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Enlargements and Morita contexts for rings with involution

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Abstract

We study Morita equivalence for idempotent rings with involution. Following the ideas of Rieffel, we define Rieffel contexts, and we also introduce Morita $*$-contexts and enlargements for rings with involution. We prove that two idempotent rings with involution have a joint enlargement if and only if they are connected by a unitary and full Rieffel context. These conditions are also equivalent to having a unitary and surjective Morita $*$-context between those rings. We also examine how the mentioned conditions are connected to the existence of certain equivalence functors between the categories of firm modules over the given rings with involution.

1. Introduction

In this paper, we study Morita theory for idempotent rings (i.e., rings R such that $RR = R$) with involution. The classical Morita theory ([16]) dealt with unital rings, but by now Morita theory has been developed for a wide range of algebraic structures, including nonunital rings ([1, 2, 27] etc). In the 1970s, Rieffel published an influential series of articles on Morita theory of C^*-algebras ([21, 22, 4, 23]). His works have inspired several mathematicians, among them Ara ([3]), who developed Morita theory for idempotent nondegenerate rings with involution, and Steinberg ([24]), who studied Morita equivalence for inverse semigroups.

Although the Morita equivalence relation is usually defined by requiring the equivalence of suitable module categories, it is often easier to work with some related bimodules and compatible bimodule homomorphisms, which form so-called Morita contexts. For example, Ara [3] uses inner product bimodules to prove that the centroids of two Morita equivalent idempotent nondegenerate rings are isomorphic. In this paper, we consider relationships between four different approaches to Morita theory: Morita contexts, inner product bimodules, enlargements and equivalence of module categories. A Morita context for rings consists of two bimodules and two bimodule homomorphisms. Rieffel showed that in the presence of a $*$-operation (an involution) one can use just one bimodule together with two inner products. The notion of an enlargement comes actually from semigroup theory ([11, 13]). In [12], Lawson showed that two semigroups with local units are Morita equivalent if and only if there exists the third semigroup where the initial semigroups can be embedded "in a nice way." Later it has been shown that joint enlargements work naturally also in the case of idempotent rings ([9]) and quantales ([14]). One of the aims of this paper is to define joint enlargements for rings with involutions in a suitable way and show that the existence of a joint enlargement is equivalent to the existence of a Morita context of a certain type.

In Section 2, we introduce the necessary notions and prove some basic facts about Rieffel contexts and Morita $*$-contexts. By a Rieffel context, we will mean a bimodule together with two so-called inner products satisfying certain axioms.

Section 3 contains our main theorem, which states that, for idempotent rings with involution, certain statements about enlargements, Rieffel contexts and Morita $*$-contexts are equivalent. In particular, we have the following result.

Theorem. *Two idempotent rings with involution are connected by a unitary and full Rieffel context if and only if they have a joint enlargement.*

After that, in Section 4 we show that, up to isomorphism, each unitary Rieffel context is induced in a canonical way by a joint enlargement of two idempotent rings with involution.

Sections 5 and 6 are devoted to module categories. We define, in a suitable way, a $*$-equivalence between rings with involution (essentially, it consists of two pairs of equivalence functors between certain module categories, which satisfy some extra conditions) and prove that any firm and full Rieffel context gives rise to a $*$-equivalence between two idempotent rings with involution. After that we prove that the existence of a firm and full Rieffel context is equivalent to the existence of certain equivalence functors between module categories.

In our final section, we compare our results with the results obtained by Ara in [3].

2. Preliminaries

In this paper, by a ring we mean an associative ring.

Definition 2.1. *A ring with an involution is a pair $(R, *)$, where R is a ring and $* : R \to R, a \mapsto a^*$ is a mapping such that*

$$(a+b)^* = a^* + b^*, \quad (a^*)^* = a, \quad (ab)^* = b^* a^* \tag{2.1}$$

for all $a, b \in R$. Note that $0^* = (0+0)^* = 0^* + 0^*$ implies $0^* = 0$.

Let $(R, *)$ and $(S, \star)$ be rings with involution. A ring homomorphism $f : R \to S$ is called a *homomorphism of rings with involution*, if $f(r^*) = f(r)^\star$ holds for every $r \in R$.

We recall that a right R module M_R is called *unitary* if $MR = M$, and *firm* if the natural mapping $\mu_M : M \otimes_R R \to M, m \otimes r \mapsto mr$ is bijective (see [20]). A bimodule is *unitary* (*firm*) if it is unitary (firm) as a left and a right module. Note that $MR = M$ if and only if μ_M is surjective; hence, every firm module is unitary. Recall that a ring R is called *idempotent* if $R = RR$.

Rieffel introduced imprimitivity bimodules over C^*-algebras ([22, Definition 6.10]) and used them to develop Morita theory for C^*-algebras ([4, 21]). Inspired by this notion we give the following definition.

Definition 2.2. *By a Rieffel context, we mean a 5-tuple $(S, T, X, \langle\, , \,\rangle, [\, , \,])$, where*

- *S and T are rings with involution,*
- *X is an (S, T)-bimodule,*
- *$\langle\, , \,\rangle : X \times X \to S$ and $[\, , \,] : X \times X \to T$ are mappings, additive in both arguments, such that the following identities hold for $x, y, z \in X$, $s \in S$ and $t \in T$:*

RC1. $\langle sx, y \rangle = s \langle x, y \rangle$;
RC2. $\langle y, x \rangle = \langle x, y \rangle^*$;
RC3. $[x, yt] = [x, y]t$;
RC4. $[y, x] = [x, y]^*$;
RC5. $\langle x, y \rangle z = x[y, z]$.

We say that $(S, T, X, \langle\, , \,\rangle, [\, , \,])$ is a Rieffel context between rings S and T. Such a context is called full (cf. [3, page 243]) if every $s \in S$ can be written as $s = \sum_{k=1}^{n} \langle x_k, y_k \rangle$ for some $n \in \mathbb{N}$, $x_k, y_k \in X$, and analogously for $[\, , \,]$. (Note that a Rieffel context is full if and only if the mappings $\langle\, , \,\rangle$ and $[\, , \,]$ are

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pseudo-surjective [26, Definition 2.2].) It is called unitary (firm) if the bimodule ${}_SX_T$ is unitary (firm). Similarly to [3], we call ${}_SX_T$ an inner product bimodule.

In addition to the axioms listed in Definition 2.2, Rieffel contexts have some more calculation rules.

Lemma 2.3. *If $(S, T, X, \langle\ ,\ \rangle, [\ ,\ ])$ is a Rieffel context, then, for every $x, y \in X$, $s \in S$ and $t \in T$,*

RC6. $\langle x, y \rangle s = \langle x, s^* y \rangle$;
RC7. $t[x, y] = [xt^*, y]$.

If this context is full, then also

RC8. $\langle x, yt^* \rangle = \langle xt, y \rangle$;
RC9. $[s^* x, y] = [x, sy]$.

Proof. The proof of **RC6** and **RC7** is inspired by [24, Proposition 2.3], and the proof of **RC8** and **RC9** by [22, Lemma 6.12]. We have

$$\langle x, s^* y \rangle = \langle s^* y, x \rangle^* = (s^* \langle y, x \rangle)^* = \langle y, x \rangle^* s^{**} = \langle x, y \rangle s,$$

$$[xt^*, y] = [y, xt^*]^* = ([y, x]t^*)^* = t^{**}[y, x]^* = t[x, y].$$

If the mapping $\langle\ ,\ \rangle$ is full and $s \in S$, then there exist $n \in \mathbb{N}$ and $x_k, y_k \in X$ such that $s = \sum_{k=1}^n \langle x_k, y_k \rangle$. Now

$$\begin{aligned}
[s^* x, y] &= \left[\sum_{k=1}^n \langle y_k, x_k \rangle x, y\right] && (\textbf{RC2}) \\
&= \sum_{k=1}^n [\langle y_k, x_k \rangle x, y] && \text{(additivity)} \\
&= \sum_{k=1}^n [y_k[x_k, x], y] && (\textbf{RC5}) \\
&= \sum_{k=1}^n [y_k[x_k, x]^{**}, y] && \text{(by (2.1))} \\
&= \sum_{k=1}^n [x_k, x]^*[y_k, y] && (\textbf{RC7}) \\
&= \sum_{k=1}^n [x, x_k][y_k, y] && (\textbf{RC4}) \\
&= \sum_{k=1}^n [x, x_k[y_k, y]] && (\textbf{RC3}) \\
&= \sum_{k=1}^n [x, \langle x_k, y_k \rangle y] && (\textbf{RC5}) \\
&= \left[x, \sum_{k=1}^n \langle x_k, y_k \rangle y\right] && \text{(additivity)} \\
&= [x, sy],
\end{aligned}$$

so **RC9** holds. The proof of **RC8** is analogous. □

We give some natural examples of Rieffel contexts.

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Example 2.4. *Consider a ring S with involution as a bimodule ${}_SS_S$ and define*

$$\langle\, s, z \,\rangle := sz^*, \quad [s, z] := s^*z$$

for every $s, z \in S$. We obtain a Rieffel context $(S, S, S, \langle\, ,\, \rangle, [\, ,\,])$. If S is an idempotent ring, then this context is full and unitary.

Example 2.5. *Let R be a ring and consider matrix rings $S = \mathsf{M}_m(R)$ and $T = \mathsf{M}_n(R)$ with transposing as an involution operation. Then $X := \mathsf{M}_{m,n}(R)$ is an (S, T)-bimodule with respect to usual addition of matrices and actions defined by matrix multiplication. Defining*

$$\langle\, A, B \,\rangle := AB^T, \quad [A, B] := A^TB$$

we see easily that the required axioms are satisfied.

If we consider the relation of having a Rieffel context between two rings with involution, then it is clear that this relation is reflexive and symmetric. Next, we show that the relation of having a full and unitary Rieffel context is also transitive. Therefore, the notion of a Rieffel context allows us to consider an equivalence relation on the class of all idempotent rings with involution. But first let us prove a small, but useful, lemma.

Lemma 2.6. *Let R and S be rings, M_R a right R module and ${}_RN_S$ an (R, S)-bimodule such that N_S is unitary. Then the tensor product $M \otimes_R N$ is a unitary right S-module.*

Proof. We know that $M \otimes_R N$ is a right S-module with an obvious S-action. Let $\sum_{k=1}^{t} m_k \otimes n_k \in M \otimes_R N$ be arbitrary. Since N_S is unitary, for every $k \in \{0, \ldots, t\}$, there exist elements $n_{k1}, \ldots, n_{ku} \in N$ and $s_{k1}, \ldots, s_{ku} \in S$ such that $n_k = n_{k1}s_{k1} + \ldots + n_{ku}s_{ku}$. Now,

$$\sum_{k=1}^{t} m_k \otimes n_k = \sum_{k=1}^{n} m_k \otimes \left(\sum_{h=1}^{u} n_{kh}s_{kh} \right) = \sum_{k=1}^{n} \sum_{h=1}^{u} (m_k \otimes n_{kh})s_{kh} \in (M \otimes_R N)S.$$

Hence, $M \otimes_R N$ is a unitary right S-module. □

The next result is a ring theoretic analogue of [24, Proposition 2.5].

Proposition 2.7. *Let S, T, R be rings with involution. If $(S, T, X, \langle\, ,\, \rangle_1, [\, ,\,]_1)$ and $(T, R, Y, \langle\, ,\, \rangle_2, [\, ,\,]_2)$ are unitary and full Rieffel contexts, then there exists a unitary and full Rieffel context between S and R.*

Proof. Let $(S, T, {}_SX_T, \langle\, ,\, \rangle_1, [\, ,\,]_1)$ and $(T, R, {}_TY_R, \langle\, ,\, \rangle_2, [\, ,\,]_2)$ be unitary and full Rieffel contexts. Consider the (S, R)-bimodule $X \otimes_T Y$. Since ${}_SX$ and Y_R are unitary modules, the bimodule $X \otimes_T Y$ is also unitary by Lemma 2.6 and its dual. Define mappings

$$\langle\, ,\, \rangle : (X \otimes_T Y) \times (X \otimes_T Y) \to S, \; \left(\sum_{k=1}^{n} x_k \otimes y_k, \sum_{h=1}^{m} x_h{}' \otimes y_h{}' \right) \mapsto \sum_{k=1}^{n} \sum_{h=1}^{m} \langle x_k \langle y_k, y'_h \rangle_2, x_h{}' \rangle_1,$$

$$[\, ,\,] : (X \otimes_T Y) \times (X \otimes_T Y) \to R, \; \left(\sum_{k=1}^{n} x_k \otimes y_k, \sum_{h=1}^{m} x_h{}' \otimes y_h{}' \right) \mapsto \sum_{k=1}^{n} \sum_{h=1}^{m} [y_k, [x_k, x_h{}']_1 y_h{}']_2.$$

By using the universal property of the tensor product a few times, it can be shown that these mappings are well-defined homomorphisms of groups.

The mappings $\langle\, ,\, \rangle$ and $[\, ,\,]$ are full because $\langle\, ,\, \rangle_1, \langle\, ,\, \rangle_2, [\, ,\,]_1, [\, ,\,]_2$ are full and the modules X_T, ${}_TY$ are unitary.

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It suffices to verify the axioms **RC1**–**RC5** on the generators $x \otimes y$ of $X \otimes_T Y$. The axioms **RC1** and **RC3** are clearly satisfied. For every $x, x' \in X$ and $y, y' \in Y$, we have

$$\begin{aligned}
\langle x \otimes y, x' \otimes y' \rangle^* &= \langle x\langle y, y'\rangle_2, x'\rangle_1^* && \text{(def. of } \langle\ ,\ \rangle\text{)}\\
&= \langle x', x\langle y, y'\rangle_2\rangle_1 && \textbf{(RC2)}\\
&= \langle x'\langle y, y'\rangle_2^*, x\rangle_1 && \textbf{(RC8)}\\
&= \langle x'\langle y', y\rangle_2, x\rangle_1 && \textbf{(RC2)}\\
&= \langle x' \otimes y', x \otimes y\rangle. && \text{(def. of } \langle\ ,\ \rangle\text{)}
\end{aligned}$$

Hence, **RC2** holds. **RC4** holds for similar reasons. Furthermore, for every $x, x', x'' \in X$ and $y, y', y'' \in Y$,

$$\begin{aligned}
\langle x \otimes y, x' \otimes y'\rangle(x'' \otimes y'') = \langle x\langle y, y'\rangle_2, x'\rangle_1 x'' \otimes y'' &= x\langle y, y'\rangle_2[x', x'']_1 \otimes y''\\
= x \otimes \langle y, y'\rangle_2[x', x'']_1 y'' &= x \otimes y[y', [x', x'']_1 y'']_2\\
&= (x \otimes y)[x' \otimes y', x'' \otimes y''].
\end{aligned}$$

So **RC5** also holds. We have constructed a unitary and full Rieffel context $(S, R, X \otimes_T Y, \langle\ ,\ \rangle, [\ ,\ ])$. □

There is also the classical notion of Morita context for rings (see, e.g., [17]).

Definition 2.8. *A Morita context connecting two rings S and T is a six-tuple $(S, T, {}_SP_T, {}_TQ_S, \theta, \phi)$, where ${}_SP_T$, ${}_TQ_S$ are bimodules and $\theta : P \otimes_T Q \to {}_SS_S$, $\phi : Q \otimes_S P \to {}_TT_T$ are bimodule homomorphisms such that $\theta(p \otimes q)p' = p\phi(q \otimes p')$ and $\phi(q \otimes p)q' = q\theta(p \otimes q')$ for all $p, p' \in P$ and $q, q' \in Q$.*

Such a context is called unitary if ${}_SP_T$ and ${}_TQ_S$ are unitary. It is called surjective if θ and ϕ are surjective.

Let S and T be rings with involution. Given any (S, T)-bimodule ${}_SX_T$, one can construct its *dual bimodule* ${}_T\overline{X}_S$ as follows (see [22, Definition 6.17]). As an abelian group, $\overline{X}$ is the same as X. We write $\overline{x}$ when we consider an element $x \in X$ as a member of $\overline{X}$. So $\overline{x} + \overline{y} = \overline{x + y}$. The T- and S-actions on $\overline{X}$ are defined by

$$t\overline{x} := \overline{xt^*}, \quad \overline{x}s := \overline{s^*x}. \tag{2.2}$$

Definition 2.9. *By a Morita $*$-context connecting rings S and T with involution we mean a Morita context $(S, T, {}_SX_T, {}_T\overline{X}_S, \theta, \phi)$ satisfying*

$$\theta(x \otimes \overline{y})^* = \theta(y \otimes \overline{x}) \quad \text{and} \quad \phi(\overline{y} \otimes x)^* = \phi(\overline{x} \otimes y)$$

for all $x, y \in X$.

The following proposition shows that the conditions appearing in Definition 2.9 are natural.

Proposition 2.10. *Let $(S, T, {}_SX_T, {}_T\overline{X}_S, \theta, \phi)$ be a Morita $*$-context. Then the tensor product ring $X \otimes_T^\phi \overline{X}$ defined by ϕ is a ring with involution and $\theta : X \otimes_T^\phi \overline{X} \to S$ is a homomorphism of rings with involution.*

Proof. Recall that the multiplication on $X \otimes_T^\phi \overline{X}$ is given by

$$(x \otimes \overline{y})(x_1 \otimes \overline{y_1}) := x \otimes \phi(\overline{y} \otimes x_1)\overline{y_1} = x \otimes \overline{y_1\phi(\overline{y} \otimes x_1)^*} = x \otimes \overline{y_1\phi(\overline{x_1} \otimes y)}.$$

By Theorem 2.11 in [26], the mapping θ is a homomorphism of rings. We define an involution on the ring $X \otimes_T^\phi \overline{X}$ by

$$(x \otimes \overline{y})^* := y \otimes \overline{x}.$$

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For every $x, x_1 \in X$ and $\overline{y}, \overline{y_1} \in \overline{X}$, we have

$$((x \otimes \overline{y})(x_1 \otimes \overline{y_1}))^* = (x \otimes \overline{y_1 \phi(\overline{x_1} \otimes y)})^* = y_1 \phi(\overline{x_1} \otimes y) \otimes \overline{x} = y_1 \otimes \phi(\overline{x_1} \otimes y)\overline{x}$$

$$= (y_1 \otimes \overline{x_1})(y \otimes \overline{x}) = (x_1 \otimes \overline{y_1})^*(x \otimes \overline{y})^*,$$

$$\theta((x \otimes \overline{y})^*) = \theta(y \otimes \overline{x}) = \theta(x \otimes \overline{y})^*.$$

Hence, we see that $X \otimes_T^\phi \overline{X}$ is indeed a ring with involution and θ is a homomorphism of rings with involution. □

Remark 2.11. *Note that the bimodules ${}_SX_T$ and ${}_T\overline{X}_S$ actually form a natural example of something that might be called a 4-module. Indeed, let S, R, T, P be rings. An abelian group X could be called a (S, R, T, P) 4-module, if there exist four actions*

$$S \times X \to X, \quad (s, x) \mapsto sx, \qquad X \times T \to X, \quad (x, t) \mapsto xt,$$

$$R \times X \to X, \quad (r, x) \mapsto rx, \qquad X \times P \to X, \quad (x, p) \mapsto xp$$

such that they induce bimodules ${}_SX_T$, ${}_SX_P$, ${}_RX_T$, ${}_RX_P$ and the conditions

$$s(rx) = r(sx), \qquad (xt)p = (xp)t$$

hold for every $x \in X$, $r \in R$, $s \in S$, $t \in T$ and $p \in P$. We believe that such structures merit studying on their own and hope it will be done some day.

Next, we adopt the notion of an enlargement to the case of rings with involution.

Definition 2.12. *Let R be a ring with an involution $\star$ and let S and T be rings with involution, which is denoted in both case by $*$. We say that R is a joint enlargement of S and T if there exist subrings S' and T' of R and ring isomorphisms $f : S \to S'$ and $g : T \to T'$ such that*

1. *S' and T' are closed with respect to $\star$;*
2. *$f(s^*) = f(s)^\star$ for every $s \in S$ and $g(t^*) = g(t)^\star$ for every $t \in T$;*
3. *$R = RS'R$, $S' = S'RS'$, $R = RT'R$, $T' = T'RT'$.*

In particular, S' and T' are rings with involution, whose involution is the restriction of $\star$ to S' and T', respectively. Note also that if S or T is idempotent, then R is also idempotent by [9, Proposition 2.2].

Lemma 2.13. *If R is a joint enlargement of S and T as in Definition 2.12, then $f^{-1}(u^\star) = (f^{-1}(u))^*$ for every $u \in S'$ and $g^{-1}(v^\star) = (g^{-1}(v))^*$ for every $v \in T'$.*

Proof. Denote $s := f^{-1}(u)$. Then we have $f(s^*) = f(s)^\star = (f(f^{-1}(u)))^\star = u^\star$, which implies $(f^{-1}(u))^* = s^* = f^{-1}(u^\star)$. A similar argument works for g^{-1}. □

3. Main theorem

Our main theorem is the following one. It can be compared with [12, Theorem 1.1] for semigroups with local units and with [9, Theorem 3.3] for idempotent rings.

Theorem 3.1. *Let S and T be idempotent rings with involution. Then the following conditions are equivalent.*

1. *S and T are connected by a firm and full Rieffel context.*
2. *S and T are connected by a unitary and full Rieffel context.*

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3. *S and T are connected by a unitary and surjective Morita $*$-context.*
4. *S and T have a joint enlargement.*

Proof. (1) $\implies$ (2). Every firm bimodule is unitary.

(2) $\implies$ (3). Assume that $(S, T, X, \langle\, , \,\rangle, [\, , \,])$ is a unitary and full Rieffel context. We are going to build a Morita $*$-context using the bimodules ${}_SX_T$ and ${}_T\overline{X}_S$. Consider the mappings

$$\hat{\theta} : X \times \overline{X} \to S, \quad (x, \overline{y}) \mapsto \langle\, x, y \,\rangle\,,$$

$$\hat{\phi} : \overline{X} \times X \to T, \quad (\overline{y}, x) \mapsto [y, x].$$

We note that the mapping $\hat{\theta}$ is T-balanced, because it is additive in both arguments and, for all $x, y \in X$ and $t \in T$,

$$\begin{aligned} \hat{\theta}(xt, \overline{y}) &= \langle\, xt, y \,\rangle && \text{(def. of } \hat{\theta}) \\ &= \langle\, x, yt^* \,\rangle && (\mathbf{RC8}) \\ &= \hat{\theta}(x, \overline{yt^*}) && \text{(def. of } \hat{\theta}) \\ &= \hat{\theta}(x, t\overline{y}). && \text{by (2.2)} \end{aligned}$$

Similarly $\hat{\phi}$ is S-balanced. By the universal property of tensor product there exist abelian group homomorphisms $\theta : X \otimes_T \overline{X} \to S$ and $\phi : \overline{X} \otimes_S X \to T$ such that

$$\theta(x \otimes \overline{y}) = \langle\, x, y \,\rangle \quad \text{and} \quad \phi(\overline{y} \otimes x) = [y, x] \tag{3.1}$$

for all $x, y \in X$. Condition **RC1** implies that θ preserves left S-action. It also preserves right S-action, because

$$\begin{aligned} \theta(x \otimes \overline{y}s) &= \theta(x \otimes \overline{s^*y}) && \text{(by (2.2))} \\ &= \langle\, x, s^*y \,\rangle && \text{(by (3.1))} \\ &= \langle\, x, y \,\rangle\, s && (\mathbf{RC6}) \\ &= \theta(x \otimes \overline{y})s && \text{by (3.1)} \end{aligned}$$

for every $x, y \in X$ and $s \in S$. Thus, θ (and similarly ϕ) is a homomorphism of bimodules. These mappings satisfy compatibility conditions, because

$$\begin{aligned} \theta(x \otimes \overline{y})x' &= \langle\, x, y \,\rangle\, x' && \text{by (3.1)} \\ &= x[y, x'] && (\mathbf{RC5}) \\ &= x\phi(\overline{y} \otimes x') && \text{by (3.1)} \end{aligned}$$

and

$$\begin{aligned} \phi(\overline{y} \otimes x)\overline{y'} &= [y, x]\overline{y'} && \text{(by (3.1))} \\ &= \overline{y'[y, x]^*} && \text{(by (2.2))} \\ &= \overline{y'[x, y]} && (\mathbf{RC4}) \\ &= \overline{\langle\, y', x \,\rangle\, y} && (\mathbf{RC5}) \\ &= \overline{y}\,\langle\, y', x \,\rangle^* && \text{(by (2.2))} \\ &= \overline{y}\,\langle\, x, y' \,\rangle && (\mathbf{RC2}) \\ &= \overline{y}\theta(x \otimes \overline{y'}). && \text{by (3.1)} \end{aligned}$$

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If $s \in S$, then there exist $n \in \mathbb{N}$, $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$ such that

$$s = \sum_{k=1}^{n} \langle x_k, y_k \rangle = \sum_{k=1}^{n} \theta(x_k \otimes \overline{y_k}) = \theta\left(\sum_{k=1}^{n} x_k \otimes y_k\right).$$

Hence, θ (and analogously ϕ) is surjective. The bimodule ${}_SX_T$ is unitary by assumption. Let $\overline{x} \in \overline{X}$. Since ${}_SX$ is unitary, there exist $n \in \mathbb{N}$, $s_1, \ldots, s_n \in S$ and $x_1, \ldots, x_n \in X$ such that $x = \sum_{k=1}^{n} s_k x_k$. Hence,

$$\overline{x} = \overline{\sum_{k=1}^{n} s_k x_k} = \sum_{k=1}^{n} \overline{s_k x_k} = \sum_{k=1}^{n} \overline{x_k} s_k^*.$$

We have shown that $\overline{X}_S$ is unitary. Analogously we see that ${}_T\overline{X}$ is unitary. Finally, we notice that, for every $x, y \in X$,

$$\theta(x \otimes \overline{y})^* = \langle x, y \rangle^* = \langle y, x \rangle = \theta(y \otimes \overline{x}),$$

$$\phi(\overline{y} \otimes x)^* = [y, x]^* = [x, y] = \phi(\overline{x} \otimes y).$$

(3) $\Longrightarrow$ (4). Let $(S, T, {}_SX_T, {}_T\overline{X}_S, \theta, \phi)$ be a unitary surjective Morita $*$-context. We consider the matrix set

$$R = \left\{ \begin{pmatrix} s & x \\ \overline{y} & t \end{pmatrix} \middle| \; s \in S, t \in T, x, y \in X \right\}$$

with componentwise addition and with the multiplication

$$\begin{pmatrix} s & x \\ \overline{y} & t \end{pmatrix} \begin{pmatrix} s_1 & x_1 \\ \overline{y_1} & t_1 \end{pmatrix} := \begin{pmatrix} ss_1 + \theta(x \otimes \overline{y_1}) & sx_1 + xt_1 \\ \overline{y}s_1 + t\overline{y_1} & \phi(\overline{y} \otimes x_1) + tt_1 \end{pmatrix} = \begin{pmatrix} ss_1 + \theta(x \otimes \overline{y_1}) & sx_1 + xt_1 \\ \overline{s_1^* y + y_1 t^*} & \phi(\overline{y} \otimes x_1) + tt_1 \end{pmatrix}.$$

The unary $\star$-operation on R is defined as

$$\begin{pmatrix} s & x \\ \overline{y} & t \end{pmatrix}^{\star} := \begin{pmatrix} s^* & y \\ \overline{x} & t^* \end{pmatrix}.$$

It is well known (see, e.g., [19, Section 2]) that R with these operations is a ring (called a Morita ring of a Morita context). Two things remain to verify.

a) The operation $\star$ is an involution on R, because

$$\left(\begin{pmatrix} s & x \\ \overline{y} & t \end{pmatrix} + \begin{pmatrix} s_1 & x_1 \\ \overline{y_1} & t_1 \end{pmatrix}\right)^{\star} = \begin{pmatrix} s + s_1 & x + x_1 \\ \overline{y + y_1} & t + t_1 \end{pmatrix}^{\star} = \begin{pmatrix} s^* + s_1^* & y + y_1 \\ \overline{x + x_1} & t^* + t_1^* \end{pmatrix} = \begin{pmatrix} s^* & y \\ \overline{x} & t^* \end{pmatrix} + \begin{pmatrix} s_1^* & y_1 \\ \overline{x_1} & t_1^* \end{pmatrix} = \begin{pmatrix} s & x \\ \overline{y} & t \end{pmatrix}^{\star} + \begin{pmatrix} s_1 & x_1 \\ \overline{y_1} & t_1 \end{pmatrix}^{\star},$$

$$\begin{pmatrix} s & x \\ \overline{y} & t \end{pmatrix}^{\star\star} = \begin{pmatrix} s^* & y \\ \overline{x} & t^* \end{pmatrix}^{\star} = \begin{pmatrix} s^{**} & x \\ \overline{y} & t^{**} \end{pmatrix} = \begin{pmatrix} s & x \\ \overline{y} & t \end{pmatrix},$$

$$\begin{aligned}
\left(\begin{pmatrix} s & x \\ \overline{y} & t \end{pmatrix} \begin{pmatrix} s_1 & x_1 \\ \overline{y_1} & t_1 \end{pmatrix}\right)^{\star} &= \begin{pmatrix} ss_1 + \theta(x \otimes \overline{y_1}) & sx_1 + xt_1 \\ \overline{s_1^* y + y_1 t^*} & \phi(\overline{y} \otimes x_1) + tt_1 \end{pmatrix}^{\star} \\
&= \begin{pmatrix} (ss_1)^* + \theta(x \otimes \overline{y_1})^* & s_1^* y + y_1 t^* \\ \overline{sx_1 + xt_1} & \phi(\overline{y} \otimes x_1)^* + (tt_1)^* \end{pmatrix} \\
&= \begin{pmatrix} s_1^* s^* + \theta(y_1 \otimes \overline{x}) & s_1^* y + y_1 t^* \\ \overline{s^{**} x_1 + x t_1^{**}} & \phi(\overline{x_1} \otimes y) + t_1^* t^* \end{pmatrix} = \begin{pmatrix} s_1^* & y_1 \\ \overline{x_1} & t_1^* \end{pmatrix} \begin{pmatrix} s^* & y \\ \overline{x} & t^* \end{pmatrix} = \begin{pmatrix} s_1 & x_1 \\ \overline{y_1} & t_1 \end{pmatrix}^{\star} \begin{pmatrix} s & x \\ \overline{y} & t \end{pmatrix}^{\star}.
\end{aligned}$$

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b) We will verify the conditions in Definition 2.12, that are related to S. Similar arguments will apply for T. The set

$$S' = \left\{ \begin{pmatrix} s & 0_X \\ \overline{0_X} & 0_T \end{pmatrix} \middle| \, s \in S \right\}$$

is a subring of R, which is also closed under $\star$-operation (for the latter we need that $0_T^* = 0_T$). It is easy to see that the mapping

$$f : S \to S', \quad s \mapsto \begin{pmatrix} s & 0_X \\ \overline{0_X} & 0_T \end{pmatrix}$$

is an isomorphism of rings. Moreover, it preserves the involution, because

$$f(s^*) = \begin{pmatrix} s^* & 0_X \\ \overline{0_X} & 0_T \end{pmatrix} = \begin{pmatrix} s^* & 0_X \\ \overline{0_X} & 0_T^* \end{pmatrix} = \begin{pmatrix} s & 0_X \\ \overline{0_X} & 0_T \end{pmatrix}^\star = f(s)^\star.$$

The equalities $R = RS'R$ and $S' = S'RS'$ hold by [9, Proposition 3.5].

(4) $\implies$ (1). Assume that R is a joint enlargement of S and T. It suffices to consider the case, where S and T are subrings of R. In that case $s^\star = s^*$ and $t^\star = t^*$ for every $s \in S$ and $t \in T$. We consider the bimodule

$${}_SX_T := S \otimes_S SRT \otimes_T T$$

with natural actions of S and T. By [15, Proposition 2.5], the bimodule ${}_SX_T$ is firm, because the rings S and T are idempotent. Consider the mapping

$$\hat{\sigma} : S \times SRT \times T \to R, \;\; (s, \rho, t) \mapsto t^* \rho^\star s^*.$$

Note that

$$\hat{\sigma}(ss_1, \rho, t) = t^* \rho^\star (ss_1)^* = t^* \rho^\star s_1^* s^* = t^* \rho^\star s_1^\star s^* = t^*(s_1\rho)^\star s^* = \hat{\sigma}(s, s_1\rho, t),$$

$$\hat{\sigma}(s, \rho t_1, t) = t^*(\rho t_1)^\star s^* = t^* t_1^\star \rho^\star s^* = t^* t_1^* \rho^\star s^* = (t_1 t)^* \rho^\star s^* = \hat{\sigma}(s, \rho, t_1 t)$$

and also $\hat{\sigma}$ is additive in all three arguments. Consequently, there exists a group homomorphism $\sigma : S \otimes_S SRT \otimes_T T \to R$ such that $\sigma(s \otimes \rho \otimes t) = t^* \rho^\star s^*$ for every $s \in S$, $\rho \in SRT$ and $t \in T$. Note that if $\rho = \sum_{k=1}^n s_k r_k t_k \in SRT$, then

$$\rho^\star = \left(\sum_{k=1}^n s_k r_k t_k \right)^\star = \sum_{k=1}^n (s_k r_k t_k)^\star = \sum_{k=1}^n t_k^* r_k^\star s_k^* \in TRS.$$

A little bit shorter argument shows that there also exists a group homomorphism $\tau : S \otimes_S SRT \otimes_T T \to R$ such that $\tau(s \otimes \rho \otimes t) = s\rho t$ for every $s \in S$, $\rho \in SRT$ and $t \in T$. We define

$$\langle x, y \rangle := \tau(x)\sigma(y), \qquad [x, y] := \sigma(x)\tau(y)$$

for every $x, y \in X$. In particular, on elementary tensors we have

$$\langle s \otimes \rho \otimes t, s_1 \otimes \rho_1 \otimes t_1 \rangle := s\rho t t_1^* \rho_1^\star s_1^* \in SSRTTTTRSS = SRTRS = SRS = S,$$

$$[s \otimes \rho \otimes t, s_1 \otimes \rho_1 \otimes t_1] := t^* \rho^\star s^* s_1 \rho_1 t_1 \in TTRSSSSRTT = TRSRT = TRT = T,$$

which implies that $\langle x, y \rangle \in S$ and $[x, y] \in T$ for every $x, y \in X$, because every element of X is a finite sum of elementary tensors. Thus we have defined mappings

$$\langle \, , \, \rangle : X \times X \to S \;\text{ and }\; [\, , \,] : X \times X \to T,$$

which are additive in both arguments, because τ and σ preserve addition and S and T have the distributivity law.

To prove that $\langle\ \rangle$ is full, we consider an element $z \in S$. Since $S = S(SRT)TT(TRS)S$, z is a finite sum of products of the form $s\rho tt_1\rho_1 s_1$, where $s, s_1 \in S$, $t, t_1 \in T$, $\rho \in SRT$ and $\rho_1 \in TRS$. It suffices to show that each such product has a preimage with respect to $\langle\, ,\, \rangle$. Indeed,

$$s\rho tt_1\rho_1 s_1 = s\rho t(t_1^*)^*(\rho_1^\star)^\star(s_1^*)^* = \left\langle s\otimes\rho\otimes t, s_1^*\otimes\rho_1^\star\otimes t_1^*\right\rangle.$$

A similar proof shows that $[\, ,\,]$ is full. Finally, we verify the conditions **RC1**–**RC5**.

RC1. For every $z \in S$,

$$\langle\, z(s\otimes\rho\otimes t), s_1\otimes\rho_1\otimes t_1\,\rangle = zs\rho tt_1^*\rho_1^\star s_1^* = z\,\langle\, s\otimes\rho\otimes t, s_1\otimes\rho_1\otimes t_1\,\rangle\,.$$

RC2. We calculate

$$\begin{aligned}
\langle\, s\otimes\rho\otimes t, s_1\otimes\rho_1\otimes t_1\,\rangle^* &= (s\rho tt_1^*\rho_1^\star s_1^*)^* && (\text{def. of } \langle\, ,\, \rangle)\\
&= (s\rho tt_1^\star\rho_1^\star s_1^\star)^\star && (t_1^* = t_1^\star,\ s_1^* = s_1^\star)\\
&= s_1^{\star\star}\rho_1^{\star\star}t_1^{\star\star}t^\star\rho^\star s^\star && ((ab)^\star = b^\star a^\star)\\
&= s_1\rho_1 t_1 t^\star\rho^\star s^\star && (a^{\star\star} = a)\\
&= s_1\rho_1 t_1 t^*\rho^\star s^* && (t^\star = t^*,\ s^\star = s^*)\\
&= \langle\, s_1\otimes\rho_1\otimes t_1, s\otimes\rho\otimes t\,\rangle\,. && (\text{def. of } \langle\, ,\, \rangle)
\end{aligned}$$

RC3. Analogous to **RC1.**
RC4. Analogous to **RC2.**
RC5. To prove this, we need to use that S and T are idempotent rings. We calculate

$$\begin{aligned}
\langle\, s\otimes\rho\otimes t, s_1\otimes\rho_1\otimes t_1\,\rangle\,(s_2\otimes\rho_2\otimes t_2) &= s\rho tt_1^*\rho_1^\star s_1^*(s_2\otimes\rho_2\otimes t_2) && (\text{def. of } \langle\, ,\, \rangle)\\
&= s\rho tt_1^*\rho_1^\star s_1^* s_2\otimes\rho_2\otimes t_2 && (\text{def. of } S\text{-action})\\
&= \left(\sum_{k=1}^n u_k u'_k\right)\rho tt_1^*\rho_1^\star s_1^* s_2\otimes\rho_2\otimes t_2 && (s = \textstyle\sum_{k=1}^n u_k u'_k)\\
&= \left(\sum_{k=1}^n u_k u'_k\rho tt_1^*\rho_1^\star s_1^* s_2\right)\otimes\rho_2\otimes t_2 && (\text{distributivity})\\
&= \sum_{k=1}^n (u_k u'_k\rho tt_1^*\rho_1^\star s_1^* s_2\otimes\rho_2\otimes t_2) && (\text{property of } \otimes)\\
&= \sum_{k=1}^n (u_k\otimes u'_k\rho tt_1^*\rho_1^\star s_1^* s_2\rho_2\otimes t_2) && (S(SRT)T(TRS)S{=}S)\\
&= \sum_{k=1}^n (u_k u'_k\otimes\rho tt_1^*\rho_1^\star s_1^* s_2\rho_2\otimes t_2) && (u'_k \in S)\\
&= \left(\sum_{k=1}^n u_k u'_k\right)\otimes\rho tt_1^*\rho_1^\star s_1^* s_2\rho_2\otimes t_2 && (\text{property of } \otimes)\\
&= s\otimes\rho tt_1^*\rho_1^\star s_1^* s_2\rho_2\otimes t_2. && (s = \textstyle\sum_{k=1}^n u_k u'_k)
\end{aligned}$$

A similar calculation shows that

$$(s\otimes\rho\otimes t)[s_1\otimes\rho_1\otimes t_1, s_2\otimes\rho_2\otimes t_2] = (s\otimes\rho\otimes t)t_1^*\rho_1^\star s_1^* s_2\rho_2 t_2 = s\otimes\rho tt_1^*\rho_1^\star s_1^* s_2\rho_2\otimes t_2.$$

This completes the proof. □

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It would be natural to call two idempotent rings with involution Morita equivalent if they satisfy the conditions of Theorem 3.1.

Remark 3.2. *The only place in the proof of Theorem 3.1, where we used the existence of an additive inverse in a ring is the fact that $0^* = 0$ (see Definition 2.1). Hence, if we define a semiring with involution by requiring identities (2.1) and $0^* = 0$, and modify the other definitions in an obvious way, then our proof shows that two idempotent semirings with involution are connected by a unitary full Rieffel context if and only if they have a joint enlargement.*

Example 3.3. *Consider again the Rieffel context from Example 2.5, but let R be an idempotent ring.*

- *The (S, T)-bimodule ${}_SX_T = \mathsf{M}_{m,n}(R)$, where $S = \mathsf{M}_m(R)$ and $T = \mathsf{M}_n(R)$, is unitary.*
 Indeed, let $D^{mn}_{kh}(r)$ denote a $(m \times n)$-matrix, which has r in position (k, h) and zeros everywhere else. Note that every matrix $[r_{kh}]^{m,n}_{k,h=1} \in X$ can be expressed as follows:

$$[r_{kh}] = \sum_{k=1}^{m}\sum_{h=1}^{n} D^{mn}_{kh}(r_{kh}) = \sum_{k=1}^{m}\sum_{h=1}^{n} D^{mn}_{kh}\left(\sum_{j=1}^{t} r_{khj}r_{khj}{}'\right)$$

$$= \sum_{k=1}^{m}\sum_{h=1}^{n}\sum_{j=1}^{t} D^{mn}_{kh}\left(r_{khj}r_{khj}{}'\right) = \sum_{k=1}^{m}\sum_{h=1}^{n}\sum_{j=1}^{t} D^{mn}_{kh}\left(r_{khj}\right) D^{nn}_{hh}\left(r_{khj}{}'\right) \in XT,$$

 where $r_{kh} = \sum_{j=1}^{t} r_{khj}r_{khj}{}'$ holds for every $k \in \{1, \ldots, m\}$ and $h \in \{1, \ldots, n\}$ due to R being idempotent. This shows that X_T is unitary. The unitarity of ${}_SX$ is analogous.
- *The rings $S = \mathsf{M}_m(R)$ and $T = \mathsf{M}_n(R)$ are idempotent.*
 The proof of this claim is similar to the previous part if we take $m = n$.
- *The mappings $\langle\, ,\, \rangle : (A, B) \mapsto AB^T$ and $[\, ,\,] : (A, B) \mapsto A^TB$ are full.*
 Indeed, by Lemma 5.6 in [26] we have that $\mathsf{M}_{m,n}(R) = \mathsf{M}_{m,1}(R)\mathsf{M}_{1,n}(R)$ holds for every m and n. Let $Y \subseteq X$ be a subset that consist of matrices which have non-zero elements only in the first row and $Z \subseteq X$ a subset of matrices with non-zero elements only in the first column. Now we see that the restrictions $\langle\, ,\, \rangle|_{Y\times Y}$ and $[\, ,\,]|_{Z\times Z}$ are full. Hence, the mappings $\langle\, ,\, \rangle$ and $[\, ,\,]$ are also full.
- *We conclude that $\mathsf{M}_m(R)$ and $\mathsf{M}_n(R)$ are Morita equivalent for every $m, n \in \mathbb{N}$.*

Remark 3.4. *Consider the implication (4) $\Longrightarrow$ (2) in Theorem 3.1. Assume that R is a joint enlargement of S and T as in Definition 2.12, with isomorphisms*

$$f : S \to S', s \mapsto \hat{s}, \quad g : T \to T', t \mapsto \hat{t}.$$

Then, it turns out that there is a simpler construction yielding a unitary full Rieffel context between S and T. We consider the bimodule

$${}_SX_T := S'RT' = \left\{ \sum_{k=1}^{n} \hat{s}_k r_k \hat{t}_k \,\middle|\, n \in \mathbb{N}, s_k \in S, r_k \in R, t_k \in T \right\} \subseteq R.$$

Addition of X is the restriction of the addition of R. The actions of S and T on X are defined by

$$s \cdot x := \hat{s}x, \quad x \cdot t := x\hat{t},$$

where $\hat{s}x$ is the product of $\hat{s}$ and x in R and $x\hat{t}$ is the product of x and $\hat{t}$ in R. It is straightforward to show that ${}_SX_T$ is an (S, T)-bimodule. Since S and T are idempotent rings, the bimodule ${}_SX_T$ is unitary.

Note that

$$(\hat{s}r\hat{t})(\hat{s}_1 r_1 \hat{t}_1)^\star = \hat{s}r\hat{t}(\hat{t}_1)^\star r_1^\star (\hat{s}_1)^\star = \hat{s}r\hat{t}(\widehat{t_1^*}) r_1^\star (\widehat{s_1^*}) \in S'RT'T'RS' = S'RT'RS' = S',$$

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and, analogously, $(\hat{s}r\hat{t})^\star(\hat{s}_1 r_1 \hat{t}_1) \in T'$. *Hence, we can define the mappings*

$$\langle\ ,\ \rangle : X \times X \to S \text{ and } [\ ,\ ] : X \times X \to T,$$

by

$$\langle\, x, y\,\rangle := f^{-1}(xy^\star),\ \ [x, y] := g^{-1}(x^\star y),$$

for every $x, y \in X$. *These mappings are additive in both arguments, because* f^{-1} *and* g^{-1} *are ring isomorphisms.*

To prove that $\langle\ ,\ \rangle$ *is full, we consider an element* $z \in S$. *Since* $S' = S'RT'T'RS'$, $\hat{z} \in S'$ *is a finite sum of products of the form* $\hat{s}r\hat{t}\hat{t}_1 r_1 \hat{s}_1$, *where* $s, s_1 \in S$, $t, t_1 \in T$, $r, r_1 \in R$. *Now*

$$f^{-1}(\hat{s}r\hat{t}\hat{t}_1 r_1 \hat{s}_1) = f^{-1}(\hat{s}r\hat{t}(\hat{t}_1)^{\star\star} r_1^{\star\star}(\hat{s}_1)^{\star\star}) = f^{-1}(\hat{s}r\hat{t}((\hat{s}_1)^\star r_1^\star(\hat{t}_1)^\star)^\star) = \left\langle\, \hat{s}r\hat{t}, (\hat{s}_1)^\star r_1^\star(\hat{t}_1)^\star \,\right\rangle.$$

It follows that $z = f^{-1}(\hat{z}) = \sum_{k=1}^{n} \langle\, x_k, y_k\,\rangle$, *where* $n \in \mathbb{N}$ *and* $x_k, y_k \in X$. *A similar proof shows that* $[\ ,\ ]$ *is full. It is easy to verify the conditions* ***RC1–RC5***.

We will say that $(S, T, S'RT', \langle\ ,\ \rangle, [\ ,\ ])$ *is the Rieffel context induced by a joint enlargement* R *of* S *and* T.

4. Rieffel contexts come from enlargements

In this section, we will show that, up to isomorphism, every unitary Rieffel context is induced by a joint enlargement. Isomorphism of classical Morita contexts is defined in [17]. Its analogue for Rieffel's contexts is the following.

Definition 4.1. *Let* S *and* T *be rings with involution. We say that a Rieffel context* $(S, T, X, \langle\ ,\ \rangle_X, [\ ,\ ]_X)$ *is isomorphic to a Rieffel context* $(S, T, Y, \langle\ ,\ \rangle_Y, [\ ,\ ]_Y)$ *if there exists a bimodule isomorphism* $h : X \to Y$ *such that, for every* $x_1, x_2 \in X$,

$$\langle\, h(x_1), h(x_2)\,\rangle_Y = \langle\, x_1, x_2\,\rangle_X \quad \text{and} \quad [h(x_1), h(x_2)]_Y = [x_1, x_2]_Y.$$

Theorem 4.2. *Every unitary Rieffel context connecting idempotent rings* S *and* T *with involution is isomorphic to a Rieffel context induced by a joint enlargement of* S *and* T.

Proof. Let $(S, T, X, \langle\ ,\ \rangle_X, [\ ,\ ]_X)$ be a unitary Rieffel context. Putting together implications $(2 \Longrightarrow 3)$ and $(3 \Longrightarrow 4)$ in Theorem 3.1 we see that it gives rise to a joint enlargement R of S and T. More precisely,

$$R = \left\{ \begin{pmatrix} s & x \\ \overline{y} & t \end{pmatrix} \middle|\ s \in S, t \in T, x, y \in X \right\}$$

is a ring with componentwise addition and with the multiplication

$$\begin{pmatrix} s & x \\ \overline{y} & t \end{pmatrix} \begin{pmatrix} s_1 & x_1 \\ \overline{y_1} & t_1 \end{pmatrix} := \begin{pmatrix} ss_1 + \langle\, x, y_1\,\rangle & sx_1 + xt_1 \\ \overline{y}s_1 + t\overline{y_1} & [y, x_1] + tt_1 \end{pmatrix} = \begin{pmatrix} ss_1 + \langle\, x, y_1\,\rangle & sx_1 + xt_1 \\ \overline{s_1^* y + y_1 t^*} & [y, x_1] + tt_1 \end{pmatrix}.$$

The unary $\star$-operation on R is defined by

$$\begin{pmatrix} s & x \\ \overline{y} & t \end{pmatrix}^\star := \begin{pmatrix} s^* & y \\ \overline{x} & t^* \end{pmatrix}.$$

The mappings

$$f : S \to R,\ \ s \mapsto \begin{pmatrix} s & 0_X \\ \overline{0_X} & 0_T \end{pmatrix} \quad \text{and} \quad g : T \to R,\ \ t \mapsto \begin{pmatrix} 0_S & 0_X \\ \overline{0_X} & t \end{pmatrix}$$

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are injective ring homomorphisms and the subrings

$$S' = f(S) = \left\{ \begin{pmatrix} s & 0_X \\ \overline{0_X} & 0_T \end{pmatrix} \,\middle|\, s \in S \right\} \quad \text{and} \quad T' = g(T) = \left\{ \begin{pmatrix} 0_S & 0_X \\ \overline{0_X} & t \end{pmatrix} \,\middle|\, t \in T \right\}$$

of R satisfy the equalities $S' = S'RS'$, $R = RS'R$, $T' = T'RT'$ and $R = RT'R$. Let the Rieffel context induced by the joint enlargement R be $(S, T, Y, \langle\, ,\, \rangle_Y, [\, ,\,]_Y)$. By Remark 3.4, this means that ${}_SY_T = S'RT'$, $\langle\, y_1, y_2\, \rangle_Y = f^{-1}(y_1y_2^\star)$ and $[y_1, y_2]_Y = g^{-1}(y_1^\star y_2)$. Observe that

$$\begin{pmatrix} s' & 0_X \\ \overline{0_X} & 0_T \end{pmatrix} \begin{pmatrix} s & x \\ \overline{y} & t \end{pmatrix} \begin{pmatrix} 0_S & 0_X \\ \overline{0_X} & t' \end{pmatrix} = \begin{pmatrix} s's & s'x \\ \overline{0_X} & 0_T \end{pmatrix} \begin{pmatrix} 0_S & 0_X \\ \overline{0_X} & t' \end{pmatrix} = \begin{pmatrix} 0_S & s'xt' \\ \overline{0_X} & 0_T \end{pmatrix}$$

for every $s, s' \in S$, $t, t' \in T$ and $x \in X$. Hence,

$${}_SY_T = \left\{ \sum_{k=1}^{n} \begin{pmatrix} 0_S & s_kx_kt_k \\ \overline{0_X} & 0_T \end{pmatrix} \,\middle|\, n \in \mathbb{N}, s_k \in S, x_k \in X, t_k \in T \right\} = \left\{ \begin{pmatrix} 0_S & x \\ \overline{0_X} & 0_T \end{pmatrix} \,\middle|\, x \in X \right\},$$

where the last equality holds because ${}_SX_T$ is a unitary bimodule.

We define a mapping $h : X \to Y$ by

$$h(x) := \begin{pmatrix} 0_S & x \\ \overline{0_X} & 0_T \end{pmatrix}.$$

Then, h is clearly an isomorphism of abelian groups. Note that

$$s \cdot h(x) = s \cdot \begin{pmatrix} 0_S & x \\ \overline{0_X} & 0_T \end{pmatrix} = \begin{pmatrix} s & 0_X \\ \overline{0_X} & 0_T \end{pmatrix} \begin{pmatrix} 0_S & x \\ \overline{0_X} & 0_T \end{pmatrix} = \begin{pmatrix} 0_S & sx \\ \overline{0_X} & 0_T \end{pmatrix} = h(sx),$$

$$h(x) \cdot t = \begin{pmatrix} 0_S & x \\ \overline{0_X} & 0_T \end{pmatrix} \begin{pmatrix} 0_S & 0_X \\ \overline{0_X} & t \end{pmatrix} = \begin{pmatrix} 0_S & xt \\ \overline{0_X} & 0_T \end{pmatrix} = h(xt)$$

for every $x \in X$, $s \in S$ and $t \in T$. Hence, h is an isomorphism of (S, T)-bimodules.

Finally, if $x_1, x_2 \in X$, then

$$\langle\, h(x_1), h(x_2)\, \rangle_Y = f^{-1}(h(x_1)h(x_2)^\star) = f^{-1}\left(\begin{pmatrix} 0_S & x_1 \\ \overline{0_X} & 0_T \end{pmatrix} \begin{pmatrix} 0_S & 0_X \\ \overline{x_2} & 0_T \end{pmatrix} \right)$$

$$= f^{-1}\left(\begin{pmatrix} \langle\, x_1, x_2\, \rangle_X & 0_X \\ \overline{0_X} & 0_T \end{pmatrix} \right) = \langle\, x_1, x_2\, \rangle_X\,,$$

and, analogously, $[h(x_1), h(x_2)]_Y = [x_1, x_2]_Y$. □

5. From a Rieffel context to an equivalence of module categories

In this section, we study relationships between Rieffel contexts and equivalences between module categories. When moving from unital rings to bigger classes of rings, one has to choose which categories of modules to use in order to develop Morita theory (see the Introduction of [5]). There are three natural choices proposed in the literature: the categories of a) unitary torsion-free modules (see, e.g., [3, 7, 18]), b) closed modules, c) firm modules ([5, 6]). In the case of an idempotent ring, all three categories are equivalent. While [3] uses unitary torsion-free (= nondegenerate) modules, we prefer to work with firm modules.

In [3, Theorem 4.1(i)] it is shown that if there exists a full nondegenerate Rieffel context between two nondegenerate idempotent rings with involution, then there is a $*$-equivalence between the categories of unitary torsion-free modules over these rings. In this section we will prove an analogue of this result by a) replacing unitary torsion-free modules by firm modules and b) abandoning the requirement that the rings and Rieffel contexts be nondegenerate. We start by modifying the necessary definitions for our purposes and by proving some lemmas that will be needed later.

Definition 5.1. *Let R be a ring. A dual R-pair consists of modules ${}_RP$ and Q_R and a mapping $\delta : P \times Q \to R$ such that*

1. *δ is additive in both arguments;*
2. *$\delta(rp, q) = r\delta(p, q)$ for every $r \in R$, $p \in P$, $q \in Q$;*
3. *$\delta(p, qr) = \delta(p, q)r$ for every $r \in R$, $p \in P$, $q \in Q$.*

Compared to the definition given on page 229 of [3], we have dropped the condition of nondegeneracy (condition (4)). Note that if S is a ring with involution and $\delta : P \times Q \to S$ is a dual S-pair, then there exists another dual S-pair

$$\overleftarrow{\delta} : \overline{Q} \times \overline{P} \to S, \qquad \overleftarrow{\delta}(\overline{q}, \overline{p}) := \delta(p, q)^*. \tag{5.1}$$

We also use a modification of Definition 1.2 in [3]. Instead of categories of unitary nondegenerate modules, we use the category ${}_S\mathsf{FMod}$ of firm left S-modules and the category FMod_S of firm right S-modules.

Definition 5.2. [Cf. [3, Definition 1.2]] *Let S and T be two idempotent rings. A functor-multiplier from S to T is a pair (F, G) such that*

1. *$F : {}_S\mathsf{FMod} \to {}_T\mathsf{FMod}$ is an additive functor;*
2. *$G : \mathsf{FMod}_S \to \mathsf{FMod}_T$ is an additive functor;*
3. *for every dual S-pair $\delta : V_1 \times W_1 \to S$, where V_1 and W_1 are firm S-modules, there exists a dual T-pair $\delta^{FG} : F(V_1) \times G(W_1) \to T$;*
4. *if $\delta_1 : V_1 \times W_1 \to S$ and $\delta_2 : V_2 \times W_2 \to S$ are dual pairs of firm modules and $f : V_1 \to V_2$, $f^\sharp : W_2 \to W_1$ are module homomorphisms such that*

$$\delta_2(f(v_1), w_2) = \delta_1(v_1, f^\sharp(w_2))$$

in S for all $v_1 \in V_1$ and $w_2 \in W_2$, then

$$\delta_2^{FG}(F(f)(a), b) = \delta_1^{FG}(a, G(f^\sharp)(b))$$

in T for every $a \in F(V_1)$ and $b \in G(W_2)$.

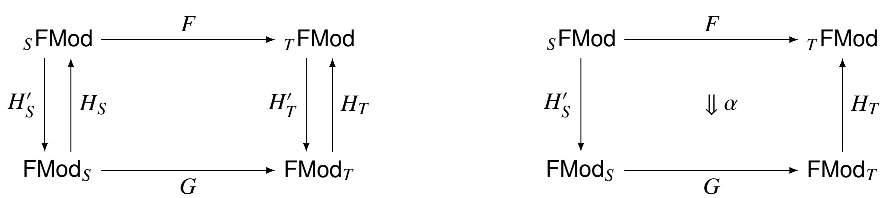
For a ring S with involution, we have the dualization functors

$$H_S : \mathsf{FMod}_S \to {}_S\mathsf{FMod}, \;\; H_S(X_S) := {}_S\overline{X},$$

and

$$H'_S : {}_S\mathsf{FMod} \to \mathsf{FMod}_S, \;\; H'_S({}_SX) := \overline{X}_S.$$

If $f : X_S \to Y_S$ is a morphism in FMod_S, then $H_S(f) : {}_S\overline{X} \to {}_S\overline{Y}$ is defined by $H_S(f)(\overline{x}) := \overline{f(x)}$, and similarly for H'_S. Note that $H_S \circ H'_S$ and $H'_S \circ H_S$ are identity functors.



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Definition 5.3 (Cf. [3, page 234]). *Let S and T be idempotent rings with involution. A $*$-functor from S to T is a triple (F, G, α), where (F, G) is a functor-multiplier from S to T and $\alpha : F \Rightarrow H_T \circ G \circ H'_S$ a natural isomorphism such that, for every dual pair $\delta : V \times W \to S$, every $a \in F(V)$ and $b \in (H'_T \circ F \circ H_S)(W)$, we have*

$$\overleftarrow{\delta}^{FG}(\alpha_V(a), b) = \delta^{FG}(a, H'_T(\alpha_{H_S(W)})(b)). \tag{5.2}$$

Note that here $(\alpha_V(a), b) \in (H_T \circ G \circ H'_S)(V) \times (H'_T \circ F \circ H_S)(W) = \overline{G(\overline{V})} \times \overline{F(\overline{W})}$,

$$\alpha_{H_S(W)} \colon (F \circ H_S)(W) \to (H_T \circ G \circ H'_S \circ H_S)(W) = (H_T \circ G)(W),$$

$$H'_T(\alpha_{H_S(W)}) \colon (H'_T \circ F \circ H_S)(W) \to (H'_T \circ H_T \circ G)(W) = G(W),$$

and $(a, H'_T(\alpha_{H_S(W)})(b)) \in F(V) \times G(W)$.

Lemma 5.4. *If (F, G, α) is a $*$-functor from S to T, $\delta : V \times W \to S$ is a dual pair, $x \in G(W)$ and $y \in (H_T \circ G \circ H'_S)(V)$, then*

$$\overleftarrow{\delta}^{FG}\left(y, H'_T(\alpha^{-1}_{H_S(W)})(x)\right) = \delta^{FG}\left(\alpha_V^{-1}(y), x\right).$$

Proof. Denote $a := \alpha_V^{-1}(y)$ and $b := H'_T(\alpha^{-1}_{H_S(W)})(x)$. Then

$$\begin{aligned}
\overleftarrow{\delta}^{FG}\left(y, H'_T(\alpha^{-1}_{H_S(W)})(x)\right) &= \overleftarrow{\delta}^{FG}(\alpha_V(a), b) \\
&= \delta^{FG}\left(a, H'_T(\alpha_{H_S(W)})(b)\right) && ((F, G, \alpha) \text{ is a } *\text{-functor}) \\
&= \delta^{FG}\left(a, H'_T(\alpha_{H_S(W)} \circ \alpha^{-1}_{H_S(W)})(x)\right) && (H'_T \text{ is a functor}) \\
&= \delta^{FG}\left(a, \mathrm{id}_{G(W)}(x)\right) \\
&= \delta^{FG}\left(\alpha_V^{-1}(y), x\right),
\end{aligned}$$

which proves the lemma. □

Note that $(\mathrm{id}_{_S\mathsf{FMod}}, \mathrm{id}_{\mathsf{FMod}_S}, \mathrm{id}_{\mathrm{id}_{_S\mathsf{FMod}}})$ is a $*$-functor from S to S. By Lemma 2.1 in [3], we know that if (F, G, α) is a $*$-functor from S to T and (F', G', β) a $*$-functor from T to R, then the triple $(F' \circ F, G' \circ G', \beta * \alpha)$ is a $*$-functor from S to R, where $\beta * \alpha$ is the horizontal composition of natural transformations α and β. Also, for every dual S-pair $\delta \colon V \times W \to S$ we have the formula

$$\delta^{F'F, G'G} = (\delta^{FG})^{F'G'} \colon \; F'(F(V)) \times G'(G(W)) \to R. \tag{5.3}$$

Definition 5.5. [Cf. [3, page 235]] *A $*$-functor (F, G, α) from S to T is called a $*$-equivalence if there exists a $*$-functor (F', G', β) from T to S such that there exist natural isomorphisms $\varphi \colon F' \circ F \Rightarrow \mathrm{id}_{_S\mathsf{FMod}}$, $\psi \colon F \circ F' \Rightarrow \mathrm{id}_{_T\mathsf{FMod}}$, for every dual pair $\delta_S \colon V \times W \to S$ we have*

$$\delta_S(\varphi_V(a), b) = \delta_S^{F'F, G'G}\left(a, H_S'\left((\beta * \alpha)_{H_S(W)} \circ \varphi^{-1}_{H_S(W)}\right)(w)\right) \tag{5.4}$$

for every $a \in (F' \circ F)(V)$ and $b \in W$ and for every dual pair $\delta_T \colon V' \times W' \to T$ we have

$$\delta_T(\psi_V(x), y) = \delta_T^{FF', GG'}\left(x, H_T'\left((\alpha * \beta)_{H_T(W)} \circ \psi^{-1}_{H_T(W)}\right)(y)\right) \tag{5.5}$$

for every $x \in (F' \circ F)(V')$ and $y \in W'$. Here $\beta * \alpha : F' \circ F \Rightarrow H_S \circ G' \circ G \circ H'_S$ is the horizontal composition of α and β and

$$H_S(W) \to^{\varphi^{-1}_{H_S(W)}} (F' \circ F \circ H_S)(W) \to^{(\beta * \alpha)_{H_S(W)}} (H_S \circ G' \circ G \circ H'_S \circ H_S)(W) = (H_S \circ G' \circ G)(W).$$

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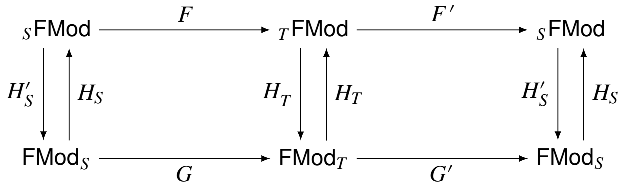

Lemma 5.6. *Let (F, G, α) be a $*$-equivalence from S to T with an inverse $*$-equivalence (F', G', β) from T to S. Then, there exist natural isomorphisms $\sigma : \mathrm{id}_{\mathsf{FMod}_S} \Rightarrow G' \circ G$ and $\rho : \mathrm{id}_{\mathsf{FMod}_T} \Rightarrow G \circ G'$.*

Proof. Let the assumptions of this lemma hold with natural isomophisms $\varphi : F' \circ F \Rightarrow \mathrm{id}_{{}_S\mathsf{FMod}}$ and $\psi : F \circ F' \Rightarrow \mathrm{id}_{{}_T\mathsf{FMod}}$. The required natural isomorphisms from the claim of the lemma have components $\sigma_W = H_S'((\beta * \alpha)_{H_S(W)} \circ \varphi^{-1}_{H_S(W)}) : W \to (G' \circ G)(W)$ (for every $W \in \mathsf{FMod}_S$) and $\rho_U = H_T'((\alpha * \beta)_{H_T(U)} \circ \psi^{-1}_{H_T(U)}) : U \to (G \circ G')(U)$ (for every $U \in \mathsf{FMod}_T$). □

Now we are ready to prove the main result of this section. It is an analogue of the classical Morita I theorem (see, e.g., [10, Theorem 18.24]).

Theorem 5.7. *Let $(S, T, {}_SX_T, \langle\, ,\, \rangle, [\, ,\,])$ be a firm and full Rieffel context connecting idempotent rings S and T with involution. Then there is a $*$-equivalence from S to T.*

Proof. We construct a functor-multiplier from S to T. If $M_S \in \mathsf{FMod}_S$, then $M \otimes_S X_T \in \mathsf{FMod}_T$, because $\mu_{M\otimes X} = \mathrm{id}_M \otimes \mu_X$, where id_M and μ_X are isomorphisms. Similarly, ${}_T\overline{X} \otimes_S N \in {}_T\mathsf{FMod}$ if ${}_SN \in {}_S\mathsf{FMod}$. This allows us to consider the tensor functors

$$F := \overline{X} \otimes_S \_ : {}_S\mathsf{FMod} \to {}_T\mathsf{FMod},$$

$$G := \_ \otimes_S X : \mathsf{FMod}_S \to \mathsf{FMod}_T.$$

The following proof is divided into several claims, which are proved separately.

Claim 1. *The pair (F, G) is a functor-multiplier.*

Conditions (1) and (2) in Definition 5.2 are obvious.

(3) Let $\delta : V \times W \to S$ be a dual S-pair. Define a mapping

$$\delta^{FG} : (\overline{X} \otimes_S V) \times (W \otimes_S X) \to T$$

by

$$\delta^{FG}(\overline{y} \otimes v, w \otimes x) := [y, \delta(v, w)x] \in T. \tag{5.6}$$

We will show that δ^{FG} is well-defined. Note that for every pair $(w', x') \in W \times X$, there is a well-defined mapping

$$d_{w',x'} : \overline{X} \times V \to T, \qquad (\overline{y}, v) \mapsto [y, \delta(v, w')x'].$$

The mapping $d_{w',x'}$ is S-balanced, because, for every $y, y' \in X$, $v, v' \in V$ and $s \in S$, we have

$$d_{w',x'}(\overline{y} + \overline{y'}, v) = d_{w',x'}(\overline{y + y'}, v) = [y + y', \delta(v, w')x'] = d_{w',x'}(\overline{y}, v) + d_{w',x'}(\overline{y'}, v),$$

$$\begin{aligned} d_{w',x'}(\overline{y}, v + v') &= [y, \delta(v + v', w')x'] = [y, (\delta(v, w') + \delta(v', w'))x'] \\ &= [y, \delta(v, w')x'] + [y, \delta(v', w')x'] = d_{w',x'}(\overline{y}, v) + d_{w',x'}(\overline{y}, v'), \end{aligned}$$

$$\begin{aligned} d_{w',x'}(\overline{y}s, v) &= d_{w',x'}(\overline{s^* y}, v) = [s^* y, \delta(v, w')x'] = [y, s\delta(v, w')x'] = [y, \delta(sv, w')x'] \\ &= d_{w',x'}(y, sv). \end{aligned}$$

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By the universal property of the tensor product, we have a well-defined group homomorphism $\overline{d_{w',x'}}: \overline{X} \otimes_S V \to T$ such that

$$\overline{d_{w',x'}}(\overline{y} \otimes v) = [y, \delta(v, w')x']$$

for every $y \in X$ and $v \in V$. Also, for every $t \in T$, $\overline{y} \in \overline{X}$ and $v \in V$, using **RC7** we have

$$\overline{d_{w',x'}}(t\overline{y} \otimes v) = \overline{d_{w',x'}}(\overline{yt^*} \otimes v) = [yt^*, \delta(v, w')x'] = t[y, \delta(v, w')x'] = t\overline{d_{w',x'}}(\overline{y} \otimes v).$$

Hence, $\overline{d_{w',x'}}$ is a homomorphism of left T-modules. Now we may define a mapping

$$\tau: \ W \times X \to \mathrm{Hom}\left(\overline{X} \otimes_S V, T\right), \qquad (w, x) \mapsto \overline{d_{w,x}}.$$

Note, that for every $w, w' \in W$, $x, x', y \in X$, $v \in V$ and $s \in S$ we have

$$\tau(w + w', x)(\overline{y} \otimes v) = \overline{d_{w+w',x}}(\overline{y} \otimes v) = [y, \delta(v, w + w')x] = [y, \delta(v, w)x] + [y, \delta(v, w')x]$$

$$= \overline{d_{w,x}}(\overline{y} \otimes v) + \overline{d_{w',x}}(\overline{y} \otimes v) = (\tau(w, x) + \tau(w', x))(\overline{y} \otimes v),$$

$$\tau(ws, x)(\overline{y} \otimes v) = \overline{d_{ws,x}}(\overline{y} \otimes v) = [y, \delta(v, ws)x] = [y, \delta(v, w)sx] = \tau(w, sx)(\overline{y} \otimes v)$$

and $\tau(w, x + x') = \tau(w, x) + \tau(w, x')$. Hence, τ is S-balanced and by the universal property of the tensor product, we may consider a well-defined group homomorphism $\overline{\tau}: W \otimes_S X \to \mathrm{Hom}\left(\overline{X} \otimes_S V, T\right)$ such that $\overline{\tau}(w \otimes x) = \overline{d_{w,x}}$ for every $w \in W$ and $x \in X$. Putting

$$\delta^{FG}(a, b) := \overline{\tau}(b)(a) \in T$$

we have a well-defined mapping $\delta^{FG}: (\overline{X} \otimes_S V) \times (W \otimes_S X) \to T$ such that

$$\delta^{FG}\left(\sum_{k=1}^{n} \overline{y}_k \otimes v_k, \sum_{h=1}^{m} w_h \otimes x_h\right) = \overline{\tau}\left(\sum_{h=1}^{m} w_h \otimes x_h\right)\left(\sum_{k=1}^{n} \overline{y}_k \otimes v_k\right) = \sum_{k=1}^{n}\sum_{h=1}^{m} [y_k, \delta(v_k, w_h)x_h].$$

It easily follows that δ^{FG} is additive in both arguments. We also have

$$\delta^{FG}(t(\overline{y} \otimes v), w \otimes x) = \delta^{FG}(\overline{yt^*} \otimes v, w \otimes x) = [yt^*, \delta(v, w)x] = t[y, \delta(v, w)x]$$

$$= t\delta^{FG}(\overline{y} \otimes v, w \otimes x),$$

$$\delta^{FG}(\overline{y} \otimes v, (w \otimes x)t) = [y, \delta(v, w)xt] = [y, \delta(v, w)x]t = \delta^{FG}(\overline{y} \otimes v, w \otimes x)t$$

for every $t \in T$, $x, y \in X$ and $v \in V$. So δ^{FG} is a dual T-pair.

(4) Let $\delta_1 : V_1 \times W_1 \to S$ and $\delta_2 : V_2 \times W_2 \to S$ be dual pairs and let $f : V_1 \to V_2$, $f^\sharp : W_2 \to W_1$ be module homomorphisms such that

$$\delta_2(f(v_1), w_2) = \delta_1\left(v_1, f^\sharp(w_2)\right)$$

in S for all $v_1 \in V_1$ and $w_2 \in W_2$. Then, for every $x, y \in X$,

$$[y, \delta_2(f(v_1), w_2)x] = [y, \delta_1(v_1, f^\sharp(w_2))x]$$

$$\Longrightarrow \delta_2^{FG}(\overline{y} \otimes f(v_1), w_2 \otimes x) = \delta_1^{FG}(\overline{y} \otimes v_1, f^\sharp(w_2) \otimes x)$$

$$\Longrightarrow \delta_2^{FG}((\mathrm{id}_{\overline{X}} \otimes f)(\overline{y} \otimes v_1), w_2 \otimes x) = \delta_1^{FG}(\overline{y} \otimes v_1, (f^\sharp \otimes \mathrm{id}_X)(w_2 \otimes x))$$

$$\Longrightarrow \delta_2^{FG}(F(f)(\overline{y} \otimes v_1), w_2 \otimes x) = \delta_1^{FG}(\overline{y} \otimes v_1, G(f^\sharp)(w_2 \otimes x)),$$

as needed. Thus, the pair (F, G) is a functor-multiplier.

Claim 2. *The functor-multiplier (F, G) is a $*$-functor.*

We need to define a natural transformation $\alpha : F \Rightarrow H_T \circ G \circ H'_S$. For every ${}_SM \in {}_S\mathsf{FMod}$, we define a mapping

$$\alpha_M : F(M) = \overline{X} \otimes_S M \to \overline{\overline{M} \otimes_S X} = (H_T \circ G \circ H'_S)(M)$$

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by

$$\alpha_M(\overline{y} \otimes m) := \overline{\overline{m} \otimes y}.$$

We will show that $\alpha := (\alpha_M)_{M \in {}_S\mathsf{FMod}} \colon F \Rightarrow H_T \circ G \circ H_S$ is a natural isomorphism.

Well-defined. Let ${}_SM \in {}_S\mathsf{FMod}$. Consider the mapping

$$\hat{\alpha}_M \colon \overline{X} \times M \to \overline{\overline{M} \otimes_S X}, \qquad (\overline{y}, m) \mapsto \overline{\overline{m} \otimes y}.$$

Note that, for every $y, y' \in X$, $m, m' \in M$ and $s \in S$, we have

$$\begin{aligned}
\hat{\alpha}_M(\overline{y} + \overline{y'}, m) &= \hat{\alpha}_M(\overline{y + y'}, m) = \overline{\overline{m} \otimes (y + y')} = \overline{\overline{m} \otimes y + \overline{m} \otimes y'} \\
&= \overline{\overline{m} \otimes y} + \overline{\overline{m} \otimes y'} = \hat{\alpha}_M(\overline{y}, m) + \hat{\alpha}_M(\overline{y'}, m), \\
\hat{\alpha}_M(\overline{y}s, m) &= \hat{\alpha}_M(\overline{s^*y}, m) = \overline{\overline{m} \otimes s^*y} = \overline{\overline{m}s^* \otimes y} = \overline{\overline{sm} \otimes y} = \hat{\alpha}_M(\overline{y}, sm)
\end{aligned}$$

and $\hat{\alpha}_M(\overline{y}, m + m') = \hat{\alpha}_M(\overline{y}, m) + \hat{\alpha}_M(\overline{y}, m')$. By the universal property of the tensor product, we see that α_M is a well-defined group homomorphism.

Homomorphism of T-modules. Let ${}_SM \in {}_S\mathsf{FMod}$. Note that, for every $t \in T$, $y \in X$ and $m \in M$, we have

$$\alpha_M(t\overline{y} \otimes m) = \alpha_M(\overline{yt^*} \otimes m) = \overline{\overline{m} \otimes yt^*} = \overline{(\overline{m} \otimes y)t^*} = t\overline{\overline{m} \otimes y} = t\alpha_M(\overline{y} \otimes m).$$

Hence, we see that α_M is a homomorphism of left T-modules.

Bijective. Let $M_S \in \mathsf{FMod}_S$. Analogously to the first point, we see that there exists a well-defined group homomorphism

$$\beta_M{}' \colon \overline{M} \otimes_S X \to \overline{X} \otimes_S M, \qquad \overline{m} \otimes y \mapsto \overline{y} \otimes m.$$

We may also consider a mapping $\beta_M \colon \overline{\overline{M} \otimes_S X} \to \overline{X} \otimes_S M$, $\overline{\overline{m} \otimes y} \mapsto \overline{y} \otimes m$, because $\beta_M = \beta_M{}' \circ u_{\overline{M} \otimes X}$, where $u_{\overline{M} \otimes X} \colon \overline{\overline{M} \otimes_S X} \to \overline{M} \otimes_S X$, $\overline{a} \mapsto a$. Clearly, β_M is the inverse of α_M in Set, which proves that α_M is bijective.

Natural transformation. Let ${}_SM, {}_SM' \in {}_S\mathsf{FMod}$ and $f \colon {}_SM \to {}_SM'$ be a homomorphism of left S-modules. We need to show that the diagram

$$\begin{array}{ccc}
\overline{X} \otimes_S M & \xrightarrow{\ \alpha_M\ } & \overline{\overline{M} \otimes X} \\
{\scriptstyle \mathrm{id}_{\overline{X}} \otimes f} \downarrow & & \downarrow {\scriptstyle (H_T \circ G \circ H_S)(f)} \\
\overline{X} \otimes_S M' & \xrightarrow{\ \alpha_M\ } & \overline{\overline{M'} \otimes_S X}
\end{array}$$

commutes. If $y \in X$ and $m \in M$, then

$$\begin{aligned}
((H_T \circ G \circ H_S)(f) \circ \alpha_M)(\overline{y} \otimes m) &= (H_T \circ G \circ H_S)(f)(\overline{\overline{m} \otimes y}) \\
&= \overline{(G \circ H_S)(f)(\overline{m}) \otimes y} = \overline{(H_S(f) \otimes \mathrm{id}_X)(\overline{m} \otimes y)} \\
&= \overline{H_S(f)(\overline{m}) \otimes y} = \overline{\overline{f(m)} \otimes y} \\
&= \alpha_{M'}(\overline{y} \otimes f(m)) = (\alpha_{M'} \circ (\mathrm{id}_X \otimes f))(\overline{y} \otimes m).
\end{aligned}$$

Therefore we see that α is indeed a natural transformation.
In conclusion, we have shown that α is a natural isomorphism.

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To verify the equality (5.2), let $\delta : V \times W \to S$ be a dual pair and consider $a \in \overline{X} \otimes_S V = F(V)$ and $b \in \overline{\overline{X}} \otimes_S \overline{\overline{W}} = ({H'}_T \circ F \circ H_S)(W)$. Then $a = \sum_{k=1}^{n} \overline{y}_k \otimes v_k$, $b = \sum_{h=1}^{m} \overline{\overline{x}_h \otimes \overline{w}_h}$ and

$$\begin{aligned}
\overleftarrow{\overleftarrow{\delta}}^{FG}(\alpha_V(a), b) &= \overleftarrow{\overleftarrow{\delta}}^{FG}\left(\alpha_V\left(\sum_{k=1}^{n} \overline{y}_k \otimes v_k\right), \sum_{h=1}^{m} \overline{\overline{x}_h \otimes \overline{w}_h}\right) \\
&= \overleftarrow{\overleftarrow{\delta}}^{FG}\left(\sum_{k=1}^{n} \overline{\overline{v}_k \otimes y_k}, \sum_{h=1}^{m} \overline{\overline{x}_h \otimes \overline{w}_h}\right) && \text{(def. of } \alpha_V) \\
&= \overleftarrow{\delta}^{FG}\left(\sum_{h=1}^{m} \overline{x}_h \otimes \overline{w}_h, \sum_{k=1}^{n} \overline{v}_k \otimes y_k\right)^{*} && \text{(by (5.1))} \\
&= \sum_{h=1}^{m}\sum_{k=1}^{n} [x_h, \overleftarrow{\delta}(\overline{w}_h, \overline{v}_k) y_k]^{*} && \text{(by (5.6))} \\
&= \sum_{h=1}^{m}\sum_{k=1}^{n} [x_h, \delta(v_k, w_h)^{*} y_k]^{*} && \text{(by (5.1))} \\
&= \sum_{h=1}^{m}\sum_{k=1}^{n} [\delta(v_k, w_h)^{*} y_k, x_h] && \textbf{(RC4)} \\
&= \sum_{h=1}^{m}\sum_{k=1}^{n} [y_k, \delta(v_k, w_h) x_h] && \textbf{(RC9)} \\
&= \delta^{FG}\left(\sum_{k=1}^{n} \overline{y}_k \otimes v_k, \sum_{h=1}^{m} w_h \otimes x_h\right) && \text{(by (5.6))} \\
&= \delta^{FG}\left(a, \sum_{h=1}^{m} \overline{\overline{w_h \otimes x_h}}\right) \\
&= \delta^{FG}\left(a, \overline{\alpha_{\overline{W}}\left(\sum_{h=1}^{m} \overline{x}_h \otimes \overline{w}_h\right)}\right) && \text{(def. of } \alpha_{\overline{W}}) \\
&= \delta^{FG}\left(a, {H_T}'(\alpha_{\overline{W}})\left(\sum_{h=1}^{m} \overline{\overline{x}_h \otimes \overline{w}_h}\right)\right) && \text{(def. of } {H'}_T) \\
&= \delta^{FG}(a, {H_T}'(\alpha_{H_S(W)})(b)). && \text{(def. of } H_S)
\end{aligned}$$

Hence, we see that (F, G, α) is indeed a $*$-functor from S to T.

Claim 3. *The $*$-functor (F, G) is a $*$-equivalence.*

Analogously, we see that the pair (F', G', β) is a $*$-functor from T to S, where

$$\begin{aligned}
F' &:= X \otimes_T __ : \ {}_T\mathsf{FMod} \to {}_S\mathsf{FMod}, \\
G' &:= __ \otimes_T \overline{X} : \ \mathsf{FMod}_T \to \mathsf{FMod}_S; \\
\beta &= (\beta_M)_{M \in {}_T\mathsf{FMod}} : \ F' \Rightarrow H_S \circ G' \circ {H_T}', \\
\beta_M(x \otimes m) &:= \overline{\overline{m} \otimes \overline{x}}.
\end{aligned}$$

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Also, for any dual T-pair $\delta_T\colon V \times W \to T$, we have

$$\delta_T^{F'G'}\colon F'(V) \times G'(W) \to S, \qquad (x \otimes v, w \otimes \overline{y}) \mapsto \langle x\,\delta_T(v,w), y\rangle. \tag{5.7}$$

To complete the proof we need to show that the requirements of Definition 5.5 are met. For every ${}_SM \in {}_S\mathsf{FMod}$, consider the mapping

$$\tau_M\colon (F' \circ F)({}_SM) = X \otimes_T \overline{X} \otimes_S M \to S \otimes_S M, \qquad x \otimes \overline{y} \otimes m \mapsto \langle x, y\rangle \otimes m.$$

As in (3.1), we have a well-defined group homomorphism $\theta\colon X \otimes_T \overline{X} \to S$, $x \otimes \overline{y} \mapsto \langle x, y\rangle$, which is a homomorphism of (S, S)-bimodules. This implies that $\tau_M = \theta \otimes \mathrm{id}_M$ is a well-defined homomorphism of left S-modules. The homomorphism τ_M is surjective, because $\langle\ ,\ \rangle$ is full. We will show that τ_M is also injective. Consider the short exact sequence of (S, S)-bimodules

$$\{0\} \to \mathrm{Ker}\theta \xrightarrow{\iota} X \otimes_T \overline{X} \xrightarrow{\theta} S \to \{0\},$$

where $\iota\colon \mathrm{Ker}\theta \to X \otimes_T \overline{X}$ is the inclusion. By [28, Result 12.8(1)], the sequence of left S-modules

$$\mathrm{Ker}\theta \otimes_S M \xrightarrow{\iota\otimes\mathrm{id}_M} X \otimes_T \overline{X} \otimes_S M \xrightarrow{\tau_M} S \otimes_S M \to \{0\} \tag{5.8}$$

is also exact. We show that $(\mathrm{Ker}\theta)S = \{0\}$. Let $\sum_{k=1}^n x_k \otimes \overline{y}_k \in \mathrm{Ker}\theta$ and $s \in S$. Since θ is surjective, there exists $\sum_{h=1}^m \xi_h \otimes \overline{\zeta}_h \in X \otimes_T \overline{X}$ such that $s = \theta(\sum_{h=1}^m \xi_h \otimes \overline{\zeta}_h)$. Now

$$\begin{aligned}
\left(\sum_{k=1}^n x_k \otimes \overline{y}_k\right)s &= \sum_{k=1}^n x_k \otimes \overline{y}_k \theta\left(\sum_{h=1}^m \xi_h \otimes \overline{\zeta}_h\right) = \sum_{k=1}^n \sum_{h=1}^m x_k \otimes \overline{y}_k \langle \xi_h, \zeta_h\rangle \\
&= \sum_{k=1}^n \sum_{h=1}^m x_k \otimes \overline{\langle \xi_h, \zeta_h\rangle^* y_k} = \sum_{k=1}^n \sum_{h=1}^m x_k \otimes \overline{\langle \zeta_h, \xi_h\rangle y_k} = \sum_{k=1}^n \sum_{h=1}^m x_k \otimes \overline{\zeta_h[\xi_h, y_k]} \\
&= \sum_{k=1}^n \sum_{h=1}^m x_k \otimes [\xi_h, y_k]^* \overline{\zeta_h} = \sum_{k=1}^n \sum_{h=1}^m x_k [\xi_h, y_k]^* \otimes \overline{\zeta_h} = \sum_{k=1}^n \sum_{h=1}^m x_k [y_k, \xi_h] \otimes \overline{\zeta_h} \\
&= \sum_{k=1}^n \sum_{h=1}^m \langle x_k, y_k\rangle \xi_h \otimes \overline{\zeta_h} = \sum_{h=1}^m \theta\left(\sum_{k=1}^n x_k \otimes y_k\right) \xi_h \otimes \overline{\zeta_h} = \sum_{h=1}^m 0\xi_h \otimes \overline{\zeta_h} = 0.
\end{aligned}$$

Hence, $(\mathrm{Ker}\theta)S = \{0\}$.

Now let $\xi \otimes m$ be an elementary tensor in $\mathrm{Ker}\theta \otimes_S M$ (then $\xi \otimes m$ is also a generator of $\mathrm{Ker}\theta \otimes_S M$). Since ${}_SM$ is unitary, there exist $s_1, \dots, s_j \in S$ and $m_1, \dots, m_j \in M$ such that $m = s_1m_1 + \dots + s_jm_j$. Then $\xi s_h = 0$ for every $h \in \{1, \dots, j\}$ and

$$\xi \otimes m = \xi \otimes \left(\sum_{h=1}^j s_h m_h\right) = \sum_{h=1}^j \xi \otimes s_h m_h = \sum_{h=1}^j \xi s_h \otimes m_h = 0.$$

It follows that $\mathrm{Ker}\theta \otimes_S M = \{0\}$. Hence, τ_M is bijective, due to the exactness of the sequence (5.8).

Let ${}_SM, {}_SM' \in {}_S\mathsf{FMod}$ and $f\colon {}_SM \to {}_SM'$ a morphism in ${}_S\mathsf{FMod}$. The diagram

$$\begin{array}{ccc}
\otimes_T \overline{X} \otimes_S M & \xrightarrow{\ \tau_M\ } & S \otimes_S M \\
\Big\downarrow{\scriptstyle \mathrm{id}_{X\otimes\overline{X}} \otimes f} & & \Big\downarrow{\scriptstyle \mathrm{id}_S \otimes f} \\
X \otimes_T \overline{X} \otimes_S M' & \xrightarrow{\ \tau_{M'}\ } & S \otimes_S M'
\end{array}$$

commutes, because

$$\begin{aligned}
((\mathrm{id}_S \otimes f) \circ \tau_M)(x \otimes \overline{y} \otimes m) &= (\mathrm{id}_S \otimes f)(\langle x, y\rangle \otimes m) = \langle x, y\rangle \otimes f(m) \\
&= \tau_{M'}(x \otimes \overline{y} \otimes f(m)) = (\tau_{M'} \circ (\mathrm{id}_{X\otimes\overline{X}} \otimes f))(x \otimes \overline{y} \otimes m).
\end{aligned}$$

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We have shown that $\tau = (\tau_M)_{M \in {_S}\mathsf{FMod}}\colon F' \circ F \Rightarrow S \otimes_S \_\_$ is a natural isomorphism. By the definition of the category ${_S}\mathsf{FMod}$, there exists a natural isomorphism $\nu = (\nu_M)_{M \in {_S}\mathsf{FMod}}\colon S \otimes_S \_\_ \Rightarrow \mathrm{id}_{{_S}\mathsf{FMod}}$, where

$$\nu_M\colon S \otimes_S M \to M, \;\; x \otimes m \mapsto sm.$$

Composing τ and ν, we obtain a natural isomorphism

$$\varphi = (\varphi_M)\colon \;\; F' \circ F \Rightarrow \mathrm{id}_{{_S}\mathsf{FMod}},$$
$$\varphi_M\colon X \otimes_T \overline{X} \otimes_S M \to {_S}M, \qquad x \otimes \overline{y} \otimes m \mapsto \langle x, y\rangle m.$$

We will show that φ satisfies condition (5.4). Similar reasoning will lead to a natural transformation $\psi : F \circ F' \Rightarrow \mathrm{id}_{{_T}\mathsf{FMod}}$ satisfying (5.5). First, we calculate the natural isomorphism

$$\beta * \alpha\colon F' \circ F = X \otimes_T \overline{X} \otimes_S \_\_ \Rightarrow (H_S \circ G' \circ H_T{}') \circ (H_T \circ G \circ H_S{}') = H_S \circ (\_\_ \otimes_S X \otimes_T \overline{X}) \circ H'_S.$$

For every ${_S}M \in {_S}\mathsf{FMod}$,

$$(\beta * \alpha)_M = \beta_{H_T G H_S{}'(M)} \circ F'(\alpha_M)\colon X \otimes_T \overline{X} \otimes_S M \to \overline{\overline{M} \otimes_S X \otimes_T \overline{X}},$$
$$x \otimes \overline{y} \otimes m \mapsto \beta_{H_T G H_S{}'(M)}(x \otimes \overline{\overline{m} \otimes y}) = \overline{\overline{m} \otimes y \otimes \overline{x}}.$$

Let $\delta\colon V \times W \to S$ be a dual pair. We will prove (5.4). Let $a \in (F' \circ F)(V)$ and $b \in W$. Then $a = \sum_{k=1}^{n} x_k \otimes \overline{y}_k \otimes v_k$ for some $x_k, y_k \in X$ and $v_k \in V$. Since W_S is firm (thus unitary) and $\langle\, ,\, \rangle$ is full, we can write $b = \sum_{h=1}^{j} w_h \langle \xi_h, \zeta_h\rangle$, where $w_h \in W$ and $\xi_h, \zeta_h \in X$. Now

$$\overline{b} = \sum_{h=1}^{j} \overline{w_h \langle \xi_h, \zeta_h \rangle} = \sum_{h=1}^{j} \langle \xi_h, \zeta_h \rangle^* \, \overline{w_h} = \sum_{h=1}^{j} \langle \zeta_h, \xi_h \rangle \, \overline{w_h} = \varphi_{H(W)} \left(\sum_{h=1}^{j} \zeta_h \otimes \overline{\xi_h} \otimes \overline{w_h} \right),$$

so we have

$$\begin{aligned}
H_S{}'((\beta * \alpha)_{H(W)} \circ \varphi^{-1}_{H(W)})(b) &= \left(H_S{}'((\beta * \alpha)_{H(W)}) \circ H'_S(\varphi^{-1}_{H(W)})\right) \left(\overline{\overline{b}}\right) && (H'_S \text{ is a functor})\\
&= H_S{}'((\beta * \alpha)_{H(W)}) \left(\overline{\varphi^{-1}_{H(W)}\left(\overline{b}\right)}\right) && (\text{def. of } H'_S)\\
&= H_S{}'((\beta * \alpha)_{H(W)}) \left(\overline{\sum_{h=1}^{j} \zeta_h \otimes \overline{\xi}_h \otimes \overline{w_h}}\right)\\
&= \overline{(\beta * \alpha)_{H(W)} \left(\sum_{h=1}^{j} \zeta_h \otimes \overline{\xi}_h \otimes \overline{w_h}\right)} && (\text{def. of } H'_S)\\
&= \overline{\sum_{h=1}^{j} \overline{\overline{\overline{w_h}} \otimes \xi_h \otimes \overline{\zeta}_h}} && (\text{def. of } (\beta * \alpha)_{H(W)})\\
&= \sum_{h=1}^{j} w_h \otimes \xi_h \otimes \overline{\zeta}_h,
\end{aligned}$$

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and therefore

$$
\begin{aligned}
&\delta^{F'F,G'G}(a, H_S'((\beta * \alpha)_{H(W)} \circ \varphi^{-1}_{H(W)})(b)) \\
&= \delta^{F'F,G'G}\left(\sum_{k=1}^{n} x_k \otimes \overline{y}_k \otimes v_k, \sum_{h=1}^{j} w_h \otimes \xi_h \otimes \overline{\zeta}_h\right) \\
&= \sum_{k=1}^{n}\sum_{h=1}^{j} \left\langle x_k \delta^{FG}(\overline{y}_k \otimes v_k, w_h \otimes \xi_h), \zeta_h\right\rangle && \text{(by (5.7))} \\
&= \sum_{k=1}^{n}\sum_{h=1}^{j} \langle x_k[y_k, \delta(v_k, w_h)\xi_h], \zeta_h\rangle && \text{(by (5.6))} \\
&= \sum_{k=1}^{n}\sum_{h=1}^{j} \langle x_k, \zeta_h[y_k, \delta(v_k, w_h)\xi_h]^*\rangle && \textbf{(RC8)} \\
&= \sum_{k=1}^{n}\sum_{h=1}^{j} \langle x_k, \zeta_h[\delta(v_k, w_h)\xi_h, y_k]\rangle && \textbf{(RC4)} \\
&= \sum_{k=1}^{n}\sum_{h=1}^{j} \langle x_k, \langle \zeta_h, \delta(v_k, w_h)\xi_h\rangle y_k\rangle && \textbf{(RC5)} \\
&= \sum_{k=1}^{n}\sum_{h=1}^{j} \langle x_k, y_k\rangle \langle \zeta_h, \delta(v_k, w_h)\xi_h\rangle^* && \textbf{(RC6)} \\
&= \sum_{k=1}^{n}\sum_{h=1}^{j} \langle x_k, y_k\rangle \langle \delta(v_k, w_h)\xi_h, \zeta_h\rangle && \textbf{(RC2)} \\
&= \sum_{k=1}^{n}\sum_{h=1}^{j} \langle x_k, y_k\rangle \delta(v_k, w_h)\langle \xi_h, \zeta_h\rangle && \textbf{(RC1)} \\
&= \delta\left(\sum_{k=1}^{n} \langle x_k, y_k\rangle v_k, \sum_{h=1}^{j} w_h\langle \xi_h, \zeta_h\rangle\right) && \text{(additivity)} \\
&= \delta\left(\varphi_V\left(\sum_{k=1}^{n} x_k \otimes \overline{y}_k \otimes v_k\right), b\right) && \text{(def. of } \varphi_V) \\
&= \delta(\varphi_V(a), b).
\end{aligned}
$$

The proof is complete. □

Corollary 5.8. *If there exists a unitary and full Rieffel context between idempotent rings with involution R and S, then the rings R and S are Morita $*$-equivalent.*

6. From a category equivalence to a Rieffel context

In this section, we will give a necessary and sufficient condition for the existence of a firm and full Rieffel context between two idempotent rings with involution in terms of equivalence of categories of firm modules.

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Lemma 6.1. *Let S and T be rings with involution. For every bimodule ${}_SX_T$, there exists a group homomorphism $\sigma_X : X \otimes_T \overline{X} \to X \otimes_T \overline{X}$ such that*

$$\sigma_X(x \otimes \overline{y}) = y \otimes \overline{x}$$

for every $x, y \in X$.

Proof. The mapping $\hat{\sigma} : X \times \overline{X} \to X \otimes_T \overline{X}$, $(x, \overline{y}) \mapsto y \otimes \overline{x}$, is T-balanced, because

$$\hat{\sigma}(xt, \overline{y}) = y \otimes \overline{xt} = y \otimes t^*\overline{x} = yt^* \otimes \overline{x} = \hat{\sigma}(x, \overline{yt^*}) = \hat{\sigma}(x, t\overline{y}),$$

and it is obviously additive in both arguments. Now the claim follows from the universal property of tensor product. □

We recall that there exist two natural transformations $\mu = (\mu_M)_{M \in \mathsf{Mod}_S} : \_\_ \otimes_S S \Rightarrow \mathrm{id}_{\mathsf{Mod}_S}$ and $\nu = (\nu_N)_{N \in {}_S\mathsf{Mod}} : S \otimes_S \_\_ \Rightarrow \mathrm{id}_{{}_S\mathsf{Mod}}$ with components

$$\mu_M : M \otimes_S S \to M, \; m \otimes s \mapsto ms,$$

$$\nu_N : S \otimes_S N \to N, \; s \otimes n \mapsto sn.$$

Lemma 6.2. *Let S and T be rings with involution. A bimodule ${}_SX_T$ is firm if and only if the dual bimodule ${}_T\overline{X}_S$ is firm.*

Proof. The mappings

$$\hat{\kappa} : X \times T \to T \otimes_T \overline{X}, \; (x, t) \mapsto t^* \otimes \overline{x},$$

$$\hat{\omega} : T \times \overline{X} \to X \otimes_T T, \; (t, \overline{x}) \mapsto x \otimes t^*$$

are T-balanced, because

$$\hat{\kappa}(xt_1, t) = t^* \otimes \overline{xt_1} = t^* \otimes t_1^*\overline{x} = t^*t_1^* \otimes \overline{x} = (t_1t)^* \otimes \overline{x} = \hat{\kappa}(x, t_1t),$$

$$\hat{\omega}(tt_1, \overline{x}) = x \otimes (tt_1)^* = x \otimes t_1^*t^* = xt_1^* \otimes t^* = \hat{\omega}(t, \overline{xt_1^*}) = \hat{\omega}(t, t_1\overline{x}),$$

and additivity in both arguments is clear. Hence, there exist group homomorphisms $\kappa : X \otimes_T T \to T \otimes_T \overline{X}$ and $\omega : T \otimes_T \overline{X} \to X \otimes_T T$ such that

$$\kappa(x \otimes t) = t^* \otimes \overline{x},$$

$$\omega(t \otimes \overline{x}) = x \otimes t^*.$$

It is easy to see that $\kappa = \omega^{-1}$, so κ is a bijective mapping. Moreover, the square

$$\begin{array}{ccc} X \otimes_T T & \xrightarrow{\mu_X} & X \\ \Big\downarrow{\scriptstyle \kappa} & & \Big\downarrow{\scriptstyle \overline{(\;)}} \\ T \otimes_T \overline{X} & \xrightarrow[\overline{X}]{} & \overline{X} \end{array}$$

commutes, because

$$(\overline{(\;)} \circ \mu_X)(x \otimes t) = \overline{xt} = t^*\overline{x} = \nu_{\overline{X}}(t^* \otimes \overline{x}) = (\nu_{\overline{X}} \circ \kappa)(x \otimes t).$$

Since κ and $\overline{(\;)}$ are bijections, μ is a bijection if and only if ν is a bijection. □

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Theorem 6.3. *For idempotent rings S and T with involution, the following are equivalent.*

1. *There exists a firm and full Rieffel context connecting S and T.*
2. *There exists a firm bimodule ${}_SX_T$ such that for the functors*

$$F = \overline{X} \otimes_S - \colon {}_S\mathsf{FMod} \to {}_T\mathsf{FMod}, \qquad G = - \otimes_S X \colon \mathsf{FMod}_S \to \mathsf{FMod}_T,$$

$$F' = X \otimes_T - \colon {}_T\mathsf{FMod} \to {}_S\mathsf{FMod}, \qquad G' = - \otimes_T \overline{X} \colon \mathsf{FMod}_T \to \mathsf{FMod}_S$$

there exist natural isomorphisms

$$\varphi : F' \circ F \Rightarrow \mathrm{id}, \;\; \psi : F \circ F' \Rightarrow \mathrm{id}, \;\; \xi : G' \circ G \Rightarrow \mathrm{id}, \;\; \zeta : G \circ G' \Rightarrow \mathrm{id}$$

such that $\zeta_{X_T} = \varphi_{{}_SX}$ and $\psi_{{}_T\overline{X}} = \xi_{\overline{X}_S}$ as mappings and the following diagrams commute:

$$\begin{array}{ccccccccc}
X \otimes \overline{X} & \xrightarrow{\mathrm{id} \otimes \mu_{\overline{X}}^{-1}} & X \otimes \overline{X} \otimes S & \xrightarrow{\mathrm{id} \otimes \mu_{\overline{X}}^{-1} \otimes \mathrm{id}} & X \otimes \overline{X} \otimes S \otimes S & \xrightarrow{\varphi_{S\otimes S}} & S \otimes S & \xrightarrow{\mu_S} & S \\
{\scriptstyle \sigma_X}\downarrow & & & & \text{(R1)} & & & & \downarrow{\scriptstyle (\ )^*} \\
X \otimes \overline{X} & \xrightarrow[\mathrm{id} \otimes \mu_{\overline{X}}^{-1}]{} & X \otimes \overline{X} \otimes S & \xrightarrow[\mathrm{id} \otimes \mu_{\overline{X}}^{-1} \otimes \mathrm{id}]{} & X \otimes \overline{X} \otimes S \otimes S & \xrightarrow[\varphi_{S\otimes S}]{} & S \otimes S & \xrightarrow[\mu_S]{} & S,
\end{array}$$

$$\begin{array}{ccccccccc}
\overline{X} \otimes X & \xrightarrow{\nu_{\overline{X}}^{-1} \otimes \mathrm{id}} & T \otimes \overline{X} \otimes X & \xrightarrow{\mathrm{id} \otimes \nu_{\overline{X}}^{-1} \otimes \mathrm{id}} & T \otimes T \otimes \overline{X} \otimes X & \xrightarrow{\zeta_{T\otimes T}} & T \otimes T & \xrightarrow{\mu_T} & T \\
{\scriptstyle \sigma_{\overline{X}}}\downarrow & & & & \text{(R2)} & & & & \downarrow{\scriptstyle (\ )^*} \\
\overline{X} \otimes X & \xrightarrow[\nu_{\overline{X}}^{-1} \otimes \mathrm{id}]{} & T \otimes \overline{X} \otimes X & \xrightarrow[\mathrm{id} \otimes \nu_{\overline{X}}^{-1} \otimes \mathrm{id}]{} & T \otimes T \otimes \overline{X} \otimes X & \xrightarrow[\zeta_{T\otimes T}]{} & T \otimes T & \xrightarrow[\mu_T]{} & T.
\end{array}$$

Proof. (1) $\Longrightarrow$ (2). Let $(S, T, {}_SX_T, \langle\, ,\, \rangle, [\, ,\,])$ be a firm and full Rieffel context. In the proof of Theorem 5.7, we have seen that

$$F = \overline{X} \otimes_S - \colon {}_S\mathsf{FMod} \to {}_T\mathsf{FMod}, \qquad G = - \otimes_S X \colon \mathsf{FMod}_S \to \mathsf{FMod}_T,$$

$$F' = X \otimes_T - \colon {}_T\mathsf{FMod} \to {}_S\mathsf{FMod}, \qquad G' = - \otimes_T \overline{X} \colon \mathsf{FMod}_T \to \mathsf{FMod}_S$$

are equivalence functors and $\varphi = (\varphi_{{}_SM}) : F' \circ F \Rightarrow \mathrm{id}_{{}_S\mathsf{FMod}}$, where

$$\varphi_{{}_SM} : X \otimes_T \overline{X} \otimes_S M \to {}_SM, \;\; x \otimes \overline{y} \otimes m \mapsto \langle\, x, y \,\rangle\, m,$$

is a natural isomorphism. Analogously we can construct natural isomorphisms ψ, ξ and ζ by defining

$$\psi_{{}_TN} : \overline{X} \otimes_S X \otimes_T N \to {}_TN, \qquad \overline{y} \otimes x \otimes n \mapsto [y, x]n,$$

$$\xi_{N_S} : N \otimes_S X \otimes_T \overline{X} \to N_S, \qquad n \otimes x \otimes \overline{y} \mapsto n\, \langle\, x, y \,\rangle\,,$$

$$\zeta_{M_T} : M \otimes_T \overline{X} \otimes_S X \to M_T, \qquad m \otimes \overline{y} \otimes x \mapsto m[y, x].$$

Now, for every $x, x', y, y' \in X$,

$$\zeta_{X_T}(x' \otimes \overline{y} \otimes x) = x'[y, x] = \langle\, x', y \,\rangle\, x = \varphi_{{}_SX}(x' \otimes \overline{y} \otimes x),$$

$$\psi_{{}_T\overline{X}}(\overline{y} \otimes x \otimes \overline{y'}) = [y, x]\overline{y'} = \overline{y'[y, x]^*} = \overline{y'[x, y]} = \overline{\langle\, y', x \,\rangle\, y}$$

$$= \overline{\langle\, x, y' \,\rangle^*\, y} = \overline{y}\, \langle\, x, y' \,\rangle = \xi_{\overline{X}_S}(\overline{y} \otimes x \otimes \overline{y'}).$$

So $\zeta_{X_T} = \varphi_{{}_SX}$ and $\psi_{{}_T\overline{X}} = \xi_{\overline{X}_S}$.

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We will prove the commutativity of rectangle (R1) (for rectangle (R2) the proof will be similar). Take $x \otimes \overline{y} \in X \otimes_T \overline{X}$. Since ${}_SX$ is unitary, we can write

$$x = \sum_{i=1}^{\hat{\imath}} s_i x_i, \quad x_i = \sum_{j=1}^{\hat{\jmath}} s_{ij} x_{ij}, \quad y = \sum_{k=1}^{\hat{k}} u_k y_k, \quad y_k = \sum_{l=1}^{\hat{l}} u_{kl} y_{kl} \tag{6.1}$$

for some $x_i, x_{ij}, y_k, y_{kl} \in X$ and $s_i, s_{ij}, u_k, u_{kl} \in S$. Then

$$\begin{aligned}
&((\mathrm{id}_X \otimes \mu_{\overline{X}}) \circ (\mathrm{id}_X \otimes \mu_{\overline{X}} \otimes \mathrm{id}_S)) \left(\sum_{k=1}^{\hat{k}} \sum_{l=1}^{\hat{l}} x \otimes \overline{y_{kl}} \otimes u_{kl}^* \otimes u_k^* \right) \\
&= (\mathrm{id}_X \otimes \mu_{\overline{X}}) \left(\sum_{k=1}^{\hat{k}} \sum_{l=1}^{\hat{l}} x \otimes \overline{y_{kl}} u_{kl}^* \otimes u_k^* \right) && (\text{def. of } \mu_{\overline{X}}) \\
&= (\mathrm{id}_X \otimes \mu_{\overline{X}}) \left(\sum_{k=1}^{\hat{k}} \sum_{l=1}^{\hat{l}} x \otimes \overline{u_{kl} y_{kl}} \otimes u_k^* \right) && (\text{by } (2.2)) \\
&= (\mathrm{id}_X \otimes \mu_{\overline{X}}) \left(\sum_{k=1}^{\hat{k}} x \otimes \overline{y_k} \otimes u_k^* \right) && (\text{by } (6.1)) \\
&= \sum_{k=1}^{\hat{k}} x \otimes \overline{y_k} u_k^* && (\text{def. of } \mu_{\overline{X}}) \\
&= \sum_{k=1}^{\hat{k}} x \otimes \overline{u_k y_k} && (\text{by } (2.2)) \\
&= x \otimes \overline{y} && (\text{by } (6.1))
\end{aligned}$$

and, analogously,

$$((\mathrm{id}_X \otimes \mu_{\overline{X}}) \circ (\mathrm{id}_X \otimes \mu_{\overline{X}} \otimes \mathrm{id}_S)) \left(\sum_{i=1}^{\hat{\imath}} \sum_{j=1}^{\hat{\jmath}} y \otimes \overline{x_{ij}} \otimes s_{ij}^* \otimes s_i^* \right) = y \otimes \overline{x}. \tag{6.2}$$

Hence,

$$\begin{aligned}
&\left((\,)^* \circ \mu_S \circ \varphi_{S\otimes S} \circ (\mathrm{id}_X \otimes \mu_{\overline{X}}^{-1} \otimes \mathrm{id}_S) \circ (\mathrm{id}_X \otimes \mu_{\overline{X}}^{-1}) \right) (x \otimes \overline{y}) \\
&= ((\,)^* \circ \mu_S \circ \varphi_{S\otimes S}) \left(\sum_{k=1}^{\hat{k}} \sum_{l=1}^{\hat{l}} x \otimes \overline{y_{kl}} \otimes u_{kl}^* \otimes u_k^* \right) \\
&= \sum_{k=1}^{\hat{k}} \sum_{l=1}^{\hat{l}} ((\,)^* \circ \mu_S \circ \varphi_{S\otimes S}) \left(x \otimes \overline{y_{kl}} \otimes u_{kl}^* \otimes u_k^* \right) && (\text{additivity}) \\
&= \sum_{k=1}^{\hat{k}} \sum_{l=1}^{\hat{l}} ((\,)^* \circ \mu_S) \left(\langle x, y_{kl} \rangle \, (u_{kl}^* \otimes u_k^*) \right) && (\text{def. of } \varphi_{S\otimes S}) \\
&= \sum_{k=1}^{\hat{k}} \sum_{l=1}^{\hat{l}} ((\,)^* \circ \mu_S) \left(\langle x, y_{kl} \rangle \, u_{kl}^* \otimes u_k^* \right) && (S\text{-action of } S \otimes S)
\end{aligned}$$

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$$
\begin{aligned}
&= \sum_{k=1}^{\hat{k}} \sum_{l=1}^{\hat{l}} \left(\langle\, x, y_{kl} \,\rangle\, u_{kl}^{*} u_{k}^{*}\right)^{*} && \text{(def. of } \mu_S \text{ and } (\,)^{*}) \\
&= \left(\sum_{k=1}^{\hat{k}} \sum_{l=1}^{\hat{l}} \langle\, x, y_{kl} \,\rangle\, u_{kl}^{*} u_{k}^{*} \right)^{*} && \text{(by (2.1))} \\
&= \left(\sum_{k=1}^{\hat{k}} \sum_{l=1}^{\hat{l}} \langle\, x, u_{kl} y_{kl} \,\rangle\, u_{k}^{*} \right)^{*} && (\textbf{RC6}) \\
&= \left(\sum_{k=1}^{\hat{k}} \left(\left\langle x, \sum_{l=1}^{\hat{l}} u_{kl} y_{kl} \right\rangle \right) u_{k}^{*} \right)^{*} && \text{(additivity)} \\
&= \left(\sum_{k=1}^{\hat{k}} \langle\, x, y_{k} \,\rangle\, u_{k}^{*} \right)^{*} && \text{(by (6.1))} \\
&= \left(\sum_{k=1}^{\hat{k}} \langle\, x, u_{k} y_{k} \,\rangle \right)^{*} && (\textbf{RC6}) \\
&= \langle\, x, y \,\rangle^{*} && \text{(by (6.1))} \\
&= \langle\, y, x \,\rangle && (\textbf{RC2})
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\mu_S \circ \varphi_{S\otimes S} \circ (\mathrm{id}_X \otimes \mu_{\overline{X}}^{-1} \otimes \mathrm{id}_S) \circ (\mathrm{id}_X \otimes \mu_{\overline{X}}^{-1}) \circ \sigma_X\right) (x \otimes \overline{y}) \\
&= \left(\mu_S \circ \varphi_{S\otimes S} \circ (\mathrm{id}_X \otimes \mu_{\overline{X}}^{-1} \otimes \mathrm{id}_S) \circ (\mathrm{id}_X \otimes \mu_{\overline{X}}^{-1})\right) (y \otimes \overline{x}) && \text{(def. of } \sigma_X) \\
&= (\mu_S \circ \varphi_{S\otimes S}) \left(\sum_{i=1}^{\hat{\imath}} \sum_{j=1}^{\hat{\jmath}} y \otimes \overline{x_{ij}} \otimes s_{ij}^{*} \otimes s_{i}^{*} \right) && \text{(by (6.2))} \\
&= \sum_{i=1}^{\hat{\imath}} \sum_{j=1}^{\hat{\jmath}} (\mu_S \circ \varphi_{S\otimes S}) \left(y \otimes \overline{x_{ij}} \otimes s_{ij}^{*} \otimes s_{i}^{*}\right) && \text{(additivity)} \\
&= \sum_{i=1}^{\hat{\imath}} \sum_{j=1}^{\hat{\jmath}} \mu_S \left(\left\langle y, x_{ij} \right\rangle (s_{ij}^{*} \otimes s_{i}^{*})\right) && \text{(def. of } \varphi_{S\otimes S}) \\
&= \sum_{i=1}^{\hat{\imath}} \sum_{j=1}^{\hat{\jmath}} \mu_S \left(\left\langle y, x_{ij} \right\rangle s_{ij}^{*} \otimes s_{i}^{*}\right) && (S\text{-action of } S \otimes S) \\
&= \sum_{i=1}^{\hat{\imath}} \sum_{j=1}^{\hat{\jmath}} \left\langle y, x_{ij} \right\rangle s_{ij}^{*} s_{i}^{*} && \text{(def. of } \mu_S) \\
&= \sum_{i=1}^{\hat{\imath}} \sum_{j=1}^{\hat{\jmath}} \left\langle y, s_{ij} x_{ij} \right\rangle s_{i}^{*} && (\textbf{RC6})
\end{aligned}
$$

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$$= \sum_{i=1}^{\hat{\imath}} \langle\, y, x_i \,\rangle\, s_i^* \qquad \text{(by (6.1))}$$

$$= \sum_{i=1}^{\hat{\imath}} \langle\, y, s_i x_i \,\rangle \qquad \textbf{(RC6)}$$

$$= \langle\, y, x \,\rangle\,. \qquad \text{(by (6.1))}$$

Thus, the rectangle (R1) commutes.

(2) $\Longrightarrow$ (1). By Theorem 3.1, it suffices to prove that there exists a firm and surjective Morita $*$-context $(S, T, {}_SX_T, {}_T\overline{X}_S, \theta, \phi)$. We will use the mappings $\varphi_{S\otimes S} : X \otimes \overline{X} \otimes S \otimes S \to S \otimes S$ and $\zeta_{T\otimes T} : T \otimes T \otimes \overline{X} \otimes X \to T \otimes T$ for this. Since ${}_T\overline{X}_S$ is a firm bimodule by Lemma 6.2, the mappings

$$\mu_{\overline{X}} : \overline{X} \otimes_S S \to \overline{X},\;\; \overline{x} \otimes s \mapsto \overline{x}s = \overline{s^* x},$$

$$\nu_{\overline{X}} : T \otimes_T \overline{X} \to \overline{X},\;\; t \otimes \overline{x} \mapsto t\overline{x} = \overline{xt^*}$$

are bijective. Idempotency of S and T means that the mappings

$$\mu_S : S \otimes_S S \to S,\;\; s \otimes u \mapsto su,$$

$$\mu_T : T \otimes_T T \to T,\;\; t \otimes v \mapsto tv$$

are surjective. We define θ as the composite

$$X \otimes \overline{X} \xrightarrow{\mathrm{id}_X \otimes \mu_{\overline{X}}^{-1}} X \otimes \overline{X} \otimes S \xrightarrow{\mathrm{id}_X \otimes \mu_{\overline{X}}^{-1} \otimes \mathrm{id}_S} X \otimes \overline{X} \otimes S \otimes S \xrightarrow{\varphi_{S\otimes S}} S \otimes S \xrightarrow{\mu_S} S$$

and ϕ as the composite

$$\overline{X} \otimes X \xrightarrow{\nu_{\overline{X}}^{-1} \otimes \mathrm{id}_X} T \otimes \overline{X} \otimes X \xrightarrow{\mathrm{id}_T \otimes \nu_{\overline{X}}^{-1} \otimes \mathrm{id}_X} T \otimes T \otimes \overline{X} \otimes X \xrightarrow{\zeta_{T\otimes T}} T \otimes T \xrightarrow{\mu_T} T.$$

From here we see immediately that θ and ϕ are surjective homomorphisms of left modules. Take $x, y \in X$. Using twice the fact that ${}_SX_T$ is unitary, we can write y as

$$y = \sum_{k=1}^{n} u_k s_k y_k = \sum_{l=1}^{m} z_l t_l v_l \qquad (6.3)$$

for some $m, n \in \mathbb{N}$, $u_k, s_k \in S$, $t_l, v_l \in T$ and $y_k, z_l \in X$. Then,

$$x \otimes \overline{y} = x \otimes \overline{\sum_{k=1}^{n} u_k s_k y_k} = \sum_{k=1}^{n} x \otimes \overline{u_k s_k y_k} = \sum_{k=1}^{n} x \otimes \overline{y_k} s_k^* u_k^*$$

$$= \sum_{k=1}^{n} (\mathrm{id}_X \otimes \mu_{\overline{X}})\,(x \otimes \overline{y_k} s_k^* \otimes u_k^*) \qquad \text{(def. of } \mu_{\overline{X}})$$

$$= \sum_{k=1}^{n} ((\mathrm{id}_X \otimes \mu_{\overline{X}}) \circ (\mathrm{id}_X \otimes \mu_{\overline{X}} \otimes \mathrm{id}_S))\,(x \otimes \overline{y_k} \otimes s_k^* \otimes u_k^*). \qquad \text{(def. of } \mu_{\overline{X}})$$

Hence,

$$\theta(x \otimes \overline{y}) = \sum_{k=1}^{n} (\mu_S \circ \varphi_{S\otimes S})(x \otimes \overline{y_k} \otimes s_k^* \otimes u_k^*) \qquad (6.4)$$

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and, analogously,

$$\phi(\overline{y} \otimes x) = \sum_{l=1}^{m} (\mu_T \circ \zeta_{T\otimes T})(v_l^* \otimes t_l^* \otimes \overline{z_l} \otimes x). \tag{6.5}$$

Note that $\rho_s : {}_S S \otimes_S S \to {}_S S \otimes_S S,\ u \otimes v \mapsto u \otimes vs$ is a homomorphism of left S-modules for every $s \in S$. Since φ is a natural transformation, the square

$$\begin{array}{ccc} {}_S X \otimes \overline{X} \otimes S \otimes S & \xrightarrow{\ \varphi_{S\otimes S}\ } & {}_S S \otimes S \\ \Big\downarrow{\scriptstyle \mathrm{id}_{X\otimes \overline{X}} \otimes \rho_s} & & \Big\downarrow{\scriptstyle \rho_s} \\ {}_S X \otimes \overline{X} \otimes S \otimes S & \xrightarrow[\ \varphi_{S\otimes S}\ ]{} & {}_S S \otimes S \end{array}$$

commutes. Hence,

$$\begin{aligned} \varphi_{S\otimes S}((x \otimes \overline{y} \otimes u \otimes v)s) &= \varphi_{S\otimes S}(x \otimes \overline{y} \otimes u \otimes vs) && (S\text{-action of } X \otimes \overline{X} \otimes S \otimes S) \\ &= (\varphi_{S\otimes S} \circ (\mathrm{id}_{X\otimes\overline{X}} \otimes \rho_s))\,(x \otimes \overline{y} \otimes u \otimes v) && (\text{def. of } \rho_s) \\ &= (\rho_s \circ \varphi_{S\otimes S})(x \otimes \overline{y} \otimes u \otimes v) && (\text{naturality of } \varphi) \\ &= \varphi_{S\otimes S}(x \otimes \overline{y} \otimes u \otimes v)s, && (\text{def. of } \rho_s) \end{aligned}$$

which means that $\varphi_{S\otimes S}$ is a homomorphism of (S, S)-bimodules. Analogously, $\zeta_{T\otimes T}$ is a homomorphism of (T, T)-bimodules. Using this and the equality $s^*y = \sum_{k=1}^{n} (s^*u_k)s_k y_k$, we see that

$$\begin{aligned} \theta((x \otimes \overline{y})s) &= \theta(x \otimes \overline{s^*y}) && (\text{by (2.2)}) \\ &= \sum_{k=1}^{n} (\mu_S \circ \varphi_{S\otimes S})(x \otimes \overline{y_k} \otimes s_k^* \otimes (s^*u_k)^*) && (\text{by (6.4)}) \\ &= \sum_{k=1}^{n} (\mu_S \circ \varphi_{S\otimes S})(x \otimes \overline{y_k} \otimes s_k^* \otimes u_k^* s) && (\text{by (2.1)}) \\ &= \sum_{k=1}^{n} (\mu_S \circ \varphi_{S\otimes S})((x \otimes \overline{y_k} \otimes s_k^* \otimes u_k^*)s) && (S\text{-action of } X \otimes \overline{X} \otimes S \otimes S) \\ &= \sum_{k=1}^{n} (\mu_S \circ \varphi_{S\otimes S})(x \otimes \overline{y_k} \otimes s_k^* \otimes u_k^*)s && (\mu_S, \varphi_{S\otimes S} \text{ are homomorphisms}) \\ &= \theta(x \otimes \overline{y})s, && (\text{by (6.4)}) \end{aligned}$$

so θ is a homomorphism of (S, S)-bimodules, and, analogously, ϕ is a homomorphism of (T, T)-bimodules.

Next we verify the compatibility conditions. Let x, y be as above and take also $x' \in X$. Consider the module homomorphisms

$$r_{x'} : {}_S S \to {}_S X,\quad s \mapsto sx', \tag{6.6}$$

$$l_x : T_T \to X_T,\quad t \mapsto xt \tag{6.7}$$

and the composites $\rho_{x'} := r_{x'} \circ \mu_S : S \otimes_S S \to X$ and $\lambda_x := l_x \circ \mu_T : T \otimes_T T \to X_T$. Since φ and ζ are natural transformations, the squares

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$$\begin{array}{ccc} {}_SX\otimes\overline{X}\otimes S\otimes S & \xrightarrow{\varphi_{S\otimes S}} & {}_SS\otimes S \\ {\scriptstyle \mathrm{id}_{X\otimes\overline{X}}\otimes\rho_{x'}}\big\downarrow & & \big\downarrow{\scriptstyle\rho_{x'}} \\ {}_SX\otimes\overline{X}\otimes X & \xrightarrow[\varphi_X]{} & {}_SX \end{array}$$

$$\begin{array}{ccc} T\otimes T\otimes\overline{X}\otimes X_T & \xrightarrow{\zeta_{T\otimes T}} & T\otimes T_T \\ {\scriptstyle \lambda_x\otimes\mathrm{id}_{\overline{X}\otimes X}}\big\downarrow & & \big\downarrow{\scriptstyle\lambda_x} \\ X\otimes\overline{X}\otimes X_T & \xrightarrow[\zeta_X]{} & X_T \end{array}$$

commute. Therefore,

$$\begin{aligned}
\theta(x\otimes\overline{y})x' &= \sum_{k=1}^{n}(\mu_S\circ\varphi_{S\otimes S})(x\otimes\overline{y_k}\otimes s_k^*\otimes u_k^*)x' && \text{(by (6.4))}\\
&= \sum_{k=1}^{n}(r_{x'}\circ\mu_S\circ\varphi_{S\otimes S})(x\otimes\overline{y_k}\otimes s_k^*\otimes u_k^*) && \text{(by (6.6))}\\
&= \sum_{k=1}^{n}(\rho_{x'}\circ\varphi_{S\otimes S})(x\otimes\overline{y_k}\otimes s_k^*\otimes u_k^*) && \text{(def. of } \rho_{x'}\text{)}\\
&= \sum_{k=1}^{n}(\varphi_X\circ(\mathrm{id}_{X\otimes\overline{X}}\otimes\rho_{x'}))(x\otimes\overline{y_k}\otimes s_k^*\otimes u_k^*) && \text{(naturality of } \varphi\text{)}\\
&= \sum_{k=1}^{n}\varphi_X(x\otimes\overline{y_k}\otimes s_k^*u_k^*x') && \text{(def. of } \rho_{x'}\text{)}\\
&= \sum_{k=1}^{n}\varphi_X(x\otimes\overline{y_k}s_k^*u_k^*\otimes x') && \text{(property of } \otimes\text{)}\\
&= \sum_{k=1}^{n}\varphi_X(x\otimes\overline{u_ks_ky_k}\otimes x') && \text{(by (2.2))}\\
&= \varphi_X(x\otimes\overline{y}\otimes x') && \text{(by (6.3))}\\
&= \zeta_X(x\otimes\overline{y}\otimes x') && (\varphi_X=\zeta_X)\\
&= \sum_{l=1}^{m}\zeta_X(x\otimes\overline{z_lt_lv_l}\otimes x') && \text{(by (6.3))}\\
&= \sum_{l=1}^{m}\zeta_X(x\otimes v_l^*t_l^*\overline{z_l}\otimes x') && \text{(by (2.2))}\\
&= \sum_{l=1}^{m}\zeta_X(xv_l^*t_l^*\otimes\overline{z_l}\otimes x') && \text{(property of } \otimes\text{)}
\end{aligned}$$

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$$= \sum_{l=1}^{m} (\zeta_X \circ (\lambda_x \otimes \mathrm{id}_{\overline{X}\otimes X}))(v_l^* \otimes t_l^* \otimes \overline{z_l} \otimes x') \qquad \text{(def. of } \lambda_x)$$

$$= \sum_{l=1}^{m} (\lambda_x \circ \zeta_{T\otimes T})(v_l^* \otimes t_l^* \otimes \overline{z_l} \otimes x') \qquad \text{(naturality of } \zeta)$$

$$= \sum_{l=1}^{m} (l_x \circ \mu_T \circ \zeta_{T\otimes T})(v_l^* \otimes t_l^* \otimes \overline{z_l} \otimes x') \qquad \text{(def. of } \lambda_x)$$

$$= \sum_{l=1}^{m} x(\mu_T \circ \zeta_{T\otimes T})(v_l^* \otimes t_l^* \otimes \overline{z_l} \otimes x') \qquad \text{(by (6.7))}$$

$$= x\phi(\overline{y} \otimes x'). \qquad \text{(by (6.5))}$$

Similarly, we can establish the compatibility condition $\phi(\overline{y} \otimes x)\overline{y'} = \overline{y}\theta(x \otimes \overline{y'})$.

The equalities $\theta(x \otimes \overline{y})^* = \theta(y \otimes \overline{x})$ and $\phi(\overline{y} \otimes x)^* = \phi(\overline{x} \otimes y)$ follow from the commutativity of the rectangles (R1) and (R2). □

7. Comparison with earlier results

In this section, we compare our results with those of Ara [3].

A module M_R is called *nondegenerate* ([3]) or *torsion-free* ([15, p. 221]) if, for every $m \in M$, $mR = 0$ implies $m = 0$. A ring R is called *nondegenerate* if it is nondegenerate as a right and left module over itself.

Recall that a module M_R is called *s-unital* ([25]) if, for every $m \in M$, there exists $r \in R$ such that $m = mr$. A ring R is called *s-unital* if it is s-unital as a right and left module over itself. It is easy to see that every s-unital module is nondegenerate.

Lemma 7.1 ([8, Proposition 1.8]). *Every unitary left module over a left s-unital ring is s-unital.*

Definition 7.2. *We say that a Rieffel context* $(S, T, X, \langle\, , \,\rangle, [\, , \,])$ *is nondegenerate if, for every* $x \in X$,

$$(\langle X, x \rangle = \{0_S\} \implies x = 0_X) \text{ and } ([x, X] = \{0_T\} \implies x = 0_X),$$

where $\langle X, x \rangle := \{\langle \xi, x \rangle \mid \xi \in X\}$ *and* $[x, X] := \{[x, \xi] \mid \xi \in X\}$.

Proposition 7.3. *Let S and T be s-unital rings with involution. Then every unitary and full Rieffel context between S and T is nondegenerate.*

Proof. Let $(S, T, X, \langle\, , \,\rangle, [\, , \,])$ be a unitary and full Rieffel context and assume that $\langle X, x \rangle = \{0_S\}$. By Lemma 7.1, X_T is s-unital. Hence, $x = xt$ for some $t \in T$. Since $[\, , \,]$ is full, there exist $y_k, y'_k \in X$ such that $t = \sum_{k=1}^{n} [y_k, y'_k]$. Now

$$x = xt = x\left(\sum_{k=1}^{n} [y_k, y'_k]\right) = \sum_{k=1}^{n} x[y_k, y'_k] = \sum_{k=1}^{n} \langle x, y_k \rangle y'_k = \sum_{k=1}^{n} \langle x, y_k \rangle^{**} y'_k$$

$$= \sum_{k=1}^{n} \langle y_k, x \rangle^* y'_k = \sum_{k=1}^{n} 0_S^* y'_k = \sum_{k=1}^{n} 0_S y'_k = 0_X.$$

A similar proof shows that $[x, X] = \{0_T\}$ implies $x = 0_X$. □

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If $(S, T, X, \langle\ ,\ \rangle, [\ ,\ ])$ is a nondegenerate Rieffel context, then according to the terminology of [3, page 243], ${}_SX_T$ is called an *inner product* (S, T)*-bimodule*.

In [3, page 235], two nondegenerate idempotent rings with involution R and S are called *Morita ∗-equivalent* if there exists a ∗-equivalence (F, G) (which is a pair of certain functors between categories of unitary torsion-free modules) from R to S.

Theorem 7.4 ([3]). *Two nondegenerate idempotent rings with involution are Morita ∗-equivalent if and only if there exists a full nondegenerate unitary Rieffel context connecting these rings.*

Proof. **Necessity.** As explained in the second paragraph of [3, page 243], by Theorem 3.1 of that article, there exists a full nondegenerate unitary Rieffel context.

Sufficiency. This is shown in [3, Theorem 4.1(i)]. □

As a consequence, we obtain the following result.

Proposition 7.5. *Two s-unital rings S and T with involution are Morita ∗-equivalent if and only if they are connected by a unitary and full Rieffel context.*

Proof. **Necessity.** Let S and T be Morita ∗-equivalent. By s-unitality, rings S and T are both idempotent and nondegenerate. Hence Theorem 7.4 applies.

Sufficiency. Assume that S and T are connected by a unitary and full Rieffel context. By Proposition 7.3, this context is nondegenerate. By Theorem 7.4, S and T are Morita ∗-equivalent. □

So for s-unital rings with involution, the theory developed in this paper is compatible with that of [3].

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