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Asymptotic analysis of the eigenvalues of an elliptic problem in an anisotropic thin multidomain

Antonio Gaudiello
DAEIMI, Università degli Studi di Cassino, via G. Di Biasio 43,
03043 Cassino, Italy (gaudiell@unina.it)

Ali Sili
Département de Mathématiques, Université du Sud Toulon-Var,
BP 20132, 83957 La Garde cedex, France
and
LATP, UMR 6632, Université de Provence, 39 rue F. Joliot-Curie,
13453 Marseille cedex 13, France (sili@univ-tln.fr)

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We study, via an asymptotic analysis, an elliptic eigenvalue problem in a 1D–1D multidomain and in a 1D–2D multidomain filled with anisotropic material. The corresponding isotropic cases were considered in a previous work by Gaudiello and Sili.

1. Introduction and main results

In what follows, $x = (x_1, \dots, x_{N-1}, x_N) = (x', x_N)$ denotes the generic point of $\mathbb{R}^N$, $N \geqslant 2$. Moreover, D, $D_{x'}$ and ∂_{x_N} stand for the transposed gradient with respect to all the components of x, for the transposed gradient with respect to the first $N-1$ components of x and for the derivative with respect to the last component of x, respectively.

For every $n \in \mathbb{N}$, let $\Omega_n \subset \mathbb{R}^N$ be a thin multidomain consisting of two vertical cylinders, one placed upon the other: the first one with height 1 and small cross-section $r_n\omega$, the second one with small thickness h_n and cross-section ω, where ω is a bounded, open, connected regular subset of $\mathbb{R}^{N-1}$ containing the origin of $\mathbb{R}^{N-1}$, and r_n and h_n are two vanishing positive parameters. Specifically,

$$\Omega_n = (r_n\omega \times [0, 1[\,) \cup (\omega \times]-h_n, 0[\,)$$

(for example, see figure 1 when $N = 2$ and figure 2 when $N = 3$).

In Ω_n we consider the following eigenvalue problem:

$$\left.\begin{aligned} -\operatorname{div}(A_n(x)DU_n) &= \lambda U_n \quad \text{in } \Omega_n, \\ U_n &= 0 \qquad \text{on } \Gamma_n = (r_n\omega \times \{1\}) \cup (\partial\omega \times]-h_n, 0[\,), \\ A_n(x)DU_n\nu &= 0 \qquad \text{on } \partial\Omega_n \setminus \Gamma_n, \end{aligned}\right\} \tag{1.1}$$

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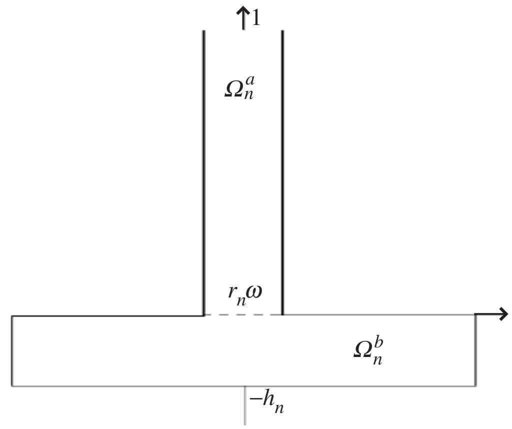

Figure 1. The thin multidomain when $N = 2$.

where ν denotes the exterior unit normal to Ω_n, $r_n\omega \times \{1\}$ is the top of the upper cylinder, $\partial\omega \times \left]-h_n, 0\right[$ is the lateral surface of the lower one, and

$$A_n(x', x_N) = \begin{cases} A\left(\dfrac{x'}{r_n}, x_N\right) & \text{a.e. in } \Omega_n^a = r_n\omega \times \left]0, 1\right[, \\ A\left(x', \dfrac{x_N}{h_n}\right) & \text{a.e. in } \Omega_n^b = \omega \times \left]-h_n, 0\right[, \end{cases} \tag{1.2}$$

$A(x) = (a_{ij}(x))_{i,j=1,\dots,N}$ being a measurable, bounded, uniformly elliptic and symmetric matrix valued function defined in $\omega \times \left]-1, 1\right[$. Note that assumption (1.2) allows us to consider different types of materials in Ω_n. For instance, one can consider a homogeneous isotropic material, a homogeneous anisotropic material, a non-homogeneous anisotropic material where the matrix is independent of x' in Ω_n^a and independent of x_N in Ω_n^b, or a cylinder Ω_n^a composed of two materials: a cylindrical hearth enveloped by a cylindrical shell made by a different material (see, for example, [15]), etc.

It is well known (see, for example, [17, theorem 6.2-1]) that, for every $n \in \mathbb{N}$, there exist an increasing divergent sequence of positive numbers $\{\lambda_{n,k}\}_{k\in\mathbb{N}}$ and a $L^2(\Omega_n)$-Hilbert orthonormal basis $\{U_{n,k}\}_{k\in\mathbb{N}}$ such that $\{\lambda_{n,k}\}_{k\in\mathbb{N}}$ forms the set of all the eigenvalues of problem (1.1) and, for every $k \in \mathbb{N}$,

$$U_{n,k} \in \mathcal{V}_n = \{V \in H^1(\Omega_n) \colon V = 0 \text{ on } \Gamma_n\}$$

is an eigenvector of (1.1) with eigenvalue $\lambda_{n,k}$. Moreover, $\{\lambda_{n,k}^{-1/2} U_{n,k}\}_{k\in\mathbb{N}}$ is a $\mathcal{V}_n$-Hilbert orthonormal basis, by equipping $\mathcal{V}_n$ with the inner product

$$(U, V) \in \mathcal{V}_n \times \mathcal{V}_n \to \int_{\Omega_n} A_n DU DV \,\mathrm{d}x.$$

The aim of this paper is to study the asymptotic behaviour, as n diverges, of the sequences $\{(\lambda_{n,k}, U_{n,k})\}_{n\in\mathbb{N}}$, for every $k \in \mathbb{N}$, when $h_n \simeq r_n^{N-1}$. In this way we obtain a more manageable eigenvalue problem for the 1D–$(N-1)$D limit domains. Specifically, the following result will be proved.

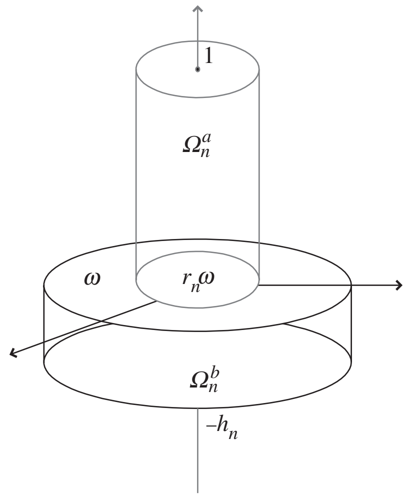

Figure 2. The thin multidomain when $N = 3$.

THEOREM 1.1. *For every $n \in \mathbb{N}$, let $\{(\lambda_{n,k}, U_{n,k})\}_{k\in\mathbb{N}}$ be a sequence as above. Assume that*

$$\lim_n r_n = 0 = \lim_n h_n, \qquad \lim_n \frac{h_n}{r_n^{N-1}} = q \in \,]0, +\infty[$$

Then, there exists an increasing diverging sequence of positive numbers $\{\lambda_k\}_{k\in\mathbb{N}}$ such that

$$\lim_n \lambda_{n,k} = \lambda_k, \quad \forall k \in \mathbb{N}, \tag{1.3}$$

and $\{\lambda_k\}_{k\in\mathbb{N}}$ is the set of all the eigenvalues of the following problem:

$$\left.\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}x_N}\left(\bar{a}^a(x_N)\frac{\mathrm{d}u^a}{\mathrm{d}x_N}(x_N)\right) &= \lambda u^a(x_N) \quad \textit{in }]0,1[\,,\\ -\operatorname{div}(\bar{A}^b(x')D_{x'}u^b(x')) &= \lambda u^b(x') \quad \textit{in } \omega,\\ u^a(1) = 0, \quad \bar{a}^a(0)\frac{\mathrm{d}u^a}{\mathrm{d}x_N}(0) = 0, \quad u^b &= 0 \quad \textit{on } \partial\omega \end{aligned}\right\} \tag{1.4}$$

if $N \geqslant 3$; of the following:

$$\left.\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}x_2}\left(\bar{a}^a(x_2)\frac{\mathrm{d}u^a}{\mathrm{d}x_2}(x_2)\right) &= \lambda u^a(x_2) \quad \textit{in }]0,1[\,,\\ -\frac{\mathrm{d}}{\mathrm{d}x_1}\left(\bar{A}^b(x_1)\frac{\mathrm{d}u^b}{\mathrm{d}x_1}(x_1)\right) &= \lambda u^b(x_1) \quad \textit{in }]c,0[\cup]0,d[\,,\\ u^a(1) = 0, \quad u^b(c) = 0, &\quad u^b(d) = 0,\\ u^a(0) = u^b(0), \quad \bar{a}^a(0)\frac{\mathrm{d}u^a}{\mathrm{d}x_2}(0) &= q\bar{A}^b(0)\left(\frac{\mathrm{d}u^b}{\mathrm{d}x_1}(0^-) - \frac{\mathrm{d}u^b}{\mathrm{d}x_1}(0^+)\right) \end{aligned}\right\} \tag{1.5}$$

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if $N=2$ and $\omega =]c,d[$, where $\bar{A}^b$ is symmetric, $\bar{a}^a$ and $\bar{A}^b$ are measurable, bounded and uniformly elliptic, and are defined by

$$\left.\begin{aligned}&\bar{a}^a\colon x_N \in]0,1[\to \int_\omega A^N(x',x_N)\begin{pmatrix} D_{x'}\hat{y}^a(x',x_N)\\ 1\end{pmatrix}\,\mathrm{d}x',\\ &\bar{A}^b\colon x'\in\omega \to \left(\int_{-1}^0 \left(a_{ij}(x',x_N) - \frac{a_{iN}(x',x_N)a_{Nj}(x',x_N)}{a_{NN}(x',x_N)}\right)\mathrm{d}x_N\right)_{i,j=1,\dots,N-1},\end{aligned}\right\} \tag{1.6}$$

and, for almost every $x_N \in]0,1[$, $\hat{y}^a(\cdot,x_N)$ is the unique solution of the following problem:

$$\left.\begin{aligned}-\operatorname{div}_{x'}\left(A'(x',x_N)\begin{pmatrix} D_{x'}\hat{y}^a(x',x_N)\\ 1\end{pmatrix}\right) &= 0 \quad \text{in } \omega,\\ A'(x',x_N)\begin{pmatrix} D_{x'}\hat{y}^a(x',x_N)\\ 1\end{pmatrix}\nu &= 0 \quad \text{on } \partial\omega,\\ \int_\omega \hat{y}^a(x',x_N)\,\mathrm{d}x' &= 0,\end{aligned}\right\} \tag{1.7}$$

where ν denotes the exterior unit normal to ω, and A' and A^N denote the sub-matrix of A composed of the first $N-1$ rows of A and of the last row of A, respectively.

There exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$ and a sequence $\{(u_k^a,u_k^b)\}_{k\in\mathbb{N}} \subset V$ (which may depend on the selected subsequence $\{n_i\}_{i\in\mathbb{N}}$) where

$$V = \{(v^a,v^b)\in H^1(]0,1[)\times H^1(\omega)\colon\\ v^a(1)=0,\ v^b = 0 \text{ on } \partial\omega \text{ (and } v^a(0)=v^b(0) \text{ if } N=2)\},$$

such that

$$\lim_i \int_{r_{n_i}\omega\times]0,1[}\left(\left|U_{n_i,k} - r_{n_i}^{-(N-1)/2}u_k^a\right|^2 + \left|D_{x'}U_{n_i,k} - r_{n_i}^{-(N-1)/2}(D_{x'}\hat{y}^a)\left(\frac{x'}{r_n},x_N\right)\frac{\mathrm{d}u_k^a}{\mathrm{d}x_N}\right|^2 + \left|\partial_{x_N}U_{n_i,k} - r_{n_i}^{-(N-1)/2}\frac{\mathrm{d}u_k^a}{\mathrm{d}x_N}\right|^2\right)\mathrm{d}x = 0 \tag{1.8}$$

and

$$\lim_i \int_{\omega\times]-h_{n_i},0[}\left(\left|U_{n_i,k} - r_{n_i}^{-(N-1)/2}u_k^b\right|^2 + \left|D_{x'}U_{n_i,k} - r_{n_i}^{-(N-1)/2}D_{x'}u_k^b\right|^2 + \left|\partial_{x_N}U_{n_i,k} + r_{n_i}^{-(N-1)/2}\sum_{j=1}^{N-1}a_{Nj}\left(x',\frac{x_N}{h_n}\right) \times a_{NN}^{-1}\left(x',\frac{x_N}{h_n}\right)\partial_{x_j}u_k^b\right|^2\right)\mathrm{d}x = 0 \tag{1.9}$$

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as $i \to +\infty$, for every $k \in \mathbb{N}$, and $u_k = (u_k^a, u_k^b)$ is an eigenvector of problem (1.4) if $N \geqslant 3$ (problem (1.5) if $N = 2$) with eigenvalue λ_k. Moreover, $\{u_k\}_{k\in\mathbb{N}}$ is a $L^2(\,]0,1[\,) \times L^2(\omega)$-orthonormal basis with respect to the inner product

$$|\omega| \int_0^1 u^a v^a \, \mathrm{d}x_N + q \int_\omega u^b v^b \, \mathrm{d}x',$$

and $\{\lambda_k^{-1/2} u_k\}_{k\in\mathbb{N}}$ is a V-Hilbert orthonormal basis with respect to the inner product

$$\int_0^1 \bar{a}^a \frac{\mathrm{d}u^a}{\mathrm{d}x_N} \frac{\mathrm{d}v^a}{\mathrm{d}x_N} \, \mathrm{d}x_N + q \int_\omega \bar{A}^b D_{x'} u^b D_{x'} v^b \, \mathrm{d}x'.$$

If $N \geqslant 3$, the eigenvalues of problem (1.4) are obtained by gathering the eigenvalues of the following two independent problems:

$$\left.\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}x_N}\left(\bar{a}^a(x_N)\frac{\mathrm{d}u^a}{\mathrm{d}x_N}(x_N)\right) &= \lambda u^a(x_N) \quad \text{in }]0,1[\,, \\ u^a(1) = 0, \qquad \bar{a}^a(0){u^a}'(0) &= 0, \end{aligned}\right\} \tag{1.10}$$

and

$$\left.\begin{aligned} -\operatorname{div}_{x'}(\bar{A}^b(x')D_{x'}u^b(x')) &= \lambda^b u^b(x') \quad \text{in } \omega, \\ u^b &= 0 \qquad\qquad \text{on } \partial\omega. \end{aligned}\right\} \tag{1.11}$$

Moreover, each eigenvalue of problem (1.10) or problem (1.11) preserves its multiplicity if it is an eigenvalue only of problem (1.10) or only of problem (1.11), otherwise its multiplicity is obtained by adding the multiplicity as eigenvalue of problem (1.10) and the multiplicity as eigenvalue of problem (1.11).

If $N = 2$, the limit problem in $]0,1[$ is coupled with the limit problem in $\omega =]c,d[$ by the junction conditions

$$u^a(0) = u^b(0) \quad \text{and} \quad \bar{a}^a(0)\frac{\mathrm{d}u^a}{\mathrm{d}x_2}(0) = q\bar{A}^b(0)\left(\frac{\mathrm{d}u^b}{\mathrm{d}x_1}(0^-) - \frac{\mathrm{d}u^b}{\mathrm{d}x_1}(0^+)\right).$$

In theorem 1.1, by virtue of (1.3), the multiplicity of $\lambda_{n,k}$ for sufficiently large n is less than or equal to the multiplicity of λ_k. Consequently, if λ_k is simple, then $\lambda_{n,k}$ is also simple for sufficiently large n. Then, arguing as in [22] (see also [3]), if λ_k is simple, fixing one of the two normalized eigenvectors u_k of the limit problem with eigenvalue λ_k, it is possible to choose, for sufficiently large n, one of the two normalized eigenvectors $U_{n,k} \in \mathcal{V}_n$ of problem (1.1) with eigenvalue $\lambda_{n,k}$ such that convergences (1.8) and (1.9) hold true for the whole sequence.

It is not necessary that the two cylinders are scaled to the same one or that the first cylinder has height 1. In fact, the result is essentially unchanged if one assumes

$$\Omega_n = (r_n\omega_a \times [0,l[\,) \cup (\omega_b \times]-h_n,0[\,),$$

with $\omega^a, \omega^b \subset \mathbb{R}^{N-1}$, $0' \in \omega^b$ and $l \in]0,+\infty[$.

This paper generalizes [6], where we considered the same problem for the Laplacian, i.e. $A = \mathrm{Id}$. Now, having reformulated the problem on a fixed domain through appropriate rescalings of the kind proposed in [2], and having introduced suitable weighted inner products, by using the min–max principle and some estimates proved in [6], we obtain an *a priori* estimate (with respect to n) of $\lambda_{n,k}$. Then, by applying some results proved in [8, 9] and based on the method of oscillating test functions introduced in [20], we derive a system for the limit eigenvalue problem depending on auxiliary unknowns that are weak limits of some quantities. Having written these unknowns in terms of solutions of some elementary problems, as proposed in [18, 19], we give the final form of the limit problem involving the diffusion along the axis of the upper cylinder and the diffusion along the horizontal section of the lower cylinder. In particular, we show that the limit matrices are uniformly elliptic. We point out that these auxiliary unknowns are specific to the anisotropic case, since they are zero in the Laplacian case. Finally, we conclude as in [6] by adapting techniques used in [22] for showing that all the eigenvalues of the limit problem have been reached.

The result of theorem 1.1 considering $A_n(x) = A(x)$ (with A also continuous) in (1.1) is clear: the limit problem is equivalent to that obtained starting from a matrix independent of x' in Ω_n^a and independent of x_N in Ω_n^b. For brevity, we do not discuss the cases where $h_n \ll r_n^{N-1}$ and $r_n^{N-1} \ll h_n$. For interested readers, these can be easily obtained by coupling the techniques adapted in this paper with those in [6].

The study of the eigenvalues in 1D–1D or 2D–2D joined elastic structures was performed in [14]. Regarding multistructures, we also refer to [1,3,13,16,21] and the references quoted therein. For a thin multistructure as considered in this paper, we refer the reader to [4,5,7–12]. Specifically, the model, as described in [8] through its integral energy and in [9] through the related constitutive equations, is a quasilinear Neumann second-order scalar problem. A Neumann second-order problem with maps in S^2 is considered in [5]. A Neumann fourth-order problem is studied in [7] in the scalar case, and in [4] in the non-convex case. Problems in linear elasticity are considered in [11] and [10], while problems in nonlinear elasticity are considered in [12].

2. Limit of the rescaled problem

Let

$$\Omega^a = \omega \times]0,1[, \qquad \Omega^b = \omega \times]-1,0[,$$

where ω is a bounded open connected regular set of $\mathbb{R}^{N-1}$, $N \geqslant 2$, containing the origin of $\mathbb{R}^{N-1}$, and $\{r_n\}_{n\in\mathbb{N}}$, $\{h_n\}_{n\in\mathbb{N}} \subset]0,1[$ are two sequences such that

$$\lim_n h_n = 0 = \lim_n r_n. \tag{2.1}$$

Let

$$A\colon x \in \omega \times]-1,1[\to A(x) = (a_{ij}(x))_{i,j=1,\dots,N} \in M_s^{N\times N} \tag{2.2}$$

be a measurable function such that there exists

$$\alpha, \beta \in]0,+\infty[\,:\, \alpha|\xi|^2 \leqslant \xi A(x)\xi, \quad |A(x)\xi^{\mathrm{T}}| \leqslant \beta|\xi|, \text{ a.e. } x \in \omega \times]-1,1[, \ \forall \xi \in \mathbb{R}^N, \tag{2.3}$$

where $M_s^{N\times N}$ denotes the set of $N \times N$ symmetric real matrices.

For every $n \in \mathbb{N}$, let H_n be the space $L^2(\Omega^a) \times L^2(\Omega^b)$ equipped with the inner product

$$(\cdot,\cdot)_n : (u,v) = ((u^a,u^b),(v^a,v^b)) \in (L^2(\Omega^a) \times L^2(\Omega^b))^2 \\ \to \int_{\Omega^a} u^a v^a \,\mathrm{d}x + \frac{h_n}{r_n^{N-1}} \int_{\Omega^b} u^b v^b \,\mathrm{d}x, \qquad (2.4)$$

and let V_n be the space

$$V_n = \{v = (v^a,v^b) \in H^1(\Omega^a) \times H^1(\Omega^b) \colon v^a = 0 \text{ on } \omega \times \{1\}, \\ v^b = 0 \text{ on } \partial\omega \times]-1,0[\,,\ v^a(x',0) = v^b(r_n x',0) \text{ a.e. } x' \in \omega\} \qquad (2.5)$$

equipped with the bilinear form

$$a_n \colon (u,v) = ((u^a,u^b),(v^a,v^b)) \in V_n^2 \\ \to \int_{\Omega^a} A \begin{pmatrix} \dfrac{1}{r_n} D_{x'} u^a \\ \partial_{x_N} u^a \end{pmatrix} \begin{pmatrix} \dfrac{1}{r_n} D_{x'} v^a \\ \partial_{x_N} v^a \end{pmatrix} \mathrm{d}x \\ + \frac{h_n}{r_n^{N-1}} \int_{\Omega^b} A \begin{pmatrix} D_{x'} u^b \\ \dfrac{1}{h_n} \partial_{x_N} u^b \end{pmatrix} \begin{pmatrix} D_{x'} v^b \\ \dfrac{1}{h_n} \partial_{x_N} v^b \end{pmatrix} \mathrm{d}x, \qquad (2.6)$$

where, from now on, if w, $z \in \mathbb{R}^k$, $w^{\mathrm{T}} z^{\mathrm{T}}$ stands for the usual inner product in $\mathbb{R}^k$, that is, wz.

We point out that the norm induced on $L^2(\Omega^a) \times L^2(\Omega^b)$ by the inner product $(\cdot,\cdot)_n$ is equivalent to the usual $(L^2(\Omega^a) \times L^2(\Omega^b))$-norm. Moreover, by virtue of (2.2) and (2.3), a_n is an inner product in V_n and its induced norm is equivalent to the $(H^1(\Omega^a) \times H^1(\Omega^b))$-norm. Consequently, V_n is continuously and compactly embedded into H_n. Furthermore, since $C_0^\infty(\Omega^a) \times C_0^\infty(\Omega^b) \subset V_n$, it results that V_n is dense in H_n. Then (see, for example, [17, theorems 6.2-1 and 6.2-2]), for every $n \in \mathbb{N}$, there exist an increasing diverging sequence of positive numbers $\{\lambda_{n,k}\}_{k\in\mathbb{N}}$ and an H_n-Hilbert orthonormal basis $\{u_{n,k}\}_{k\in\mathbb{N}}$ such that $\{\lambda_{n,k}\}_{k\in\mathbb{N}}$ forms the set of all the eigenvalues of the following problem:

$$u_n \in V_n, \qquad a_n(u_n,v) = \lambda(u_n,v)_n, \quad \forall v \in V_n, \qquad (2.7)$$

and, for every $k \in \mathbb{N}$, $u_{n,k} \in V_n$ is an eigenvector of (2.7) with eigenvalue $\lambda_{n,k}$. Moreover, $\{\lambda_{n,k}^{-1/2} u_{n,k}\}_{k\in\mathbb{N}}$ is a V_n-Hilbert orthonormal basis. Furthermore, for every $k \in \mathbb{N}$, $\lambda_{n,k}$ is characterized by the following min–max principle:

$$\lambda_{n,k} = \min_{\mathcal{E}_k \in \mathcal{F}_k} \max_{v \in \mathcal{E}_k, v \neq 0} \frac{a_n(v,v)}{(v,v)_n}, \qquad (2.8)$$

where $\mathcal{F}_k$ is the set of the subspaces $\mathcal{E}_k$ of V_n with dimension k.

The min–max principle and proposition 2.1 of [6] immediately involve the following *a priori* estimates for the eigenvalues of problem (2.7).

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Proposition 2.1. *For every $n, k \in \mathbb{N}$, let $\lambda_{n,k}$ be as above. Then, it results that*

$$0 < \lambda_{n,k} \leqslant \beta k^2 \pi^2, \quad \forall n, k \in \mathbb{N}, \tag{2.9}$$

where β is given in (2.3).

Proof. Let

$$\begin{aligned}\tilde{a}_n : (u, v) = ((u^a, u^b), (v^a, v^b)) \in V_n^2 \\ \to \int_{\Omega^a} \frac{1}{r_n^2} D_{x'} u^a D_{x'} v^a + \partial_{x_N} u^a \partial_{x_N} v^a \, \mathrm{d}x \\ + \frac{h_n}{r_n^{N-1}} \int_{\Omega^b} D_{x'} u^b D_{x'} v^b + \frac{1}{h_n^2} \partial_{x_N} u^b \partial_{x_N} v^b \, \mathrm{d}x.\end{aligned}$$

Since it results in

$$a_n(v, v) \leqslant \beta \tilde{a}_n(v, v), \quad \forall v \in V_n,$$

by the min–max principle (2.8) and proposition 2.1 of [6], it follows that

$$\begin{aligned}\lambda_{n,k} &= \min_{\mathcal{E}_k \in \mathcal{F}_k} \max_{v \in \mathcal{E}_k, v \neq 0} \frac{a_n(v, v)}{(v, v)_n} \\ &\leqslant \beta \min_{\mathcal{E}_k \in \mathcal{F}_k} \max_{v \in \mathcal{E}_k, v \neq 0} \frac{\tilde{a}_n(v, v)}{(v, v)_n} \\ &\leqslant \beta k^2 \pi^2, \quad \forall n, k \in \mathbb{N}.\end{aligned}$$

□

By using a diagonal argument, proposition 2.1 provides the existence of an increasing sequence of positive integer numbers $\{n_i\}_{i \in \mathbb{N}}$ and an increasing sequence of non-negative numbers $\{\lambda_k\}_{k \in \mathbb{N}}$, such that

$$\lim_i \lambda_{n_i,k} = \lambda_k, \quad \forall k \in \mathbb{N}.$$

For characterizing the sequence $\{\lambda_k\}_{k \in \mathbb{N}}$, when

$$\lim_n \frac{h_n}{r_n^{N-1}} = q \in \,]0, +\infty[\,, \tag{2.10}$$

let us introduce the space:

$$\begin{aligned}H = \{v = (v^a, v^b) \in L^2(\Omega^a) \times L^2(\Omega^b) \colon \\ v^a \text{ is independent of } x', \ v^b \text{ is independent of } x_N\}\end{aligned} \tag{2.11}$$

equipped with the inner product

$$[\cdot, \cdot]_q \colon (u, v) = ((u^a, u^b), (v^a, v^b)) \in H^2 \to |\omega| \int_0^1 u^a v^a \, \mathrm{d}x_N + q \int_\omega u^b v^b \, \mathrm{d}x', \tag{2.12}$$

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and the space

$$V = \begin{cases} \{v = (v^a, v^b) \in H^1(\Omega^a) \times H^1(\Omega^b)\colon \\ \quad v^a \text{ is independent of } x',\ v^b \text{ is independent of } x_N, \\ \qquad\qquad v^a(1) = 0,\ v^b = 0 \text{ on } \partial\omega,\ v^a(0) = v^b(0)\} \quad \text{if } N = 2, \\ \{v = (v^a, v^b) \in H^1(\Omega^a) \times H^1(\Omega^b) : \\ \quad v^a \text{ is independent of } x',\ v^b \text{ is independent of } x_N, \\ \qquad\qquad\qquad v^a(1) = 0,\ v^b = 0 \text{ on } \partial\omega\} \quad \text{if } N \geqslant 3, \end{cases} \tag{2.13}$$

equipped with the following bilinear form:

$$\alpha_q\colon ((u^a, u^b), (v^a, v^b)) \in V^2 \\ \to \int_0^1 \bar{a}^a(x_N) \frac{\mathrm{d}u^a}{\mathrm{d}x_N} \frac{\mathrm{d}v^a}{\mathrm{d}x_N}\,\mathrm{d}x_N + q\int_\omega \bar{A}^b(x') D_{x'} u^b D_{x'} v^b\,\mathrm{d}x', \tag{2.14}$$

where $\bar{a}^a$ and $\bar{A}^b$ are defined in (1.6). Note that, for almost every $x_N \in\,]0,1[$, problem (1.7) admits a unique weak solution by virtue of (2.3).

The proof of theorem 2.2 shows that $\bar{A}^b(x') \in M_s^{N\times N}$ for almost every $x' \in \omega$, $\bar{a}^a \in L^\infty(\,]0,1[\,)$, $\bar{A}^b \in (L^\infty(\omega))^{N-1}$ and $\bar{a}^a$ and $\bar{A}^b$ are uniformly elliptic. Consequently, α_q is an inner product in V and the induced norm is equivalent to the $(H^1(\,]0,1[\,)\times H^1(\omega))$-norm. Since the norm induced on H by the inner product $[\cdot,\cdot]_q$ is equivalent to the $(L^2(\,]0,1[\,)\times L^2(\omega))$-norm, it follows that V is continuously and compactly embedded into H. Moreover, since

$$C_0^\infty(\,]0,1[\,) \times C_0^\infty(\omega) \subset V \quad \text{if } N \geqslant 3, \\ C_0^\infty(\,]0,1[\,) \times \{v \in C_0^\infty(\omega)\colon v(0) = 0\} \subset V \quad \text{if } N = 2,$$

we obtain that V is dense in H. Then, for the following eigenvalue problem:

$$u \in V, \qquad \alpha_q(u, v) = \lambda[u, v]_q, \quad \forall v \in V, \tag{2.15}$$

all classic results hold true (see [17, theorems 6.2-1 and 6.2-2]).

The main result of this paper is the following.

THEOREM 2.2. *For every $n \in \mathbb{N}$, let $\{\lambda_{n,k}\}_{k\in\mathbb{N}}$ be the increasing diverging sequence of all the eigenvalues of problem (2.2)–(2.7), and $\{u_{n,k}\}_{k\in\mathbb{N}}$ be a H_n-Hilbert orthonormal basis such that $\{\lambda_{n,k}^{-1/2} u_{n,k}\}_{k\in\mathbb{N}}$ is a V_n-Hilbert orthonormal basis and, for every $k \in \mathbb{N}$, $u_{n,k}$ is an eigenvector of problem (2.2)–(2.7) with eigenvalue $\lambda_{n,k}$. Assume (2.1) and (2.10).*

Then, there exists an increasing diverging sequence of positive numbers $\{\lambda_k\}_{k\in\mathbb{N}}$ such that

$$\lim_n \lambda_{n,k} = \lambda_k, \quad \forall k \in \mathbb{N},$$

and $\{\lambda_k\}_{k\in\mathbb{N}}$ is the set of all the eigenvalues of problem (2.11)–(2.15). Moreover, there exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$ and a $(H, [\cdot,\cdot]_q)$-Hilbert orthonormal basis $\{u_k\}_{k\in\mathbb{N}}$ (depending possibly on the selected subsequence $\{n_i\}_{i\in\mathbb{N}}$) such that, for every $k \in \mathbb{N}$, $u_k \in V$ is an eigenvector of *problem (2.11)–(2.15) with eigenvalue* λ_k*, and*

$$u_{n_i,k} \to u_k \text{ strongly in } H^1(\Omega^a) \times H^1(\Omega^b), \qquad \forall k \in \mathbb{N}, \tag{2.16}$$

$$\frac{1}{r_{n_i}} D_{x'} u^a_{n_i,k} \to \frac{\mathrm{d}u^a_k}{\mathrm{d}x_N} D_{x'} \hat{y}^a \text{ strongly in } (L^2(\Omega^a))^{N-1}, \quad \forall k \in \mathbb{N}, \tag{2.17}$$

$$\frac{1}{h_{n_i}} \partial_{x_N} u^b_{n_i,k} \to -\sum_{j=1}^{N-1} \frac{a_{Nj}}{a_{NN}} \partial_{x_j} u^b_k \text{ strongly in } L^2(\Omega^b), \quad \forall k \in \mathbb{N}, \tag{2.18}$$

as $i \to +\infty$*, where* $\hat{y}^a$ *is given by (1.7). Furthermore,* $\{\lambda_k^{-1/2} u_k\}_{k\in\mathbb{N}}$ *is a* (V, α_q)*-Hilbert orthonormal basis.*

Proof. We have that,

$$u_{n,k} \in V_n, \qquad a_n(u_{n,k}, v) = \lambda_{n,k}(u_{n,k}, v)_n, \quad \forall v \in V_n, \ \forall n, k \in \mathbb{N}, \tag{2.19}$$

and

$$(u_{n,k}, u_{n,h})_n = \delta_{h,k}, \quad \forall n, k, h, \in \mathbb{N}, \tag{2.20}$$

where $\delta_{h,k}$ is the Kronecker delta.

By choosing $v = u_{n,k}$ in (2.19) and by taking into account (2.20) and proposition 2.1, we have that

$$a_n(u_{n,k}, u_{n,k}) = \lambda_{n,k}(u_{n,k}, u_{n,k})_n = \lambda_{n,k} \leqslant \beta k^2 \pi^2, \quad \forall n, k \in \mathbb{N}.$$

Consequently, using (2.1), (2.3), (2.10), [8, proposition 2.1] and a diagonal argument, it is easy to see that there exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$, an increasing sequence of non-negative numbers $\{\lambda_k\}_{k\in\mathbb{N}}$ and a sequence $\{u_k\}_{k\in\mathbb{N}} \subset V$ (depending possibly on the selected subsequence $\{n_i\}_{i\in\mathbb{N}}$) such that Change OK?

$$\lim_i \lambda_{n_i,k} = \lambda_k, \quad \forall k \in \mathbb{N}, \tag{2.21}$$

$$u_{n_i,k} \rightharpoonup u_k \text{ weakly in } H^1(\Omega^a) \times H^1(\Omega^b) \text{ and strongly in } L^2(\Omega^a) \times L^2(\Omega^b), \quad \forall k \in \mathbb{N}, \tag{2.22}$$

as $i \to +\infty$. Then, adapting the proofs of [9, theorem 1.1] and [9, theorem 1.2], from (2.10), (2.21), (2.22), (2.19) and (2.20) it follows that (2.16) holds true and that there exists a sequence

$$\{(y^a_k, y^b_k)\}_{k\in\mathbb{N}} \subset L^2(0,1; H^1_m(\omega)) \times L^2(\omega; H^1_m(]-1,0[)) \tag{2.23}$$

such that

$$\frac{1}{r_{n_i}} D_{x'} u^a_{n_i,k} \to D_{x'} y^a_k \text{ strongly in } (L^2(\Omega^a))^{N-1}, \quad \forall k \in \mathbb{N}, \tag{2.24}$$

$$\frac{1}{h_{n_i}} \partial_{x_N} u^b_{n_i,k} \to \partial_{x_N} y^b_k \text{ strongly in } L^2(\Omega^b), \qquad \forall k \in \mathbb{N}, \tag{2.25}$$

as $i \to +\infty$,

$$[u_k, u_h]_q = \delta_{h,k}, \quad \forall k, h \in \mathbb{N} \tag{2.26}$$

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and, for every $k \in \mathbb{N}$, $u_k = (u_k^a, u_k^b)$, $y_k = (y_k^a, y_k^b)$ solves the following system:

$$\left.\begin{aligned} & u_k = (u_k^a, u_k^b) \in V, \quad (y_k^a, y_k^b) \in L^2(0,1; H_m^1(\omega)) \times L^2(\omega; H_m^1(\,]-1,0[\,)) \\ & \int_{\Omega^a} A' \begin{pmatrix} D_{x'} y_k^a \\ \dfrac{\mathrm{d}u_k^a}{\mathrm{d}x_N} \end{pmatrix} D_{x'} z^a \,\mathrm{d}x = 0, \\ & \int_{\Omega^b} A^N \begin{pmatrix} D_{x'} u_k^b \\ \partial_{x_N} y_k^b \end{pmatrix} \partial_{x_N} z^b \,\mathrm{d}x = 0, \\ & \int_{\Omega^a} A^N \begin{pmatrix} D_{x'} y_k^a \\ \dfrac{\mathrm{d}u_k^a}{\mathrm{d}x_N} \end{pmatrix} \frac{\mathrm{d}v^a}{\mathrm{d}x_N} \,\mathrm{d}x + q \int_{\Omega^b} A' \begin{pmatrix} D_{x'} u_k^b \\ \partial_{x_N} y_k^b \end{pmatrix} D_{x'} v^b \,\mathrm{d}x = \lambda_k [u_k, v]_q, \\ & \forall v = (v^a, v^b) \in V, \quad \forall (z^a, z^b) \in L^2(0,1; H_m^1(\omega)) \times L^2(\omega; H_m^1(\,]-1,0[\,)), \end{aligned}\right\} \tag{2.27}$$

where, for $S = \omega$ or $S = \,]-1,0[\,$, $H_m^1(S)$ denotes the space of elements of $H_m^1(S)$ with zero average.

To obtain a suitable representation of y_k^a, we begin by showing that the function $\hat{y}^a$ defined by (1.7) satisfies the following property:

$$\hat{y}^a \in L^\infty(0,1; H_m^1(\omega)). \tag{2.28}$$

This property is known if $A \in (C^\infty(\bar{\omega} \times [0,1]))^{(N-1)\times(N-1)}$. Otherwise, using a convolution argument, one can build a sequence

$$\{A_\varepsilon\}_\varepsilon \subset (C^\infty(\bar{\omega} \times [0,1]))^{(N-1)\times(N-1)}$$

of equibounded and uniformly elliptic matrices with the same ellipticity constant, such that

$$A_\varepsilon \to A' \quad \text{strongly in } (L^2(\omega \times \,]0,1[\,))^{(N-1)\times(N-1)}. \tag{2.29}$$

For a.e. $x_N \in \,]0,1[\,$, let $y_\varepsilon(\cdot, x_N)$ be the unique solution of the following problem:

$$\left.\begin{aligned} & y_\varepsilon(\cdot, x_N) \in H_m^1(\omega), \\ & \int_\omega A_\varepsilon \begin{pmatrix} D_{x'} y_\varepsilon \\ 1 \end{pmatrix} D_{x'} \hat{z}^a \,\mathrm{d}x' = 0, \quad \forall \hat{z}^a \in H_m^1(\omega). \end{aligned}\right\} \tag{2.30}$$

By choosing $\hat{z}^a = y_\varepsilon(\cdot, x_N)$ as a test function in (2.30), one immediately obtains that $\{y_\varepsilon\}_\varepsilon$ is bounded in $L^\infty(0,1; H_m^1(\omega))$. Consequently, there exists a function $\bar{y} \in L^\infty(0,1; H_m^1(\omega))$ such that, on extraction of a suitable subsequence (not relabelled),

$$y_\varepsilon \rightharpoonup \bar{y} \quad \text{weakly}^* \text{ in } L^\infty(0,1; H_m^1(\omega)), \tag{2.31}$$

as ε tends to zero. By passing to the limit as ε tends to zero, in (2.30), having multiplied the equation by $\varphi \in C_0^\infty(\,]0,1[\,)$ and having integrated with respect to x_N on $]0,1[\,$, it follows from (2.29) and (2.31) that

$$\begin{gathered} \bar{y} \in L^\infty(0,1; H_m^1(\omega)), \\ \int_{[0,1]\times\omega} A' \begin{pmatrix} D_{x'} \bar{y} \\ 1 \end{pmatrix} D_{x'} \hat{z}^a \varphi \,\mathrm{d}x = 0, \quad \forall \hat{z}^a \in H_m^1(\omega),\ \forall \varphi \in C_0^\infty(\,]0,1[\,), \end{gathered}$$

which involves that, for almost every $x_N \in]0,1[$, $\bar{y}(\cdot, x_N)$ is the unique solution of the following problem:

$$\left.\begin{aligned} &\bar{y}(\cdot, x_N) \in H^1_m(\omega), \\ &\int_\omega A' \begin{pmatrix} D_{x'}\bar{y} \\ 1 \end{pmatrix} D_{x'}\hat{z}^a \, \mathrm{d}x' = 0, \quad \forall \hat{z}^a \in H^1_m(\omega). \end{aligned}\right\} \tag{2.32}$$

Finally, comparing (2.32) with the weak formulation of problem (1.7),

$$\left.\begin{aligned} &\hat{y}^a(\cdot, x_N) \in H^1_m(\omega), \\ &\int_\omega A' \begin{pmatrix} D_{x'}\hat{y}^a \\ 1 \end{pmatrix} D_{x'}\hat{z}^a \, \mathrm{d}x' = 0, \quad \forall \hat{z}^a \in H^1_m(\omega), \end{aligned}\right\} \tag{2.33}$$

one deduces that, for almost every $x_N \in]0,1[$, $\hat{y}^a(\cdot, x_N) = \bar{y}(\cdot, x_N)$ a.e. in ω, that is,

$$\hat{y}^a = \bar{y} \in L^\infty(0,1; H^1_m(\omega)),$$

and so (2.28) is proved.

Now, for every $k \in \mathbb{N}$, set

$$\tilde{y}^a_k = \hat{y}^a \frac{\mathrm{d}u^a_k}{\mathrm{d}x_N}, \quad \text{a.e. in } \Omega^a.$$

The properties of u^a_k, (2.28) and (2.33) entail that $\tilde{y}^a_k$ solves the following problem:

$$\left.\begin{aligned} &\tilde{y}^a_k \in L^2(0,1; H^1_m(\omega)), \\ &\int_{\Omega^a} A' \begin{pmatrix} D_{x'}\tilde{y}^a_k \\ \dfrac{\mathrm{d}u^a_k}{\mathrm{d}x_N} \end{pmatrix} D_{x'}z^a \, \mathrm{d}x = 0, \quad \forall z^a \in L^2(0,1; H^1_m(\omega)). \end{aligned}\right\} \tag{2.34}$$

Since, by (2.3), this problem admits a unique solution, by comparing (2.34) with the first equation in (2.27), one deduces that

$$y^a_k = \hat{y}^a \frac{\mathrm{d}u^a_k}{\mathrm{d}x_N}, \quad \text{a.e. in } \Omega^a, \ \forall k \in \mathbb{N}. \tag{2.35}$$

By combining (2.23) with (2.35), one obtains (2.17).

In what concerns y^b_k, for every $v^b \in H^1(\omega)$, set

$$\begin{aligned} y_{v^b} = &-\sum_{j=1}^{N-1} \int_0^{x_N} \frac{a_{Nj}(x',t)}{a_{NN}(x',t)} \, \mathrm{d}t \partial_{x_j} v^b \\ &\qquad + \sum_{j=1}^{N-1} \int_{-1}^0 \int_0^{x_N} \frac{a_{Nj}(x',t)}{a_{NN}(x',t)} \, \mathrm{d}t \, \mathrm{d}x_N \partial_{x_j} v^b, \quad \text{a.e. in } \Omega^b. \end{aligned}$$

By (2.3), y_{v^b} is well defined, and it evidently solves the following algebraic equation:

$$A^N \begin{pmatrix} D_{x'}v^b \\ \partial_{x_N} y_{v^b} \end{pmatrix} = 0, \quad \text{a.e. in } \Omega^b.$$

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Consequently, y_{v^b} also solves the following problem:

$$\left.\begin{aligned} y_{v^b} \in L^2(\omega; H^1_m(\,]-1,0[\,)), \\ \int_{\Omega^b} A^N \begin{pmatrix} D_{x'} v^b \\ \partial_{x_N} y_{v^b} \end{pmatrix} \partial_{x_N} z^b \,\mathrm{d}x = 0, \quad \forall z^b \in L^2(\omega; H^1_m(\,]-1,0[\,)). \end{aligned}\right\} \tag{2.36}$$

Since, by (2.3), this problem admits a unique solution, by comparing (2.36) when $v^b = u^b_k$ with the second equation in (2.27), we deduce that

$$\begin{aligned} y^b_k = &-\sum_{j=1}^{N-1} \int_0^{x_N} \frac{a_{Nj}(x',t)}{a_{NN}(x',t)} \,\mathrm{d}t \partial_{x_j} u^b_k \\ &+ \sum_{j=1}^{N-1} \int_{-1}^0 \int_0^{x_N} \frac{a_{Nj}(x',t)}{a_{NN}(x',t)} \,\mathrm{d}t \,\mathrm{d}x_N \partial_{x_j} u^b_k, \quad \text{a.e. in } \Omega^b, \ \forall k \in \mathbb{N}. \end{aligned} \tag{2.37}$$

By combining (2.25) with (2.37), we obtain (2.18).

By replacing (2.35) and (2.37) in the last equation in (2.27), we obtain

$$\begin{gathered} u_k = (u^a_k, u^b_k) \in V, \\ \int_{\Omega^a} A^N \begin{pmatrix} D_{x'} \hat{y}^a \\ 1 \end{pmatrix} \frac{\mathrm{d}u^a_k}{\mathrm{d}x_N} \frac{\mathrm{d}v^a}{\mathrm{d}x_N} \,\mathrm{d}x \\ + q \int_{\Omega^b} A' \begin{pmatrix} D_{x'} u^b_k \\ -\displaystyle\sum_{j=1}^{N-1} \frac{a_{Nj}}{a_{NN}} \partial_{x_j} u^b_k \end{pmatrix} D_{x'} v^b \,\mathrm{d}x = \lambda_k [u_k, v]_q, \quad \forall v = (v^a, v^b) \in V, \end{gathered}$$

or, equivalently,

$$\left.\begin{gathered} u_k = (u^a_k, u^b_k) \in V, \\ \int_0^1 \bar{a}^a \frac{\mathrm{d}u^a_k}{\mathrm{d}x_N} \frac{\mathrm{d}v^a}{\mathrm{d}x_N} \,\mathrm{d}x_N + q \int_\omega \bar{A}^b D_{x'} u^b_k D_{x'} v^b \,\mathrm{d}x' = \lambda_k [u_k, v]_q, \quad \forall v = (v^a, v^b) \in V, \end{gathered}\right\} \tag{2.38}$$

where $\bar{a}^a$ and $\bar{A}^b$ are defined in (1.6). Then, by taking into account (2.26), we have proved that $\{u_k\}_{k\in\mathbb{N}}$ is an orthonormal sequence in $(H, [\cdot,\cdot]_q)$ and, for every $k \in \mathbb{N}$, u_k is an eigenvector for problem (2.15) with eigenvalue λ_k. Moreover, (2.26) and (2.38) give

$$\alpha_q(\lambda_k^{-1/2} u_k, \lambda_h^{-1/2} u_h) = \delta_{h,k}, \quad \forall k, h \in \mathbb{N}. \tag{2.39}$$

Now, note that $\bar{A}^b(x') \in M^{N\times N}_s$ for almost every $x' \in \omega$, $\bar{a}^a \in L^\infty(\,]0,1[\,)$, $\bar{A}^b \in (L^\infty(\omega))^{N-1}$, by virtue of (2.2), (2.3) and (2.28). Moreover, $\bar{a}^a$ and $\bar{A}^b$ are uniformly elliptic. In fact, from (1.6), (2.33) and (2.3), it follows that

$$\begin{aligned} \bar{a}^a(x_N) &= \int_\omega A^N \begin{pmatrix} D_{x'} \hat{y}^a \\ 1 \end{pmatrix} \mathrm{d}x' \\ &= \int_\omega A^N \begin{pmatrix} D_{x'} \hat{y}^a \\ 1 \end{pmatrix} \mathrm{d}x' + \int_\omega A' \begin{pmatrix} D_{x'} \hat{y}^a \\ 1 \end{pmatrix} D_{x'} \hat{y}^a \,\mathrm{d}x' \end{aligned}$$

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$$
\begin{aligned}
&= \int_\omega A \begin{pmatrix} D_{x'}\hat{y}^a \\ 1 \end{pmatrix} \begin{pmatrix} D_{x'}\hat{y}^a \\ 1 \end{pmatrix} \mathrm{d}x' \\
&\geqslant \alpha \int_\omega \left| \begin{pmatrix} D_{x'}\hat{y}^a \\ 1 \end{pmatrix} \right|^2 \mathrm{d}x' \\
&\geqslant \alpha|\omega|, \quad \text{a.e. } x_N \in\,]0,1[\,,
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{A}^b(x')\xi\xi &= \int_{-1}^0 A' \begin{pmatrix} \xi \\ -\displaystyle\sum_{j=1}^{N-1} \frac{a_{Nj}}{a_{NN}}\xi_j \end{pmatrix} \xi \, \mathrm{d}x_N \\
&= \int_{-1}^0 \left[A' \begin{pmatrix} \xi \\ -\displaystyle\sum_{j=1}^{N-1} \frac{a_{Nj}}{a_{NN}}\xi_j \end{pmatrix} \xi + A^N \begin{pmatrix} \xi \\ -\displaystyle\sum_{j=1}^{N-1} \frac{a_{Nj}}{a_{NN}}\xi_j \end{pmatrix} \sum_{j=1}^{N-1} \frac{-a_{Nj}}{a_{NN}}\xi_j \right] \mathrm{d}x_N \\
&= \int_{-1}^0 A \begin{pmatrix} \xi \\ -\displaystyle\sum_{j=1}^{N-1} \frac{a_{Nj}}{a_{NN}}\xi_j \end{pmatrix} \begin{pmatrix} \xi \\ -\displaystyle\sum_{j=1}^{N-1} \frac{a_{Nj}}{a_{NN}}\xi_j \end{pmatrix} \mathrm{d}x_N \\
&\geqslant \alpha \int_{-1}^0 \left| \begin{pmatrix} \xi \\ -\displaystyle\sum_{j=1}^{N-1} \frac{a_{Nj}}{a_{NN}}\xi_j \end{pmatrix} \right|^2 \mathrm{d}x_N \\
&\geqslant \alpha|\xi|^2, \quad \text{a.e. } x' \in \omega,\ \forall \xi \in \mathbb{R}^{N-1}.
\end{aligned}
$$

Then, α_q is an inner product in V and its induced norm is equivalent to the $(H^1(\,]0,1[\,) \times H^1(\omega))$-norm. Hence, taking into account (2.39), it results that every λ_k is a strictly positive number and $\{\lambda_k^{-1/2} u_k\}_{k\in\mathbb{N}}$ is an orthonormal sequence in (V, α_q). Moreover, note that

$$
\lim_k \lambda_k = +\infty \tag{2.40}
$$

in fact, or (2.40) holds true, or $\{\lambda_k\}_{k\in\mathbb{N}}$ is a finite set. In the second case, by virtue of (2.26), problem (2.15) would admit an eigenvalue of infinite multiplicity. But this is not possible, due to Fredholm's alternative theorem.

By arguing as in step 2 of the proof of theorem 2.5 of [6] (see also [3, 22]), we can prove that there does not exist $(\bar{u}, \bar{\lambda}) \in V \times \mathbb{R}$ satisfying the following problem:

$$
\left.
\begin{aligned}
&\bar{u} \in V, \\
&\alpha_q(\bar{u}, v) = \bar{\lambda}[\bar{u}, v]_q, \quad \forall v \in V, \\
&[\bar{u}, u_k]_q = 0, \qquad\qquad \forall k \in \mathbb{N}, \\
&[\bar{u}, \bar{u}]_q = 1.
\end{aligned}
\right\} \tag{2.41}
$$

This property shows that the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ forms the whole set of the eigenvalues of problem (2.15).

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By arguing as in [6], we can prove that the set of finite combinations of elements of $\{\lambda_k^{-1/2} u_k\}_{k\in\mathbb{N}}$ is dense in (V, α_q), which entails that the set of the finite combinations of elements of $\{u_k\}_{k\in\mathbb{N}}$ is dense in $(H, [\cdot,\cdot]_q)$, since V is continuously embedded into H, with dense inclusion.

In conclusion, since the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ can be characterized by the min–max principle (see, for example, [17, theorem 6.2-2]), for every $k \in \mathbb{N}$, convergence (2.21) holds true for the whole sequence $\{\lambda_{n,k}\}_{n\in\mathbb{N}}$. □

Proof of theorem 1.1. For every $n \in \mathbb{N}$, let $\{(\lambda_{n,k}, U_{n,k})\}_{k\in\mathbb{N}}$ be a sequence as at the beginning of § 1. Problem (1.1) can be reformulated on a fixed domain through an appropriate rescaling that maps Ω_n into $\Omega = \omega \times \,]-1,1[$. Namely, by setting

$$\tilde{u}_{n,k}(x) = \begin{cases} \tilde{u}^a_{n,k}(x', x_3) = U_{n,k}(r_n x', x_3), & (x', x_3) \text{ a.e. in } \Omega^a = \omega \times \,]0,1[\,, \\ \tilde{u}^b_{n,k}(x', x_3) = U_{n,k}(x', h_n x_3), & (x', x_3) \text{ a.e. in } \Omega^b = \omega \times \,]-1,0[\,, \end{cases} \quad \forall n, k \in \mathbb{N}, \tag{2.42}$$

we have that, for every $n \in \mathbb{N}$, $\{\lambda_{n,k}\}_{k\in\mathbb{N}}$ forms the set of all the eigenvalues of problem (2.7), $u_{n,k} = r_n^{(N-1)/2} \tilde{u}_{n,k} \in V_n$ is an eigenvector of (2.7) with eigenvalue $\lambda_{n,k}$, $\{u_{n,k}\}_{k\in\mathbb{N}}$ is an H_n-Hilbert orthonormal basis and $\{\lambda_{n,k}^{-1/2} u_{n,k}\}_{k\in\mathbb{N}}$ is a V_n-Hilbert orthonormal basis, where V_n and H_n are as defined at the beginning of this section. Then theorem 1.1 is an immediate consequence of theorem 2.2. □

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