The reciprocation of one quadric into another. By Professor H. F. BAKER.

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It is a familiar fact that if two quadric forms, in $(n+1)$ homogeneous variables, be each expressible as a sum of squares, of $(n+1)$ independent linear functions of the variables, then they are polar reciprocals of one another, in regard to any one of 2^n quadrics. The question arises whether this is true for any two quadrics. Segre* states that this is an unsettled question. A solution is given, however, by Terracini† for any two non-degenerate quadrics, supposed to have been reduced to the Weierstrass canonical form. The present note has the purpose of pointing out that a solution is derivable from a remark made by Frobenius‡; this requires a knowledge of the roots of the equation satisfied by the matrix of the two quadrics§.

§ 1. We consider two quadrics in $(n+1)$ homogeneous variables, both of non-vanishing discriminant. The matrix of the coefficients of one of these being m^{-1}, the matrix of the coefficients of the tangential-form of this will be m; let the matrix of the coefficients of the point-form of the other quadric be M. If the quadric which is sought, by means of which the two given quadrics are polar reciprocals of one another, be of matrix p, we are to substitute, for the tangential coordinates u in the first form, mu^2, the values given by $u = px$, and the result is to be the second form Mx^2. Now

$$mu^2 = mpx \,.\, px = \bar{p}mp \,.\, x^2,$$

where $\bar{p}$ denotes, as usual, the matrix obtained from p by transposition of rows and columns. In other words, the problem is, given m and M, to find a *symmetrical* matrix p such that

$$pmp = M;$$

by hypothesis m and M are symmetrical.

* *Encykl. Math.* III, C. 7, p. 864, 1912.

† *Ann. d. Mat.* XXX, 1921, p. 155.

‡ *Berlin. Sitzungsber.* 1896, p. 7. Frobenius refers to Kelland and Tait, *Quaternions* (1873), chap. X, and to Sylvester, *Papers*, III, pp. 562–567 (1882). Sylvester refers to Babbage's *Calculus of Functions*.

§ This equation may be the same for cases of different invariant factors. For example, if m be the matrix of the coefficients of the form $2xy + z^2 + t^2$, and M_1, M_2 be the respective matrices of the coefficients of the two forms

$$2\theta xy + x^2 + \theta z^2 + \phi t^2, \quad 2\theta xy + x^2 + \phi z^2 + \phi t^2,$$

both the matrices $M_1 m$, $M_2 m$ satisfy the equation

$$(N-\theta)^2 (N-\phi) = 0.$$

§ 2. Now we can, without loss of generality, suppose the quadric form $m^{-1}x^2$ reduced to a sum of $(n+1)$ independent squares, at starting, and find the form to which Mx^2 is reduced by the same transformation. Thus it is clear that there is no loss of generality in supposing $m = 1$. Then the problem is to find p such that

$$p^2 = M.$$

But, if the equation satisfied by M be of order $(r+1)$, it is an obvious suggestion of the theory of algebraic numbers to attempt to solve this equation by substituting for p a polynomial in M of order r; the number of conditions to be satisfied by the coefficients in this polynomial is then equal to the number of these coefficients. Frobenius (*loc. cit.*) has given an explicit formula for this polynomial when the roots of the equation satisfied by M are known; this may be regarded as the well-known Lagrange's interpolation formula extended to the case of repeated roots. It is clear that if M is symmetrical, this polynomial will likewise represent a symmetrical matrix. The formula is quoted below (§ 5).

§ 3. Without this reduction of the matrix m to a unit matrix, the equation $pmp = M$ may be solved by finding q such that

$$q^2 = Mm,$$

and then taking $p = qm^{-1}$. To prove this it is only necessary to show that qm^{-1} is a symmetrical matrix; for, from $q^2 = Mm$, we have

$$pmpm = Mm,$$

and hence $$pmp = M.$$

Now the equation $q^2 = Mm$ leads, because m, M are symmetrical, to

$$\bar{q}^2 = mM;$$

thus, if $$q = A_r(Mm)^r + \ldots + A_1(Mm) + A_0,$$

we have, for $\bar{q}$,

$$\bar{q} = A_r(mM)^r + \ldots + A_1(mM) + A_0;$$

and hence, as

$$m(Mm)^k = m\,Mm\,Mm\ldots Mm = (mM)^k\,m,$$

we have $$mq = \bar{q}m,$$

or $$qm^{-1} = m^{-1}\bar{q},$$

which, since m is symmetrical, shows that qm^{-1} is symmetrical.

We may illustrate this procedure by deducing Terracini's result. The forms to which Weierstrass reduces any two quadrics have matrices such as

$$\begin{vmatrix} a_1 & . & . & . \\ . & a_2 & . & . \\ . & . & . & . \\ . & . & . & . \end{vmatrix}, \quad \begin{vmatrix} b_1 & . & . & . \\ . & b_2 & . & . \\ . & . & . & . \\ . & . & . & . \end{vmatrix}$$

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wherein a_1 is a matrix whose diagonal coincides with that of the whole matrix, and b_1 a similarly placed matrix of the same order; likewise a_2, b_2 are similarly placed matrices of equal order; and so on; each of the couples (a_1, b_1), (a_2, b_2), ... is associated with one of the elementary divisors of the matrix formed by the original family of quadrics, that is, of the matrix $M + \rho m^{-1}$. As the product of two matrices of these reduced forms is the matrix

$$\begin{vmatrix} a_1b_1 & \cdot & \cdot & \cdot \\ \cdot & a_2b_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

it is clear on consideration that it is sufficient to consider only the two matrices a_1, b_1, the other couples being similarly treated. Thus we may suppose, in the equation $pmp = M$, limiting ourselves to four rows and columns, and recalling the forms obtained by Weierstrass, that

$$m = \begin{vmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{vmatrix}, \quad M = \begin{vmatrix} \cdot & \cdot & 1 & \theta \\ \cdot & 1 & \theta & \cdot \\ 1 & \theta & \cdot & \cdot \\ \theta & \cdot & \cdot & \cdot \end{vmatrix},$$

the elements not written being zeros. Here $m^{-1} = m$, and we have

$$Mm = \begin{vmatrix} \theta & 1 & \cdot & \cdot \\ \cdot & \theta & 1 & \cdot \\ \cdot & \cdot & \theta & 1 \\ \cdot & \cdot & \cdot & \theta \end{vmatrix}.$$

Thus $(Mm - \theta)^4 = 0$; therefore, denoting $Mm - \theta$ by N, a matrix, q, such that $q^2 = \theta + N$, may be obtained by expanding $(\theta + N)^{\frac{1}{2}}$, in powers of N, as far as the term in q^3. Namely we have

$$q = \theta^{\frac{1}{2}} (1 + \tfrac{1}{2}\theta^{-1} N - \tfrac{1}{8}\theta^{-2}N^2 + \tfrac{1}{16}\theta^{-3}N^3);$$

this, however, is

$$q = \theta^{\frac{1}{2}} \left[\begin{vmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{vmatrix} + \tfrac{1}{2}\theta^{-1} \begin{vmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} - \tfrac{1}{8}\theta^{-2} \begin{vmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} + \tfrac{1}{16}\theta^{-3} \begin{vmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} \right]$$

$$= \theta^{\frac{1}{2}} \begin{vmatrix} 1 & \frac{1}{2}\theta^{-1} & -\frac{1}{8}\theta^{-2} & \frac{1}{16}\theta^{-3} \\ \cdot & 1 & \frac{1}{2}\theta^{-1} & -\frac{1}{8}\theta^{-2} \\ \cdot & \cdot & 1 & \frac{1}{2}\theta^{-1} \\ \cdot & \cdot & \cdot & 1 \end{vmatrix};$$

thus $p, = qm^{-1}$, is given by

$$p = \theta^{\frac{1}{2}} \begin{vmatrix} \frac{1}{16}\theta^{-3} & -\frac{1}{8}\theta^{-2} & \frac{1}{2}\theta^{-1} & 1 \\ -\frac{1}{8}\theta^{-2} & \frac{1}{2}\theta^{-1} & 1 & . \\ \frac{1}{2}\theta^{-1} & 1 & . & . \\ 1 & . & . & . \end{vmatrix}.$$

Terracini obtains this result by trying, as a form for p, the matrix

$$p = \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_2 & \lambda_3 & \lambda_4 & . \\ \lambda_3 & \lambda_4 & . & . \\ \lambda_4 & . & . & . \end{vmatrix};$$

then, with the proper m, we have

$$pmp = \begin{vmatrix} (1234) & (234) & (34) & (4) \\ (234) & (34) & (4) & . \\ (34) & (4) & . & . \\ (4) & . & . & . \end{vmatrix},$$

where
$$\begin{aligned} (1234) &= (\lambda_1, \lambda_2, \lambda_3, \lambda_4)(\lambda_4, \lambda_3, \lambda_2, \lambda_1) = 2\lambda_1\lambda_4 + 2\lambda_2\lambda_3, \\ (234) &= (\lambda_2, \lambda_3, \lambda_4)(\lambda_4, \lambda_3, \lambda_2) = 2\lambda_2\lambda_4 + \lambda_3^2, \\ (34) &= (\lambda_3, \lambda_4)(\lambda_4, \lambda_3) = 2\lambda_3\lambda_4, \\ (4) &= (\lambda_4)(\lambda_4) = \lambda_4^2; \end{aligned}$$

to identify this pmp with the matrix M we require

$$\lambda_4^2 = \theta, \;\; 2\lambda_3\lambda_4 = 1, \;\; 2\lambda_2\lambda_4 + \lambda_3^2 = 0, \;\; 2\lambda_1\lambda_4 + 2\lambda_2\lambda_3 = 0,$$

and these give, for $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, the values obtained otherwise above.

§ 4. We may compare the procedures under § 2 and § 3. To put the matrix m in unit form, it is necessary to take a matrix α such that

$$m = \alpha\bar{\alpha};$$

the equation to be solved is then

$$p\alpha\bar{\alpha}p = M,$$

or, if $r = \bar{\alpha}p\alpha$, $\quad r^2 = \bar{\alpha}M\alpha.$

Conversely, when r is found from this, p can be found. To identify this with the procedure in § 3, it is necessary to show that

$$\bar{\alpha}\,(M\alpha\bar{\alpha})^{\frac{1}{2}} = (\bar{\alpha}M\alpha)^{\frac{1}{2}}\,\bar{\alpha};$$

this can be proved precisely as it was proved that

$$m\,(Mm)^{\frac{1}{2}} = (mM)^{\frac{1}{2}}\,m.$$

§ 5. The formula of Frobenius, referred to above, for the matrix p such that $p^2 = M$, is as follows: Let the equation of least order satisfied by M be

$$(M-\theta)^a (M-\phi)^b \dots = 0,$$

where $\theta, \phi, \dots$ are all different. If the determinant $| M-\rho |$ divide by $(\rho-\theta)^l$, and all the first minors divide by $(\rho-\theta)^{l_1}$, then $a = l - l_1$; and similarly for $b, \dots$*; thus, to find the equation, we may divide the determinant $| M-\rho |$ by the highest common factor (as regards ρ) of the first minors of this determinant, and then replace ρ by M. The process is rational when M is given; the determination of $\theta, \phi, \dots$ is a subsequent step. Now expand the fraction

$$\frac{M^{\frac{1}{2}}}{(M-\phi)^b \dots},$$

where the denominator is obtained by omission of the factor $(M-\theta)^a$, as if M were a single number, in powers of $M-\theta$; denote the terms of this expansion, up to the term in $(M-\theta)^{a-1}$ inclusive, with an arbitrary $\pm$ sign prefixed, by $[\theta]$. Then the polynomial in M, consisting of the sum of such terms as

$$[\theta] (M-\phi)^b \dots,$$

computed for all the roots $\theta, \phi, \dots$ in turn, is such that, when squared and then reduced by means of the equation satisfied by M, it reduces to M. This polynomial, U, is of order less than the order of the equation satisfied by M. The verification is by expanding $M - U^2$ in powers of $M-\theta$, $M-\phi, \dots$ in turn, which shows that this divides by the left side of the equation satisfied by M. If k be the number of different roots of $| M-\rho | = 0$, there are 2^k possibilities for U. The product of any two such terms of U as $[\theta] (M-\phi)^b \dots$ contains all the factors of $(M-\theta)^a (M-\phi)^b \dots$; thus U^2 is the sum of the squares of the separate terms.

As a simple illustration, suppose

$$M = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix};$$

then $M^3 = 1$, and we find

$$M^{\frac{1}{2}} = \tfrac{1}{3}(aM^2 + bM + c),$$

where, with $\epsilon_1{}^2 = \epsilon_2{}^2 = \epsilon_3{}^2 = 1$, $\omega^3 = 1$,

$$a = \epsilon_1 + \epsilon_2 + \epsilon_3, \quad b = \epsilon_1 + \epsilon_2\omega + \epsilon_3\omega^2, \quad c = \epsilon_1 + \epsilon_2\omega^2 + \epsilon_3\omega.$$

§ 6. Certainly for some particular values of m and M, the preceding does not give the general solution of the equation $pmp = M$.

* *Proc. Lond. Math. Soc.* xxx, 1899, p. 196.

For instance, when $m = M = 1$, and the matrices are, respectively, of orders 2 and 3, we have solutions, with arbitrary parameters,

$$p = \begin{vmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{vmatrix},$$

$$p = (a^2+b^2+c^2)^{-1} \begin{vmatrix} b^2+c^2-a^2, & -2ab, & -2ac \\ -2ab, & c^2+a^2-b^2, & -2bc \\ -2ac, & -2bc, & a^2+b^2-c^2 \end{vmatrix}$$

these being symmetrical orthogonal transformations.

With somewhat more generality, if the matrix, x, of an orthogonal transformation, for which $x\bar{x} = 1$, be capable of the form

$$x = \beta^{-1} p \alpha,$$

where p involves arbitrary parameters, but α and β are definite, the matrix p being symmetrical, the equation $x\bar{x} = 1$ is the same as

$$\beta^{-1} p \alpha \bar{\alpha} p \bar{\beta}^{-1} = 1,$$

or

$$pmp = M,$$

whereby the symmetric matrix $m, = \alpha\bar{\alpha}$, is changed, with the symmetric matrix p, involving parameters, into the symmetric matrix $M, = \beta\bar{\beta}$. A case of this, the only one existing for matrices of order 3, is

$$x = \tfrac{1}{4} \begin{vmatrix} 1 & 0 & 2 \\ 0 & 2\sqrt{2} & 0 \\ -3i & 0 & 2i \end{vmatrix} \begin{vmatrix} 0 & 0 & 1 \\ 0 & -1 & -m \\ 1 & -m & \tfrac{1}{2}(1-m^2) \end{vmatrix} \begin{vmatrix} 1 & 0 & -i \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & i \end{vmatrix}.$$

This arises from the symmetric orthogonal form quoted above by taking $a = m$, $b = 2\sqrt{2}$, $c = im$. It corresponds to the fact that the conics

$$2xz + y^2 = 0, \quad 2xz + y^2 + z^2 = 0$$

are polar reciprocals not only with regard to

$$2xz + y^2 + \tfrac{1}{2}z^2 = 0,$$

but also with regard to

$$2xz + y^2 + \tfrac{1}{2}z^2 - \tfrac{1}{2}(mz + 2y)^2 = 0,$$

for any value of m.

§ 7. We have deduced Terracini's result from Frobenius' work. But the converse view may be taken. And, by a combination of both, we are in a position to write down at once a general mth root of any matrix, supposed first put in reduced form—namely, by means of components such as that called q in § 3 above, the fractional coefficients being replaced by the coefficients in the expansion of $(1+t)^{\frac{1}{m}}$, and $\theta^{\frac{1}{2}}$ by $\theta^{\frac{1}{m}}$. Or, indeed, to write down explicitly any algebraic function of the matrix.

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