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# $F_\sigma$ EQUIVALENCE RELATIONS AND LAVER FORCING

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**Abstract.** Following the topic of the book Canonical Ramsey Theory on Polish Spaces by V. Kanovei, M. Sabok, and J. Zapletal we study Borel equivalences on Laver trees. We prove that equivalence relations Borel reducible to an equivalence relation on $2^\omega$ given by some $F_\sigma$ $P$-ideal on $\omega$ can be canonized to the full equivalence relation or to the identity relation.

This has several consequences, e.g., Silver type dichotomy for the Laver ideal and equivalences Borel reducible to equivalence relations given by $F_\sigma$ $P$-ideals.

**Introduction.** The aim of this paper is to prove canonization results for the Laver ideal in the spirit of [1].

Let us recall that a Borel equivalence relation $E$ on a standard Borel space $X$ is an equivalence relation which is a Borel subset of $X \times X$. All equivalence relations in this text will be assumed to be Borel. We say that an equivalence relation $E$ on $X$ is Borel reducible to an equivalence $F$ on $Y$, $E \leq_B F$, if there exists a Borel function $f : X \to Y$ such that $xEy$ iff $f(x)Ff(y)$. They are bireducible, $E \approx_B F$, if $E \leq_B F$ and $F \leq_B E$. For a Borel subset $A \subseteq X$, $E \restriction A$ is the Borel equivalence relation $E \cap A \times A$, the restriction of $E$ on $A$.

For a Borel ideal $\mathcal{I}$ on $\omega$ we denote by $E_\mathcal{I}$ the Borel equivalence relation on $2^\omega$ where $xE_\mathcal{I}y$ iff $\{n : x(n) \neq y(n)\} \in \mathcal{I}$. Obviously, if $\mathcal{I}$ is $\Sigma^0_\alpha$ ($\Pi^0_\alpha$), then also $E_\mathcal{I}$ is $\Sigma^0_\alpha$ ($\Pi^0_\alpha$).

An ideal $\mathcal{I}$ on $\omega$ is called $P$-ideal if $\forall (A_n)_{n\in\omega} \subseteq \mathcal{I} \exists A \in \mathcal{I} \forall n (A_n \subseteq_* A)$.

Let us recall that a Laver tree $T \subseteq \omega^{<\omega}$ is a tree with stem $s$, the maximal node such that every other node is compatible with it, such that every node above $s$ (and including $s$) splits into infinitely many immediate successors. The set of all branches of $T$ is denoted as $[T]$.

We can now state the main result of this paper.

THEOREM 0.1. *Let $T$ be a Laver tree, $\mathcal{I}$ be an $F_\sigma$ $P$-ideal on $\omega$, and $E \subseteq [T] \times [T]$ be an equivalence relation Borel reducible to $E_\mathcal{I}$. Then, there is a Laver subtree $S \leq T$ such that $E \restriction [S]$ is equal either to* $\mathrm{id}([S])$ *or to* $[S] \times [S]$.

We note that the subtree $S$, in general, cannot be found as a direct extension of $T$.

The list of equivalence relations Borel bireducible with $E_\mathcal{I}$ for $\mathcal{I}$, an $F_\sigma$ $P$-ideal, includes, for instance, $E_{\ell_p}$ equivalences for $p \in [1, \infty)$ on $\mathbb{R}^\omega$, where $xE_{\ell_p}y \equiv$

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$x - y \in \ell_p$; or $E_2$ (which is, in fact, bireducible with $E_{\ell_1}$) on $2^\omega$, where $xE_2y \equiv \sum\{1/(n+1) : x(n) \neq y(n)\} < \infty$.

**§1. Preliminaries.** To get into the context, we now present the general program of [1]: let $X$ be a Polish space, $I$ a $\sigma$-ideal on $X$, and $E \subseteq X^2$ a Borel equivalence relation. By $I^+$ we denote the set $\mathcal{P}(X) \setminus I$, i.e., the coideal associated with $I$.

- We say that $E$ is in the *spectrum* of $I$ if there exists a Borel set $B \in I^+$ such that $\forall C \in (I^+ \cap \text{Borel}(B))\ E \restriction C$ has the same complexity as $E$ on the whole space, i.e., $E \restriction C$ is Borel bireducible with $E \restriction X$.
- On the other hand, $I$ *canonizes* $E$ to a relation $F \leq_B E$ if for every Borel $B \in I^+$ there is some Borel $C \in (I^+ \cap \text{Borel}(B))$ such that $E \restriction C$ is bireducible with $F$.

Before proving the main theorem we state the existing knowledge about the spectrum of the Laver ideal and some results about $F_\sigma$ $P$-ideals and the Laver ideal that we will need in the proof of the theorem.

The following theorem connects $F_\sigma$ $P$-ideals with lower semicontinuous submeasures. Recall that a submeasure $\mu : \mathcal{P}(\omega) \to [0, \infty]$ is *lower semicontinuous* if it is lower semicontinuous in the Cantor space topology on $\mathcal{P}(\omega)$ (identified with $2^\omega$).

THEOREM 1.1 (Solecki [2]). *Let $\mathcal{I}$ be an $F_\sigma$ $P$-ideal on $\omega$. Then, there exists a lower semicontinuous submeasure $\mu : \mathcal{P}(\omega) \to [0, \infty]$ such that $\mathcal{I} = \text{Exh}(\mu) = \text{Fin}(\mu)$, where $\text{Exh}(\mu) = \{A \subseteq \omega : \lim_{n\to\infty} \mu(A \setminus n) = 0\}$ and $\text{Fin}(\mu) = \{A \subseteq \omega : \mu(A) < \infty\}$.*

We add some notation concerning Laver trees and the Laver ideal. We say that a Laver tree $S$ is a *direct extension* of a Laver tree $T$, $S \leq_0 T$ in symbols, if $S \subseteq T$ and the stem of $S$ is the same as the stem of $T$. If $s \in T$ is a node above the stem then by $T_s$ we denote the induced subtree with $s$ as the stem, i.e., $T_s = \{t \in T : t \text{ is compatible with } s\}$.

We use the definition of the Laver ideal $I$ from [4, p. 200]; $I \subseteq \mathcal{P}(\omega^\omega)$ is the $\sigma$-ideal generated by sets $A_g = \{f \in \omega^\omega : \exists^\infty n(f(n) \in g(f \restriction n))\}$, where $g$ is a function from $\omega^{<\omega}$ to $\omega$.

The following proposition, resp. its corollary, will be used extensively. The proof is in [3].

PROPOSITION 1.2. *Let $A \subseteq \omega^\omega$ be analytic. Then, either $A$ contains all branches of some Laver tree or $A \in I$.*

We will provide a proof of the following corollary. Recall that a barrier $B$ in a Laver tree $T$ is a subset of nodes such that $\forall x \in [T] \exists n(x \restriction n \in B)$.

COROLLARY 1.3. *Let $T$ be a Laver tree and let $A \subseteq [T]$ be analytic. Then, there exists a direct extension $S \leq_0 T$ such that either $[S] \subseteq A$ or $[S] \cap A = \emptyset$.*

PROOF. It follows from the proposition above that there is always $S \leq T$ such that $[S] \subseteq A$ or $[S] \cap A = \emptyset$. Such $S$ is, in general, not a direct extension though. The use of "direct extension property" will give us the desired tree. Let $t$ be the stem of $T$. If there exist infinitely many immediate successors $s$ of $t$ such that there exists a direct extension $S \leq_0 T_s$ with the property that either $[S] \subseteq A$ or $[S] \cap A = \emptyset$, then for infinitely many of them it holds that $[S] \subseteq A$, or for infinitely many of them it holds that $[S] \cap A = \emptyset$, and we use them. So suppose that not, we erase these finitely many exceptions and proceed to the next level and do the same. At the end

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we obtain a Laver tree $T' \leq_0 T$. We apply the proposition above and get a node $t \in T'$ and a direct extension $S \leq_0 T'_t$ such that either $[S] \subseteq A$ or $[S] \cap A = \emptyset$. That is a contradiction since such a node was erased during the construction of $T'$. ⊣

We now state some results that have been obtained about the spectrum of Laver in [1]. We will need just Corollary 1.6 in our proof, however, we state the general canonization result from which it follows.

THEOREM 1.4 ([1]). *Let $I$ be a $\sigma$-ideal on a Polish space $X$ such that the quotient forcing $P_I$ is proper, nowhere ccc, and adds a minimal forcing extension. Then, $I$ has total canonization for equivalence relations classifiable by countable structures.*

As the Laver ideal fulfils these conditions we immediately get the following corollary.

COROLLARY 1.5 ([1]). *Let $T$ be a Laver tree and $E$ an equivalence classifiable by countable structures. Then, there is a Laver subtree on which $E$ is either the identity relation or the full relation.*

COROLLARY 1.6 ([1]). *Let $T$ be a Laver tree and $E$ a countable equivalence relation (i.e., with countable classes). Then, there is a Laver subtree on which $E$ is either the identity relation or the full relation.*

Zapletal found the following $F_\sigma$ equivalence relation (with $K_\sigma$ classes) that is in the spectrum of Laver.

DEFINITION 1.7. *For $x, y \in \omega^\omega$, we set $xKy$ if $\exists b \forall n \exists m_x, m_y \leq b(y(n+m_y) \geq x(n) \wedge x(n+m_x) \geq y(n))$.*

The following lemma gives us basic properties of $K$. The proof may be found in [1], we provide here the proof of the last item as it is stated slightly differently in [1]. Notice the difference between $E_{\ell_p}$ for $p \in [1, \infty)$ and $E_{\ell_\infty}$ as the former can be canonized according to the main theorem.

LEMMA 1.8.

(a) *For any two Laver trees $T, S$ there are branches $x_1, x_2 \in [T]$ and $y_1, y_2 \in [S]$ such that $x_1 K y_1$ and $x_2 \not K y_2$.*
(b) *$K$ is in the spectrum of Laver.*
(c) *$K$ is Borel bireducible with $E_{\ell_\infty} \subseteq \mathbb{R}^\omega \times \mathbb{R}^\omega$, where $xE_{\ell_\infty}y \equiv x - y \in \ell_\infty$.*

PROOF. We refer to [1] for the proof of the first two items and prove (c).

- $E_{\ell_\infty} \leq_B K$: we will prove $E_{\ell_\infty} \leq_B E_{\ell_\infty} \restriction (\mathbb{R}^+)^\omega \leq_B K$. To prove the first inequality, consider $f : \mathbb{R}^\omega \to (\mathbb{R}^+)^\omega$ such that if $x(n) \geq 0$, then, $f(x)(2n) = x(n)$ and $f(x)(2n+1) = 0$ and if $x(n) < 0$, then, $f(x)(2n) = 0$ and $f(x)(2n+1) = |x(n)|$.

  For the second, let $\pi : \omega^2 \to \omega$ be a bijection and $(I_{\pi(i,j)})_{i,j}$ a partition of $\omega$ into intervals such that $|I_{\pi(i,j)}| = j+1$ and $I_{\pi(i,j)} = \{p_0^{i,j}, p_1^{i,j}, \ldots, p_j^{i,j}\}$. We define $g : (\mathbb{R}^+)^\omega \to \omega^\omega$ as follows: let $x \in (\mathbb{R}^+)^\omega$ be given, $g(x)(p_k^{i,j}) = \min\{j, \lfloor x(i) \rfloor + k\}$ for $k < j$ and $g(x)(p_j^{i,j}) = j$.

  If $xE_{\ell_\infty}y$ and $\forall n(|x(n) - y(n)| \leq m)$, then $\forall k, ij \exists m_1, m_2 \leq m(g(x)(p_k^{i,j}) \leq g(y)(p_{k+m_1}^{i,j}) \wedge g(y)(p_k^{i,j}) \leq g(x)(p_{k+m_2}^{i,j}))$, thus $g(x)Kg(y)$. Just observe that $g(x)(p_k^{i,j}) \leq j$ and either $j - k \leq m$ and we have $g(y)(p_j^{i,j}) = j$, or since

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$x(i) - y(i) = m_1 \leq m$ we have $g(y)(p^{i,j}_{k+m_1}) = y(i) + k + m_1 = x(i) + k = g(x)(p^{i,j}_k)$.

Suppose $x \not\!E_{\ell_\infty} y$, let $m$ be arbitrary, and let $n$ be such that $|x(n) - y(n)| > m$, let us assume that $y(n) - x(n) > m$. Then, $\forall b \leq m(g(x)(p^{n,m}_{k+b}) < g(y)(p^{n,m}_k))$. Since $m$ was arbitrary, we have $g(x) \not\!K g(y)$.

- $K \leq_B E_{\ell_\infty}$: let $(s_n)_n$ be an enumeration of $\omega^{<\omega}$. We define $f : \omega^\omega \to \mathbb{R}^\omega$ as follows: $f(x)(n) = \min\{b : \exists y \supseteq s_n(xKy \wedge b$ is the bound from the definition that works$)\}$. One can easily check that $f$ is Borel. Let $xKy$ such that a bound $b$ works for this pair and let $n$ be arbitrary. Let $z \supseteq s_n$ be arbitrary such that $xKz$ and $b_1$ works for the pair and $yKz$ and $b_2$ works for the pair. Then, one can check that $|b_1 - b_2| \leq b$, so $f(x) E_{\ell_\infty} f(y)$.

  Suppose that $x \not\!K y$ and let $m$ be arbitrary. Then, there exists $n$ such that $x(n) > y(n + k)$, for $k < m$ (or vice versa). Let $s_i = x \restriction (n + 1)$, then $f(x)(n) = 0$, however, $f(y)(n) \geq m$, thus, $f(x) \not\!E_{\ell_\infty} f(y)$. ⊣

**§2. Main result.** We can now start proving the main theorem, we provide its statement here again for the convenience.

THEOREM 2.1. *Let $T$ be a Laver tree, $\mathcal{I}$ be an $F_\sigma$ $P$-ideal on $\omega$, and $E \subseteq [T] \times [T]$ be an equivalence relation Borel reducible to $E_\mathcal{I}$. Then, there is a Laver subtree $S \leq T$ such that $E \restriction [S]$ is equal either to* $\mathrm{id}([S])$ *or to* $[S] \times [S]$.

PROOF. Let $f : [T] \to 2^\omega$ be the Borel reduction and let $\mu$ be the lower semicontinuous submeasure for $\mathcal{I}$ guaranteed by Theorem 1.1. The submeasure $\mu$ induces a pseudometric (which may attain infinite value though), which we denote $d$, i.e., $d(x, y) = \mu(x \triangle y)$ for $x, y \in 2^\omega$. Moreover, we define $d^k_n(x, y)$ as $\mu(x \restriction (n, k) \triangle y \restriction (n, k))$. When $n$ or $k$ is omitted it means that $n = 0$, resp. $k = \infty$.

We need to refine $T$ to obtain a Laver tree with some special properties. This will be done in a series of claims. To simplify the notation, after applying each one of these claims we will still denote the obtained tree as $T$.

CLAIM 2.2. *There exist a direct extension $T' \leq_0 T$ and a function $p : T' \to 2^{<\omega}$ which is monotone and preserves length of sequences, i.e., if $s \subseteq t$, then $p(s) \subseteq p(t)$, and $|s| = |p(s)|$, such that $\forall x \in [T'](f(x) = \bigcup_n p(x \restriction n))$.*

*In other words, $f$ on $[T']$ is 1-Lipschitz.*

PROOF OF THE CLAIM. We will find a direct extension of $T$ and $p$ defined on it from the statement of the claim. For simplicity, we assume the stem of $T$ is the empty sequence. ⊣

Consider the following sets

$$A_i = \{x \in [T] : f(x)(0) = i\},$$

for $i \in \{0, 1\}$. They are Borel and according to Corollary 1.3 one of them contains a direct extension $S$ of $T$. We replace $T$ by $S$, set $p(\emptyset) = i$ and fix the first level above the stem. Then, for any immediate successor $s$ of the stem we again consider sets $A^1_i = \{x \in [S_s] : f(x)(1) = i\}$. One of them contains direct extension and we continue similarly. The final tree is obtained by fusion. ⊣

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Claim 2.3. *There exists a direct extension $T' \leq_0 T$ such that for each node $s \in T'$ and the set of its immediate successors $\{s_n : n \in \omega\}$ we have that for every $i \leq m \leq n$ and $\forall x \in [T'_{s_m}] \forall y \in [T'_{s_n}]$ we have that $p(x \restriction |s| + i + 1) = p(y \restriction |s| + i + 1)$.*

Proof. Let $s$ be the stem of $T$. Pick an arbitrary immediate successor $s_0$ of $s$. Let $S_0'$ be the set of all immediate successors of $s$ except $s_0$. Let $S_0$ be an infinite subset of $S_0'$ such that $\forall t_0, t_1 \in S_0(p(t_0) = p(t_1))$. Fix an arbitrary element $s_1 \in S_0$ and denote $S_1'$ as the set $S_0 \setminus \{s_1\}$. Using pigeonhole principle and throwing some nodes of $T$ away we can find an infinite subset $S_1 \subseteq S_1'$ such that for every $t_0, t_1 \in S_1$ and every immediate successor $v_1$ of $t_1$ and every immediate successor $v_2$ of $t_2$ we have $p(v_1) = p(v_2)$.We pick an arbitrary element $s_2 \in S_1$ and continue similarly. At the end we end up with a set $\{s_n : n \in \omega\}$ of immediate successors of $s$ and a Laver tree $T_0 \leq_0 T$ such that for every $i \leq m \leq n$ and for every $u, v \in T_0$ such that $u \supseteq s_m, v \supseteq s_n$ and $|u| = |v| = |s| + i$ we have $p(u) = p(v)$.

We fix the level above the stem and repeat the same construction we did for $s$ for all $s_m$, $m \in \omega$. Then, we fix another level and repeat the construction for nodes on that level. By a fusion we get a Laver tree $T_\omega$ such that for every node $t \in T_\omega$ above or equal to the stem and for the set of its immediate successors $\{t_n : n \in \omega\}$ we have that for every $i \leq m \leq n$ and for every $u, v \in T_\omega$ such that $u \supseteq t_m, v \supseteqq t_n$ and $|u| = |v| = |t| + i$ we have $p(u) = p(v)$. $T_\omega$ is the required $T'$ from the statement of the claim. ⊣

Definition 2.4. *Let $s \in T$ be a node above (or equal to) the stem of $T$ and let $\{s_n : n \in \omega\}$ be the set of its immediate successors. Define then, $x_s \in 2^\omega$ as follows: $x_s(n) = p(x \restriction n + 1)(n)$, where $x \in [T_{s_m}]$ and $m + n \geq |s|$. This definition does not depend on $m$ (provided $m + n \geq |s|$) and $x \in [T_{s_m}]$.*

Observation 2.5. *It follows from the previous fact that for every $s \in T$ and the set of its immediate successors $\{s_n : n \in \omega\}$ we have $x_s = \lim_{n\to\infty} x_{s_n}$.*

Let $s \in T$ be any node above (or equal to) the stem of $T$. Let $\{s_0, s_1, \dots\}$ be the set of its immediate successors. We reduce this set so that precisely one of the following two possibilities happens: $\forall n(x_{s_n} E_{\mathcal{I}} x_s)$ or $\forall n(x_{s_n} \not{E}_{\mathcal{I}} x_s)$.

Definition 2.6. *If the former case holds then we mark $s$ as "convergent", if the latter then we mark it as "divergent".*

*Moreover, for every $s \in S$ strictly above the stem we define $\varepsilon_s$ as follows: if the immediate predecessor $t$ of $s$ is marked as convergent, then we set $\varepsilon_s = d(x_t, x_s)$; otherwise, we set $\varepsilon_s = \infty$.*

**Splitting into cases**

We split into two complementary cases (i.e., one holds if and only if the other does not).

- **Case 1** There exists $S \leq T$ such that every $s \in S$ above the stem is marked as convergent.
- **Case 2** For every $s \in T$ above the stem there is a barrier $B \subseteq T_s$ of elements above $s$ that were marked as divergent.

*Proof of canonization assuming* Case 1. We will do a fusion. Let us denote the stem of $S$ as $s$. We will inductively build $U_n, S_n, m_n$ for every $n$ such that $S_n \leq_0 S_{n-1}$, $U_n \subseteq S_m$, for every $n \leq m$, is an $n + 1$-element subtree $\{u_0, \dots, u_n\}$ of $S$ and

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$(m_n)_n$ is an increasing sequence of elements of $\omega$. At the end we will get a direct extension $U = \bigcup_n U_n = \bigcap_n S_n$ together with pairwise disjoint sets $C_{u_1}, C_{u_2}, \ldots$, where $C_{u_i} \subseteq (m_{i-1}, m_i)$, such that $\forall x \in [U]\ (f(x) \triangle x_s) \cap (m_{i-1}, m_i) = C_{u_i}$ if $u_i \subseteq x$ and $\mu(\bigcup_{\{i>0: u_i \not\subseteq x\}} f(x) \cap (m_{i-1}, m_i)) < 1$. The following conditions will be satisfied during the $n$-th step of the fusion.

- For every $0 < i \leq n$ and any branch $x \in [S_n]$ going through $u_i$ we have $|d^{m_i}_{m_{i-1}}(f(x), x_s) - \varepsilon_{u_i}| < 1/2^i$; more precisely, there will be some finite set $C_{u_i} \subseteq (m_{i-1}, m_i)$ always defined as $(x_{u_i} \triangle x_s) \cap (m_{i-1}, m_i)$ such that for any branch $x \in [S_n]$ going through $u_i$ we will have $(f(x) \triangle x_s) \cap (m_{i-1}, m_i) = C_{u_i}$ and $|\mu(C_{u_i}) - \varepsilon_{u_i}| < 1/2^i$. And for every branch $y \in [U_n]$ not going through $u_i$ but going through some other $u_j$ we have $d^{m_i}_{m_{i-1}}(f(y), x_s) < 1/2^i$; thus, it will follow from the triangle inequality that $|d^{m_i}_{m_{i-1}}(f(x), f(y)) - \varepsilon_{u_i}| < 1/2^{i-1}$; resp. $\mu((f(x) \triangle f(y) \triangle C_{u_i}) \cap (m_{i-1}, m_i)) < 1/2^i$.
- For every $i \leq n$ $d_{m_n}(x_{u_i}, x_s) < 1/2^{n+2}$.

Suppose at first that such $U$ has been already constructed. Let us consider the set

$$A = \{x \in [U] : \mu(\bigcup_{i=|s|+1}^{\infty} C_{x \restriction i}) < \infty\}.$$

It is Borel and by Corollary 1.3 either there is a Laver subtree $V \leq_0 U$ such that $[V] \subseteq A$ or there is a Laver subtree $V \leq_0 U$ such that $[V] \cap A = \emptyset$. In the former case, $V$ is a Laver subtree such that $\forall x, y \in [V](xEy)$; while in the latter case, $V$ is a Laver subtree such that $\forall x, y \in [V](x \not{E} y)$. This follows immediately from the condition above. Let $x, y \in [V]$ be two different branches splitting on the $n$-th level. Then, $\max\{\mu(\bigcup_{i=n}^{\infty} C_{x\restriction i}), \mu(\bigcup_{i=n}^{\infty} C_{y \restriction i})\} - \sum_{j=n-|s|+1}^{\infty} 1/2^j \leq d(f(x), f(y)) \leq \mu(\bigcup_{i=n}^{\infty} C_{x\restriction i}) + \mu(\bigcup_{i=n}^{\infty} C_{y\restriction i}) + \sum_{j=n-|s|+1}^{\infty} 1/2^{j-1}$.

Let $s$ be the stem of $S$. Set $S_0 = S$, $U_0 = \{s\}$ and $m_0 = |s|$. Before treating the general step, let us describe the case $n = 1$. We pick some immediate successor of the stem $s$, denote it as $u_1$ and we set $U_1 = \{s = u_0, u_1\}$. Since $d(x_{u_1}, x_s) = \varepsilon_{u_1}$, there is some $m > m_0$ such that $d^m(x_{u_1}, x_s) > \varepsilon_{u_1} - 1/2$. There is some $m_1 \geq m$ such that $d_{m_1}(x_{u_1}, x_s) < 1/2^3$. Then, there exist direct extensions $E_1 \leq_0 S_{0u_1}$ and $E_0 \leq_0 S_0$ such that for all branches $x \in [E_1]$ we have $f(x)(m) = x_{u_1}(m)$ for $m \leq m_1$, and for all branches $y \in [E_0]$ we have $f(y)(m) = x_s(m)$ for $m \leq m_1$. We set $S_1 = E_0 \cup E_1$, i.e., we replace $S_{0u_1}$ in $S_0$ by its direct extension $E_1$ and we replace $S_0 \setminus S_{0u_1}$ by its direct extension $E_0$. The required conditions are satisfied and we proceed to a general step.

Now let us suppose that we have already found $S_{n-1}, U_{n-1} = \{s = u_0, u_1, \ldots, u_{n-1}\}$ and $m_0 = |s|, m_1, \ldots, m_{n-1}$. Choose some next node $u_n \in S_{n-1}$ for the fusion so that it is an immediate successor of some $u_i$ $i < n$ and $x_{u_i} \restriction m_{n-1} = x_{u_n} \restriction m_{n-1}$ (recall Observation 2.5). Set $U_n = U_{n-1} \cup \{u_n\}$. There is some $m$ such that $d^m_{m_{n-1}}(x_{u_n}, x_{u_i}) > \varepsilon_{u_n} - 1/2^{n+1}$. Since we have from the inductive assumption that $d^m_{m_{n-1}}(x_{u_i}, x_s) < 1/2^{n+1}$, we get from the triangle inequality $d^m_{m_{n-1}}(x_{u_n}, x_s) > \varepsilon_{u_m} - 1/2^n$. Let $m_n$ be the $\max\{m, \max\{k_i : i \leq n\}\}$, where $k_i$ is any number such that $d_{k_i}(x_{u_i}, x_s) < 1/2^{n+2}$. Note that such $k_i$ exists because $x_{u_i} E_{\mathcal{I}} x_s$. We then find direct extensions $E_i \leq_0 S_{n-1,u_i}$ such that for every branch

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$x \in [E_i]$ we have $f(x)(m) = x_{u_i}(m)$ for $m \leq m_n$. We refine them so that they are mutually disjoint, i.e., $E_i \cap E_j = \emptyset$ for $i \neq j$ and we set $S_n = \bigcup_{i \leq n} E_i$. The induction step is done, all required conditions are satisfied. That finishes the proof of this case.

*Proof of canonization assuming* Case 2. We will assume that we have a Laver tree $S \leq T$ such that for every $s \in S$ above the stem if $s$ is marked as convergent then there is a barrier $B \subseteq S_s$ of elements above $s$ that are marked as divergent (if we assume that Case 1 does not hold then we may take $S = T$).

The following lemma will be the main tool.

LEMMA 2.7. *For any Laver subtree $P \leq S$ there is its direct extension $Q \leq_0 P$ such that for any two branches $x, y \in [Q]$ splitting from the stem of $Q$ we have $d(f(x), f(y)) > 1$.*

Once the lemma is proved the rest will be rather easy. We will do a fusion in which we will be fixing levels. We will construct direct extensions of $S$ $S = V_0 \geq_0 V_1 \geq_0 V_2 \geq_0 \dots$ such that for $i < j$ the $i$-th level of $V_i$ is equal to the $i$-th level of $V_j$ in such a way that the resulting tree $S \geq_0 V = \bigcap_i V_i$ will have the property that for any two different branches $x, y \in [V]$ we will have $d(f(x), f(y)) > 1$.

This is not hard to do. We start with the stem $s$ of $S = V_0$. We find a direct extension $V_1 \leq_0 V_0$ guaranteed by the lemma. We fix the first level $\{s_0, s_1, \dots\}$ (the set of all immediate successors of $s$) above the stem. Then, for every immediate successor $s_i \in V_1$ of $s$ we apply the lemma with $V_{1s_i}$ as $P$ and obtain a direct extension $Q_i$. We set $V_2 = \bigcup_i Q_i \leq_0 V_1$, fix the second level above the stem, and continue similarly.

Then, we are done by the following claim and Corollary 1.6.

CLAIM 2.8. *$E$ on $[V]$ is countable.*

PROOF. Suppose for contradiction that there is some $x \in [V]$ that has uncountably many equivalent branches $(y_\alpha)_{\alpha<\omega_1} \subseteq [V]$. For every $\alpha$ there is $n$ such that $d_n(f(x), f(y_\alpha)) < 1/2$. Since the set of all $y_\alpha$s is uncountable, we may assume that one single $n$ works for them all. But let $y_{\alpha_0}, y_{\alpha_1}$ be two of such branches that split above the $n$-th coordinate. It follows that from our construction that $d(f(y_{\alpha_0}), f(y_{\alpha_1})) > 1$ and since $f$ is 1-Lipschitz, we have that, in fact, $d_n(f(y_{\alpha_0}), f(y_{\alpha_1})) > 1$, so for one of them, let us say $y_{\alpha_0}$, must hold that $d_n(f(x), f(y_{\alpha_0})) > 1/2$, a contradiction. ⊣

So what remains is to prove the lemma.

PROOF OF THE LEMMA. Let $P \leq S$ be given. Denote $s$ its stem. There are two cases. ⊣

- $s$ is marked as divergent: pick its immediate successor $s_0$. Since $s$ is marked as divergent, there is $n_0$ such that $d^{n_0}(x_{s_0}, x_s) > 1$ and there are direct extensions $Q_0 \leq_0 P_{s_0}$, $P_0 \leq_0 P \setminus P_{s_0}$ such that $\forall x \in [Q_0] \forall y \in [P_0] \forall m \leq n_0 (f(x)(m) = x_{s_0}(m) \wedge f(y)(m) = x_s(m))$.

  We then pick next immediate successor $s_1 \in P_0$ of $s$. There is again some $n_1$ such that $d^{n_1}(x_{s_1}, x_s) > 1$ and we find direct extensions $Q_1 \leq_0 P_{0s_1}$, $P_1 \leq_0 P_0 \setminus P_{0s_1}$ such that $\forall x \in [Q_1] \forall y \in [P_1] \forall m \leq n_1 (f(x)(m) = x_{s_0}(m) \wedge f(y)(m) = x_s(m))$.

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We continue similarly until we pick infinitely many immediate successors of $s$ and find corresponding direct extensions $Q_i$. Then, we set $Q = \bigcup_i Q_i$. It is easy to check that it has the required properties.

- $s$ is marked as convergent: there is a barrier $B \subseteq P$ of elements that were marked as divergent. We may assume that for every $b \in B$ and every $s \leq t < b$, $t$ is marked as convergent. We will do a similar fusion to that in the proof of canonization assuming Case 1. We will inductively build $Q_n, P_n, m_n$ such that $P_n \leq_0 P_{n-1}$, $Q_n = (\{q_0 = s, \dots, q_n\} \cup R) \subseteq P_m$ for $n \leq m$ and $(m_n)_n$ is an increasing sequence of elements of $\omega$. Let $\{q_i : i \in C\} \subseteq \{q_0, \dots, q_n\}$ be the (possibly empty) set of those elements that are immediate successors of some element from $B$. Then, $R = \bigcup_{i \in C} P_{i,q_i}$. The final tree is again obtained as $Q = \bigcup_i Q_i = \bigcap_i P_i$. Conditions that must be satisfied during the $n$-th step of the fusion are the following.
  - For every $i < n$ if $i \notin C$, i.e., $q_i$ is not an immediate successor of an element from $B$, then for any branch $x \in [P_n]$ going through $q_i$ we have $d^{m_n}_{m_{n-1}}(f(x), x_s) < 1/2^n$. And if $n \in C$, i.e., $q_n$ is an immediate successor of an element from $B$, then for any branch $y \in [P_n]$ going through $q_n$ we have $d^{m_n}_{m_{n-1}}(f(y), x_s) > 2$; thus, it will follow from the triangle inequality that $d^{m_n}_{m_{n-1}}(f(x), f(y)) > 1$.
  - For every $i \leq n$ if $i \notin C$, i.e., $q_i$ is not an immediate successor of an element from $B$, then $d_{m_n}(x_{q_i}, x_s) < 1/2^{n+2}$.

  Suppose at first that such $Q$ has been constructed. We need to prove that for any two branches $x, y \in [Q]$ with $s$ as the last common node we have $d(f(x), f(y)) > 1$. It follows from the assumption that $x$ goes through some $u_i$, which is an immediate successor of some element from $B$, similarly $y$ goes through some different $u_j$ with the same property. Assume $i < j$. Then, since $x, y \in [Q] \subseteq [P_i]$, we have guaranteed during the $i$-th step of the induction that $d^{m_i}_{m_{i-1}}(f(x), f(y)) > 1$ and we are done.

  In the first step of the induction we set $Q_0 = \{s\}$, $P_0 = P$ and $m_0 = |s|$; the set $R$ is empty.

  Suppose we have already found $Q_{n-1}, P_{n-1}, m_{n-1}$. We choose some $q_n$ that is an immediate successor of some $q_i$. We have two cases.
  - $q_i \notin B$, i.e., $q_n$ is not an immediate successor of an element from $B$. Then, we set $m_n = \max\{k_j : j \leq n, j \notin C\}$, where $k_j$, for $j \notin C$, is any number such that $d_{k_j}(x_{q_j}, x_s) < 1/2^{n+2}$. Note that such $k_j$ exists because $x_{q_j} E_{\mathcal{I}} x_s$. We then find direct extensions $E_j \leq_0 P_{n-1,q_j}$ for $j \leq n, j \notin C$ such that for every branch $x \in [E_j]$ we have $f(x)(m) = x_{q_j}(m)$ for $m \leq m_n$ (again recall Observation 2.5 to see that this is possible). We refine them so that they are mutually disjoint, i.e., $E_j \cap E_l = \emptyset$ for $i \neq j$ and we set $P_n = (\bigcup_{j \notin C} E_j) \cup R$.
  - $q_i \in B$, i.e., $q_n$ is an immediate successor of an element from $B$. We add $n$ to $C$. There is some $m$ such that $d^m_{m_{n-1}}(x_{q_n}, x_{q_i}) > 2 + 1/2^{n+1}$ since $q_i \in B$ is marked as divergent. Since from the inductive assumption we have $d^m_{m_{n-1}}(x_{q_i}, x_s) < 1/2^{n+1}$, we get from the triangle inequality that $d^m_{m_{n-1}}(x_{q_n}, x_s) > 2$. We then set $m_n = \max\{m, \max\{k_i : i \leq n, i \notin C\}\}$, where $k_i$s are defined exactly the same as in the first case. We then again find direct extensions $E_i \leq_0 P_{n-1,q_i}$ for $i = n$ and $i < n, i \notin C$ such that for every

branch $x \in [E_i]$ we have $f(x)(m) = x_{q_i}(m)$ for $m \leq m_n$. We refine them so that they are mutually disjoint, i.e., $E_i \cap E_j = \emptyset$ for $i \neq j$. We add $E_n$ to $R$ and we set $P_n = (\bigcup_{i \notin C} E_i) \cup R$.

In both cases it is easy to check that all required conditions are satisfied. ⊣

## §3. Corollaries.

Theorem 3.1. *Let $E \subseteq \omega^\omega \times \omega^\omega$ be an equivalence relation containing $K$, i.e., $E \supseteq K$, which is Borel reducible to $E_{\mathcal{I}}$ for some $F_\sigma$ $P$-ideal. Then there exists a Laver large set contained in one equivalence class.*

Recall that $K$ was defined in Definition 1.7.

Proof. Consider the set

$$X = \{x \in \omega^\omega : [x]_E \text{ contains all branches of some Laver tree}\}.$$

We use Theorem 2.1 to prove that $X$ is nonempty. Suppose it is empty, then by Theorem 2.1 there exists a Laver tree $T$ such that $E \restriction [T] = \mathrm{id}([T])$. However, there must be two branches $x, y \in [T]$ such that $xKy$ and since $K \subseteq E$, also $xEy$, a contradiction.

Thus, $X$ is nonempty. We show that it is also $E$-equivalent, i.e., there is no pair $x, y \in X$ such that $x \not{E} y$. Suppose the contrary. Then, $[x]_E$ contains all branches of some Laver tree $T_x$ and $[y]_E$ contains all branches of Laver tree $T_y$ and there are branches $b_x \in T_x$ and $b_y \in T_y$ such that $b_x K b_y$ and since $K \subseteq E$, also $b_x E b_y$, a contradiction.

So $X$ is a single equivalence class, containing all branches of some Laver tree $T$, and thus, it is Borel. If it were not Laver large, then the complement would be a Borel Laver positive set, so by Proposition 1.2 it would contain all branches of some Laver tree $S$. But we would again have that there are a branch $x \in [T] \subseteq X$ and a branch $y \in [S]$ such that $xKy$, thus $xEy$, a contradiction. ⊣

Theorem 3.2 (Silver dichotomy - under "$\forall x \in \mathbb{R}(\omega_1^{L[x]} < \omega_1)$"). *Let $E \subseteq \omega^\omega \times \omega^\omega$ be an equivalence relation Borel reducible to $E_{\mathcal{I}}$ for an $F_\sigma$ $P$-ideal $\mathcal{I}$. Then, either $\omega^\omega = (\bigcup_{n\in\omega} E_n) \cup J$, where $E_n$ for every $n$ is an equivalence class of $E$ and $J$ is a set in the Laver ideal, or there exists a Laver tree $T$ such that $E \restriction [T] = \mathrm{id}([T])$.*

This is just a combination of Theorem 2.1 and the results from the section on Silver dichotomy from [1]. It is not known if the assumption "$\forall x \in \mathbb{R}(\omega_1^{L[x]} < \omega_1)$" is necessary.

Corollary 3.3 (under the same assumption). *Let $E \subseteq \omega^\omega \times \omega^\omega$ be an equivalence relation Borel reducible to $E_{\mathcal{I}}$ for $F_\sigma$ $P$-ideal $\mathcal{I}$ and let $X \subseteq \omega^\omega$ be an arbitrary Laver-positive subset (not necessarily definable) such that $\forall x, y \in X (x \not{E} y)$. Then, there exists a Laver tree $T$ such that $E \restriction [T] = \mathrm{id}([T])$.*

Proof. Just use the Silver dichotomy from the previous theorem and notice that the first possibility cannot happen. If $\omega^\omega = (\bigcup_{n\in\omega} E_n) \cup J$, as in the statement of the previous theorem, then $X \setminus J$ is still not in the Laver ideal and is uncountable. ⊣

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