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ENDOGENOUS BUSINESS CYCLES IN OVERLAPPING-GENERATIONS ECONOMIES WITH MULTIPLE CONSUMPTION GOODS

Carine Nourry
IUF
Université de la Méditerranée
and
GREQAM

Alain Venditti
CNRS–GREQAM
and
EDHEC

We consider an overlapping-generations economy with two consumption goods. There are two sectors that produce a pure consumption good and a mixed good that can be either consumed or used as capital. We prove that the existence of Pareto-optimal expectations-driven fluctuations is compatible with standard sectoral technologies if the share of the pure consumption good is low enough. Following Reichlin's [*Journal of Economic Theory* 40 (1986), 89–102] influential conclusion, this result suggests that some fiscal policy rules can prevent business-cycle fluctuations in the economy by driving it to the optimal steady state as soon as they are announced.

Keywords: Two-Sector OLG Model, Multiple Consumption Goods, Dynamic Efficiency, Endogenous Fluctuations, Local Indeterminacy

1. INTRODUCTION

A widespread perception among economists is that macroeconomic busi-ness-cycle fluctuations are driven not only by shocks on technologies or preferences, but also by changes in expectations about the fundamentals. A major strand of the literature focusing on fluctuations derived from agents' beliefs is based on the concept of sunspot equilibria, which dates back to the early work of Shell (1977), Azariadis (1981), and Cass and Shell (1983). As shown by Woodford (1986), the

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existence of sunspot equilibria is closely related to the equilibrium indeterminacy under perfect foresight, i.e., the existence of a continuum of equilibrium paths converging toward one steady state from the same initial value of the state variable.

Since the contribution of Reichlin (1986), the possible coexistence of dynamic efficiency, i.e., Pareto-optimal equilibrium paths, and local indeterminacy in overlapping-generations (OLG) models has been widely discussed in the literature. This is an important question in terms of stabilization policies. Indeed, when local indeterminacy occurs under dynamic *efficiency*, the introduction of a public policy based on taxes and transfers could at the same time stabilize the economy and reach the Pareto-optimal steady state along which all generations get an equal level of welfare. In contrast, when local indeterminacy occurs under dynamic *inefficiency*, stabilization policies targeting the steady state leave room for welfare losses. Although Reichlin (1986) has shown that locally indeterminate dynamically efficient equilibria can occur in an aggregate model with a Leontief technology, Cazzavillan and Pintus (2007) have recently proved that this result is not robust to the introduction of any positive capital–labor substitution.

In Galor (1992) type two-sector OLG models, with one pure consumption good and one pure investment good, the conclusion is not so clear-cut. Indeed, Drugeon et al. (2010) and Nourry and Venditti (2011) have proved that local indeterminacy is ruled out when the steady state is dynamically efficient, provided the sectoral technologies are not too close to Leontief functions. The intuition for this result is quite simple. Starting from an arbitrary equilibrium, consider another one in which the agents expect a higher rate of investment at time t, leading to some higher capital stock at time $t+1$. This expectation will be self-fulfilling, provided the amount of saving at date t is large enough to support the increase of the investment good output that directly provides the capital stock of the next period. When the equilibrium is dynamically efficient, the share of first period consumption is large enough to generate a stationary capital stock lower than the Golden Rule and thus prevents the agent from saving enough. At the same time, when the sectoral technologies have elasticities of capital–labor substitution far enough from zero, any transfer of capital in the investment good sector is followed by a decrease of labor, so that the final output cannot generate a large enough increase of capital.[1] As a result, the initial expectation cannot be realized as an equilibrium, and, under dynamic efficiency, local indeterminacy, together with fluctuations based on local sunspots, is ruled out.

However, in a multisector framework, the assumption of a unique consumption good is highly peculiar and is likely to generate singular properties. When heterogeneous sectors are introduced, it is quite common to assume instead that multiple consumption goods coexist.[2] In such a case, the existence of additional substitution mechanisms between the different consumption goods suggests that new conclusions could be obtained. Unfortunately, the literature on OLG models has almost exclusively focused on Galor-type formulations. Among the few contributions dealing with a different framework, the most noteworthy is a paper of Kalra (1996), which studies the existence of cyclical equilibria in a generalization of the standard two-sector model by assuming that both goods are consumed.[3]

Besides the pure consumption good, the second sector produces a mixed good that can either be consumed or be used as capital. Such a formulation, which is more in line with standard national accounting data than the usual Galor-type formulation,[4] allows for the existence of intratemporal substitution in consumption. Kalra then shows that local indeterminacy through the occurrence of a Hopf bifurcation is more likely to occur than in the standard Galor-type model.[5] However, as he does not discuss the dynamic efficiency property of the equilibrium, nothing is said about the question initially raised by Reichlin (1986).

The purpose of this paper is then to complement the analysis of Kalra and to explore the existence of dynamically efficient endogenous fluctuations. A simple intuition suggests that new conclusions can be obtained. With two consumption goods, everything else equal, the stock of capital in the next period that is supported by the amount of saving at date t is lower than in the standard case because it is based on a mixed good output decreased by the part that is used for consumption. In such a case, a given amount of saving may be compatible with the expectation of a higher capital stock at time $t+1$ if a larger part of the mixed good output is consumed. Based on this new channel, local indeterminacy and the existence of expectations-driven fluctuations can become compatible with dynamic efficiency under standard sectoral technologies with higher elasticities of capital–labor substitution.

We consider a simplified version of Kalra's model assuming a unitary elasticity of intratemporal substitution between the two consumption goods. We provide a simple condition for dynamic efficiency. Then we show that, according to the previous intuition, when the pure consumption good is capital-intensive, local indeterminacy and expectations-driven fluctuations occur for dynamically efficient competitive equilibria with higher sectoral elasticities of capital–labor substitution than in the standard model, provided the share of the pure consumption good in utility is low enough. We thus show that Reichlin's (1986) result is more robust in a two-sector OLG economy with multiple consumption goods than in the standard Galor-type formulation.

The paper is organized as follows: Section 2 presents the model. Section 3 proves the existence of a steady state and provides a condition for dynamic efficiency. Section 4 contains our main results on the coexistence of local indeterminacy and dynamic efficiency. Some results under dynamic inefficiency are also provided in order to establish a better understanding of the relationship between Pareto optimality and the dynamic properties of competitive equilibria. Concluding comments are in Section 5 and the proofs are gathered in a final Appendix.

2. THE MODEL

2.1. The Production Side

We consider an economy with two produced goods, one pure consumption good y_0 that cannot be used as capital, and one mixed good y that can be either consumed or invested, i.e., used as capital. There are two inputs, capital and labor. We assume complete depreciation of capital within one period and inelastic supply of labor. We then get

$$y_t - z_t = k_{t+1}, \tag{1}$$

with z_t the consumption part of the mixed good in period t and k_{t+1} the total amount of capital in period $t + 1$.

Each good is produced with a constant–returns to scale technology such that $y_0 = f^0(k^0, l^0)$ and $y = f^1(k^1, l^1)$, with $k^0 + k^1 \leq k$, k being the total stock of capital, and $l^0 + l^1 \leq \ell$, ℓ being the total amount of labor.

Assumption 1. Each production function $f^i : \mathbf{R}^2_+ \to \mathbf{R}_+$, $i = 0, 1$, is C^2, increasing in each argument, concave, homogeneous of degree one, and such that for any $x > 0$, $f^i_1(0, x) = f^i_2(x, 0) = +\infty$, $f^i_1(+\infty, x) = f^i_2(x, +\infty) = 0$.

For any given (k, y, ℓ), profit maximization in the representative firm in each sector is equivalent to solving the following problem of optimal allocation of productive factors between the two sectors:

$$\begin{aligned} \tau(k, y, \ell) = &\max_{k^0,k^1,l^0,l^1 \geq 0} f^0(k^0, l^0), \\ \text{s.t.} \quad & y \leq f^1(k^1, l^1), k^0 + k^1 \leq k, \text{ and } l^0 + l^1 \leq \ell. \end{aligned} \tag{2}$$

The social production function $\tau(k, y, \ell)$ gives the maximal output of the consumption good. Under Assumption 1, $\tau(k, y, \ell)$ is homogeneous of degree one, concave, and twice continuously differentiable.[6] Denoting as w the wage rate, r the gross rental rate of capital, and p the price of the mixed good, all in terms of the price of the pure consumption good, we derive

$$r = \tau_1(k, y, \ell), \quad p = -\tau_2(k, y, \ell), \quad w = \tau_3(k, y, \ell). \tag{3}$$

2.2. The Consumption Side

In each period t, N_t agents are born, and they live for two periods. In their first period of life (when *young*), the agents are endowed with one unit of labor, which they supply inelastically to firms. Their income is equal to the real wage. They allocate this income between current consumption and savings, which are invested in the firms. In their second period of life (when *old*), they are retired and their income resulting from the return on the savings is entirely consumed. Each agent is assumed to have one child, so that population is constant, i.e., $N_t = N$.

The preferences of a representative agent born at time t are defined over his consumption bundle for each of the two produced goods, c^0_t, c^1_t, when young, and d^0_{t+1}, d^1_{t+1}, when old. They are summarized by the utility function $U(c^0_t, c^1_t, d^0_{t+1}, d^1_{t+1}) = u(C_t, D_{t+1}/B)$ with

$$C_t = \left(c^0_t\right)^\theta \left(c^1_t\right)^{1-\theta}, \quad D_{t+1} = \left(d^0_{t+1}\right)^\theta \left(d^1_{t+1}\right)^{1-\theta} \tag{4}$$

and[7]

$$u(C_t, D_{t+1}/B) = \left[C_t^{1-1/\gamma} + \delta(D_{t+1}/B)^{1-1/\gamma}\right]^{\gamma/(\gamma-1)}, \quad \textbf{(5)}$$

where $0 < \theta \leq 1$, $\delta > 0$ is the discount factor, $\gamma > 0$ is the elasticity of intertemporal substitution in consumption, and $B > 0$ is a scaling parameter. C_t and D_{t+1} can be interpreted as composite goods derived from the two consumption goods y_t^0 and z_t with

$$y_t^0 = T(k_t, y_t, \ell_t) = N\left(c_t^0 + d_t^0\right) \quad \text{and} \quad z_t = N\left(c_t^1 + d_t^1\right). \quad \textbf{(6)}$$

It follows that θ is the share of the pure consumption good and $1 - \theta$ the share of the mixed good in the composite goods C and D.

A young agent born in period t has first to solve two static problems of optimal composition of his two composite goods. Denoting by π_t the consumer price index in terms of the pure consumption good, we get the following optimization program for C_t:

$$\max_{c_t^0, c_t^1 \geq 0} \left(c_t^0\right)^\theta \left(c_t^1\right)^{1-\theta} \text{ s.t. } c_t^0 + p_t c_t^1 = \pi_t C_t. \quad \textbf{(7)}$$

The corresponding program for D_{t+1} is similar, with $d_{t+1}^0 + p_{t+1} d_{t+1}^1 = \pi_{t+1} D_{t+1}$. Solving the associated first-order conditions gives

$$c_t^0 = \theta \pi_t C_t, \quad c_t^1 = \frac{(1-\theta)\pi_t C_t}{p_t}, \quad \pi_t = \left(\frac{p_t}{1-\theta}\right)^{1-\theta} \theta^{-\theta}, \quad \textbf{(8)}$$

with similar expressions for d_{t+1}^0 and d_{t+1}^1, namely

$$d_{t+1}^0 = \theta \pi_{t+1} D_{t+1}, \quad d_{t+1}^1 = \frac{(1-\theta)\pi_{t+1} D_{t+1}}{p_{t+1}}. \quad \textbf{(9)}$$

Under perfect foresight, and considering w_t and R_{t+1} as given, a young agent also has to solve an intertemporal allocation problem in order to maximize his utility function over his life cycle:

$$\max_{C_t, D_{t+1}, \phi_t \geq 0} u(C_t, D_{t+1}/B) \text{ s.t. } w_t = \pi_t C_t + \phi_t \text{ and } R_{t+1}\phi_t = \pi_{t+1} D_{t+1}.$$

Solving the first-order conditions gives

$$C_t = \frac{w_t/\pi_t}{1 + \delta^\gamma \left[R_{t+1}\pi_t/(B\pi_{t+1})\right]^{\gamma-1}} \equiv \alpha(v_t) w_t/\pi_t \quad \textbf{(10)}$$

with $v_t = R_{t+1}\pi_t/(B\pi_{t+1})$ and $\alpha(v_t) \in (0, 1)$ the propensity to consume of the young, or equivalently the share of first-period consumption spending over the wage income. We also get the saving function

$$\phi_t = \phi(w_t, v_t) \equiv (1 - \alpha(v_t)) w_t. \quad \textbf{(11)}$$

In the rest of the paper we introduce the following standard assumption:

Assumption 2. $\gamma > 1$.

Such a restriction implies that the saving function (11) is increasing with respect to the gross rate of return R.

3. DYNAMIC EFFICIENCY OF COMPETITIVE EQUILIBRIUM

3.1. Perfect-Foresight Competitive Equilibrium

Total labor is given by the number of young households and is normalized to one; i.e., $\ell = N = 1$. From now on, let $\tau(k, y, 1) = T(k, y)$ and $\tau_i(k, y, 1) = T_i(k, y)$, $i = 1, 2, 3$. Using (1), (3), and (11), a perfect-foresight competitive equilibrium is defined as a sequence $\{k_t, y_t\}_{t\geq 0}$ that satisfies $\{1 - \alpha[R_{t+1}\pi_t/(B\pi_{t+1})]\}w_t = k_{t+1}$ and $y_t - z_t = k_{t+1}$ with $R_{t+1} = r_{t+1}/p_t$. Using (6), (8), and (9), we conclude that a perfect-foresight competitive equilibrium satisfies the following system of two difference equations:

$$k_{t+1} + \frac{T_3(k_t, y_t)\left(1 - \alpha\left\{-\frac{T_1(k_{t+1}, y_{t+1})}{T_2(k_t, y_t)B}\left[\frac{T_2(k_t, y_t)}{T_2(k_{t+1}, y_{t+1})}\right]^{1-\theta}\right\}\right)}{T_2(k_t, y_t)} = 0 \qquad \textbf{(12)}$$

$$k_{t+1} - \frac{y_t}{\theta} - \frac{(1-\theta)\left[T_3(k_t, y_t) + T_1(k_t, y_t)k_t\right]}{\theta T_2(k_t, y_t)} = 0$$

with k_0 given.

It is worth noting at this point that if $\theta = 1$ the second difference equations reduces to $k_{t+1} = y_t$ and we get the standard two-sector OLG model with one pure consumption good and one pure investment good studied in Drugeon et al. (2010) and Nourry and Venditti (2011).[8]

3.2. A Normalized Steady State

A steady state $(k_t, y_t) = (k^*, y^*)$ for all t satisfies

$$k^* + \frac{T_3(k^*, y^*)\left[1 - \alpha\left(-\frac{T_1(k^*, y^*)}{T_2(k^*, y^*)B}\right)\right]}{T_2(k^*, y^*)} = 0 \qquad \textbf{(13)}$$

$$k^* - \frac{y^*}{\theta} - \frac{(1-\theta)\left[T_3(k^*, y^*) + T_1(k^*, y^*)k^*\right]}{\theta T_2(k^*, y^*)} = 0.$$

We consider a family of economies parameterized by the elasticity of intertemporal substitution in consumption γ. We follow the same procedure as in Drugeon et al. (2010): we use the scaling parameter B and the share θ to ensure the existence of a normalized steady state (NSS) (k^*, y^*), which remains invariant as γ is varied.

Let us define the maximum admissible value of capital $\bar{k}$ solution of

$$\bar{k} - f^1(\bar{k}, 1) = 0.$$

Under Assumption 1 we have indeed $f^1(k, 1) > k$ if $k < \bar{k}$, whereas $f^1(k, 1) < k$ if $k > \bar{k}$. Obviously, the NSS must be such that $(k^*, y^*) \in (0, \bar{k}) \times (0, \bar{k})$. We then get

PROPOSITION 1. *Under Assumptions 1 and 2, let* $(k^*, y^*) \in (0, \bar{k}) \times (0, \bar{k})$. *Then there exist unique values* $\theta(k^*, y^*) \in (0, 1)$ *and* $B(k^*, y^*, \gamma) > 0$ *such that* (k^*, y^*) *is a steady state if and only if* $\theta = \theta(k^*, y^*)$ *and* $B = B(k^*, y^*, \gamma)$.

Proof. See Appendix A.1. ■

In the rest of the paper we assume $\theta = \theta(k^*, y^*)$ and $B = B(k^*, y^*, \gamma)$, so that the share of capital in total income, as given by

$$s = s(k^*, y^*) = \frac{r^*k^*}{T(k^*, y^*) + p^*y^*}, \tag{14}$$

and $\alpha = \alpha(R^*/B(k^*, y^*, \gamma))$ remain constant as γ is made to vary.

3.3. Dynamic Efficiency

From the homogeneity of τ, assuming that $k^*T_2/T_3 = (T_2/T_1)(k^*T_1/T_3) = -s/[R(1-s)]$, we derive the stationary gross rate of return along the NSS:

$$R^* = \frac{s}{(1-\alpha)(1-s)}. \tag{15}$$

It is well known that in OLG models, if the capital–labor ratio exceeds the Golden-Rule level, the economy is dynamically inefficient. In our two-sector model, the Golden-Rule level, denoted $\hat{k}$, is characterized from the total stationary consumption, which is given by the sum of the social production function and the consumption part of the mixed good, namely $C + D = T(k, y) + p[y - k]$. Denoting $R(k, y) = -T_1(k, y)/T_2(k, y)$, $\hat{k}$ satisfies $R(\hat{k}, \hat{y}) \equiv \hat{R} = 1$ as usual. Underaccumulation of capital is obtained if and only if $R^* > 1$. As in Drugeon et al. (2010), we get

PROPOSITION 2. *Under Assumption 1, let* $\underline{\alpha} = 1 - s/(1-s)$. *Then an intertemporal competitive equilibrium converging toward the NSS is dynamically efficient if* $\alpha \in (\underline{\alpha}, 1)$ *and dynamically inefficient if* $\alpha \in (0, \underline{\alpha})$.

Dynamic inefficiency can be avoided if the amount of savings is not too large.

4. ENDOGENOUS BUSINESS CYCLES

Our aim is to show that a two-sector OLG model with two consumption goods provides new results on the existence of endogenous business cycles derived from agents' beliefs. This type of fluctuations are based on the concept of sunspot equilibria. As shown by Woodford (1986), the existence of sunspot equilibria is closely related to the equilibrium local indeterminacy, i.e., the existence of a continuum of equilibrium paths converging toward the NSS from the same initial capital stock. In our framework, local indeterminacy occurs when the characteristic roots associated with the linearization of equations (12) around the NSS are less than 1 in absolute value.

We focus on two particular aspects of the model: the share θ of the pure consumption good in the composite goods C and D, and the share α of first-period consumption spending in total income. Let us also introduce the relative capital intensity difference across sectors and the elasticity of the rental rate of capital, respectively

$$b \equiv \frac{l^1}{y}\left(\frac{k^1}{l^1} - \frac{k^0}{l^0}\right) \quad \text{and} \quad \varepsilon_{rk} = -T_{11}(k^*, y^*)k^*/T_1(k^*, y^*). \qquad \textbf{(16)}$$

Note that ε_{rk} is negatively linked to the sectoral elasticities of capital–labor substitution [see Drugeon (2004)].

4.1. A Labor-Intensive Pure Consumption Good

Let us consider first the case $b > 0$. We derive from the homogeneity of τ and equations (3) that at the NSS $b < b_{\text{Max}}$, with

$$b_{\text{Max}} = \frac{s}{R^*\{1 - \theta\,[(1-s)\alpha + s]\}} > 0. \qquad \textbf{(17)}$$

We get the following results:

PROPOSITION 3. *Under Assumptions 1 and 2, any equilibrium path is unique and monotone when the pure consumption good is labor-intensive* ($b \in (0, b_{\text{Max}})$).

Proof. See Appendix A.2. ■

This proposition implies that endogenous fluctuations and local indeterminacy are ruled out when $b \in (0, b_{\text{Max}})$. It complements the analysis of Kalra (1996) in the case of a labor-intensive pure consumption good. First, it confirms that endogenous period-2 cycles through the existence of negative characteristic roots and a flip bifurcation are ruled out. Second, it proves that endogenous quasi-periodic cycles through the existence of a Hopf bifurcation are also ruled out in the case of a unitary elasticity of intratemporal substitution between the two consumption goods. Therefore, Proposition 2 in Kalra (1996) requires an elasticity sufficiently lower than one.

4.2. A Capital-Intensive Pure Consumption Good

Let us consider from now on the case $b < 0$ and focus in a first step on dynamically efficient equilibria with $\alpha > \underline{\alpha}$. We assume $\theta < 1/2$, to provide conditions for the existence of local indeterminacy through the occurrence of a Hopf bifurcation. The motivation for considering such a configuration comes from the fact that when θ is close to 1, a Hopf bifurcation cannot occur under dynamic efficiency [see Nourry and Venditti (2011)]. We also restrict the share of capital to get a positive value for the bound $\underline{\alpha} = 1 - s/(1-s)$ and to be compatible with standard empirical values:

Assumption 3. $s \in [1/3, 1/2)$.

PROPOSITION 4. *Under Assumptions 1–3, let* $\alpha > s/(1-s)(> \underline{\alpha})$ *and* $b \in (-(1-\alpha)s/[\alpha(1-\theta+\theta s)], 0)$. *Then there exist* $\bar{\theta} \in (s/[(1-s)\alpha + s], 1/2]$, $\tilde{\epsilon}_{rk} < \underline{\epsilon}_{rk} < \bar{\epsilon}_{rk}$, *and* $\gamma_H, \gamma_F > \gamma_T > 1$ *such that when* $\theta \in (s/[(1-s)\alpha + s], \bar{\theta})$, *the following results hold:*

(1) *If* $\epsilon_{rk} \in (\underline{\epsilon}_{rk}, \bar{\epsilon}_{rk})$, *then* $\gamma_F > \gamma_H$ *and the NSS is locally indeterminate when* $\gamma \in (\gamma_T, \gamma_H)$ *and undergoes a Hopf bifurcation when* $\gamma = \gamma_H$. *Moreover, there generically exist locally indeterminate (resp. locally unstable) quasi-periodic cycles when* $\gamma \in (\gamma_H, \gamma_H + \epsilon)$ *(resp.* $\gamma \in (\gamma_H - \epsilon, \gamma_H)$*) with* $\epsilon > 0$, *i.e., when the bifurcation is super- (resp. sub-) critical.*
(2) *If* $\epsilon_{rk} \in (\tilde{\epsilon}_{rk}, \underline{\epsilon}_{rk})$, *then* $\gamma_H > \gamma_F$ *and the NSS is locally indeterminate when* $\gamma \in (\gamma_T', \gamma_F)$ *and undergoes a flip bifurcation when* $\gamma = \gamma_F$. *Moreover, there generically exist locally indeterminate (resp. saddlepoint stable) period-2 cycles when* $\gamma \in (\gamma_F, \gamma_F + \varepsilon)$ *(resp.* $\gamma \in (\gamma_F - \varepsilon, \gamma_F)$*) with* $\varepsilon > 0$, *i.e., when the bifurcation is super- (resp. sub-) critical.*

Proof. See Appendix A.3. ■

Remark 1. Whether the bifurcation is super- or subcritical is driven by the sign of some coefficient computed from the second- and third-order approximations to the dynamical system (12). This property determines whether the bifurcation leads to the occurrence of locally indeterminate or unstable (resp. saddlepoint stable) quasi-periodic (resp. period-2) cycles near the bifurcation value.

Proposition 4 shows that when the share θ of the pure consumption good in the composite goods is low enough, local indeterminacy and expectations-driven fluctuations arise when $\epsilon_{rk} \in (\tilde{\epsilon}_{rk}, \bar{\epsilon}_{rk})$, i.e., for strictly positive but intermediary values of the sectoral elasticities of capital–labor substitution. We then get more general conclusions for a wider range of elasticities of capital–labor substitution than in the case $\theta = 1$, in which the occurrence of local indeterminacy is only obtained through a flip bifurcation and requires consideration of sectoral technologies very close to Leontief functions [see Nourry and Venditti (2011)]. The intuition for the existence of dynamically efficient expectations-driven fluctuations with multiple consumption goods given in the Introduction is thus confirmed.

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We thus show that Reichlin's (1986) result is more robust in a two-sector OLG economy with multiple consumption goods than in the Galor-type formulation. The introduction of a public policy based on taxes and transfers can at the same time stabilize the economy and reach the Pareto-optimal steady state along which all generations get an equal level of welfare.[9]

To establish a better understanding of the relationship between Pareto optimality and the dynamic properties of competitive equilibria, let us finally focus on dynamically inefficient equilibria with $\alpha < \underline{\alpha}$. We successively consider the existence of local indeterminacy through flip and Hopf bifurcations. As mentioned in Remark 1, the two following Propositions will encompass the super- and subcritical bifurcations. Let us denote

$$\bar{b} = \min\left\{\frac{1}{R^*(1-\theta)-\theta}, -\frac{(1-\alpha)s}{\alpha\left[1-\theta+\theta s\right]}\right\}.$$

PROPOSITION 5. *Under Assumptions 1–3, let* $\alpha < \underline{\alpha}$. *If one of the following conditions is satisfied,*

(i) $\theta \in (s/[1-\alpha(1-s)], 1/2)$ *and* $b < \bar{b}$,
(ii) $\theta \in (1/2, [s^2+\alpha(1-s)]/[2s^2+\alpha(1-s)^2])$ *and* $b \in (1/[R^*(1-2\theta)], \bar{b})$, *then there exists* $\gamma_F > 1$ *such that the NSS is locally indeterminate when* $\gamma > \gamma_F$ *and undergoes a flip bifurcation when* $\gamma = \gamma_F$. *Moreover, there generically exist locally indeterminate (resp. saddlepoint stable) period-2 cycles when* $\gamma \in (\gamma_F - \varepsilon, \gamma_F)$ *(resp.* $\gamma \in (\gamma_F, \gamma_F + \varepsilon)$*) with* $\varepsilon > 0$, *i.e., when the bifurcation is super- (resp. sub-) critical.*

Proof. See Appendix A.4. ■

When compared with Proposition 4, Proposition 5 shows that considering dynamically efficient equilibria implies restrictions for the existence of local indeterminacy that are not necessary under dynamic inefficiency. Indeed, expectations-driven fluctuations now occur without any condition on the elasticity of the rental rate of capital ε_{rk} and are thus compatible with any values of the sectoral elasticities of capital–labor substitution.

This conclusion is also confirmed when local indeterminacy is appraised through the existence of a Hopf bifurcation. Let us indeed denote

$$\tilde{\theta} = \frac{s+(1-\alpha)(1-s)+(1-s)\sqrt{(\underline{\alpha}-\alpha)(1-\alpha)}}{s+3(1-\alpha)(1-s)}$$

and

$$\tilde{b} = \min\left\{-\frac{s}{R^*\left[\theta\left[(1-s)\alpha+s\right]-s\right]}, \frac{(1-\theta)R^*-\theta+\theta\sqrt{1-R^*}}{R^*\left[(1-\theta)^2R^*-\theta(2-3\theta)\right]}\right\}.$$

To simplify the formulation, we focus on the standard value $s = 1/3$ for the share of capital in total income. We then get:

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PROPOSITION 6. *Under Assumptions 1 and 2, let* $s = 1/3$. *There exist* $\tilde{\alpha} \in (0, \underline{\alpha})$ *and* $\gamma_F > \gamma_H > 1$ *such that if* $\alpha \in (\tilde{\alpha}, \underline{\alpha})$, $\theta \in (s/[(1-s)\alpha + s], \tilde{\theta})$ *and* $b < \tilde{b}$, *the NSS is locally indeterminate when* $\gamma \in (\gamma_H, \gamma_F)$, *undergoes a flip bifurcation when* $\gamma = \gamma_F$, *and undergoes a Hopf bifurcation when* $\gamma = \gamma_H$. *Moreover, there generically exist locally indeterminate (resp. saddlepoint stable) period-2 cycles when* $\gamma \in (\gamma_F, \gamma_F + \varepsilon)$ *(resp.* $\gamma \in (\gamma_F - \varepsilon, \gamma_F)$*) with* $\varepsilon > 0$, *i.e., when the bifurcation is super- (resp. sub-) critical, and locally indeterminate (resp. locally unstable) quasi-periodic cycles when* $\gamma \in (\gamma_H - \epsilon, \gamma_H)$ *[resp.* $\gamma \in (\gamma_H, \gamma_H + \epsilon)$*] with* $\epsilon > 0$, *i.e., when the bifurcation is super- (resp. sub-) critical.*

Proof. See Appendix A.5. ∎

Propositions 5 and 6 prove that expectations-driven fluctuations are more likely to occur under dynamic inefficiency and thus confirm the crucial role of saving behavior. Moreover, in this case, any stabilization policy targeting the steady state leaves room for welfare losses associated with productive inefficiency.

5. CONCLUDING COMMENTS

We have considered a two-sector OLG economy with two consumption goods that enter the utility function in both periods of life through a composite good. To simplify, we have assumed that the share of each consumption good in the composite one is constant. We have proved that although dynamically efficient competitive equilibria are less likely to be locally indeterminate than dynamically inefficient ones, the existence of Pareto-optimal expectations-driven fluctuations becomes compatible with standard sectoral technologies if the share of the pure consumption good is low enough.

NOTES

1. In contrast, when the technologies are close to Leontief, capital and labor are complementary and both can be transfered into the investment good sector, so that the increase of the final output may compensate for the lack of savings and lead to an increase of capital compatible with the expectations. As shown in Nourry and Venditti (2011), local indeterminacy then becomes compatible with dynamic efficiency.
2. See, for instance, Benhabib and Nishimura (1981) for an optimal-growth infinite-horizon model with many consumption goods.
3. See also Drugeon (2010) for the analysis of saddlepoint stability in a similar model with endogenous labor.
4. National accounting data are based on subdivisions of the productive sectors in which many goods are both final and intermediary.
5. For instance, Reichlin (1992) proves the existence of a Hopf bifurcation in a standard OLG model with Leontief technologies, and Kalra (1996) extends this conclusion to the case of technologies with substitutable factors.
6. See Benhabib and Nishimura (1981).
7. All the results in this paper can be obtained with a general concave and homothetic utility function $u(C_t, D_{t+1}/B)$.

8. See also Galor (1992) and Venditti (2005).
9. See Nourry and Venditti (2011) for an illustration of such a policy in the Galor-type OLG model.

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APPENDIX

A.1. PROOF OF PROPOSITION 1

Let $(k^*, y^*) \in (0, \bar{k}) \times (0, \bar{k})$. Solving the second equation in (13) with respect to θ and using the homogeneity of τ gives

$$\theta\,(k^*, y^*) = \frac{T(k^*, y^*)}{T(k^*, y^*) - T_2(k^*, y^*)(y^* - k^*)} \in (0, 1).$$

Using (10) and solving the first equation in (13) with respect to B gives

$$B(k^*, y^*, \gamma) = -\frac{T_1(k^*, y^*)}{T_2(k^*, y^*)}\left(\frac{-k^* T_2(k^*, y^*)}{\delta^\gamma\,[T_3(k^*, y^*) + k^* T_2(k^*, y^*)]}\right)^{\frac{1}{1-\gamma}}.$$

Thus (k^*, y^*) is a NSS if and only if $\theta = \theta(k^*, y^*)$ and $B = B(k^*, y^*, \gamma)$. ■

A.2. PROOF OF PROPOSITION 3

From (10), we derive

$$\alpha'(v) = (1-\gamma)\alpha(v)[1-\alpha(v)]/v. \tag{A.1}$$

Under Assumption 1, we get from the first-order conditions of program (2) $T_{12} = -T_{11}b < 0$, $T_{22} = T_{11}b^2 < 0$, $T_{31} = -T_{11}a > 0$, and $T_{32} = T_{11}ab$, with $a \equiv k^0/l^0 > 0$, b as defined by (16), and $T_{11} < 0$. Consider ϵ_{rk} as given by (16) together with $T_1 k^*/T_3 = s/(1-s)$, $-T_1/T_2 = R^* = s/(1-\alpha)(1-s)$, and the fact that the homogeneity of $\tau(k, y, \ell)$ implies $a = [(1-\alpha)(1-s)(1-\theta b) - (1-\theta)b]\,k^*/[(1-\alpha)(1-s)]$. Total differentiation of (12) using (3), (10), (14), and (A.1) evaluated at the NSS gives the characteristic polynomial $\mathcal{P}_\gamma(\lambda) = \lambda^2 - \lambda\mathcal{T}_\theta(\gamma) + \mathcal{D}_\theta(\gamma)$ with

$$\mathcal{D}_\theta(\gamma) = \frac{s\left\{b\Big[(1-s)\alpha(\gamma-1)\theta + \theta(1-s)\alpha - s(1-\theta)\Big] + (1-\alpha)(1-s)\right\}}{\alpha(\gamma-1)(1-\alpha)(1-s)^2\theta b},$$

$$\mathcal{T}_\theta(\gamma) = \frac{1 + \alpha\,(\gamma-1)\,\epsilon_{rk}\left\{[1-(1-\theta)\,R^* b]^2 + \theta^2 b^2 R^*\right\} + \theta R^* b\epsilon_{rk}\Big[(1-\theta)\,\frac{b\alpha}{s} + \theta b\alpha + 1 - \alpha\Big]}{\alpha(\gamma-1)\,[1-(1-\theta)\,R^* b]\,\theta b\epsilon_{rk}}.$$

When $\theta = \theta(k^*, y^*)$ and $B = B(k^*, y^*, \gamma)$, the NSS and α remain constant as γ is made to vary, and as in Grandmont et al. (1998), we can study the variations of $\mathcal{T}_\theta(\gamma)$ and $\mathcal{D}_\theta(\gamma)$ in the $(\mathcal{T}, \mathcal{D})$ plane. Solving $\mathcal{T}$ and $\mathcal{D}$ with respect to $\alpha(\gamma-1)$ yields the linear relationship

$$\mathcal{D} = \Delta(\mathcal{T}) = \mathcal{S}_\theta\mathcal{T} + \mathcal{M}, \tag{A.2}$$

where $\mathcal{M}$ is a constant term and the slope $\mathcal{S}_\theta$ of $\Delta(\mathcal{T})$ is

$$\mathcal{S}_\theta = \frac{\epsilon_{rk} s R^*\,[1-(1-\theta)\,R^* b]\,[(1-s)\alpha + s]\,(\theta-\theta_1)(b-b_1)}{(1-s)\left\{s + \theta R^* b\epsilon_{rk}\alpha\,[1-\theta+\theta s]\,(b-b_2)\right\}} \tag{A.3}$$

with

$$\theta_1 = \frac{s}{(1-s)\alpha + s},\; b_1 = -\frac{s}{R^*\,[(1-s)\alpha + s]\,(\theta-\theta_1)},\; \text{and } b_2 = -\frac{(1-\alpha)s}{\alpha\,(1-\theta+\theta s)}. \tag{A.4}$$

For a given $\theta = \theta(k^*, y^*)$, as γ spans the interval $(1, +\infty)$, $\mathcal{T}_\theta(\gamma)$ and $\mathcal{D}_\theta(\gamma)$ vary linearly along the line $\Delta(\mathcal{T})$.

As $\gamma \in (1, +\infty)$, the fundamental properties of $\Delta(\mathcal{T})$ are characterized from the consideration of its extremities. The starting point of the pair $(\mathcal{T}_\theta(\gamma), \mathcal{D}_\theta(\gamma))$ is indeed obtained when $\gamma = +\infty$:

$$\lim_{\gamma\to+\infty} \mathcal{D}_\theta(\gamma) = \mathcal{D}^\infty = \frac{s}{(1-\alpha)(1-s)}, \qquad \textbf{(A.5)}$$

$$\lim_{\gamma\to+\infty} \mathcal{T}_\theta(\gamma) = \mathcal{T}_\theta^\infty = \frac{[1-R^*b(1-\theta)]^2+\theta^2b^2R^*}{\theta b[1-R^*b(1-\theta)]},$$

whereas the end point is obtained when γ converges to 1 from above:

$$\lim_{\gamma\to1^+} \mathcal{D}_\theta(\gamma) = \mathcal{D}_\theta^1 = \pm\infty \Leftrightarrow b(\theta-\theta_1)(b-b_1) \gtrless 0, \qquad \textbf{(A.6)}$$

$$\lim_{\gamma\to1^+} \mathcal{T}_\theta(\gamma) = \mathcal{T}_\theta^1 = \pm\infty \Leftrightarrow b\mathcal{S}_\theta \lessgtr 0.$$

$\mathcal{D}^\infty \gtrless 1$ for any $\theta \in (0,1)$ if and only if $\alpha \gtrless \underline{\alpha}$, with $\underline{\alpha} > 0$ if and only if $s < 1/2$, whereas $\mathcal{T}_\theta^\infty$ depends on the value of θ. Depending on θ, the pairs $(\mathcal{T}_\theta^\infty, \mathcal{D}^\infty)$ are located on a horizontal line above (below) the line $\mathcal{D} = 1$ when $\alpha > (<)\underline{\alpha}$. Therefore $\Delta(\mathcal{T})$ is a half-line starting on the horizontal line $(\mathcal{T}_\theta^\infty, \mathcal{D}^\infty)$ and pointing upward or downward, depending on $\mathcal{D}_\theta^1 = \pm\infty$.

Let us now prove Proposition 3. We conclude from (A.6) that if $b \in (0, b_{\text{Max}})$ then $\mathcal{D}_\theta^1 = +\infty$, $\mathcal{T}_\theta^\infty > 0$, and $\mathcal{S}_\theta > 0$. The characteristic roots are always positive. Let us finally compute

$$\mathcal{P}_\infty(1) = 1 - \mathcal{T}_\theta^\infty + \mathcal{D}^\infty = -\frac{[1-R^*b]\{1-b[R^*(1-\theta)+\theta]\}}{b\theta[1-R^*b(1-\theta)]}.$$

Straightforward computations show that $\mathcal{P}_\infty(1) > 0$ if and only if $\alpha > \underline{\alpha}$, $\theta > (1-s)/[(1-s)\alpha+s]$, and $b \in (1/R^*, b_{\text{Max}})$. Moreover, when $s < 1/2$ and $\alpha < \underline{\alpha}$, if $\mathcal{D}_\theta(\gamma) = 1$ we get $\mathcal{T}_\theta(\gamma) > 2$. All this implies that for any $\alpha \in (0,1)$, any equilibrium path is unique and monotone when $b \in (0, b_{\text{Max}})$. ■

A.3. PROOF OF PROPOSITION 4

Let $b < 0$, so that $\mathcal{P}_\infty(1) > 0$. Because $\mathcal{T}_\theta^\infty < 0$, we need to compute

$$\mathcal{P}_\infty(-1) = 1 + \mathcal{T}_\theta^\infty + \mathcal{D}^\infty = \frac{[1-R^*b(1-2\theta)]\{1-b[R^*(1-\theta)-\theta]\}}{b\theta[1-R^*b(1-\theta)]}.$$

Consider the bounds defined in (A.4) and let

$$\theta_2 = \frac{s}{1-\alpha(1-s)},\ b_3 = \frac{1}{R^*(1-2\theta)},\ \text{and } b_4 = \frac{1}{R^*(1-\theta)-\theta}. \qquad \textbf{(A.7)}$$

When $\alpha > \underline{\alpha}$, $\mathcal{P}_\infty(-1) > 0$ if and only if (i) $\theta \in (1/2, \theta_2)$ and $b < b_3$ or (ii) $\theta > \theta_2$ and $b \in (b_4, b_3)$. Moreover, we derive from (A.6) that $\mathcal{D}_\theta^1 = -\infty$ if and only if (a) $\theta < \theta_2$ or (b) $\theta > \theta_2$ and $b \in (b_1, 0)$. Finally, we get from (A.3) that under the conditions (a) or (b), $\mathcal{S}_\theta < 0$ if and only if $b > b_2 (> b_1)$ and $\epsilon_{rk} > \hat{\epsilon}_{rk} = -s/[\theta R^*\alpha b(1-\theta+\theta s)(b-b_2)]$.

Using the expressions for $\mathcal{D}_\theta(\gamma)$ and $\mathcal{T}_\theta(\gamma)$ allows to show that when $\mathcal{D}_\theta(\gamma) = 1$, $\mathcal{T}_\theta(\gamma) > -2$ if and only if

$$1 + \epsilon_{rk} \left\{ \frac{s\,[(1-s)\alpha + s]\,(\theta - \theta_1)(b - b_1)}{b(1-s)^2\theta(\underline{\alpha} - \alpha)} \mathcal{P}(b) + \frac{\theta R^* b\alpha[1 - \theta + \theta s]}{s}(b - b_2) \right\} < 0 \tag{A.8}$$

with

$$\mathcal{P}(b) = b^2 R^* \left[(1-\theta)^2 R^* - \theta(2 - 3\theta)\right] - 2b\left[(1-\theta)R^* - \theta\right] + 1 > 0 \tag{A.9}$$

for all $\theta \in (0, 1]$. Under $b < b_3$ and conditions (a) or (b), $\mathcal{T}_\theta(\gamma) < -2$ and $\mathcal{S}_\theta > 0$ as long as $b \leq b_2$ or $b \in (b_2, 0)$ and $\epsilon_{rk} \in (0, \hat{\epsilon}_{rk})$.

Based on all this, we now prove Proposition 4. Let $\alpha > s/(1-s)(> \underline{\alpha})$, so that $\theta_2 > 1/2 > \theta_1$.

(1) Assume that $\theta = \theta_1 + \varepsilon \leq 1/2$ with $\varepsilon > 0$ small and $b > b_2(> b_1)$. It follows that $\mathcal{P}_\infty(-1) < 0$ and the term between brackets in (A.8) is negative. Moreover, $\mathcal{T}_\theta(\gamma) \geq -2$ if and only if

$$\epsilon_{rk} \geq -\frac{1}{\dfrac{s\,[(1-s)\alpha + s]\,(b - b_1)}{b(1-s)^2\theta(\underline{\alpha} - \alpha)}\mathcal{P}(b)\epsilon + \dfrac{\theta R^* b\alpha(1 - \theta + \theta s)}{s}(b - b_2)}$$
$$\equiv \underline{\epsilon}_{rk}(> \hat{\epsilon}_{rk}). \tag{A.10}$$

Moreover, there exists $\bar{\epsilon}_{rk} > \underline{\epsilon}_{rk}$ such that $\mathcal{T}_\theta(\gamma) = 2$ when $\epsilon_{rk} = \bar{\epsilon}_{rk}$ and the half-line is given by $\bar{\Delta}$. Therefore $\mathcal{T}_\theta(\gamma) \in (-2, 2)$ as long as $\epsilon_{rk} \in (\underline{\epsilon}_{rk}, \bar{\epsilon}_{rk})$. Let us denote as $\hat{\varepsilon}$ the value of ε such that the denominator of the ratio in (A.10) is equal to zero. The maximal admissible value of ε is such that $\bar{\varepsilon} = \min\{\hat{\varepsilon}, 1/2 - \theta_1\}$. It follows that when $\theta \in (\theta_1, \bar{\theta})$ with $\bar{\theta} = \theta_1 + \bar{\varepsilon}$ and $\epsilon_{rk} \in (\underline{\epsilon}_{rk}, \bar{\epsilon}_{rk})$, we get a half-line above Δ_1, as shown in Figure 1. This proves the first part of Proposition 4.

(2) Note from (A.3) that $\lim_{\epsilon_{rk} \to \hat{\epsilon}_{rk}} \mathcal{S}_\theta = \infty$. When $\epsilon_{rk} = \hat{\epsilon}_{rk}$, the half-line is given by $\hat{\Delta}$ and there exists $\tilde{\epsilon}_{rk} \in (\hat{\epsilon}_{rk}, \underline{\epsilon}_{rk})$ such that when $\theta \in (\theta_1, \bar{\theta})$ and $\epsilon_{rk} = \tilde{\epsilon}_{rk}$, we get a half-line such that $\tilde{\Delta}$. As a result, if $\epsilon_{rk} \in (\tilde{\epsilon}_{rk}, \underline{\epsilon}_{rk})$, the half-line is given by Δ_2, as shown on Figure 1. This proves the second part of Proposition 4.

The bifurcation values γ_H, γ_F, and γ_T are, respectively, defined as the solutions of

$$\mathcal{D}_\theta(\gamma) = 1, \mathcal{P}_\gamma(-1) = 1 + \mathcal{T}_\theta(\gamma) + \mathcal{D}_\theta(\gamma) = 0, \mathcal{P}_\gamma(1) = 1 - \mathcal{T}_\theta(\gamma) + \mathcal{D}_\theta(\gamma) = 0$$

with the corresponding values of θ and ϵ_{rk}. ■

A.4. PROOF OF PROPOSITION 5

Consider the bounds defined in (A.4) and (A.7). $\alpha < \underline{\alpha}$ implies $\mathcal{D}^\infty < 1$. We get $\theta_2 < 1/2 < \theta_1$ and $\mathcal{P}_\infty(-1) > 0$ if and only if (i) $\theta \in (\theta_2, 1/2)$ and $b < b_4$ or (ii) $\theta > 1/2$ and $b \in (b_3, b_4)$. Moreover, $\mathcal{D}_\theta^1 = +\infty$ if and only if $\theta > \theta_1$ and $b < b_1$ with $b_1 < b_3 < b_4$. This implies that when $\mathcal{P}_\infty(-1) > 0$, $\mathcal{D}_\theta^1 = -\infty$, and any Hopf bifurcation is ruled out.

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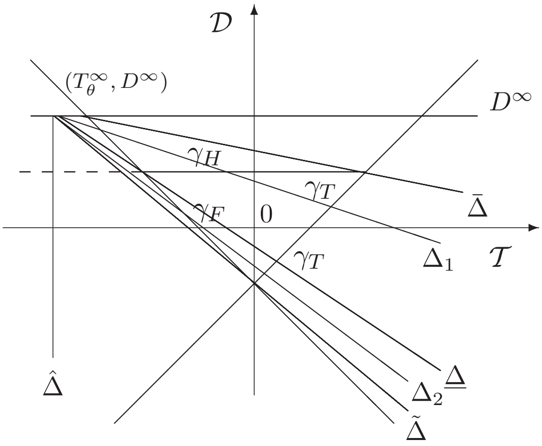

FIGURE 1. Hopf and flip bifurcations with $\alpha > s/(1-s)(> \underline{\alpha})$.

Also, when $\mathcal{D}_\theta(\gamma) = -1$, $\mathcal{T}_\theta(\gamma) < 0$ if $b < b_2$ with $b_2 > b_3$ if

$$\theta < \theta_3 = \frac{s^2 + \alpha(1-s)}{2s^2 + \alpha(1-s)^2} \in (1/2, 1).$$

Based on all this, we now prove Proposition 5. We focus here on the case in which $\mathcal{P}_\infty(1) > 0$ and $\mathcal{P}_\infty(-1) > 0$. As $b < 0$, we already know that $\mathcal{P}_\infty(1) > 0$.

(i) Assume first that $\theta \in (\theta_2, 1/2)$ and $b < b_4$. This implies that $\mathcal{P}_\infty(-1) > 0$ and thus $\mathcal{D}_\theta^1 = -\infty$. A flip bifurcation occurs if $\mathcal{T}_\theta(\gamma) < 0$ when $\mathcal{D}_\theta(\gamma) = -1$. This is obtained provided $b < b_2$. Part (i) is then proved by assuming $b < \bar{b}$ with $\bar{b} = \min\{b_2, b_4\}$. We indeed get the half-line Δ as shown in Figure 2.

(ii) Assume now that $\theta \in (1/2, \theta_3)$ and $b \in (b_3, \bar{b})$. We get as previously $\mathcal{P}_\infty(-1) > 0$, $\mathcal{D}_\theta^1 = -\infty$ and $\mathcal{T}_\theta(\gamma) < 0$ when $\mathcal{D}_\theta(\gamma) = -1$, which proves part (ii).

In both cases the flip bifurcation value γ_F is defined as the solution of $\mathcal{P}_\gamma(-1) = 1 + \mathcal{T}_\theta(\gamma) + \mathcal{D}_\theta(\gamma) = 0$. ∎

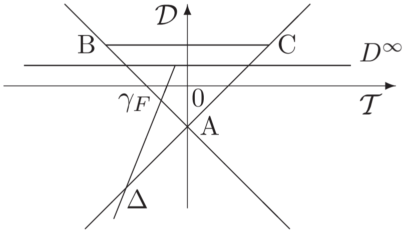

FIGURE 2. Flip bifurcation with $\alpha < \underline{\alpha}$.

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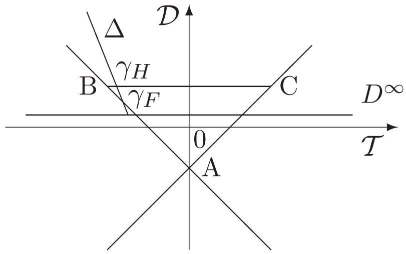

FIGURE 3. Flip and Hopf bifurcations with $\alpha \in (\tilde{\alpha}, \underline{\alpha})$.

A.5. PROOF OF PROPOSITION 6

Consider the bounds defined in (A.4) and (A.7). Let $\theta > \theta_1$ and $b < b_1 (< b_3 < b_4)$, so that $\mathcal{D}_\theta^1 = +\infty$ and $\mathcal{P}_\infty(-1) < 0$. Moreover, $\mathcal{S}_\theta < 0$ as $b_1 < b_2$. Local indeterminacy may arise as shown by Δ in Figure 3 only if $\mathcal{T}_\theta^\infty \in (-2, 0)$, i.e., only if $\mathcal{P}(b) < 0$ with $\mathcal{P}(b)$ as given by (A.9).

Let us then assume

$$\theta_\pm = \frac{s + (1-\alpha)(1-s) \pm (1-s)\sqrt{(\underline{\alpha}-\alpha)(1-\alpha)}}{s + 3(1-\alpha)(1-s)}$$

$$\text{and } b_+ = \frac{(1-\theta)R^* - \theta + \theta\sqrt{1-R^*}}{R^*\left[(1-\theta)^2 R^* - \theta(2-3\theta)\right]}.$$

It is easy to show that $(1-\theta)^2 R^* - \theta(2-3\theta) < 0$ if and only if $\theta \in (\theta_-, \theta_+)$ with $\theta_1 > \theta_-$. Moreover, $\theta_1 < \theta_+$ if and only if $g(\alpha) = [(1-s)\alpha) + s]\sqrt{(\underline{\alpha}-\alpha)(1-\alpha)} - \alpha^2(1-s) + \alpha(1+2s) - 2s > 0$, with $g(0) < 0$ and $g(\underline{\alpha}) \leq 0$ under Assumption 3. When $s = 1/3$, we get $\lim_{\alpha \to \underline{\alpha}} g'(\alpha) < 0$. It follows that there exists $\alpha_1 \in (0, \underline{\alpha})$ such that when $\alpha \in (\alpha_1, \underline{\alpha})$, $g(\alpha) > 0$. As a result, we conclude that when $\alpha \in (\alpha_1, \underline{\alpha})$, $\theta \in (\theta_1, \theta_+)$ and $b < b_+$, we get $\mathcal{P}(b) < 0$. Therefore, assuming $\theta \in (\theta_1, \theta_+)$ and $b < \min\{b_1, b_+\}$, we derive from (A.8) that there exists $\tilde{\alpha} \in [\alpha_1, \underline{\alpha})$ such that if $\alpha \in (\tilde{\alpha}, \underline{\alpha})$, $\mathcal{T}_\theta(\gamma) > -2$ when $\mathcal{D}_\theta(\gamma) = 1$. Proposition 6 is then proved under the following conditions: $\alpha \in (\tilde{\alpha}, \underline{\alpha})$, $\theta \in (s/[(1-s)\alpha + s], \tilde{\theta})$, and $b < \tilde{b}$ with $\tilde{\theta} = \theta_+$ and $\tilde{b} = \min\{b_1, b_+\}$. We indeed get the half-line Δ as shown in Figure 3. The bifurcation values γ_H and γ_F are finally defined as the solutions of

$$\mathcal{D}_\theta(\gamma) = 1 \text{ and } \mathcal{P}_\gamma(-1) = 1 + \mathcal{T}_\theta(\gamma) + \mathcal{D}_\theta(\gamma) = 0. \qquad \blacksquare$$