y,R)) \hookrightarrow L^{2^*}(B(y,R))$ we have

$$\begin{aligned} \int_{B(y,R)} |w_k|^s &\leqslant \left(\int_{B(y,R)} |w_k|^2\right)^{(1-\theta)s/2} \left(\int_{B(y,R)} |w_k|^{2^*}\right)^{\theta s/2^*} \\ &\leqslant C\left(\int_{B(y,R)} |w_k|^2\right)^{(1-\theta)s/2} \left(\int_{B(y,R)} |w_k|^2 + |\nabla w_k|^2\right)^{\theta s/2}, \end{aligned}$$

where $\theta = (s-2)/2sN$. Now suppose $\theta s \geqslant 2$ that is, $s \geqslant 4/N + 2 = \overline{s}$, then

$$\int_{B(y,R)} |w_k|^s \leqslant C\left(\int_{B(y,R)} |w_k|^2\right)^{(1-\theta)s/2} \left(\int_{B(y,R)} |w_k|^2 + |\nabla w_k|^2\right) \|w_k\|^{\theta s-2}_{H^1(\mathbb{R}^N)}.$$

Now, covering $\mathbb{R}^N$ by balls of radius R, in such a way that each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls, we find

$$\int_{\mathbb{R}^N} |w_k|^s \leqslant (N+1)\sup_{y\in\mathbb{R}^N}\left(\int_{B(y,R)} |w_k|^2\right)^{(1-\theta)s/2} \|w_k\|^{\theta s}_{H^1(\mathbb{R}^N)}.$$

But $w_k \in H^1(\mathbb{R}^N)$ and so by the assumption of lemma, $w_k \to 0$ in $L^s(\mathbb{R}^N)$ for $s \geqslant \overline{s}$. If $2 < s < \overline{s}$, $s = 2\theta + \overline{s}(1-\theta)$ for some $\theta \in (0,1)$, hence by the Holder's inequality,

$$\|w_k\|^s_{L^{\overline{s}}(\mathbb{R}^N)} \leqslant \|w_k\|^{\theta}_{L^2(\mathbb{R}^N)}\|w_k\|^{1-\theta}_{L^s(\mathbb{R}^N)}$$

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and the claim then follows from the case already established. Now using (f_2), we conclude

$$\begin{aligned}\int_{\mathbb{R}^N} f(u_k)u_k &\leqslant C\int_{\{|u_k|\geqslant 1\}}|u_k|^p + C\int_{\{|u_k|\leqslant 1\}}|u_k|^q\\ &\leqslant C\int_{\{|u_k|\geqslant \varepsilon\}}|u_k|^p - C\int_{\{\varepsilon\leqslant|u_k|\leqslant 1\}}|u_k|^p + C\int_{\{\varepsilon\leqslant|u_k|\leqslant 1\}}|u_k|^q + C\int_{\{|u_k|\leqslant \varepsilon\}}|u_k|^q\\ &\leqslant C\int_{\{|u_k|\geqslant \varepsilon\}}|u_k|^p - C\int_{\{\varepsilon\leqslant|u_k|\leqslant 1\}}|u_k|^p + C\int_{\{\varepsilon\leqslant|u_k|\leqslant 1\}}|u_k|^p + C\int_{\{|u_k|\leqslant \varepsilon\}}|u_k|^q\\ &= C\int_{\{|u_k|\geqslant \varepsilon\}}|u_k|^p + C\int_{\{|u_k|\leqslant \varepsilon\}}|u_k|^q\\ &\leqslant C\int_{\{|u_k|\geqslant \varepsilon\}}|w_k|^p + C\int_{\{|u_k|\leqslant \varepsilon\}}|u_k|^{q-2^*}|u_k|^{2^*}\\ &\leqslant C\|w_k\|^p_{L^p(\mathbb{R}^N)} + C\varepsilon^{q-2^*}\|u_k\|^{2^*}_{L^{2^*}(\mathbb{R}^N)}\end{aligned}$$

by which, since $w_k \to 0$ and $q > 2^*$

$$\int_{\mathbb{R}^N} f(u_k)u_k \leqslant C\varepsilon^{q-2^*}.$$

Because $\varepsilon \in (0,1)$ is arbitrary we get the conclusion. □

LEMMA 3.4. *Every $(PS)_d$-sequence (u_k) for I_Ω restricted the $\mathcal{N}_\Omega$ contains a bounded subsequence which is a $(PS)_d$-sequence for I_Ω in $\mathcal{D}^{1,2}(\Omega)$.*

Proof. Let (u_k) be a $(PS)_d$-sequence for I_Ω on $\mathcal{N}_\Omega$, by (2.3) with replaced $\mathbb{R}^N$ by Ω and $I_\Omega(u_k) \to d$ we have that (u_k) is bounded. To complete the proof we show that $I'_{\mathcal{N}_\Omega}(u_k) \to 0$ imply

$$I'_\Omega(u_k) \to 0 \quad \text{in } (\mathcal{D}^{1,2}(\Omega))'. \tag{3.1}$$

Write

$$I'_\Omega(u_k) = I'_{\mathcal{N}_\Omega}(u_k) + t_k J'_\Omega(u_k). \tag{3.2}$$

By property (f_2) and remark 1.1, Holder's inequality, Sobolev inequality and the boundednes of (u_k), for any $v \in \mathcal{D}^{1,2}(\Omega)$,

$$|\int_\Omega [f'(u_k)u_k - f(u_k)]v| \leqslant C\int_\Omega (|u_k|^{2^*-1})|v| \leqslant C\|u_k\|^{2^*-1}_{L^{2^*}}\|v\|_{L^{2^*}} \leqslant C\|v\|_\Omega.$$

Therefore

$$|\langle J'_\Omega(u_k), v\rangle_\Omega| = |2\langle u_k, v\rangle_\Omega - \int_\Omega [f'(u_k)u_k + f(u_k)]v| \leqslant C\|v\|\ , \quad \forall v \in \mathcal{D}^{1,2}(\Omega).$$

This proves that$(J'_\Omega(u_k))$ is bounded in $(\mathcal{D}^{1,2}(\Omega))'$.

As $|J'_\Omega(u_k)u_k| \leqslant \|J'_\Omega(u_k)\|\|u_k\|_\Omega < C$, after passing to a subsequence, we have that $|J'_\Omega(u_k)u_k| \to \varrho \geqslant 0$. We will show that $\varrho > 0$. From lemma 2.4 (d) and $u_k \in \mathcal{N}_V$, we have

$$0 < \rho^2 \leqslant \|u_k\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} f(u_k)u_k, \tag{3.3}$$

then by lemma 3.3 there is $\delta > 0$ such that

$$\sup_{y\in\mathbb{R}^N} \int_{B(y,R)} |u_k|^2 > \delta,$$

and so there exists a sequence (y_k) such that

$$\int_{B(y_k,R)} |u_k|^2 \geqslant \delta. \tag{3.4}$$

Now consider $\tilde{u}_k = u_k(\cdot - y_k)$, which is bounded and passing to a subsequence, $\tilde{u}_k \rightharpoonup u$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and $\tilde{u}_k \to u$ in $L^2_{loc}(\mathbb{R}^N)$. We claim that $u \not\equiv 0$. Indeed if $\|\tilde{u}_k\|_{L^2(B(0,R))} \to 0$ as $k \to \infty$ we have a contradiction with (3.4). Hence, $u \not\equiv 0$ and there exists a subset Λ of positive measure such that $u(x) \not\equiv 0$ for every $x \in \Lambda$. Property (f_1) implies that $f'(s)s^2 - f(s)s > 0$ if $s \neq 0$. So, from Fatou's lemma, it follows that

$$\begin{aligned}
\varrho &= \liminf_{k\to\infty} |J'(u_k)u_k| = \liminf_{k\to\infty} \left\{ 2\|u_k\|^2 - \int_\Omega [f'(u_k)u_k^2 + f(u_k)u_k] \right\} \\
&= \liminf_{k\to\infty} \int_\Omega [f'(u_k)u_k^2 - f(u_k)u_k] \\
&= \liminf_{k\to\infty} \int_{\mathbb{R}^N} [f'(u_k)u_k^2 - f(u_k)u_k] = \liminf_{k\to\infty} \int_{\mathbb{R}^N} [f'(\tilde{u}_k)\tilde{u}_k^2 - f(\tilde{u}_k)\tilde{u}_k] \\
&\geqslant \liminf_{k\to\infty} \int_\Lambda [f'(\tilde{u}_k)\tilde{u}_k^2 - f(\tilde{u}_k)\tilde{u}_k] \geqslant \int_\Lambda \liminf_{k\to\infty} [f'(\tilde{u}_k)\tilde{u}_k^2 - f(\tilde{u}_k)\tilde{u}_k] \\
&= \int_\Lambda [f'(u)u^2 - f(u)u] > 0,
\end{aligned}$$

hence the claim that $\varrho > 0$ is proved. Taking the inner product of (3.2) with u_k it holds that

$$0 = I'_\Omega(u_k)u_k = \langle I'_{\mathcal{N}_\Omega}(u_k), u_k\rangle_\Omega + t_k J'_{\mathcal{N}_\Omega}(u_k)u_k = o_k(1) + t_k J'_{\mathcal{N}_\Omega}(u_k)u_k,$$

so $t_k \to 0$ and from (3.2) we deduce $I'_\Omega(u_k) \to 0$ as $I'_{\mathcal{N}_\Omega}(u_k) \to 0$. This completes the proof of the lemma. □

Lemma 3.5 (Splitting). *Let (u_k) be a sequence in $\mathcal{N}_\Omega$ such that*

$$I_\Omega(u_k) \to d \quad \textit{and} \quad I'_{\mathcal{N}_\Omega}(u_k) \to 0 \quad \textit{in } (\mathcal{D}^{1,2}(\Omega))'.$$

Replacing u_k by a subsequence if necessary, there exist a solution u_0 of $(\mathcal{P})$, a number $m \in \mathbb{N}$, m function $w_1, \ldots, w_m$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and m sequences of points $(y_k^j) \in \mathbb{R}^N$, $1 \leqslant j \leqslant m$, satisfying:

(a) $u_k \to u_0$ *in* $\mathcal{D}^{1,2}(\Omega)$ *or*

(b) w_j *are nontrivial solutions of* $(\mathcal{P}_{\mathbb{R}^N})$;

(c) $|y_k^j| \to +\infty$ *e* $|y_k^j - y_k^i| \to +\infty$ $i \neq j$;

(d) $u_k - \sum_{i=1}^{m} w_j(\cdot - y_k^j) \to u_0$ *in* $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

(e) $d = I_\Omega(u_0) + \sum_{i=1}^{m} I_{\mathbb{R}^N}(w_j)$.

Proof. By lemma 3.4 (u_k) is bounded and we can extract a subsequence, which converges to u_0 weakly in $\mathcal{D}^{1,2}(\Omega)$. We verify that u_0 solves $(\mathcal{P})$. Indeed, by lemma 3.4 for $\varphi \in C_0^\infty(\Omega)$, it follows

$$I'_\Omega(u_k)\varphi = \int_\Omega \nabla u_k \nabla \varphi \mathrm{d}x - \int_\Omega f(u_k)\varphi \mathrm{d}x \to 0 \quad \text{as } k \to \infty. \tag{3.5}$$

By (b) of lemma 3.1 and the fact that for $p < 2^*$, $u_k \to u_0$ strongly in $L^p_{loc}(\Omega)$ and using the mean value theorem

$$f(u_k(x)) - f(u_0(x)) = f'(u_k(x) + \theta(x)u_0(x))(u_k(x) - u_0(x)) \text{ with } 0 < \theta(x) < 1\,,$$

from (f_2) and (1.2) we get

$$\int_\Omega |f(u_k) - f(u_0)|\varphi \mathrm{d}x \leqslant \int_{supp\varphi} (|u_k| + |u_0|)^{2^*-2}(u_k - u_0)\varphi \mathrm{d}x \to 0 \quad \text{as } k \to \infty,$$

and so

$$\int_\Omega \nabla u_k \nabla \varphi \mathrm{d}x - \int_\Omega f(u_k)\varphi \mathrm{d}x \to \int_\Omega \nabla u_0 \nabla \varphi \mathrm{d}x - \int_\Omega f(u_0)\varphi \mathrm{d}x \text{ as } \; k \to \infty. \tag{3.6}$$

By (3.5) and (3.6), u_0 solves $(\mathcal{P})$ and immediately $u_0 \in \mathcal{N}_\Omega$. Now set $u_k^1 = u_k - u_0$ and define $u_k^1 = 0$ in $\mathbb{R}^N \setminus \Omega$, so u_k^1 converges to zero weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and as we will see in remark 3.6, $I'_\Omega(u_k^1)u_k^1 \to 0$ and so

$$I'_\Omega(u_k^1)u_k^1 = \int_\Omega |\nabla u_k^1|^2 - \int_\Omega f(u_k^1)u_k^1 \to 0 \quad \text{as } k \to \infty. \tag{3.7}$$

Lemma 3.2, (a) and (b), imply that

$$\|u_k^1\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = \|u_k\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} - \|u_0\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} + o(1) \tag{3.8}$$

$$I_{\mathbb{R}^N}(u_k^1) = I_\Omega(u_k) - I_\Omega(u_0) + o(1). \tag{3.9}$$

Assume $u_k^1 \not\to 0$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, otherwise we have the claim, then from (3.7)

$$0 < \eta \leqslant \|u_k^1\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} f(u_k^1)u_k^1 + o(1). \tag{3.10}$$

Arguing as in lemma 3.4, there is (y_k) and $\delta > 0$ such that

$$\int_{B(y_k,R)} |u_k^1|^2 > \delta. \tag{3.11}$$

Now consider $\tilde{u}_k = u_k^1(\cdot - y_k^1)$, which is bounded, so passing to a subsequence there is $\tilde{u}_k \rightharpoonup u$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and $\tilde{u}_k \to u$ in $L^2_{loc}(\mathbb{R}^N)$. We claim that $u \not\equiv 0$. Indeed if $\|\tilde{u}_k\|_{L^p(B(0,R))} \to 0$ as $k \to \infty$ this contradicts (3.11) and the claim is proved. Hence by the boundedness of u_k^1, there exists $w_1 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $u_k^1(x - y_k^1) \to w_1 \neq 0$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and the sequence $(y_k^1) \in \mathbb{R}^N$ with $y_k^1 \to \infty$ as $k \to \infty$, since if (y_k^1) were bounded, by passing to subsequence, we should find y^1 that $y_k^1 \to y^1$ and

$$\int_{B(y^1,R)} |u_k^1|^2 > \delta\,. \tag{3.12}$$

As above u_k^1 is bounded, so passing to a subsequence there is u^1 such that $u_k^1 \rightharpoonup u^1$ in $\mathcal{D}^{1,2}(B(y^1,R))$ and $u^1 \not\equiv 0$, which is contradictory with u_k^1 converging weakly to 0 in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Moreover, w_1 is a weak solution of $(\mathcal{P}_{\mathbb{R}^N})$ and the proof of this is remark 3.6, which is stated in what follows. Define $u_k^2 := u_k^1 - w_1(\cdot - y_k^1)$ then, by arguing as before, u_k^2 satisfies

$$I_{\mathbb{R}^N}(u_k^2) \to d - I_\Omega(u_0) - I_{\mathbb{R}^N}(w_1)$$

and if $u_k^2 \not\to 0$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ (otherwise we have the claim) then there exists a sequence $\{y_k^2\} \in \mathbb{R}^N$ with $\{y_k^2\} \to \infty$ as $k \to \infty$ and $u_k^2(x - y_k^1) \to w_2 \neq 0$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, such that w_2 is a weak solution of $(\mathcal{P}_{\mathbb{R}^N})$. Moreover, any nontrivial critical point u of $I_{\mathbb{R}^N}$ satisfies $I_{\mathbb{R}^N}(u) \geqslant c > 0$, so iterating the above procedure we construct sequences w_i and (y_k^j). Since for every i, $I_{\mathbb{R}^N}(w_i) \geqslant c$, the iteration must terminate at some finite index m. □

Remark 3.6. It holds that $w_1 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ is a weak solution of $(\mathcal{P}_{\mathbb{R}^N})$.

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$, using the mean value theorem and (f_2), by (b) of lemma 3.1, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla u_k^1(x - y_k^1)\nabla\varphi - f(u_k^1(x - y_k^1))\varphi \,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^N} \nabla u_k^1(z)\nabla\varphi(z + y_k^1) - f(u_k^1(z))\varphi(z + y_k^1) \,\mathrm{d}z \\
&\quad = \int_{\mathbb{R}^N} [f(u_k) - f(u_0) - f(u_k^1)]\varphi(z + y_k^1) \,\mathrm{d}z + o(1) \\
&\quad \leqslant \int_{B_R} [f(u_0 + u_k^1) - f(u_0)]\varphi(z + y_k^1) \,\mathrm{d}z
\end{aligned}$$

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$$+\int_{\mathbb{R}^N\setminus B_R}[f(u_0+u_k^1)-f(u_k^1)]\varphi(z+y_k^1)\,\mathrm{d}z$$
$$-\int_{B_R} f(u_k^1)\varphi(z+y_k^1)\,\mathrm{d}z-\int_{\mathbb{R}^N\setminus B_R} f(u_0)\varphi(z+y_k^1)\,\mathrm{d}z+o(1)$$
$$\leqslant C\|(|u_0|^{2^*-2}+|u_k^1|^{2^*-2})\varphi(\cdot+y_k^1)\|_{L^{p'}(\mathbb{R}^N)}a_{k,R}$$
$$+C\|(|u_0|^{2^*-2}+|u_k^1|^{2^*-2})\varphi(\cdot+y_k^1)\|_{L^{p'}\cap L^{q'}(\mathbb{R}^N)}b_R+o(1)$$

where $a_{k,R}=\|u_k^1\|_{L^p(B_R)}$, $b_R=\|u_0\|_{L^{p'}\cap L^{q'}(\mathbb{R}^N\setminus B_R)}$. Since $b_R\to 0$ as $R\to\infty$, and given R, $a_{k,R}\to 0$ as $k\to\infty$, by the above estimate we get

$$\int_{\mathbb{R}^N}\nabla u_k^1(x-y_k^1)\nabla\varphi-f(u_k^1(x-y_k^1))\varphi\,\mathrm{d}x\to 0$$

as $k\to\infty$. On the other hand, by (a) of lemma 3.1, it is easy to see that

$$\int_{\mathbb{R}^N}\nabla u_k^1(x-y_k^1)\nabla\varphi-f(u_k^1(x-y_k^1))\varphi\,\mathrm{d}x\to\int_{\mathbb{R}^N}\nabla w_1\nabla\varphi-f(w_1))\varphi\,\mathrm{d}x.$$

So we get the claim and the proof of the lemma is complete. □

Corollary 3.7 (Compactness). *I_Ω satisfies the Palais-Smale condition on $\mathcal{N}_\Omega$ at every level $d\in(c,2c)$.*

Proof. Let (u_k) be a $(PS)_d$-sequence for I_Ω on $\mathcal{N}_\Omega$. If $d\in(c,\overline{c})$ and (u_k) does not have a convergent subsequence then, by the splitting lemma,

$$\overline{c}>d=I(u_0)+\sum_{i=1}^{m}I_{\mathbb{R}^N}(w_j)\geqslant\begin{cases} mc & \text{if } u_0=0,\\ c_\Omega+mc\geqslant(m+1)c & \text{if } u_0\neq 0.\end{cases}\tag{3.13}$$

Then in both cases, $m<2$ and so $m=1$. The hypothesis $2c>d\geqslant(m+1)c$ implies that it is not possible to have $m=1$ and $u_0\neq 0$, therefore $u_0=0$, which yields $I(u_n)\to I_{\mathbb{R}^N}(w_1)=d$ giving a contradiction with the uniqueness of solution of $(\mathcal{P}_{\mathbb{R}^N})$. Hence, I_Ω satisfies the Palais-Smale condition on $\mathcal{N}_\Omega$ at every $d\in(c,2c)$. □

Remark 3.8. If u is a solution of $(\mathcal{P})$ with $I_\Omega(u)\in[c,2c)$, then u does not change sign. In fact, if u is a solution of $(\mathcal{P})$ then

$$0=I'_\Omega(u)u^\pm=J_\Omega(u^\pm),$$

where $u^+:=\max\{u,0\}$ and $u^-:=\min\{u,0\}$, so $u^\pm\in\mathcal{N}_\Omega$. Now if $u^+\neq 0$ and $u^-\neq 0$, then

$$I_\Omega(u)=I_\Omega(u^+)+I_\Omega(u^-)\geqslant 2c.$$

4. Existence of a positive solution

Now, for any $R>0$, $y\in\partial B_2(y_0)$, let us define

$$\varepsilon_R:=\int_{\mathbb{R}^N}f(w_0^R)w_y^R.$$

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Lemma 4.1. *There exists $C > 0$ such that*

$$\varepsilon_R = \int_{\mathbb{R}^N} f(w_0^R) w_y^R \leqslant C R^{-(N-2)} \tag{4.1}$$

for all $y \in \partial B_2(y_0)$ and $R \geqslant 1$.

Proof. It is sufficient to take $r = 2^*$ and $s = 1$ in lemma 2.11. □

Note that the previous lemma implies $\varepsilon_R \to 0$ as $R \to \infty$, uniformly for y in $\partial B_2(y_0)$.

Lemma 4.2. *There exists $C > 0$ such that for all $s, t \geqslant 1/2$, $y \in \partial B_2(y_0)$ and $R \geqslant 1$,*

$$\int_{\mathbb{R}^N} f(s w_0^R) t w_y^R \geqslant C R^{-(N-2)}. \tag{4.2}$$

Proof. For $|x| < 1$ and $R \geqslant 1$, we have

$$1 + |x - R(y - y_0)| < 1 + |x| + R|(y - y_0)| < 4R. \tag{4.3}$$

Now by (f_1), (4.3) and the decay estimates (2.1) there exists $C > 0$ such that

$$\begin{aligned}
\int_{\mathbb{R}^N} f(s w_0^R) t w_y^R &= st \int_{\mathbb{R}^N} \left[\frac{f(s w_0^R)}{s w_0^R} \right] w_0^R w_y^R \\
&\geqslant \frac{1}{4} \int_{\mathbb{R}^N} \left[\frac{f(1/2 w_0^R)}{1/2 w_0^R} \right] w_0^R w_y^R \\
&\geqslant \frac{1}{4} \int_{B_1(R y_0)} \left[\frac{f(1/2 w_0^R)}{1/2 w_0^R} \right] w_0^R w_y^R \\
&\geqslant \frac{1}{4} \left[\min_{x \in B_1(0)} \frac{f(1/2 w(x))}{1/2 w(x)} \right] \int_{x \in B_1(0)} w(x) w(x - R(y - y_0)) \\
&\geqslant C \int_{B_1(0)} (1 + |x|)^{-(N-2)} w(x - R(y - y_0)) \\
&\geqslant C R^{-(N-2)}.
\end{aligned}$$

□

If we set $s, t = 1$ in the previous lemma we have

$$\varepsilon_R \geqslant C R^{-(N-2)}. \tag{4.4}$$

Lemma 4.3. *For every $b > 1$ there is a constant $C > 0$ such that*

$$|\int_{\Omega} [s f(w_0^R \psi) - f(s w_0^R \psi)] w_y^R \psi| \leqslant C |s - 1| \; \varepsilon_R,$$

for all $s \in [0, b]$, $y \in \partial B_2(y_0)$ and $R \geqslant 1$.

Proof. Fix $u \in \mathbb{R}$ and consider the function $g(s) := sf(u) - f(su)$. By (1.2),

$$\begin{aligned} g'(s) := f(u) - f'(su)u &\leqslant |f(u)| + C(s^{2^*-1}|u|^{2^*}) \\ &\leqslant C|u|^{2^*} \quad \forall s \in [0,1]. \end{aligned}$$

Hence, by the mean value theorem,

$$\begin{aligned} |sf(u) - f(su)| = |g(s) - g(1)| &= |g'(t)||s-1| \\ &\leqslant C|u|^{2^*}|s-1|. \end{aligned}$$

This inequality yields

$$\begin{aligned} &\int_\Omega |sf(w_0^R\psi) - f(sw_0^R\psi)|w_y^R\psi \\ &\quad \leqslant |s-1|[C\int_\Omega (|w_0^R\psi|^{2^*})w_y^R\psi], \\ &\quad = |s-1|C\int_{\mathbb{R}^N} (|w_0^R|^{2^*} w_y^R(\psi)^{2^*+1}). \end{aligned}$$

Now applying lemma 2.11 and using that $|\psi| \leqslant 1$, we obtain

$$\int_{\mathbb{R}^N} |sf(w_0^R\psi) - f(sw_0^R\psi)|w_y^R\psi \leqslant |s-1|O(\varepsilon_R) \leqslant C|s-1|\ \varepsilon_R$$

for all $s \in [0,b]$, $y \in \partial B_2(y_0)$ and $R \geqslant 1$, as claimed. □

PROPOSITION 4.4. *There exists $R_1 > 0$ and, for each $R > R_1$, a number $\eta = \eta_R > 0$, $\eta_R = o_R(1)$ such that*

$$I_\Omega(T^R_{\lambda,y}U^R_{\lambda,y}) \leqslant 2c - \eta,$$

for all $\lambda \in [0,1]$, $y \in \partial B_2(y_0)$.

Proof. Let us denote for simplicity $s := T^R_{\lambda,y}\lambda$ and $t := T^R_{\lambda,y}(1-\lambda)$, then we have

$$\begin{aligned} &I_\Omega(sw_0^R\psi + tw_y^R\psi) \\ &\quad = \frac{1}{2}\int_\Omega |\nabla(sw_0^R\psi + tw_y^R\psi)|^2 - \int_\Omega F(sw_0^R\psi + tw_y^R\psi) \\ &\quad = \frac{s^2}{2}\int_\Omega |\nabla(w_0^R\psi)|^2 + \frac{t^2}{2}\int_\Omega |\nabla(w_y^R\psi)|^2 + st\int_\Omega \nabla(w_0^R\psi)\nabla(w_y^R\psi) \\ &\qquad - \int_\Omega F(sw_0^R\psi) - \int_\Omega F(tw_y^R\psi) - \int_\Omega F(sw_0^R\psi + tw_y^R\psi) \\ &\qquad - F(sw_0^R\psi) - F(tw_y^R\psi) \end{aligned}$$

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$$= \frac{s^2}{2}\int_\Omega |\nabla(w_0^R\psi)|^2 - \int_\Omega F(sw_0^R\psi) \tag{4.5}$$

$$+ \frac{t^2}{2}\int_\Omega |\nabla(w_y^R\psi)|^2 - \int_\Omega F(tw_y^R\psi) \tag{4.6}$$

$$+ st\int_\Omega \nabla(w_0^R\psi)\nabla(w_y^R\psi) \tag{4.7}$$

$$- \int_\Omega F(sw_0^R\psi + tw_y^R\psi) - F(sw_0^R\psi) - F(tw_y^R\psi) - f(sw_0^R\psi)tw_y^R\psi - f(tw_y^R\psi)sw_0^R\psi \tag{4.8}$$

$$- \int_\Omega f(sw_0^R\psi)tw_y^R\psi - \int_\Omega f(tw_y^R\psi)sw_0^R\psi\,. \tag{4.9}$$

The sum in (4.5) is equal to $I_{\mathbb{R}^N}(sw_0^R) + o(\varepsilon_R)$ since

$$\begin{aligned}(4.5) &= I_{\mathbb{R}^N}(sw_0^R) - I_{\mathbb{R}^N}(sw_0^R) + \frac{s^2}{2}\int_\Omega |\nabla(w_0^R\psi)|^2 - \int_\Omega F(sw_0^R\psi)\\ &= I_{\mathbb{R}^N}(sw_0^R) + \frac{s^2}{2}\int_{B_{2K}(0)} |\nabla(w_0^R\psi)|^2 - |\nabla w_0^R|^2 - \int_{B_{2K}(0)} F(sw_0^R) - F(sw_0^R\psi)\end{aligned}$$

and by (2.6) lemma 2.10, (4.1), (4.4) and s bounded by T_0 we have

$$\frac{s^2}{2}\int_{B_{2K}(0)} |\nabla w_0^R\psi|^2 - |\nabla w_0^R|^2 = o(\varepsilon_R).$$

On the other hand, by the mean value theorem, (f_2) and lemma 2.10 we have

$$\begin{aligned}\int_{B_{2K}(0)} F(sw_0^R) - F(sw_0^R\psi) &= \int_{B_{2K}(0)} f(sw_0^R + \theta(x)sw_0^R\psi)(sw_0^R - sw_0^R\psi)\\ &\leqslant C\int_{B_{2K}(0)} (|w_0^R|^{2^*-1})w_0^R = C\int_{B_{2K}(0)} |w_0^R|^{2^*} = o(\varepsilon_R).\end{aligned}$$

The sum gives that $(4.5) = I_{\mathbb{R}^N}(sw_0^R) + o(\varepsilon_R)$ and since w_0^R is a least energy solution of the limit problem $(\mathcal{P}_{\mathbb{R}^N})$, by lemma 2.4 (b), we have that $I_{\mathbb{R}^N}(sw_0^R) \leqslant c$. Similarly, we have the same for the sum in (4.6) and so

$$(4.5) + (4.6) \leqslant 2c + o(\varepsilon_R).$$

As to (4.8), in lemma 2.9 let $2^* - 2 < \nu < q - 2$ and so $1 + \nu/2 > 2^*/2$. Now by lemma 2.11 it holds that

$$\begin{aligned}&- \int_{\mathbb{R}^N} F(sw_0^R\psi + tw_y^R\psi) - F(sw_0^R\psi) - F(tw_y^R\psi) - f(sw_0^R\psi)tw_y^R\psi - f(tw_y^R\psi)sw_0^R\psi\\ &\leqslant C(st)^{1+\nu/2}\int_{\mathbb{R}^N} (w_y^R\psi w_0^R\psi)^{1+\nu/2}\\ &\leqslant C(st)^{1+\nu/2}\int_{\mathbb{R}^N} (w_y^R w_0^R)^{1+\frac{\nu}{2}} \leqslant CR^{-(N-2)(1+\nu/2)} = o(\varepsilon_R)\end{aligned}$$

so we have shown that

$$(4.8) \leqslant o(\varepsilon_R).$$

Now using analogous arguments to (4.9) we have

$$\int_\Omega f(sw_0^R\psi)tw_y^R\psi + \int_\Omega f(tw_y^R\psi)sw_0^R\psi = \int_{\mathbb{R}^N} f(sw_0^R)tw_y^R + \int_{\mathbb{R}^N} f(tw_y^R)sw_0^R + o(\varepsilon_R)$$

and so we can write the sum of the remaining terms as

$$\begin{aligned}
(4.7) + (4.9) &\leqslant st\int_\Omega \nabla w_0^R\psi\nabla w_y^R\psi - \int_{\mathbb{R}^N} f(sw_0^R)tw_y^R - \int_{\mathbb{R}^N} f(tw_y^R)sw_0^R + o(\varepsilon_R)\\
&= \frac{st}{2}\int_{\mathbb{R}^N} f(w_y^R)w_0^R + \frac{st}{2}\int_{\mathbb{R}^N} f(w_0^R)w_y^R - \int_{\mathbb{R}^N} f(sw_0^R)tw_y^R\\
&\quad - \int_{\mathbb{R}^N} f(tw_y^R)sw_0^R + o(\varepsilon_R)\\
&= \frac{t}{2}\int_{\mathbb{R}^N}[sf(w_0^R) - f(sw_0^R)]w_y^R + \frac{s}{2}\int_{\mathbb{R}^N}[tf(w_y^R) - f(tw_y^R)]w_0^R\\
&\quad - \frac{1}{2}\int_{\mathbb{R}^N} f(sw_0^R)tw_y^R - \frac{1}{2}\int_{\mathbb{R}^N} f(tw_y^R)sw_0^R + o(\varepsilon_R).
\end{aligned}$$

By lemma 4.3 there is a constant $C > 0$ such that

$$\frac{t}{2}\int_{\mathbb{R}^N}[sf(w_0^R) - f(sw_0^R)]w_y^R + \frac{s}{2}\int_{\mathbb{R}^N}[tf(w_y^R) - f(tw_y^R)]w_0^R \leqslant C(|s-1| + |t-1|)\,\varepsilon_R$$

for all $s, t \in [0, T_0], y \in \partial B_2(y_0)$ and R large enough. Moreover, from lemma 4.2, there is a constant $C_0 > 0$ such that

$$\frac{1}{2}\int_{\mathbb{R}^N} f(sw_0^R)tw_y^R + \frac{1}{2}\int_{\mathbb{R}^N} f(tw_y^R)sw_0^R \geqslant C_0\,\varepsilon_R$$

for all $s, t \geqslant 1/2$, $y \in \partial B_2(y_0)$ and R large enough. By lemma 2.14, if $\lambda = 1/2$, then $s, t \to 1$ as $R \to \infty$. So taking $R_0 > 0$ sufficiently large and $\delta \in (0, 1/2)$ sufficiently small such that for all $\lambda \in [1/2 - \delta, 1/2 + \delta]$, $C(|s-1| + |t-1|) \leqslant C_0/2$, we have

$$(4.7) + (4.9) \leqslant -C_0/2\,\varepsilon_R + o(\varepsilon_R)$$

for all $y \in \partial B_2(y_0)$ and $R > R_0$. Summing up, so far we have proved that

$$I_\Omega(sw_0^R + tw_y^R) \leqslant 2c - C_0/2\,\varepsilon_R + o(\varepsilon_R), \tag{4.10}$$

for all $y \in \partial B_2(y_0)$ and $R > R_0$.

On the other hand, for all $\lambda \in [0, 1/2 - \delta] \cup [1/2 + \delta, 1]$, $y \in \partial B_2(y_0)$ and R sufficiently large, since if $T_{\lambda,y}^R \leqslant 2$ then $s = T_{\lambda,y}^R\lambda \in [0, 1 - 2\delta]$ or $t = T_{\lambda,y}^R(1-\lambda) \in [1, 1 - 2\delta]$ and if $T_{\lambda,y}^R \geqslant 2$ then $s = T_{\lambda,y}^R\lambda \in [1 + 2\delta, \infty]$ or $t = T_{\lambda,y}^R(1-\lambda) \in [1 + 2\delta, \infty]$, in fact, one of s or t is in $[0, 1 - 2\delta] \cup [1 + 2\delta, \infty]$ and so (4.5)+(4.6) $\leqslant 2c - \gamma + O(\varepsilon_R)$. By lemma 2.4(b), there exists $\gamma \in (0, c)$ such that

$$I_{\mathbb{R}^N}(rw_0^R) \leqslant c - \gamma \quad \forall r \in [0, 1 - 2\delta] \cup [1 + 2\delta, \infty]$$

also with our previous estimates we have (4.7) $+ \cdots +$ (4.9)$= O(\varepsilon_R)$, and so

$$I_\Omega(sw_0^R + tw_y^R) \leqslant 2c - \gamma + O(\varepsilon_R). \tag{4.11}$$

Inequalities (4.10) and (4.11), together, yield the statement of the proposition. □

Lemma 4.5. *For any $\delta > 0$, there exists $R_2 > 0$ such that*

$$I_\Omega(T^R_{\lambda,y} U^R_{\lambda,y}) < c + \delta,$$

for $\lambda = 0$ and every $y \in \partial B_2(y_0)$ and $R \geqslant R_2$.

Proof. $T^R_{\lambda,y}$ is bounded uniformly in λ, y and R. As w^R_y is a ground state of problem $(\mathcal{P}_{\mathbb{R}^N})$, like in (4.5) we have

$$\begin{aligned} I_\Omega(T^R_{0,y} U^R_{0,y}) & \\ &\leqslant I_{\mathbb{R}^N}(T^R_{0,y} w^R_y) + o(\varepsilon_R) \\ &\leqslant \max_{s>0} I_{\mathbb{R}^N}(s w^R_y) + o(\varepsilon_R) \leqslant c + o(\varepsilon_R). \end{aligned}$$

This proves the lemma. □

Let us consider $\beta : \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$ a barycenter map as defined in [**13**] (see also [**20**]), that is, a map obtained as follows:

$$\mu(u)(x) := \frac{1}{|B_1(x)|} \int_{B_1(x)} |u(y)| \mathrm{d}y, \quad \mu(u) \in L^\infty \cap C(0, +\infty),$$

$$\hat{u}(x) := \left[\mu(u)(x) - \frac{\|\mu(u)\|_\infty}{2} \right]^+, \quad \hat{u} \in C_0(\mathbb{R}^N)$$

and hence, the barycenter of a function $u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}$ is defined by

$$\beta(u) = \frac{1}{\|\hat{u}\|_1} \int_{\mathbb{R}^N} x \hat{u}(x) \mathrm{d}x \, .$$

It is a continuous function with the properties

$$\beta(u(\cdot - y)) = \beta(u) + y \quad \forall\, y \in \mathbb{R}^N, \tag{4.12}$$

$$\beta(Tu) = \beta(u) \quad \forall\, T > 0. \tag{4.13}$$

Note that $\beta(u) = 0$ if u is radial.

Remark 4.6. For the sake of completeness, we recall some facts shown in [**20**] concerning the barycenter of translated copies of w. Since w is radially symmetric, positive and decreasing in $(0, +\infty)$, this implies that $\mu(w)$ is also decreasing with respect to $|x|$. Moreover, as proved in theorem 2.1 in [**3**] $\mu(w) \to 0$ as $|x| \to +\infty$, then, arguing as in [**20**], we obtain that there exists a unique $r_0 > 0$ such that for every $|x| = r_0$, $\mu(w)(x) = \|\mu(w)\|_\infty/2$. We consider the set

$$E(w) := \left\{x \in \mathbb{R}^N, \mu(w)(x) \geqslant \frac{\|\mu(w)\|_\infty}{2}\right\},$$

then by (4.12) it holds

$$E(w) = B_{r_0}(0) \Rightarrow E(w(\cdot - Ry)) = B_{r_0}(Ry), \tag{4.14}$$

for every $R \in \mathbb{R}^+$. If we fix R such that $R > 2K + 1 + r_0$, then since $y \in \partial B_2(y_0)$, it results that $|x| > 2K + r_0$ for every $x \in B_1(Ry)$. Hence,

$$\begin{aligned}\mu(\psi w_y^R)(Ry) &= \frac{1}{|B_1(Ry)|}\int_{B_1(Ry)} \psi(x)w(x - Ry)\mathrm{d}x = \frac{1}{|B_1(Ry)|}\int_{B_1(Ry)} w(x - Ry)\mathrm{d}x \\ &= \frac{1}{|B_1(0)|}\int_{B_1(0)} w(\sigma)\mathrm{d}\sigma = \mu(w)(0) = \|\mu(w)\|_\infty.\end{aligned}$$

Since, $\|\psi\|_\infty \leqslant 1$, it results

$$|\mu(\psi w_y^R)(x)| \leqslant |\mu(w)(x - Ry)| \leqslant \|\mu(w)\|_\infty, \tag{4.15}$$

giving that

$$\|\mu(\psi w_y^R)\|_\infty = |\mu(\psi w_y^R)(Ry)| = \|\mu(w)\|_\infty. \tag{4.16}$$

Furthermore, for every $x \in B_{r_0}(Ry)$, any $z \in B_1(x)$ satisfies $|z| > 2K$, showing that $B_1(x) \subseteq \mathbb{R}^N \setminus B_{2K}(0)$, and using again the definition of the cut-off function ψ, we have

$$\begin{aligned}\mu(\psi w_y^R)(x) &= \frac{1}{|B_1(x)|}\int_{B_1(x)} \psi(z)w(z - Ry)\mathrm{d}z = \frac{1}{|B_1(x)|}\int_{B_1(x)} w(z - Ry)\mathrm{d}z \\ &= \mu(w)(x - Ry).\end{aligned}$$

From (4.14) it follows that, for every $x \in B_{r_0}(Ry)$, $\mu(w)(x - Ry) > \|\mu(w)\|_\infty/2$, thus by the previous equality

$$\mu(\psi w_y^R)(x) > \frac{1}{2}\|\mu(w)\|_\infty \quad \text{for every} \quad x \in B_{r_0}(Ry),$$

so that $\widehat{\psi w_y^R} \neq 0$ if $x \in B_{r_0}(Ry)$. If $x \notin B_{r_0}(Ry)$, then by (4.14), (4.15) and (4.16) it results

$$\mu(\psi w_y^R) \leqslant \mu(w(\cdot - Ry)) < \frac{1}{2}\|\mu(w(\cdot - Ry))\|_\infty = \frac{1}{2}\|\mu(\psi w_y^R)\|_\infty.$$

Therefore, $\widehat{\psi w_y^R} \neq 0$ if and only if $x \in B_{r_0}(Ry)$, but, in this set $\psi \equiv 1$, so that $\widehat{\psi w_y^R} = \widehat{w}(\cdot - Ry)$ and hence

$$\beta(\psi w_y^R) = \beta(w(\cdot - Ry)) = Ry. \tag{4.17}$$

Lemma 4.7. *There exists $\delta > 0$ such that*

$$\beta(u) \neq 0, \quad \forall u \in \mathcal{N}_\Omega \cap I_\Omega^{c+\delta}$$

where $I_\Omega^c = \{u \in H_0^1(\Omega), \quad I_\Omega(u) \leqslant c\}$.

Proof. Arguing by contradiction, assume that for each $k \in \mathbb{N}$ there exists $v_k \in \mathcal{N}_\Omega$ such that $I_\Omega(v_k) < c_\Omega + 1/k$ and $\beta(v_k) = 0$. By Ekeland's variational principle, there exists a $(PS)_d$-sequence (u_k) for I_Ω on $\mathcal{N}_\Omega$ at the level $d = c_\Omega$ such that $\|u_k - v_k\| \to 0$ ([**25**], theorem 8.5). As c_Ω is not attained, Lemma 3.5 (splitting) implies that there exists a sequence (y_k) in $\mathbb{R}^N$ such that $|y_k| \to \infty$ and $\|u_k - w(\cdot - y_k)\| \to 0$, where w is the (positive or negative) radial ground state of $(\mathcal{P}_{\mathbb{R}^N})$. Setting $\tilde{v}_k(x) := v_k(x + y_k)$, and using property (4.12) and the continuity of the barycenter, we conclude that

$$-y_k = \beta(v_k) - y_k = \beta(\tilde{v}_k) \to \beta(w) = 0,$$

but this is a contradiction. □

Proof of theorem 1.2. We will show that I_Ω has a critical value in $(c, 2c)$. By lemma 4.7, we may fix $\delta \in (0, c/4)$ such that

$$\beta(u) \neq 0, \quad \forall\, u \in \mathcal{N}_\Omega \cap I_\Omega^{c+\delta}.$$

Proposition 4.4 and lemma 4.5 allow us to choose $R > 0$ sufficiently large and its corresponding $\eta_R = \eta \in (0, c/4)$ such that

$$I_\Omega(T_{\lambda,y}^R U_{\lambda,y}^R) \leqslant \begin{cases} 2c - \eta & \text{for all } \ \lambda \in [0,1] \quad \text{ and all } y \in \partial B_2(y_0) \\ c + \delta & \text{for } \lambda = 0 \text{ and all } y \in \partial B_2(y_0). \end{cases}$$

For this fixed $R > 0$, define $\alpha : B_2(y_0) \to \mathcal{N}_\Omega \cap I_\Omega^{2c-\eta}$ by

$$\alpha(\lambda y_0 + (1-\lambda) y) := T_{\lambda,y}^R U_{\lambda,y}^R \quad \text{ with } \lambda \in [0,1], \quad y \in \partial B_2(y_0).$$

Arguing by contradiction, assume that I_Ω does not have a critical value in $(c, 2c)$. As, by corollary 3.7, I_Ω satisfies the Palais-Smale condition on $\mathcal{N}_\Omega$ at every level in $(c, 2c)$, there exists $\varepsilon > 0$ such that

$$\|\nabla_{\mathcal{N}_\Omega} I_\Omega(u)\| \geqslant \varepsilon, \quad \forall\ u \ \in \mathcal{N}_\Omega \cap I_\Omega^{-1}[c+\delta, 2c - \eta].$$

Then using a Deformation lemma for C^1 manifolds (see [**11**]), it yields a continuous function

$$\rho : \mathcal{N}_\Omega \cap I_\Omega^{2c-\eta} \to \mathcal{N}_\Omega \cap I_\Omega^{c+\delta}$$

such that $\rho(u) = u$ for all $u \in \mathcal{N}_\Omega \cap I_\Omega^{c+\delta}$. Now we define $\Gamma(x) := (\beta \circ \rho \circ \alpha)(x)$. By lemma 4.7, $\Gamma(x) \neq 0$ and so the function $h : B_2(y_0) \to \partial B_2(0)$ given by

$$h(x) := 2 \frac{\Gamma(x)}{|\Gamma(x)|}$$

is well defined and continuous. Moreover, if $y \in \partial B_2(y_0)$, by lemma 4.5

$$\alpha(y) = T^R_{0,y} U^R_{0,y} = T^R_{0,y} \psi w^R_y \in \mathcal{N}_\Omega \cap I^{c+\delta}_\Omega$$

and hence by (4.17) in remark 4.6,

$$(\beta \circ \rho \circ \alpha)(y) = \beta(w^R_y) = Ry,$$

since $\beta(T^R_{0,y}\psi w^R_y) = \beta(w^R_y)$ (see [**20**], estimate (5.15)). Note also that $h(y) = 2Ry/|Ry| = 2y/|y|$.

Therefore, if we consider the homeomorphism $\tilde{h} : \partial B_2(y_0) \to \partial B_2(0)$ defined by $\tilde{h}(y) := 2y/|y|$, then $(\tilde{h}^{-1} \circ h)(y) = y$ for every $y \in \partial B_2(y_0)$, however, by Brouwer Fixed Point theorem such a map does not exist, so I_Ω must have a critical point $u \in \mathcal{N}_\Omega$ with $I_\Omega(u) \in (c, 2c)$. By remark 3.8 u does not change sign, so if $u \geqslant 0$ with the maximum principle, we get $u > 0$ is a solution of $(\mathcal{P})$. On the other hand if $u \leqslant 0$, then by the oddness of f, $f(u) \leqslant 0$ and so $-u$ is a positive solution. This proves that problem $(\mathcal{P})$ has a positive solution.

Now we can write $(\mathcal{P})$ as

$$-\Delta u = au$$

where $a = f(u)/u$ and if we show $a \in L^{N/2}_{loc}(\mathbb{R}^N)$ then by Brezis-Kato theorem u is in $L^p_{loc}(\mathbb{R}^N)$ for all $1 \leqslant p < \infty$ and so $u \in W^{2,p}_{loc}(\mathbb{R}^N)$ and by Sobolev embedding $u \in C^{0,1-N/p}_{loc}(\mathbb{R}^N)$. Now let $p > N$ we have u is locally Hölder continuous and since f is of class C^1, we have $f(u)$ is locally Hölder continuous and so by elliptic regularity theorems, $u \in C^2(\mathbb{R}^N)$ and so u is classic solution. In order to complete the proof we show that $a \in L^{N/2}_{\rm loc}(\mathbb{R}^N)$. By (f_2) we have

$$|a(x)| = \frac{f(u)}{u} \leqslant C|u|^{2^*-2}$$

and so

$$\int_\Gamma |a(x)|^{\frac{N}{2}} \leqslant C \int_\Gamma |u|^{((2^*-2)N)/2} = C \int_\Gamma |u|^{2^*} < \infty$$

for any open set $\Gamma \subset\subset \mathbb{R}^N$. Hence the theorem is proved. □

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