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Existence of a solution for a free boundary problem in the thermoelectrical modelling of an aluminium electrolytic cell†

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We present a coupled system of elliptic equations describing the steady state of the thermoelectrical behaviour of an aluminium electrolytic cell. The thermal model is a free boundary problem which consists of the heat equation with Joule heating as a source. We neglect the Joule heating in the ledge, and allow for temperature-dependent electrical conductivity. We also formulate a numerical approximation using a finite element method. An iterative algorithm and numerical results are presented. The existence of a weak solution is also proved.

1 Introduction

In this paper, a free boundary problem motivated by the thermoelectrical modelling of an aluminium reduction cell is studied. Production of aluminium by electrolytic reduction of alumina (Al_2O_3) dissolved in a bath based on molten cryolite (NA_3AlF_6) is known as the Hall–Héroult process (see Grjotheim & Kvande [10]). This complex process takes place in a rectangular cell with an inner lining of prebaked carbon cathodic blocks with embedded steel collector bars (see [3] and the references therein), surrounding the aluminium and the electrolytic bath, which are the liquid parts of the cell. In Figure 1 a cross-section of an aluminium electrolytic cell is depicted.

As the aluminium is forming, a solidified bath layer, the so-called ledge, protects the cell sidewall from corrosive electrolyte and reduces the heat loss from the cell. Moreover, its profile strongly influences the electromagnetic effects causing oscillations of the aluminium bath interface which are related to current efficiency. Consequently one of the objectives of cell sidewall design is to promote the formation of a good ledge profile. We emphasize that its profile is an unknown of the problem, i.e. it is a free boundary.

In the present paper, we study the thermoelectrical behaviour of the cell cathode including the liquid metal and the bath as well. Mathematically, the problem is to solve a coupled system of elliptic equations consisting of the heat equation with Joule

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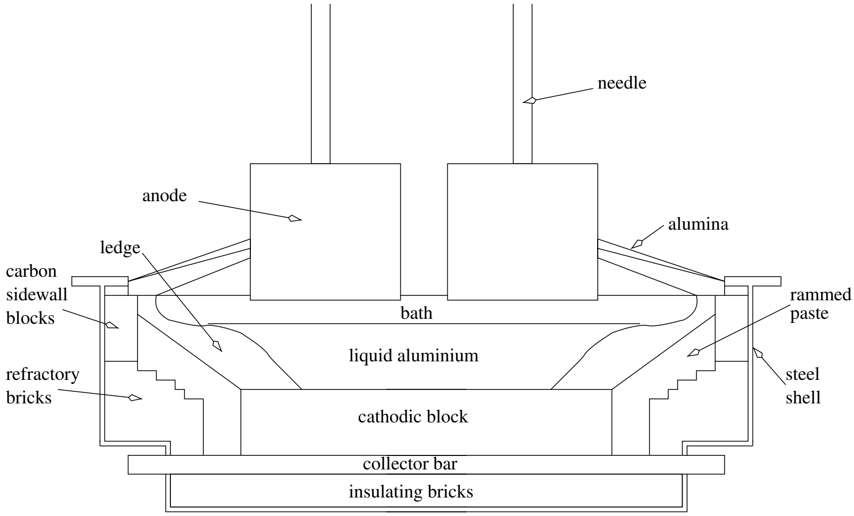

FIGURE 1. Cross-section of an electrolytic cell.

heating as a source and the current conservation equation with temperature-dependent electrical conductivity. Moreover, we assume that below the melting temperature the electric conductivity is small enough for the corresponding Joule heating to be negligible at the ledge. The thermal model is a stationary two-phase Stefan problem with a source at the solid conductors and the liquid zones of the cell; therefore, it is a free boundary problem. This is the main difference between this problem and, say, the thermistor problem (see Howison *et al.* [13] and Chen & Friedman [8]). In Gariepy *et al.* [9], for a related model of In Situ Vitrification, a degenerate elliptic equation is considered for the electric potential, and the concept of 'capacity solutions' is used.

Several articles about the thermoelectrical behaviour of an aluminium electrolytic cell have been published during the last few years (see Arita *et al.* [1] and Bermúdez *et al.* [3], for instance). The new contribution of our work is to include the free boundary in the source term of the heat equation.

The outline of this paper is as follows: after introducing the physical problem in § 2, we obtain a weak formulation in § 3. We introduce a Galerkin problem in § 4, and we present some numerical results for real industrial situations in § 5. Then we formulate a mathematical model in Section 6 and an existence theorem is proved.

2 Statement of the problem

Let Ω be the two-dimensional open set corresponding to a half cross-section of the cathode of the cell. Let us denote by Ω_E the subset of Ω occupied by the block, the collector bar and the so-called 'rammed paste', which is a mixture of pitch and coke (see [11]), namely the solid conductors of the cell (see Figure 2).

We denote by $V = V(x)$ and $T = T(x)$ the electric potential and the temperature, respectively, where $x = (x_1, x_2)$ is the spatial variable. Moreover, we denote by $\hat{\Omega}$ the

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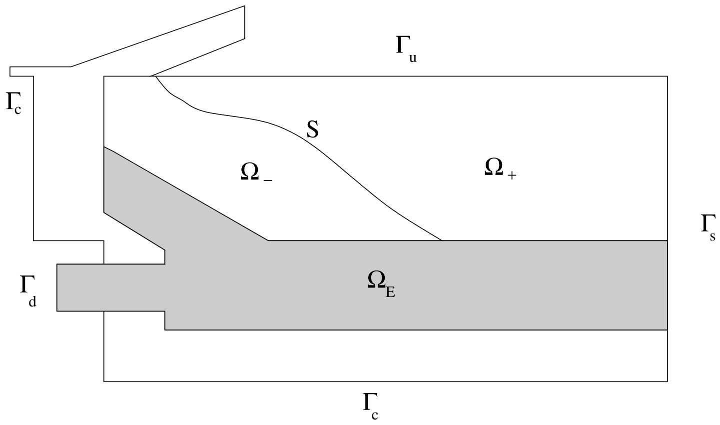

FIGURE 2. The domain Ω.

domain occupied by the electrolytic bath and the aluminium; the liquid part is

$$\Omega_+ = [T > T_s],$$

and the solid ledge is

$$\Omega_- = [T < T_s],$$

where T_s is the melting temperature and [.] denotes the set of points of $\hat{\Omega}$ satisfying the condition in the brackets. We denote by S the interface between the solid and liquid phases in $\hat{\Omega}$. In conclusion, Ω consists of Ω_E which includes the block, the collector bar and the rammed paste, $\hat{\Omega}$ which includes the materials undergoing the change of phase, i.e. the bath and the metal, and the rest of the cathode which includes the refractory and insulating bricks, the steel shell etc.

We assume the electrical conductivity, denoted by σ, is smooth within the different materials, i.e.

$$\sigma(x,T) = \begin{cases} \sigma_E(T) & \text{if } x \in \Omega_E, \\ \sigma_+(T) & \text{if } x \in \hat{\Omega} \text{ and } T > T_s, \\ \sigma_s & \text{if } x \in \hat{\Omega} \text{ and } T \leqslant T_s. \end{cases} \tag{2.1}$$

Moreover, we assume that σ_s is a constant small enough that the Joule heating is negligible in Ω_-.

The boundary of Ω is divided into:

- Γ_s: the axis of symmetry.
- Γ_u: the interface between the bath and the air.
- Γ_d: the boundary of the collector bar.
- Γ_c: the boundary between the shell of the cell and the air.

We define

$$\hat{\Omega}_E = int(\bar{\Omega}_E \cup \bar{\hat{\Omega}}).$$

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The thermoelectrical model consists of the energy equation for the temperature coupled with the charge conservation equation for the electric potential. Namely, the problem is to find T and V defined in Ω and $\hat{\Omega}_E$, respectively, such that

$$-\nabla \cdot (\sigma(x,T)\nabla V) = 0 \text{ in } \hat{\Omega}_E, \tag{2.2}$$

$$-\nabla \cdot (k(x,T)\nabla T) = \chi_{\hat{\Omega}_E}\, \sigma(x,T)|\nabla V|^2 \text{ in } \Omega, \tag{2.3}$$

$$V = 0 \text{ on } \Gamma_u, \tag{2.4}$$

$$\sigma(x,T)\frac{\partial V}{\partial n} = j \text{ on } \Gamma_d, \tag{2.5}$$

$$\sigma(x,T)\frac{\partial V}{\partial n} = 0 \text{ on } \partial\hat{\Omega}_E \backslash (\Gamma_d \cup \Gamma_u), \tag{2.6}$$

$$T = T_d \text{ on } \Gamma_d, \tag{2.7}$$

$$k(x,T)\frac{\partial T}{\partial n} = \alpha(T_c - T) \text{ on } \Gamma_c \cup \Gamma_u, \tag{2.8}$$

$$k(x,T)\frac{\partial T}{\partial n} = 0 \text{ on } \Gamma_s. \tag{2.9}$$

In (2.2)–(2.9) we see the following physical parameters: k is the thermal conductivity, j the electric current density, α the convective heat transfer coefficient, T_c the convective temperature of the surroundings and T_d the temperature of the cathodic bar on Γ_d.

Notice that the electric and thermal problems are coupled; the latter depends on the electric potential while the electric conductivity is a function of temperature. Moreover, the potential at the boundary Γ_d is not known and the current density j in (2.5) is prescribed in operational cells. In equation (2.8), the heat flux through the boundary $\Gamma_c \cup \Gamma_u$ is due to losses by convection or heat exchange with the air. The homogeneous Neumann boundary condition (2.9) holds by symmetry.

3 A weak formulation

To obtain a variational formulation, we choose a sufficiently regular function z which vanishes on Γ_u and we apply standard arguments based on integration by parts. Then, from (2.2), (2.5) and (2.6) we obtain

$$\int_{\hat{\Omega}_E} \sigma(x,T)\nabla V \cdot \nabla z\, dx = \int_{\Gamma_d} jz\, d\Gamma. \tag{3.1}$$

On the other hand, choosing a sufficiently regular function z which vanishes on Γ_d, from (2.3), (2.8) and (2.9) we have

$$\int_\Omega k(x,T)\nabla T \cdot \nabla z\, dx + \int_{\Gamma_c \cup \Gamma_u} \alpha(T - T_c)\, z\, d\Gamma = \int_{\hat{\Omega}_E} \sigma(x,T)|\nabla V|^2 z\, dx. \tag{3.2}$$

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Assuming that the second term of (3.2) is only non-zero in $\Omega_E \cup \Omega_+$, (3.2) can be written as follows:

$$\int_\Omega k(x,T)\nabla T\cdot\nabla z dx + \int_{\Gamma_c\cup\Gamma_u}\alpha(T-T_c)\,zd\Gamma$$
$$= \int_{\Omega_E}\sigma(x,T)|\nabla V|^2\,zdx + \int_{\hat\Omega}\chi_{\Omega_+}\sigma(x,T)|\nabla V|^2\,zdx. \tag{3.3}$$

Throughout this paper we use standard notation for Sobolev spaces and norms (see Troianiello [18] for instance). Moreover, we define

$$H^1_\gamma(\Omega) := \{z\in H^1(\Omega)\,:\; z|_\gamma = 0\}, \tag{3.4}$$

γ being a nonempty subset of $\partial\Omega$. So, we are led to look for V and a pair (T,χ_{Ω_+}) satisfying (3.1) and (3.3), respectively. Recasting this within appropriate spaces, the problem becomes:

Problem P
Find T in $H^1(\Omega)$, V in $H^1_{\Gamma_u}(\hat\Omega_E)$ and q in $L^\infty(\hat\Omega)$, such that $T|_{\Gamma_d} = T_d$ and

$$\int_{\hat\Omega_E}\sigma(x,T)\nabla V\cdot\nabla zdx = \int_{\Gamma_d} jzd\Gamma, \qquad \forall z\in H^1_{\Gamma_u}(\hat\Omega_E) \tag{3.5}$$

$$\int_\Omega k(x,T)\nabla T\cdot\nabla zdx + \int_{\Gamma_c\cup\Gamma_u}\alpha(T-T_c)zd\Gamma$$
$$= \int_{\Omega_E}\sigma(x,T)|\nabla V|^2\,zdx + \int_{\hat\Omega} q\,\sigma(x,T)|\nabla V|^2\,zdx, \quad \forall z\in H^1_{\Gamma_d}(\Omega)\cap L^\infty(\hat\Omega_E), \tag{3.6}$$

$$q\in H(T-T_s), \tag{3.7}$$

where H denotes the multivalued Heaviside function given by

$$H(r) = \begin{cases} 0 & \text{if } r<0,\\ [0,1] & \text{if } r=0,\\ 1 & \text{if } r>0.\end{cases} \tag{3.8}$$

4 Finite element discretization

In this section we approximate the problem **P** using a finite element method. The functions V and T are approximated by continuous piecewise linear finite elements on a triangular mesh. Thus we obtain a nonlinear discrete problem which is solved by an iterative algorithm.

Associated with a family of triangular meshes $\mathcal{T}_h$ of the domain Ω, we consider the finite element spaces W_h and W_{Eh} given by

$$W_h = \{T_h\in C(\bar\Omega)\,:\, T_h|_K\in P_1(K), \forall K\in\mathcal{T}_h\}, \tag{4.1}$$
$$W_{Eh} = \{V_h\in C(\bar{\hat\Omega}_E)\,:\, V_h|_K\in P_1(K), \forall K\in\mathcal{T}_h\}, \tag{4.2}$$

where $P_1(K)$ denotes the space of polynomials of degree $\leqslant 1$ defined on an element K. Then we introduce the discretized problem: Find T_h in W_h, V_h in W_{Eh} and q_h in W_h, such that $T_h|_{\Gamma_d} = T_d$ and

$$\int_{\hat{\Omega}_E} \sigma(x, T_h)\nabla V_h \cdot \nabla z_h dx = \int_{\Gamma_d} j z_h d\Gamma, \quad \forall z_h \in W_{Eh};\, z_h|_{\Gamma_u} = 0 \tag{4.3}$$

$$\int_\Omega k(x, T_h)\nabla T_h \cdot \nabla z_h dx + \int_{\Gamma_c \cup \Gamma_u} \alpha(T_h - T_c) z_h d\Gamma$$

$$= \int_{\Omega_E} \sigma(x, T_h)|\nabla V_h|^2 z_h dx + \int_{\hat{\Omega}} q_h\, \sigma(x, T_h)|\nabla V_h|^2 z_h dx, \quad \forall z_h \in W_h;\, z_h|_{\Gamma_d} = 0, \tag{4.4}$$

$$q_h(p) \begin{cases} \in H(T_h(p) - T_s), & \text{for all vertices } p \text{ in } \bar{\hat{\Omega}} \\ = 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

Notice that the multivalued Heaviside function H is relating temperature to the 'Lagrange multiplier' q_h in (4.5). To deal with this nonlinearity, we use an iterative algorithm introduced in Bermúdez & Moreno [4] and based on the equivalence

(i) $q_h(p) \in H(T_h(p) - T_s)$,

(ii) $q_h(p) = H_\lambda(T_h(p) - T_s + \lambda\, q_h(p)), \quad \forall \lambda > 0$,

where

$$H_\lambda(s) = \begin{cases} 0 & \text{if } s \leqslant 0, \\ \frac{s}{\lambda} & \text{if } 0 \leqslant s \leqslant \lambda, \\ 1 & \text{if } s \geqslant \lambda\,. \end{cases} \tag{4.6}$$

Therefore, it is quite natural to try the following iterative algorithm to solve (4.3)–(4.5). At iteration n, the functions T_h^n and q_h^n are known. Then we compute V_h^{n+1} and T_h^{n+1} as the solution of the linear problems

$$\int_{\hat{\Omega}_E} \sigma\left(x, T_h^n\right) \nabla V_h^{n+1} \cdot \nabla z_h dx = \int_{\Gamma_d} j z_h d\Gamma, \quad \forall z_h \in W_{Eh};\, z_h|_{\Gamma_u} = 0 \tag{4.7}$$

$$\int_\Omega k\left(x, T_h^n\right) \nabla T_h^{n+1} \cdot \nabla z_h dx + \int_{\Gamma_c \cup \Gamma_u} \alpha\left(T_h^{n+1} - T_c\right) z_h d\Gamma$$

$$= \int_{\Omega_E} \sigma\left(x, T_h^n\right) \left|\nabla V_h^{n+1}\right|^2 z_h dx + \int_{\hat{\Omega}} q_h^n\, \sigma\left(x, T_h^n\right) \left|\nabla V_h^{n+1}\right|^2 z_h dx, \quad \forall z_h \in W_h;\, z_h|_{\Gamma_d} = 0, \tag{4.8}$$

$$T_h^{n+1}\big|_{\Gamma_d} = T_d, \tag{4.9}$$

respectively, and we update q_h^{n+1} by

$$q_h^{n+1}(p) = \rho H_\lambda\left(T_h^{n+1}(p) - T_s + \lambda q_h^n(p)\right) + (1 - \rho) q_h^n(p),$$

where ρ is a relaxation parameter.

This iterative process stops when the term T_h^{n+1} is sufficiently close to the previous one.

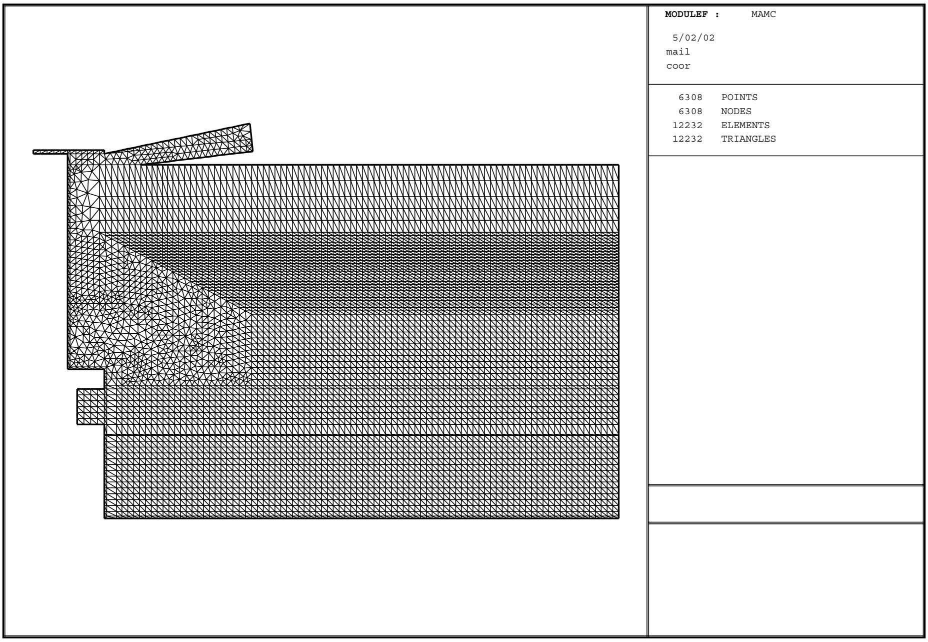

FIGURE 3. Geometry and mesh of the cell.

5 Numerical results

In this section we present numerical results for a real industrial situation, namely, the two-dimensional section of an electrolytic cell. Figure 3 shows the geometry and the mesh used for finite element discretization. It was made with the Modulef library (see Toit [17]).

Isolines for the temperature in the domain are given in Figure 4. We remark that the strong electromagnetic field promotes a convective transport in the liquid phases of the cell, causing a uniform temperature in these zones. In order to take into account this effect, we enhance the thermal conductivity of the liquid phases (see Bruggemen & Danka [2]).

Finally, Figure 5 shows the free boundary which is the surface of the solidified bath and metal.

6 A mathematical model

In the sequel, we study the existence of a solution for the thermoelectrical problem with a source term similar to the one used in the thermistor problems (see Howison *et al.* [13]) but only non-zero in $\Omega_+ \cup \Omega_E$. Let us state the following:

Problem $\mathscr{P}$
Find T in $H^1(\Omega)$, V in $H^1_{\Gamma_u}(\hat{\Omega}_E) \cap L^\infty(\hat{\Omega}_E)$ and q in $L^\infty(\hat{\Omega})$, such that $T|_{\Gamma_d} = T_d$ and

$$\int_{\hat{\Omega}_E} \sigma(x,T)\nabla V \cdot \nabla z dx = \int_{\Gamma_d} jz d\Gamma, \quad \forall z \in H^1_{\Gamma_u}(\hat{\Omega}_E) \tag{6.1}$$

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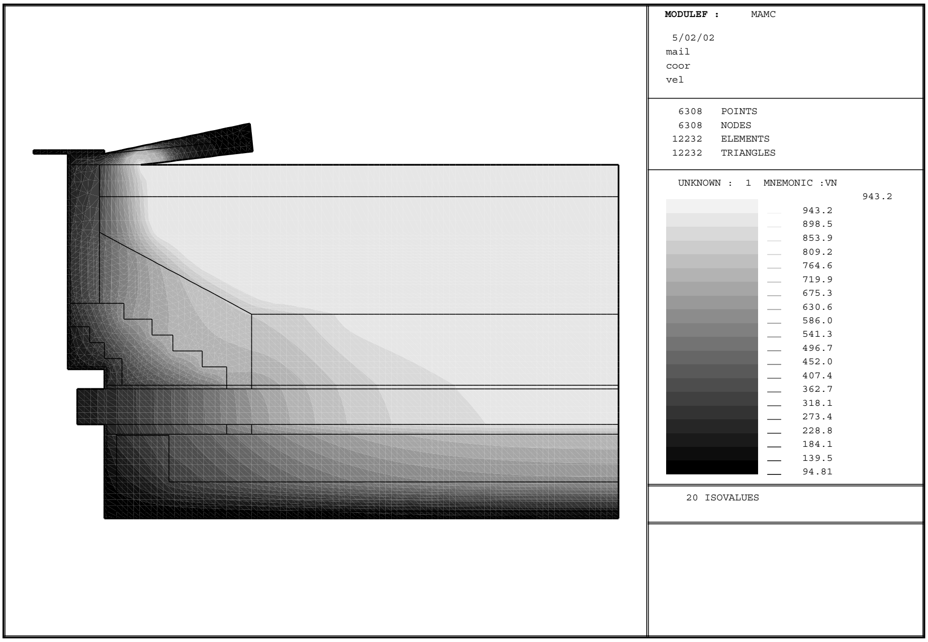

FIGURE 4. Isolines for temperature.

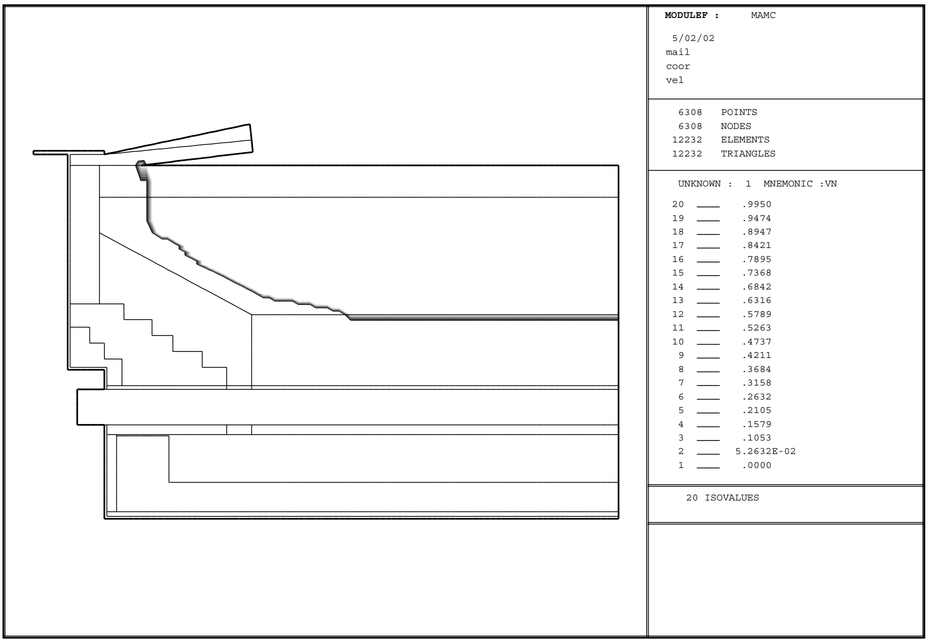

FIGURE 5. Ledge profile.

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$$\int_{\Omega} k(x,T)\nabla T\cdot\nabla z dx + \int_{\Gamma_c\cup\Gamma_u}\alpha(T-T_c)z d\Gamma$$
$$= -\int_{\Omega_E}\sigma(x,T)V\,\nabla V\cdot\nabla z dx - \int_{\hat{\Omega}} q\,\sigma(x,T)V\,\nabla V\cdot\nabla z dx,\quad \forall z\in H^1_{\Gamma_d}(\Omega), \tag{6.2}$$

$$q\in H(T-T_s), \tag{6.3}$$

where H denotes the multivalued Heaviside function given by (3.8).

The main result of this paper is the proof of existence of a solution of the problem $\mathscr{P}$. This proof consists of regularizing the Heaviside function to obtain an approximated problem which is solved by a fixed point technique. Finally, we get the existence of the required solution using compactness arguments. To obtain that, the following assumptions on the data are made:

(H1) $\sigma:\Omega\times\mathbb{R}\to\mathbb{R}$, defined in (2.1), is a Carathéodory function and there exist two positive constants, σ_s and σ_{max}, such that $\sigma_s\leqslant\sigma(x,\xi)\leqslant\sigma_{max}$ a.e. $x\in\hat{\Omega}_E$ and for all $\xi\in\mathbb{R}$. Moreover, $\sigma_+(T)=\sigma_s$ if $T\leqslant T_s$.

(H2) $k:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function and there exist two positive constants, k_{min} and k_{max}, such that $k_{min}\leqslant k(x,\xi)\leqslant k_{max}$ a.e. $x\in\Omega$ and for all $\xi\in\mathbb{R}$.

(H3) T_s and T_d are positive constants.

(H4) $T_c\in L^\infty(\Gamma_c\cup\Gamma_u)$.

(H5) $j\in L^2(\Gamma_d)$, with $j(x)\geqslant 0$ a.e. $x\in\Gamma_d$.

(H6) $\alpha\in L^\infty(\Gamma_c\cup\Gamma_u)$, with $0<\alpha_{min}\leqslant\alpha(x)\leqslant\alpha_{max}$ a.e. $x\in\Gamma_c\cup\Gamma_u$.

6.1 An approximated problem

We argue as in Brezis *et al.* [6], and first introduce the following regularized problem depending on a positive parameter ε.

Problem $\mathscr{P}_\varepsilon$
For a fixed $\varepsilon>0$, find $T_\varepsilon\in H^1(\Omega)$ and $V_\varepsilon\in H^1_{\Gamma_u}(\hat{\Omega}_E)\cap L^\infty(\hat{\Omega}_E)$ such that $T_\varepsilon|_{\Gamma_d}=T_d$ and

$$\int_{\hat{\Omega}_E}\sigma(x,T_\varepsilon)\nabla V_\varepsilon\cdot\nabla z dx = \int_{\Gamma_d} jz d\Gamma,\quad \forall z\in H^1_{\Gamma_u}(\hat{\Omega}_E), \tag{6.4}$$

$$\int_{\Omega} k(x,T_\varepsilon)\nabla T_\varepsilon\cdot\nabla z dx + \int_{\Gamma_c\cup\Gamma_u}\alpha(T_\varepsilon-T_c)z d\Gamma$$
$$= -\int_{\Omega_E}\sigma(x,T_\varepsilon)V_\varepsilon\nabla V_\varepsilon\cdot\nabla z dx - \int_{\hat{\Omega}} H_\varepsilon(T_\varepsilon-T_s)\sigma(x,T_\varepsilon)V_\varepsilon\nabla V_\varepsilon\cdot\nabla z dx,\quad \forall z\in H^1_{\Gamma_d}(\Omega), \tag{6.5}$$

where H_ε is the regularization of H given by (4.6).

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We prove existence of solution of the problem $\mathscr{P}_\varepsilon$ by means of a fixed point argument. Indeed, we will apply the Schauder fixed point theorem to the operator

$$\begin{array}{llll} \mathscr{L}: L^2(\Omega) & \to H^1_{\Gamma_u}(\hat{\Omega}_E) & \to L^2(\Omega) \\ \quad T_0 & \mapsto V_0 & \mapsto T_1, \end{array} \tag{6.6}$$

V_0 and T_1 being the solutions of the following problems, respectively:

Problem $\mathscr{V}_0$
Given $T_0 \in L^2(\Omega)$, find V_0 in $H^1_{\Gamma_u}(\hat{\Omega}_E)$ such that

$$\int_{\hat{\Omega}_E} \sigma(x, T_0)\nabla V_0 \cdot \nabla z dx = \int_{\Gamma_d} j\, z\, d\Gamma, \quad \forall z \in H^1_{\Gamma_u}(\hat{\Omega}_E). \tag{6.7}$$

Remark 6.1 The unique solution of the above problem $\mathscr{V}_0$ belongs to $W^{1,p}(\hat{\Omega}_E)$, for a certain $p > 2$ (see [12]), and consequently it belongs to $L^\infty(\hat{\Omega}_E)$.

Problem $\mathscr{T}_1$
For fixed $\varepsilon > 0$ and given $T_0 \in L^2(\Omega)$ and $V_0 \in H^1_{\Gamma_u}(\hat{\Omega}_E) \cap L^\infty(\hat{\Omega}_E)$, find T_1 in $H^1(\Omega)$ such that

$$T_1|_{\Gamma_d} = T_d; \tag{6.8}$$

$$\int_\Omega k(x, T_0)\nabla T_1 \cdot \nabla z dx + \int_{\Gamma_c \cup \Gamma_u} \alpha(T_1 - T_c)\, z d\Gamma$$
$$= -\int_{\Omega_E} \sigma(x, T_0)V_0\nabla V_0 \cdot \nabla z dx - \int_{\hat{\Omega}} H_\varepsilon(T_0 - T_s)\sigma(x, T_0)V_0\nabla V_0 \cdot \nabla z dx, \quad \forall z \in H^1_{\Gamma_d}(\Omega). \tag{6.9}$$

Remark 6.2 The unique solution of the above problem belongs to $W^{1,q}(\Omega)$, $q > 2$.

Let us begin to establish some *a priori* estimates.

Proposition 6.1 *Let V_0 be a solution of the problem $\mathscr{V}_0$. Under the assumptions* (H1) *and* (H5), *the following global estimate holds:*

$$\| V_0 \|_{1,2,\hat{\Omega}_E} \leqslant \tilde{C}\frac{\|j\|_{2,\Gamma_d}}{\sigma_s}, \tag{6.10}$$

where $\tilde{C}$ is the constant due to the continuity of the trace.

Proof Taking $z = V_0$ as a test function in (6.7) and using the lower bound σ_s of σ, the estimate results from the Cauchy–Schwarz inequality and the continuity of the trace. □

To obtain an upper bound for solutions of the problem $\mathscr{V}_0$, we need the following lemma which is proved in Kinderlehrer & Stampacchia [14, p. 63].

Lemma 6.1 *Let $\mu(t)$, $k_0 \leqslant t < \infty$, be a nonnegative, nonincreasing function such that*

$$\mu(h) \leqslant \frac{C}{(h-k)^\sigma}|\mu(k)|^\beta,\ h > k > k_0,$$

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where C, σ and β are positive constants with $\beta > 1$. Then

$$\mu(k_0 + d) = 0,$$

where

$$d^\sigma = C|\mu(k_0)|^{\beta-1} 2^{\frac{\sigma\beta}{\beta-1}}.$$

Proposition 6.2 *Let V_0 be a solution of the problem $\mathscr{V}_0$. Under the assumptions* (H1) *and* (H5)*, the following global estimate holds:*

$$0 \leqslant V_0 \leqslant C_\infty \frac{\|j\|_{2,\Gamma_d}}{\sigma_s} \text{ a.e. in } \hat{\Omega}_E, \tag{6.11}$$

where C_∞ is a positive constant which does not depend upon ε.

Proof Taking $z = V_0^- = max\{-V_0, 0\}$ as a test function in (6.7), we have

$$-\int_{\hat{\Omega}_E} \sigma(x, T_0)|\nabla V_0^-|^2 dx = \int_{\Gamma_d} jV_0^- d\Gamma.$$

Since the left-hand side is nonpositive and the right-hand side is nonnegative, we deduce $V_0^- = 0$ on Γ_d, and also, $\int_{\hat{\Omega}_E} |\nabla V_0^-|^2 dx = 0$. Then applying Poincaré inequality we obtain $V_0^- = 0$ a.e. in $\hat{\Omega}_E$.

We shall prove the upper bound with a similar technique already used in Murthy & Stampacchia [15]. For each $k > 0$, we define $A(k) = \{x \in \hat{\Omega}_E : V_0 \geqslant k\}$ and we choose $z = (V_0 - k)^+$ as a test function in (6.7). By the same argument already used in the proof of the estimate (6.10), we get

$$\sigma_s \int_{\hat{\Omega}_E} |\nabla(V_0 - k)^+|^2 dx \leqslant \|j\|_{2,\Gamma_d} \left(\int_{\Gamma_d \cap \partial A(k)} |(V_0 - k)^+|^2 d\Gamma \right)^{1/2}.$$

Applying the Hölder inequality, with $s > 2$, we obtain

$$\sigma_s \int_{\hat{\Omega}_E} |\nabla(V_0 - k)^+|^2 dx \leqslant \|j\|_{2,\Gamma_d} \|(V_0 - k)^+\|_{s,\Gamma_d} \left[meas(\Gamma_d \cap \partial A(k))^{1-\frac{2}{s}}\right]^{1/2}.$$

From the continuity of the trace and using the Young inequality, we conclude

$$\| (V_0 - k)^+ \|_{1,2,\hat{\Omega}_E} \leqslant \tilde{C} \frac{\|j\|_{2,\Gamma_d}}{\sigma_s} \left[meas(\Gamma_d \cap \partial A(k))^{1-\frac{2}{s}}\right]^{1/2}.$$

Defining $\mu(k) = meas(A(k)) + meas(\Gamma_d \cap \partial A(k))$, then

$$\| (V_0 - k)^+ \|^2_{1,2,\hat{\Omega}_E} \leqslant \tilde{C}^2 \frac{\|j\|^2_{2,\Gamma_d}}{\sigma_s^2} \mu(k)^{1-\frac{2}{s}}. \tag{6.12}$$

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Taking into account that $A(h) \subset A(k)$ when $h > k > 0$, we have

$$(h-k)^s \mu(h) \leqslant \int_{A(h)} |(V_0-k)^+|^s dx + \int_{\Gamma_d \cap \partial A(h)} |(V_0-k)^+|^s d\Gamma$$
$$\leqslant \int_{A(k)} |(V_0-k)^+|^s dx + \int_{\Gamma_d \cap \partial A(k)} |(V_0-k)^+|^s d\Gamma.$$

Applying the inequality $(a+b)^\gamma \leqslant (a^\gamma + b^\gamma)$, with $\gamma = 2/s < 1$, we obtain

$$(h-k)^2 \mu(h)^{\frac{2}{s}} \leqslant \|(V_0-k)^+\|^2_{s,\hat{\Omega}_E} + \|(V_0-k)^+\|^2_{s,\Gamma_d} \leqslant K_1 \| (V_0-k)^+ \|^2_{1,2,\hat{\Omega}_E}$$

where hereafter K_1 is a constant only dependent of the Sobolev imbedding and the continuity of the trace. Hence, using (6.12),

$$\mu(h)^{\frac{2}{s}} \leqslant \frac{K_1}{(h-k)^2} \tilde{C}^2 \frac{\|j\|^2_{2,\Gamma_d}}{\sigma_s^2} \mu(k)^{1-\frac{2}{s}}.$$

Then, Lemma 6.1 yields the result with $s > 4$ and

$$C_\infty = K_1^{1/2} \tilde{C} [meas(\hat{\Omega}_E) + meas(\Gamma_d)]^{(s-4)/(2s)} 2^{(s-2)/(s-4)}.$$

□

Proposition 6.3 *Let T_1 be a solution of the problem $\mathscr{T}_1$. Under the assumptions* (H1)–(H6)*, the following global estimate holds:*

$$\| T_1 \|_{1,2,\Omega} \leqslant C_1, \tag{6.13}$$

C_1 being a constant independent of ε.

Proof Let T_d be the constant function belonging to $W^{1,\infty}(\Omega)$. Taking $z = T_1 - T_d$ as a test function in (6.9), we obtain

$$\int_\Omega k(x,T_0)|\nabla(T_1-T_d)|^2\, dx + \int_{\Gamma_c \cup \Gamma_u} \alpha(T_1-T_d)^2\, d\Gamma$$
$$= \int_{\Gamma_c \cup \Gamma_u} \alpha(T_c - T_d)\,(T_1-T_d)\, d\Gamma - \int_{\Omega_E} \sigma(x,T_0) V_0 \nabla V_0 \cdot \nabla(T_1-T_d)\, dx$$
$$- \int_{\hat{\Omega}} H_\varepsilon(T_0-T_s)\sigma(x,T_0)V_0\nabla V_0 \cdot \nabla(T_1-T_d)\, dx.$$

Using the Cauchy–Schwarz inequality and the fact that $0 \leqslant H_\varepsilon \leqslant 1$, we deduce

$$min\{k_{min}, \alpha_{min}\}\ \|T_1 - T_d\|^2_{1,2,\Omega} \leqslant \alpha_{max}\, C\ \|T_c - T_d\|_{2,\Gamma_c\cup\Gamma_u} \|T_1 - T_d\|_{1,2,\Omega}$$
$$+ 2\,\sigma_{max}\, \|V_0\|_{\infty,\hat{\Omega}_E} \|\nabla V_0\|_{2,\hat{\Omega}_E} \|T_1 - T_d\|_{1,2,\Omega},$$

C being the constant related to the continuity of the trace. The desired estimate follows from (6.10), (6.11) and taking into account that $\|T_1\|_{1,2,\Omega} \leqslant \|T_1 - T_d\|_{1,2,\Omega} + \|T_d\|_{1,2,\Omega}$. □

Finally, we are able to prove the following:

Theorem 6.1 *Under assumptions* (H1)–(H6)*, there exists a solution* $(T_\varepsilon, V_\varepsilon)$ *of the problem* $\mathscr{P}_\varepsilon$.

Proof Let us consider $\mathscr{L}$ given by (6.6).

Step 1 $\mathscr{L}$ is well defined
Indeed, there exists a unique solution of the linear problem $\mathscr{V}_0$, by applying the Lax–Milgram theorem. Furthermore, we have a $W^{1,p}$-estimate, with $p > 2$, due to the regularity for solutions to mixed boundary value problems (see Groger [12]). Hence, applying the DeGiorgi–Stampacchia theorem (see Rodrigues [16, p. 170]), there exists a unique solution of the problem $\mathscr{T}_1$, belonging to $H^1(\Omega) \cap C^{0,\alpha}(\bar{\Omega})$, with $0 < \alpha < 1$.

Step 2 $\mathscr{L}$ is compact
Indeed, since $\mathscr{L}$ maps $L^2(\Omega)$ into a ball due to Proposition 6.3 and taking into account that $H^1(\Omega)$ is compactly imbedded in $L^2(\Omega)$, it is sufficient to prove the continuity of $\mathscr{L}$. To this purpose, let $\{T_0^n\} \to T_0$ in $L^2(\Omega)$ and let V_0^n and V_0 be the corresponding solutions of problems $\mathscr{V}_0^n$ and $\mathscr{V}_0$, respectively. By estimate (6.10), we can select a subsequence, still denoted by n, such that $V_0^n \rightharpoonup V$ in $H^1(\Omega)$. Passing to the limit in $(6.7)_n$, and using the uniqueness, we conclude that $V = V_0$ and also that the whole sequence converges.

Analogously, we consider the corresponding solutions T_1^n and T_1 of the problems $\mathscr{T}_1^n$ and $\mathscr{T}_1$, respectively. From the estimate (6.13) we can extract a subsequence, still denoted by n, such that $T_1^n \rightharpoonup T$ in $H^1(\Omega)$. Therefore, using the Lebesgue theorem and passing to the limit in $(6.9)_n$, we obtain, by uniqueness, that $T = T_1$ and also the convergence of the whole sequence.

Finally, the existence of the solution of the problem $\mathscr{P}_\varepsilon$ results from the Schauder fixed point theorem. □

Remark 6.3 Regarding the function H_ε, we only considered its continuity and the property $0 \leqslant H_\varepsilon \leqslant 1$ for proving the existence of the solution of the problem $\mathscr{P}_\varepsilon$.

6.2 Existence of a weak solution

In this section we prove existence of a solution for the problem $\mathscr{P}$ using classical compactness techniques. Indeed, we have

Theorem 6.2 *Under assumptions* (H1)–(H6)*, there exists a solution* (T, V, q) *of the problem* $\mathscr{P}$*. Moreover,* $T \in C^{0,\alpha}(\bar{\Omega})$ *with* $0 < \alpha < 1$.

Proof To obtain the existence of solution, we need to pass to the limit in the problem $\mathscr{P}_\varepsilon$. For a fixed $\varepsilon > 0$, let $(V_\varepsilon, T_\varepsilon)$ be a solution of the coupled problem $\mathscr{P}_\varepsilon$ given by Theorem 6.1. From (6.10) and (6.13), we can extract subsequences of ε still denoted by ε such that

$$V_\varepsilon \rightharpoonup V \text{ in } H^1(\hat{\Omega}_E) \text{ weakly,} \tag{6.14}$$

$$V_\varepsilon \to V \text{ in } L^2(\hat{\Omega}_E) \text{ and a.e. in } \hat{\Omega}_E, \tag{6.15}$$

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$$T_\varepsilon \rightharpoonup T \text{ in } H^1(\Omega) \text{ weakly,} \tag{6.16}$$

$$T_\varepsilon \to T \text{ in } L^2(\Omega) \text{ and a.e. in } \Omega. \tag{6.17}$$

Moreover, $\{H_\varepsilon(T_\varepsilon - T_s)\}$ is bounded in $L^\infty(\hat{\Omega})$, then there exists q in $L^\infty(\hat{\Omega})$ such that

$$H_\varepsilon(T_\varepsilon - T_s) \rightharpoonup q \text{ in } L^\infty(\hat{\Omega}) \text{ weak}*. \tag{6.18}$$

Step 1

We prove that $q \in H(T - T_s)$. Using the same argument of Carrillo-Menendez *et al.* [7], q belongs to the convex and weak$*$ closed set $\mathcal{N}$ defined by

$$\mathcal{N} = \{f \in L^\infty(\hat{\Omega}) : 0 \leqslant f \leqslant 1 \text{ a.e. in } \Omega\}.$$

On the other hand, for each $x \in [T < T_s]$ we have $T_\varepsilon(x) < T_s$ for ε small enough and then $H_\varepsilon(T_\varepsilon - T_s) = 0$. Thus, we get

$$H_\varepsilon(T_\varepsilon - T_s) \to 0 \text{ a.e. in } [T < T_s],$$

and, applying the Lebesgue theorem, we deduce $H_\varepsilon(T_\varepsilon - T_s) \to 0$ in $L^2([T < T_s])$ and also in $L^2([T < T_s])$ weak$*$ and by uniqueness of the limit $q = 0$ a.e. in $[T < T_s]$ which completes the proof.

Step 2

Let us pass to the limit in (6.4). Applying the DeGiorgi–Stampacchia theorem we have uniqueness of the limit solution, and then we conclude that V satisfies (6.1) and also the convergence of the whole sequence. Notice that V belongs to $L^\infty(\hat{\Omega}_E)$ (see Groger [12]).

Step 3

We prove that the convergence (6.14) is strong. Indeed, taking $z = V_\varepsilon - V$ as a test function in (6.4) and (6.1) and subtracting we obtain

$$\int_{\hat{\Omega}_E} (\sigma(x, T_\varepsilon)\nabla V_\varepsilon - \sigma(x, T)\nabla V) \cdot \nabla(V_\varepsilon - V)\, dx = 0.$$

Therefore, we obtain

$$\int_{\hat{\Omega}_E} [\sigma(x, T_\varepsilon)\nabla(V_\varepsilon - V) + (\sigma(x, T_\varepsilon) - \sigma(x, T))\nabla V] \cdot \nabla(V_\varepsilon - V)\, dx = 0.$$

Using (H1) we have

$$\sigma_s \int_{\hat{\Omega}_E} |\nabla(V_\varepsilon - V)|^2\, dx \leqslant \int_{\hat{\Omega}_E} (\sigma(x, T) - \sigma(x, T_\varepsilon))\nabla V \cdot \nabla(V_\varepsilon - V)\, dx.$$

Recalling that $\sigma(x, T_\varepsilon) \to \sigma(x, T)$ in $L^{2p/(p-2)}(\hat{\Omega}_E)$ and $\nabla V \in L^p(\hat{\Omega}_E)$ with $p > 2$, and taking into account (6.14) we achieve the result.

Step 4
To pass the limit (6.5), the only non-trivial term is

$$\int_{\hat{\Omega}} H_\varepsilon(T_\varepsilon - T_s)\sigma(x, T_\varepsilon)V_\varepsilon \nabla V_\varepsilon \cdot \nabla z dx.$$

Regarding (6.18), it remains to verify for all $z \in H^1_{\Gamma_d}(\Omega)$

$$\sigma(x, T_\varepsilon)V_\varepsilon \nabla V_\varepsilon \cdot \nabla z \to \sigma(x, T)V\nabla V \cdot \nabla z \text{ in } L^1(\hat{\Omega}).$$

Indeed, taking into account the following decomposition

$$\begin{aligned}
&\int_{\hat{\Omega}} |\sigma(x, T_\varepsilon)V_\varepsilon \nabla V_\varepsilon \cdot \nabla z - \sigma(x, T)V\nabla V \cdot \nabla z| dx \\
&\quad \leqslant \int_{\hat{\Omega}} |(\sigma(x, T_\varepsilon) - \sigma(x, T))\, V_\varepsilon \nabla V_\varepsilon \cdot \nabla z| dx + \int_{\hat{\Omega}} |\sigma(x, T)(V_\varepsilon - V)\nabla V_\varepsilon \cdot \nabla z| dx \\
&\qquad + \int_{\hat{\Omega}} |\sigma(x, T)V\, \nabla(V_\varepsilon - V) \cdot \nabla z| dx,
\end{aligned}$$

and applying the Hölder inequality, we have

$$\begin{aligned}
&\int_{\hat{\Omega}} |\sigma(x, T_\varepsilon)V_\varepsilon \nabla V_\varepsilon \cdot \nabla z - \sigma(x, T)V\nabla V \cdot \nabla z| dx \\
&\leqslant \left(\|\sigma(x, T_\varepsilon) - \sigma(x, T)\|_{\frac{2p}{p-2},\hat{\Omega}_E} \|V_\varepsilon\|_{\infty,\hat{\Omega}_E} + \sigma_{max} \|V_\varepsilon - V\|_{\frac{2p}{p-2},\hat{\Omega}_E} \right) \|\nabla V_\varepsilon\|_{p,\hat{\Omega}_E} \|\nabla z\|_{2,\Omega} \\
&\quad + \sigma_{max} \|V\|_{\infty,\hat{\Omega}_E} \|\nabla(V_\varepsilon - V)\|_{2,\hat{\Omega}_E} \|\nabla z\|_{2,\Omega}.
\end{aligned}$$

Using the $L^{2p/(p-2)}(\hat{\Omega}_E)$ convergence of $\sigma(x, T_\varepsilon) \to \sigma(x, T)$ and $V_\varepsilon \to V$, the fact that V_ε and V belong to $L^\infty(\hat{\Omega}_E)$, the ε-independent $W^{1,p}(\hat{\Omega}_E)$ estimate of V_ε (see Groger [12]) and the strong convergence $V_\varepsilon \to V$ in $H^1(\hat{\Omega}_E)$, we conclude the desired result.

The regularity $T \in C^{0,\alpha}(\bar{\Omega})$, with $0 < \alpha < 1$ follows from the DeGiorgi–Stampacchia estimate (see Rodrigues [16, p. 170]). □

7 Conclusions

A numerical method for solving a thermoelectrical model arising in an aluminium electrolytic cell is presented. The two partial differential equations governing the temperature and the electric potential are discretized by using triangular finite elements with three degrees of freedom. Here the free boundary given as the surface of the ledge is handled by considering an additional unknown function. To solve the discretized problem, an iterative algorithm is proposed and a computed code has been written. An industrial electrolytic cell has been simulated and both isothermal and isopotential lines are in good agreement with those obtained in Bermúdez & Salgado [5] by using a domain decomposition method.

Regarding the theoretical study of the full thermoelectrical model, unfortunately to our knowledge the proof of existence of the solution of the coupled thermoelectrical problem with a free boundary is an open problem. However, the fact that our simulation has

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produced satisfactory results suggests that there is a well-defined weak solution to this problem. Nevertheless, to give an insight into the problem, we have proved the existence of solution of a thermoelectrical problem with a source term similar to the one used in the thermistor problems but vanishing in the ledge.

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