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TWISTED ALEXANDER IDEALS AND THE ISOMORPHISM PROBLEM FOR A FAMILY OF PARAFREE GROUPS

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Abstract In 1969, Baumslag introduced a family of parafree groups $G_{i,j}$ which share many properties with the free group of rank 2. The isomorphism problem for the family $G_{i,j}$ is known to be difficult; a few small partial results have been found so far. In this paper, we compute the twisted Alexander ideals of the groups $G_{i,j}$ associated with non-abelian representations into $SL(2,\mathbb{Z}_2)$. Using the twisted Alexander ideals, we prove that several pairs of groups among $G_{i,j}$ are not isomorphic. As a consequence, we solve the isomorphism problem for sub-families containing infinitely many groups $G_{i,j}$.

Keywords: group isomorphism problem; twisted Alexander ideal; parafree groups

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1. Introduction

The isomorphism problem is a fundamental problem in group theory in which one has to decide whether two finitely presented groups are isomorphic. Because the general isomorphism problem is unsolvable, people often restrict the problem to a special class of groups. Recall that a group G is called n-*parafree* if it is residually nilpotent and has the same nilpotent quotient as the free group F_n and *parafree* if n-parafree for some n. As parafree groups share many properties with free groups, the isomorphism problem for parafree groups is known to be difficult.

In this paper, we study a family of parafree groups $G_{i,j}$ which was introduced by Baumslag in [**2**, **3**]:

$$G_{i,j} := \langle a, b, c \mid a = [c^i, a][c^j, b]\rangle,$$

where $[x, y] := x^{-1}y^{-1}xy$ and i, j are positive integers. The isomorphism problem for the family $G_{i,j}$ has attracted considerable interest. The family $G_{i,j}$ is mentioned in [**6**] by Magnus and Chandler to demonstrate the difficulty of the isomorphism problem for torsion-free one-relator groups. They also note that as of 1980 it was unknown whether any pair of the groups $G_{i,j}$ was non-isomorphic. Later, several approaches were used to attack the isomorphism problem for the family $G_{i,j}$. In 1994, Lewis and Liriano [13] distinguished a number of parafree groups in the family $G_{i,j}$ by counting the homomorphisms between $G_{i,j}$ and the finite groups $SL(2,\mathbb{Z}/4)$ and $SL(2,\mathbb{Z}/5)$. A group-theoretical attack by Fine, Rosenberger and Stille [7] was able to show that $G_{i,1} \ncong G_{1,1}$ for $i > 1$ and $G_{i,1} \ncong G_{j,1}$ for distinct primes i, j. More recently, by using a computational approach along the lines of [13], Baumslag *et al.* [5] showed that all the groups $G_{i,j}, 1 \leq i, j \leq 10$ are distinct.

In our previous work [11], we used the Alexander ideal, an algebraic invariant of groups which originated from topology, to study the isomorphism problem for families of groups. Our approach in [11] completely solves the isomorphism problem for the Baumslag–Solitar groups and a family of parafree groups $K_{i,j} := \langle a, s, t |\ a^i[s, a] = t^j \rangle$ introduced by Baumslag and Cleary in [4]. However, as noted in [11], the Alexander ideals of all the group $G_{i,j}$ are trivial.

In this paper, we develop our approach in [11] further to attack the isomorphism problem for the family of groups $G_{i,j}$ by using the twisted Alexander ideals. The twisted Alexander ideal is a non-abelian generalization of the classical Alexander polynomial. It turns out that, for certain values of i, j, the twisted Alexander ideals of $G_{i,j}$ are non-trivial. By comparing the twisted Alexander ideals, we obtain sub-families of $G_{i,j}$ which contain infinitely many pairwise non-isomorphic groups. Our result is completely disjoint from the known results in [5, 7, 13].

Theorem 1.1.

(i) *Let p, q be two positive odd integers such that* $\gcd(p, q) = 1$. *For any $d, d' \geq 1$, the following holds:*

$$G_{p(2d-1),q(2d-1)} \cong G_{p(2d'-1),q(2d'-1)} \quad \text{if and only if } d = d'.$$

(ii) *Let p, q be two positive integers such that* $\gcd(p, q) = 1$ *and* $3|(p + q)$. *Then, for any $d, d' > 1$ and $3 \nmid d, 3 \nmid d'$, the following holds:*

$$G_{pd,qd} \cong G_{pd',qd'} \quad \text{if and only if } d = d'.$$

The rest of this paper consists of four sections. In § 2, we give a brief review of the background on twisted Alexander ideals of a group. In § 3, we classify conjugacy classes of non-abelian representations from $G_{i,j}$ into $G_{i,j}$ and prove a series of technical lemmas about Laurent polynomials. Section 4 contains the computation of the twisted Alexander ideals of $G_{i,j}$ associated to non-abelian representations into $SL(2,\mathbb{Z}_2)$. Section 5 is devoted to applications of the twisted Alexander ideals to the isomorphism problem for the family $G_{i,j}$. In particular, we show that several pairs of groups among $G_{i,j}$ are non-isomorphic and, as a consequence, we obtain Theorem 1.1 above.

2. Background on twisted Alexander ideals

The Alexander polynomial (see [1, 10]) is a topological invariant of knots which can be computed from the information on the fundamental group of its complement. The twisted Alexander ideals are non-abelian generalizations of the classical Alexander polynomials.

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The twisted Alexander ideals for knots were introduced by Lin in [**14**]. In this paper, we use a version of twisted Alexander ideals by Wada [**15**] which is defined for a finitely presented group. There is an effective algorithm to compute the twisted Alexander ideals by using Fox's free differential calculus [**8**, **9**], which we will describe briefly below.

Suppose that $F_k = \langle x_1, \ldots, x_k | \rangle$ is the free group on k generators. Let $\epsilon : \mathbb{Z}F_k \to \mathbb{Z}$ be the augmentation homomorphism defined by $\epsilon(\sum n_i g_i) = \sum n_i$. The jth partial Fox derivative is a linear operator $\partial/\partial x_j : \mathbb{Z}F_k \to \mathbb{Z}F_k$ which is uniquely determined by the following rules:

$$\frac{\partial}{\partial x_j}(1) = 0 \quad \frac{\partial}{\partial x_j}(x_i) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j; \end{cases}$$
$$\frac{\partial}{\partial x_j}(uv) = \frac{\partial}{\partial x_j}(u)\epsilon(v) + u\frac{\partial}{\partial x_j}(v).$$

As consequences of the above rules we get:

(i) $$\frac{\partial}{\partial x_i}(x_i^n) = 1 + x_i + x_i^2 + \cdots + x_i^{n-1} \quad \text{for all } n \geq 1;$$

(ii) $$\frac{\partial}{\partial x_i}(x_i^{-n}) = -x_i^{-1} - x_i^{-2} - \cdots - x_i^{-n} \quad \text{for all } n \geq 1.$$

Let $G = \langle x_1, \ldots, x_k | r_1, \ldots, r_l \rangle$ be a finitely presented group. We denote by $\mathrm{ab}(G)$ the maximal abelian quotient of G. From the sequence

$$F_k \xrightarrow{\phi} G \xrightarrow{\alpha} \mathrm{ab}(G)$$

we get the sequence

$$\mathbb{Z}F_k \xrightarrow{\tilde{\phi}} \mathbb{Z}G \xrightarrow{\tilde{\alpha}} \mathbb{Z}[\mathrm{ab}(G)].$$

Under the assumption that $\mathrm{ab}(G)$ is torsion-free, we fix an isomorphism $\chi : \mathrm{ab}(G) \to \mathbb{Z}^r$. So the group ring $\mathbb{Z}[\mathrm{ab}(G)]$ can be identified with $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_r^{\pm 1}]$.

Given a homomorphism $\rho : G \to GL(n, R)$, where R is an unique factorization domain, we get the induced homomorphism of group ring $\tilde{\rho} : \mathbb{Z}G \to M(n, R)$.

Denote by $\tilde{\rho} \otimes \tilde{\alpha} : \mathbb{Z}G \to M(n, R[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_r^{\pm 1}])$ the tensor product homomorphism which is defined by

$$(\tilde{\rho} \otimes \tilde{\alpha})\big(\sum n_i g_i\big) = \sum n_i \rho(g_i)\alpha(g_i) \quad \text{where } n_i \in \mathbb{Z}, g_i \in G. \tag{2.1}$$

We also need the composition map $\Phi := (\tilde{\rho} \otimes \tilde{\alpha}) \circ \tilde{\phi} : \mathbb{Z}F_k \to M(n, R[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_r^{\pm 1}])$. More details of the construction of the twisted Alexander ideals can be found in [**12**, § 2].

We regard $(\Phi(\partial/\partial x_j r_i))_{i=1,\ldots,\ell, j=1,\ldots,k}$ as an $n\ell \times nk$ matrix whose entries belong to $R[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_r^{\pm 1}]$, which we call the *twisted Alexander matrix of G associated with the n-dimensional representation ρ*.

The dth *twisted Alexander ideal of G associated with the n-dimensional representation* ρ is defined as the ideal generated by all the $(nk - d)$-minors of the twisted Alexander matrix. Recall that a k-minor of a $m \times n$ matrix A, with $0 < k \leq \min(m, n)$, is the determinant of some $k \times k$ matrix obtained from A by deleting $(m - k)$ rows and $(n - k)$ columns.

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Theorems 2.1 and 2.2 of [**12**] say that the twisted Alexander ideal does not depend on the choice of the presentation of G. It depends only on the group G, the conjugacy class of the representation ρ and the choice of the isomorphism χ. Since we have freedom in choosing the isomorphism χ, the twisted Alexander ideal is an invariant of (G, ρ) defined up to a *monomial automorphism* of $R[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_r^{\pm 1}]$; that is, an automorphism of the form $\varphi(t_i) = t_1^{a_{i1}} t_2^{a_{i2}} \cdots t_r^{a_{ir}}, i = 1, 2, \ldots r$, where $(a_{ij}) \in GL(n, \mathbb{Z})$. In summary, we have the following.

Theorem 2.1. *The twisted Alexander ideal is an invariant of the pair* (G, ρ) *consisting of a group* G *and a conjugacy class of the representation* $\rho : G \to GL(n, R)$*, up to a monomial automorphism of* $R[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_r^{\pm 1}]$.

3. Auxiliary results

This section is devoted to some auxiliary results to set up the computation of the twisted Alexander ideals. Note that, for all i, j, the maximal abelian quotient of $G_{i,j}$ has the following presentation:

$$\mathrm{ab}(G_{i,j}) := \left\langle a, b, c \mid a = [c^i, a][c^j, b], [a, b] = [b, c] = [c, a] = 1 \right\rangle = \langle b, c \| \ [b, c] = 1 \rangle .$$

Therefore, $\mathrm{ab}(G_{i,j})$ is isomorphic to a free abelian group of rank 2:

$$\begin{array}{ccc} \mathrm{ab}(G_{i,j}) & \cong & \langle x \rangle \oplus \langle y \rangle \\ a & \mapsto & 0 \\ b & \mapsto & x \\ c & \mapsto & y. \end{array}$$

We will fix an identification of $\mathbb{Z}[\mathrm{ab}(G_{i,j})]$ with the ring of Laurent polynomials $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ by mapping b to x and c to y. We denote by L the ring of Laurent polynomials with $\mathbb{Z}_2$ coefficients $\mathbb{Z}_2[x^{\pm 1}, y^{\pm 1}]$.

As each group $G_{i,j}$ is given by three generators and one relation, the twisted Alexander matrix is of size 2×6. We will compute the fourth twisted Alexander ideal of $G_{i,j}$, that is, the ideal in L generated by all the 2-minors of the twisted Alexander matrix. From now on, when we refer to twisted Alexander ideals of $G_{i,j}$, we always mean the fourth twisted Alexander ideals.

We only consider the 'twisting' given by non-abelian representations, since the abelian case reduces to the usual Alexander ideal, which is trivial as noted above. As we know from Theorem 2.1, the twisted Alexander ideal only depends on the conjugacy class of ρ. We have the following.

Proposition 3.1. *There are exactly three conjugacy classes of non-abelian representations* $\rho : G_{i,j} \to SL(2, \mathbb{Z}_2)$*, for every* i, j.

Proof. It is well known that the group $SL(2, \mathbb{Z}_2)$ is isomorphic to the symmetric group S_3 and its structure is very simple.

If ρ is a representation, then $\rho(a) = [\rho(c^i), \rho(a)][\rho(c^j), \rho(b)]$. Because the commutator subgroup of $SL(2, \mathbb{Z}_2)$ is abelian, we get $\rho(a) = [\rho(c^j), \rho(b)][\rho(c^i), \rho(a)]$. From that we easily find that $\rho(a) = \rho(c^i)[\rho(c^j), \rho(b)]\rho(c^{-i})$. Therefore, any representation ρ is uniquely

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Table 1. *Conjugacy classes of non-abelian representations.*

Type of Repr.	$\rho(b)$	$\rho(c)$
1	$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$
2	$\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$
3	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$

specified by the images $\rho(b)$ and $\rho(c)$. Direct calculation gives us that, independent of i, j, there are three conjugacy classes that each contain six representations. The representative of each conjugacy class is given in Table 1. □

We now compute the Fox derivatives of the relation of $G_{i,j}$. Choosing the relation $r = a[c^j, b]^{-1}[c^i, a]^{-1} = ab^{-1}c^{-j}bc^j a^{-1}c^{-i}ac^i$, we find

$$\frac{\partial r}{\partial a} = 1 - a[c^j, b]^{-1}a^{-1} + a[c^j, b]^{-1}a^{-1}c^{-i}.$$

Now, as $r = 1$ in $G_{i,j}$, we simplify to get

$$\tilde{\phi}\left(\frac{\partial r}{\partial a}\right) = 1 - c^{-i}a^{-1}c^i + c^{-i}a^{-1}.$$

Similarly, we get the other Fox derivatives:

$$\begin{aligned}\tilde{\phi}\left(\frac{\partial r}{\partial b}\right) &= -ab^{-1} + ab^{-1}c^{-j},\\ \tilde{\phi}\left(\frac{\partial r}{\partial c}\right) &= -ab^{-1}(c^{-1} + \cdots + c^{-j}) + ab^{-1}c^{-j}b(1 + c + \cdots + c^{j-1})\\ &\quad - c^{-i}a^{-1}(1 + c + \cdots + c^{i-1}) + (c^{-1} + \cdots + c^{-i}).\end{aligned}$$

To simplify the twisted Alexander ideals, we need to employ several technical lemmas about greatest common divisors and divisibilities of Laurent polynomials. Before going into computing the twisted Alexander ideals, we will present here a series of utility lemmas. We first remind the readers of some basic properties of the Laurent polynomial ring L which will be used later.

(a) L is a unique factorization domain, that is, it has no zero-divisor and every non-zero non-unit element can be written as a product of irreducible elements, uniquely up to order and units.

(b) Any two elements of L have a greatest common divisor which is unique up to multiplication by units.

(c) Units of L are elements of the form $x^m y^n$, where m, n are integers.

(d) A polynomial in L has $\mathbb{Z}_2$-coefficients, so we are free to write 1 or (-1) for any non-zero coefficient when it is convenient.

Lemma 3.2. *In the Laurent polynomial ring L, for all integers m, n which are not simultaneously zero, we have*

$$\gcd(1+y^m, 1+y^n) = 1 + y^{\gcd(m,n)}.$$

Proof. If one of the numbers m or n is zero, the assertion is obviously true. Note that $(1+y^{-n})$ and $(1+y^n)$ differ by a unit factor: $(1+y^{-n}) = y^{-n}(1+y^n)$. Therefore, we only need to prove the case where m and n are both positive.

We proceed by induction on $m+n$. The case $m+n=2$ is obviously true. Suppose that the lemma holds for all $m+n<t$; we now show that it also holds for $m+n=t$. If $m=n$ the lemma is also obviously true, so we may assume that $m>n$. We see that

$$\gcd(1+y^m, 1+y^n) = \gcd(y^n(1+y^{m-n}) + (1+y^n), 1+y^n) = \gcd(1+y^{m-n}, 1+y^n).$$

By induction hypothesis, $\gcd(1+y^{m-n}, 1+y^n) = 1+y^{\gcd(m-n,n)} = 1+y^{\gcd(m,n)}$. So the lemma is proved. □

Lemma 3.3. *Suppose that $3 \nmid s$. Then, in the Laurent polynomial ring L, we have*

$$\gcd(1+y^t+y^{2t}, 1+y^s) = 1 \quad \textit{for any } t.$$

Proof. As $3 \nmid s$, using Lemma 3.2 we obtain:

$$\gcd((1+y^t)(1+y^t+y^{2t}), 1+y^s) = \gcd(1+y^{3t}, 1+y^s) = 1+y^{\gcd(3t,s)} = 1+y^{\gcd(t,s)}. \tag{3.1}$$

On the other hand, since $\gcd(1+y^t, 1+y^t+y^{2t}) = \gcd(1+y^t, y^{2t}) = 1$, we also have

$$\gcd((1+y^t)(1+y^t+y^{2t}), 1+y^s) = \gcd(1+y^t, 1+y^s)\gcd(1+y^t+y^{2t}, 1+y^s).$$

It follows from Lemma 3.2 that

$$\gcd((1+y^t)(1+y^t+y^{2t}), 1+y^s) = (1+y^{\gcd(t,s)})\gcd(1+y^t+y^{2t}, 1+y^s). \tag{3.2}$$

From (3.1) and (3.2), it follows that $\gcd(1+y^t+y^{2t}, 1+y^s) = 1$. □

Lemma 3.4. *Suppose that $m = kd$ and $n = ld$ where $d = \gcd(m,n)$. Assume further that $3 \nmid k$, $3 \nmid l$, and $3 \mid (k+l)$. Then, in the ring L, the following holds:*

$$\gcd(1+y^m+y^{2m}, 1+y^{m+n}) = 1+y^d+y^{2d}.$$

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Proof. As $3|(k+l)$, by Lemma 3.2 we have

$$\gcd((1+y^m)(1+y^m+y^{2m}), 1+y^{m+n}) = \gcd(1+y^{3kd}, 1+y^{(k+l)d}) = 1+y^{d\gcd(3k,k+l)}$$
$$= 1+y^{3d\gcd(k,k+l)} = 1+y^{3d}. \tag{3.3}$$

On the other hand, as in the proof of Lemma 3.3 above,

$$\gcd((1+y^m)(1+y^m+y^{2m}), 1+y^{m+n}) = (1+y^d)\gcd(1+y^m+y^{2m}, 1+y^{m+n}). \tag{3.4}$$

Combining 3.3 and 3.4, we get

$$(1+y^d)\gcd(1+y^m+y^{2m}, 1+y^{m+n}) = 1+y^{3d} = (1+y^d)(1+y^d+y^{2d}).$$

It follows that $\gcd(1+y^m+y^{2m}, 1+y^{m+n}) = 1+y^d+y^{2d}$. □

Corollary 3.5. *Suppose that $3\nmid k$. Then $(1+y^d+y^{2d})|(1+y^{kd}+y^{2kd})$ for any positive integer d.*

Proof. We can always reduce to the case where k is odd, since if k is even then $1+y^{kd}+y^{2kd} = (1+y^{kd/2}+y^{kd})^2$. The corollary then follows by applying Lemma 3.4 for $m = kd$, $n = d$ if $k \equiv 2 \bmod 3$ and for $m = kd$, $n = 2d$ if $k \equiv 1 \bmod 3$. □

Lemma 3.6. *Suppose that m, l, d are positive integers satisfying $m = ld$ and $3\nmid m$. The following hold in L:*

(i) *if l is even then $(1+y+y^2)(1+y^d+y^{2d})|(1+y^m+y^{2m})$;*

(ii) *if $l \equiv 1 \bmod 6$ then $(1+y+y^2)(1+y^d+y^{2d})|((1+y^m+y^{2m})-(1+y^d+y^{2d}))$;*

(iii) *if $l \equiv 5 \bmod 6$ then $(1+y+y^2)(1+y^d+y^{2d})|((1+y^m+y^{2m})-y^{-2d}(1+y^d+y^{2d}))$.*

Proof. If l is even then $m = 2td$, and we get

$$1+y^m+y^{2m} = (1+y^{td}+y^{2td})^2.$$

As $3\nmid m = 2td$, it follows from Corollary 3.5 that $(1+y^{td}+y^{2td})$ is divisible by both $(1+y+y^2)$ and $(1+y^d+y^{2d})$. So (i) holds.

For the proof of (ii), we assume that $l = 6t+1$ and first consider the case where $3\nmid t$. By Corollary 3.5, $(1+y^{td}+y^{2td})$ is divisible by both $(1+y+y^2)$ and $(1+y^d+y^{2d})$. It follows that $1+y^{6td} = (1+y^{td})^2(1+y^{td}+y^{2td})^2$ is divisible by $(1+y+y^2)(1+y^d+y^{2d})$. We deduce that

$$(1+y^m+y^{2m})-(1+y^d+y^{2d}) = y^d(1+y^{6td})+y^{2d}(1+y^{12td})$$

is divisible by $(1+y+y^2)(1+y^d+y^{2d})$ and (ii) holds in this case.

In the case $3|t$, by Lemma 3.2, $(1+y^{td})$ is divisible by both $1+y^3 = (1+y)(1+y+y^2)$ and $1+y^{3d} = (1+y^d)(1+y^d+y^{2d})$. Therefore, $(1+y^{td})^2$ is divisible by $(1+y+y^2)(1+y^d+y^{2d})$ and so is $1+y^{6td} = (1+y^{td})^2(1+y^{td}+y^{2td})^2$. As in the previous paragraph, we deduce that (ii) also holds.

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For the proof of (iii), we assume that $l = 6t + 5$ and first consider the case where $3 \nmid (t+1)$. By Corollary 3.5, $(1 + y^{(t+1)d} + y^{2(t+1)d})$ is divisible by both $(1 + y + y^2)$ and $(1 + y^d + y^{2d})$. It follows that $1 + y^{6(t+1)d} = (1 + y^{(t+1)d})^2(1 + y^{(t+1)d} + y^{2(t+1)d})^2$ is divisible by $(1 + y + y^2)(1 + y^d + y^{2d})$. We deduce that

$$(1 + y^m + y^{2m}) - y^{-2d}(1 + y^d + y^{2d}) = y^{-d}(1 + y^{6(t+1)d}) + y^{-2d}(1 + y^{12(t+1)d})$$

is divisible by $(1 + y + y^2)(1 + y^d + y^{2d})$ and (iii) holds in this case.

In the case $3|(t+1)$, by Lemma 3.2, $(1 + y^{(t+1)d})$ is divisible by both $1 + y^3 = (1 + y)(1 + y + y^2)$ and $1 + y^{3d} = (1 + y^d)(1 + y^d + y^{2d})$. Therefore, $(1 + y^{(t+1)d})^2$ is divisible by $(1 + y + y^2)(1 + y^d + y^{2d})$ and so is $1 + y^{6(t+1)d} = (1 + y^{(t+1)d})^2(1 + y^{(t+1)d} + y^{2(t+1)d})^2$. Arguing as in the previous paragraph, we deduce that (iii) also holds. □

4. Computations of the twisted Alexander ideals

In each of the following subsections, we will present the computation of the twisted Alexander ideal of the group $G_{i,j}$ associated with each type of conjugacy class of representation $\rho : G_{i,j} \to SL(2, \mathbb{Z}_2)$.

4.1. Representation of type 1

To state the next result, we need to introduce the following Laurent polynomials in the ring L:

$$f_{2n} := 1 + y^2 + \cdots + y^{2n} \text{ for } n \geq 1, \quad f_0 := 1 \quad \text{and} \quad f_{2n} := 0 \text{ for } n < 0.$$

The rest of this subsection is devoted to the proof of the following.

Proposition 4.1. *Let I be the twisted Alexander ideal of $G_{i,j}$ associated with a representation of type* 1. *We write $d := \gcd(i, j)$ and f_{2n} defined as above. Then*:

(i) *$I = L$ in the case where either j or i is even*;

(ii) *$I = (f_{2(d-1)})$ in the case where both i, j are odd and $4|(i - j)$*;

(iii) *$I = (1 + y^{2d}, f_{2(d-1)} + xyf_{2(d-1)})$ in the case where both i, j are odd and $4 \nmid (i - j)$.*

Proof. To find the twisted Alexander matrix we have to compute $\Phi(\partial r/\partial a), \Phi(\partial r/\partial b)$ and $\Phi(\partial r/\partial a)$ where Φ is defined in (2.1). We divide into several cases.

Case 1: j is even. From Table 1 above, we know $\rho(b)$ and $\rho(c)$. We find that $\rho(a) = \rho(c^i)[\rho(c^j), \rho(b)]\rho(c^{-i}) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We deduce that $\Phi(\partial r/\partial a) = y^{-i}\rho(c)^{-i}$ and therefore $\det(\Phi(\partial r/\partial a)) = y^{-i}$.

As y^{-i} is a unit, we see that in this case I is the whole ring L.

Case 2: i is even and j is odd. In this case, we compute $\rho(a) = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Next, we find that

$$\Phi\left(\frac{\partial r}{\partial a}\right) = \begin{pmatrix} 1 & 1 + y^{-i} \\ 1 + y^{-i} & y^{-i} \end{pmatrix} \quad \text{and} \quad \Phi\left(\frac{\partial r}{\partial b}\right) = x^{-1}\begin{pmatrix} 1 & y^{-j} \\ 1 + y^{-j} & 1 + y^{-j} \end{pmatrix}.$$

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Note that multiplying a column by a unit does not affect the twisted Alexander ideal (see [**12**, p. 298]). *From now on, in every case, we always ignore the factor* x^{-1} *in* $\Phi(\partial r/\partial b)$. Consider two 2-minors:

$$\det\begin{pmatrix} 1 & y^{-j} \\ 1+y^{-i} & 1+y^{-j}\end{pmatrix} = 1+y^{-i-j}, \quad \det\begin{pmatrix} 1+y^{-i} & 1 \\ y^{-i} & 1+y^{-j}\end{pmatrix} = 1+y^{-i-j}+y^{-j}.$$

We conclude that the I contains y^{-j}, which is a unit. Therefore, part (i) is proved.

Case 3: i and j are odd. We find that $\rho(a) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1\end{smallmatrix}\right)$. From that we get

$$\Phi\left(\frac{\partial r}{\partial a}\right) = \begin{pmatrix} 1+y^{-i} & 1 \\ 1+y^{-i} & y^{-i}\end{pmatrix}, \quad \Phi\left(\frac{\partial r}{\partial b}\right) = \begin{pmatrix} y^{-j} & 1 \\ 1 & y^{-j}\end{pmatrix} \quad \text{and}$$

$$\begin{aligned}\Phi\left(\frac{\partial r}{\partial c}\right) &= -\rho(ab^{-1})x^{-1}(\rho(c^{-1})y^{-1}+\cdots+\rho(c^{-j})y^{-j}) \\ &\quad + \rho(ab^{-1}c^{-j}b)y^{-j}(\rho(1)+\rho(c)y\cdots+\rho(c^{j-1})y^{j-1}) \\ &\quad - \rho(c^{-i}a^{-1})y^{-i}(\rho(1)+\rho(c)y+\cdots+\rho(c^{i-1})y^{i-1}) \\ &\quad + (\rho(c^{-1})y^{-1}+\cdots+\rho(c^{-i})y^{-i}). \end{aligned} \tag{4.1}$$

As $\rho(c) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right), \rho(c)^2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right)$, we have the following identities:

$$\rho(1)+\rho(c)y\cdots+\rho(c^{2n})y^{2n} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} f_{2n} + \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} y f_{2(n-1)}; \tag{4.2}$$

$$\rho(c^{-1})y^{-1}+\cdots+\rho(c^{-2n+1})y^{-2n+1} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} y^{-2n+2} f_{2(n-2)} + \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} y^{-2n+1} f_{2(n-1)}. \tag{4.3}$$

As i, j are odd, we use (4.2) and (4.3) to simplify (4.1) as follows:

$$\begin{aligned}\Phi\left(\frac{\partial r}{\partial c}\right) &= -\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} x^{-1}y^{-j}f_{j-1} - \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} x^{-1}y^{-j+1}f_{j-3} + \begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix} y^{-j}f_{j-1} \\ &\quad + \begin{pmatrix} 1 & 1 \\ 1 & 0\end{pmatrix} y^{-j+1}f_{j-3} - \begin{pmatrix} 1 & 0 \\ 1 & 1\end{pmatrix} y^{-i}f_{i-1} - \begin{pmatrix} 0 & 1 \\ 1 & 1\end{pmatrix} y^{-i+1}f_{i-3} \\ &\quad + \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} y^{-i+1}f_{i-3} + \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} y^{-i}f_{i-1}.\end{aligned}$$

Now, we change the coefficients to 1 and combine all into the following matrix:

$$\begin{pmatrix} \begin{array}{c} x^{-1}y^{-j}f_{j-1}+y^{-j}f_{j-1}+y^{-j+1}f_{j-3} \\ +y^{-i}f_{i-1}+y^{-i+1}f_{i-3}\end{array} & \begin{array}{c} x^{-1}y^{-j+1}f_{j-3}+y^{-j+1}f_{j-3}+y^{-j}f_{j-1} \\ +y^{-i}f_{i-1}+y^{-i+1}f_{i-3}\end{array} \\ x^{-1}y^{-j+1}f_{j-3}+y^{-j+1}f_{j-3}+y^{-i+1}f_{i-3} & x^{-1}y^{-j}f_{j-1}+y^{-j}f_{j-1}+y^{-i}f_{i-1}\end{pmatrix}.$$

Next, we perform the elementary operations to simplify the twisted Alexander matrix (see [**12**, p. 298], for a list of elementary operations). Notice that the second column of

$\Phi(\partial r/\partial a)$ is $\binom{1}{y^{-i}}$. We use this column to make all the entries in the first row of any other column $\binom{X}{Y}$ to be zero by replacing it with

$$\begin{pmatrix} X \\ Y \end{pmatrix} + X\begin{pmatrix} 1 \\ y^{-i} \end{pmatrix} = \begin{pmatrix} 0 \\ Y + y^{-i}X \end{pmatrix}.$$

So we bring the twisted Alexander matrix to the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1+y^{-2i} & y^{-i} & 1+y^{-i-j} & y^{-i}+y^{-j} & A & B \end{pmatrix},$$

where

$$\begin{aligned} A &= y^{-i-j}(x^{-1}f_{j-1} + f_{j-1} + yf_{j-3}) + y^{-2i}(f_{i-1} + yf_{i-3}) \\ &\quad + x^{-1}y^{-j+1}f_{j-3} + y^{-j+1}f_{j-3} + y^{-i+1}f_{i-3}, \\ B &= y^{-i-j}(x^{-1}yf_{j-3} + yf_{j-3} + f_{j-1}) + y^{-2i}(f_{i-1} + yf_{i-3}) \\ &\quad + x^{-1}y^{-j}f_{j-1} + y^{-j}f_{j-1} + y^{-i}f_{i-1}. \end{aligned}$$

Since multiplying a generator of an ideal by unit of the ring does not affect the ideal, we change each generator of the form $(1+y^{-n})$ to $y^n(1+y^{-n}) = 1+y^n$ and $(y^{-i}+y^{-j})$ to $y^i(y^{-i}+y^{-j}) = 1+y^{i-j}$. So we get $I = (1+y^{2i}, 1+y^{i+j}, 1+y^{i-j}, A, B)$.

Writing $i = kd, j = ld$, where $\gcd(k,l) = 1$, as i, j are odd, we easily deduce that $\gcd(2i, i+j, i-j) = 2d$. It follows from Lemma 3.2 that $I = (1+y^{2d}, A, B)$. We will simplify A and B to write I in a simpler form.

Note that I is unchanged if we replace the generator A by $A' = A + f \in L$, where f is a multiple of $(1+y^{2d})$. In the proof of this Proposition, we use the notation $A \equiv A'$ if and only if $(A - A')$ is a multiple of $(1+y^{2d})$. In the following, we are allowed to replace the generator A (or B) by A' (or B') such that $A \equiv A'$ (or $B \equiv B'$).

As $(1+y^{2i})$ is a multiple of $(1+y^{2d})$, for any polynomial g we have

$$y^{-2i}g = y^{-2i}(1+y^{2i})g + g \equiv g.$$

So we can replace the factor y^{-2i} of any term in the generators A or B by 1. Similarly, other terms such as $y^{2i}, y^{\pm 2j}, y^{\pm i \pm j}, \ldots$ appearing in A or B can also be replaced by 1.

With this in mind, we find that

$$\begin{aligned} A &\equiv (x^{-1}f_{j-1} + f_{j-1} + yf_{j-3}) + (f_{i-1} + yf_{i-3}) \\ &\quad + x^{-1}y^{-j+1}f_{j-3} + y^{-j+1}f_{j-3} + y^{-i+1}f_{i-3} \end{aligned}$$

and

$$B \equiv (x^{-1}yf_{j-3} + yf_{j-3} + f_{j-1}) + (f_{i-1} + yf_{i-3}) + x^{-1}y^{-j}f_{j-1} + y^{-j}f_{j-1} + y^{-i}f_{i-1}.$$

After regrouping, we get:

$$\begin{aligned} A &\equiv x^{-1}(f_{j-1} + y^{-j+1}f_{j-3}) + (f_{j-1} + y^{-j+1}f_{j-3}) \\ &\quad + (f_{i-1} + y^{-i+1}f_{i-3}) + (yf_{i-3} + yf_{j-3}), \qquad (4.4) \\ B &\equiv x^{-1}(yf_{j-3} + y^{-j}f_{j-1}) + (yf_{j-3} + y^{-j}f_{j-1}) + (yf_{i-3} + y^{-i}f_{i-1}) + (f_{i-1} + f_{j-1}). \end{aligned}$$

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As noted before, multiplying B by the unit y^{-j} does not change the twisted Alexander ideal. So we may assume that

$$\begin{aligned}B &= x^{-1}(y^{-2j}f_{j-1} + y^{-j+1}f_{j-3}) + (y^{-2j}f_{j-1} + y^{-j+1}f_{j-3}) \\ &\quad + (y^{-j-i}f_{i-1} + y^{i-j}y^{-i+1}f_{i-3}) + (y^{i-j}y^{-i}f_{i-1} + y^{-j}f_{j-1}).\end{aligned}$$

After replacing y^{-2j}, y^{-j-i} and y^{i-j} by 1, we get

$$\begin{aligned}B &\equiv x^{-1}(f_{j-1} + y^{-j+1}f_{j-3}) + (f_{j-1} + y^{-j+1}f_{j-3}) + (f_{i-1} + y^{-i+1}f_{i-3}) \\ &\quad + (y^{-i}f_{i-1} + y^{-j}f_{j-1}).\end{aligned} \tag{4.5}$$

We simplify the terms in A and B as follows:

$$\begin{aligned}f_{j-1} + y^{-j+1}f_{j-3} &= (1 + y^2 + \cdots + y^{j-1}) + y^{-2j}y^{j+1}(1 + y^2 + \cdots + y^{j-3}) \\ &\equiv (1 + y^2 + \cdots + y^{j-1}) + y^{j+1}(1 + y^2 + \cdots + y^{j-3}) \\ &= f_{2(j-1)}.\end{aligned}$$

Now, since l is odd, we write

$$\begin{aligned}f_{2(j-1)} &= \frac{1 + y^{2j}}{1 + y^2} = \frac{(1 + y^{2d})(1 + y^{2d} + \cdots y^{2(l-1)d})}{1 + y^2} \\ &= \frac{1 + y^{2d}}{1 + y^2} + \frac{1 + y^{2d}}{1 + y^2}(y^{2d} + y^{4d} + \cdots + y^{2(l-1)d}) \\ &= f_{2(d-1)} + (1 + y^{2d})\left[\frac{y^{2d}(1 + y^{2d})}{1 + y^2} + \frac{y^{6d}(1 + y^{2d})}{1 + y^2} + \cdots + \frac{y^{2(l-2)d}(1 + y^{2d})}{1 + y^2}\right] \\ &\equiv f_{2(d-1)}.\end{aligned}$$

So we obtain

$$f_{j-1} + y^{-j+1}f_{j-3} \equiv f_{2(d-1)}. \tag{4.6}$$

By similar computation, we get

$$f_{i-1} + y^{-i+1}f_{i-3} \equiv f_{2(d-1)}. \tag{4.7}$$

Moreover, we have

$$(yf_{i-3} + yf_{j-3}) = y\frac{(1 + y^{i-1})}{1 + y^2} + y\frac{(1 + y^{j-1})}{1 + y^2} = \frac{(y^i + y^j)}{1 + y^2}, \tag{4.8}$$

$$(y^{-i}f_{i-1} + y^{-j}f_{j-1}) = y^{-i}\frac{(1 + y^{i+1})}{1 + y^2} + y^{-j}\frac{(1 + y^{j+1})}{1 + y^2} = y^{-i-j}\frac{(y^i + y^j)}{1 + y^2} \equiv \frac{(y^i + y^j)}{1 + y^2}. \tag{4.9}$$

By substituting the identities in (4.6)–(4.9) into (4.4) and (4.5), we deduce that

$$A \equiv x^{-1}f_{2(d-1)} + \frac{(y^i + y^j)}{1 + y^2} \quad \text{and} \quad B \equiv x^{-1}f_{2(d-1)} + \frac{(y^i + y^j)}{1 + y^2}.$$

To simplify further, we need to divide into two sub-cases.

Sub-case 3a: $4|(i-j)$. As i,j are odd, it follows that $4|(k-l)$. Without loss of generality, we assume that $i-j>0$. Then,

$$\begin{aligned}\frac{(y^i+y^j)}{1+y^2} &= y^j\frac{(1+y^{i-j})}{1+y^2} = y^j\frac{1+y^{2d}}{1+y^2}(1+y^{2d}+y^{4d}\cdots+y^{2((k-l)/2-1)d})\\ &= y^j(1+y^{2d})\left[\frac{(1+y^{2d})}{1+y^2}+\frac{y^{4d}(1+y^{2d})}{1+y^2}+\cdots+\frac{y^{2((k-l)/2-2)d}(1+y^{2d})}{1+y^2}\right].\end{aligned}$$

So in this case $(y^i+y^j)/(1+y^2)$ is a multiple of $(1+y^{2d})$ and therefore $A\equiv B\equiv x^{-1}f_{2(d-1)}$. As a result, we have $I=(1+y^{2d},f_{2(d-1)})$. As $f_{2(d-1)}\mid(1+y^{2d})$, we deduce (ii).

Sub-case 3b: $4\not|(i-j)$. As i,j are odd, in this case $i-j\equiv 2 \bmod 4$. Similar to the previous sub-case, we get

$$\begin{aligned}\frac{(y^i+y^j)}{1+y^2} &= y^j\frac{(1+y^{i-j})}{1+y^2} = y^j\frac{1+y^{2d}}{1+y^2}(1+y^{2d}+y^{4d}\cdots+y^{2((k-l)/2-1)d})\\ &= y^j\frac{(1+y^{2d})}{1+y^2}+y^j(1+y^{2d})\left[\frac{y^{2d}(1+y^{2d})}{1+y^2}+\cdots+\frac{y^{2((k-l)/2-2)d}(1+y^{2d})}{1+y^2}\right]\\ &\equiv y^jf_{2(d-1)}.\end{aligned}$$

So in this case $A\equiv B\equiv x^{-1}f_{2(d-1)}+y^jf_{2(d-1)}$. Moreover, note that $(1+y^2)f_{2(d-1)}=1+y^{2d}$, so $y^2f_{2(d-1)}\equiv f_{2(d-1)}$. As j is odd, we deduce that $y^jf_{2(d-1)}\equiv yf_{2(d-1)}$.

After multiplying both A and B by the unit x, we can write $I=(1+y^{2d},f_{2(d-1)}+xyf_{2(d-1)})$ and (iii) follows. □

4.2. Representation of type 2

The next proposition allows us to find the twisted Alexander ideal associated with a representation of type 2.

Proposition 4.2. *The twisted Alexander ideal of $G_{i,j}$ associated with a representation of type 2 can be obtained from that of a representation of type 1 by the change of variables $x\mapsto xy^{-1}, y\mapsto y$.*

Proof. It is not hard to check that the map ψ below is a well-defined automorphism of $G_{i,j}$:

$$\psi: G_{i,j}\to G_{i,j}\quad\text{defined by}\quad \psi(b)=c^{-1}b,\ \psi(c)=c,\ \psi(a)=a.$$

Notice that if ρ is a representation of type 2 then $\rho\circ\psi$ is of type 1. By a result in [15] (p. 246), the twisted Alexander ideal of $G_{i,j}$ associated with ρ is the image of the one associated with $\rho\circ\psi$ under the map

$$\begin{array}{rccc}\psi_*:\mathbb{Z}[\mathrm{ab}(G_{i,j})] & \to & \mathbb{Z}[\mathrm{ab}(G_{i,j})] \\ x & \mapsto & xy^{-1} \\ y & \mapsto & y.\end{array}\tag{4.10}$$

So the Proposition is proved. □

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4.3. Representation of type 3

In the case of type 3 representation, we obtain the following result.

Proposition 4.3. *We put* $d := \gcd(i,j)$ *and let* I *be the twisted Alexander ideal of* $G_{i,j}$ *associated with a representation of type* 3*. Then:*

(i) $I = L$ *in the case where* $3|j$ *or both* $3\not| j$ *and* $3\not| i+j$ *hold;*

(ii) $I = (1+y^d+y^{2d})$ *in the case where* $3\not| j$, $3|(i+j)$ *and* l *is even;*

(iii) $I = ((1+y^d+y^{2d})/(1+y+y^2))$ *in the case where* $3\not| j$, $3|(i+j)$ *and* l *is odd.*

Proof. We write $i = kd$, $j = ld$ and divide into the following cases.

Case 1: $3|j$. Since $\rho(c)$ has order 3, we have

$$\rho(a) = \rho(c^i)[\rho(c^j), \rho(b)]\rho(c^{-i}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Similar to the first case in the proof of Proposition 4.1, we also deduce that $\Phi(\partial r/\partial a) = y^{-i}\rho(c)^{-i}$. So we get $I = L$.

Case 2: $3\not| j$ and $3\not| i+j$. We need to consider four sub-cases.

Sub-case 2a: $j \equiv 1 \bmod 3$, $i \equiv 0 \bmod 3$. We first find that $\rho(a) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. From that we get

$$\Phi\left(\frac{\partial r}{\partial a}\right) = \begin{pmatrix} y^{-i} & 1+y^{-i} \\ 1+y^{-i} & 1 \end{pmatrix} \quad \text{and} \quad \Phi\left(\frac{\partial r}{\partial b}\right) = \begin{pmatrix} 1+y^{-j} & y^{-j} \\ 1 & 1+y^{-j} \end{pmatrix}.$$

We compute the 2-minors:

$$\det \Phi\left(\frac{\partial r}{\partial b}\right) = y^{-2j}(1+y^j+y^{2j}), \quad \det \begin{pmatrix} 1+y^{-i} & 1+y^{-j} \\ 1 & 1 \end{pmatrix} = y^{-i}(1+y^{i-j}).$$

As $(i-j) \equiv 2 \bmod 3$, it follows from Lemma 3.3 that $\gcd(1+y^j+y^{2j}, 1+y^{i-j}) = 1$. Since I contains both $(1+y^j+y^{2j})$ and $(1+y^{i-j})$, we deduce that $I = L$. So assertion (i) holds.

Sub-case 2b: $j \equiv 1 \bmod 3$, $i \equiv 1 \bmod 3$. In this case we also have $\rho(a) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. From that we get

$$\Phi\left(\frac{\partial r}{\partial a}\right) = \begin{pmatrix} 0 & 1+y^{-i} \\ 1+y^{-i} & 1+y^{-i} \end{pmatrix} \quad \text{and} \quad \Phi\left(\frac{\partial r}{\partial b}\right) = \begin{pmatrix} 1+y^{-j} & y^{-j} \\ 1 & 1+y^{-j} \end{pmatrix}.$$

We find the determinants

$$\det \Phi\left(\frac{\partial r}{\partial a}\right) = y^{-2i}(1+y^{2i}), \quad \det \Phi\left(\frac{\partial r}{\partial b}\right) = y^{-2j}(1+y^j+y^{2j}).$$

As $2i \equiv 2 \bmod 3$, it follows from Lemma 3.3 that $\gcd(1+y^j+y^{2j}, 1+y^{2i}) = 1$. We deduce that $I = L$ and assertion (i) follows.

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Sub-case 2c: $j \equiv 2 \bmod 3$, $i \equiv 0 \bmod 3$. In this case $\rho(a) = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and we can compute

$$\Phi\left(\frac{\partial r}{\partial a}\right) = \begin{pmatrix} 1 & 1+y^{-i} \\ 1+y^{-i} & y^{-i} \end{pmatrix} \quad \text{and} \quad \Phi\left(\frac{\partial r}{\partial b}\right) = \begin{pmatrix} 1+y^{-j} & 1 \\ y^{-j} & 1+y^{-j} \end{pmatrix}.$$

We compute the 2-minors:

$$\det \Phi\left(\frac{\partial r}{\partial b}\right) = y^{-2j}(1+y^j+y^{2j}), \quad \det \begin{pmatrix} 1 & 1 \\ 1+y^{-i} & 1+y^{-j} \end{pmatrix} = y^{-i}(1+y^{i-j}).$$

As $(i-j) \equiv 1 \bmod 3$, it follows from Lemma 3.3 that $\gcd(1+y^j+y^{2j}, 1+y^{i-j}) = 1$. Therefore, we deduce that $I = L$ as required.

Sub-case 2d: $j \equiv 2 \bmod 3, i \equiv 2 \bmod 3$. We also have $\rho(a) = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and we can find that

$$\Phi\left(\frac{\partial r}{\partial a}\right) = \begin{pmatrix} 1+y^{-i} & 1+y^{-i} \\ 1+y^{-i} & 0 \end{pmatrix} \quad \text{and} \quad \Phi\left(\frac{\partial r}{\partial b}\right) = \begin{pmatrix} 1+y^{-j} & 1 \\ y^{-j} & 1+y^{-j} \end{pmatrix}.$$

We compute two determinants:

$$\det \Phi\left(\frac{\partial r}{\partial a}\right) = y^{-2i}(1+y^{2i}), \quad \det \Phi\left(\frac{\partial r}{\partial b}\right) = y^{-2j}(1+y^j+y^{2j}).$$

As $2i \equiv 1 \bmod 3$, it follows from Lemma 3.3 that $\gcd(1+y^j+y^{2j}, 1+y^{2i}) = 1$. We obtain $I = L$ as required.

Case 3: $j \equiv 1 \bmod 3$, $i \equiv 2 \bmod 3$. In this case, we have $\rho(a) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.
We compute $\Phi(\partial r/\partial b) = \left(\begin{smallmatrix} 1+y^{-j} & y^{-j} \\ 1 & 1+y^{-j} \end{smallmatrix}\right)$ and $\Phi(\partial r/\partial a) = \left(\begin{smallmatrix} y^{-i} & 1 \\ 1 & 1+y^{-i} \end{smallmatrix}\right)$.
To simplify $\Phi(\partial r/\partial c)$ in (4.1), we need to introduce the following polynomials:

$$g_{3n} := 1 + y^3 + \cdots + y^{3n} \text{ for } n \geq 1, \quad g_0 := 1 \quad \text{and} \quad g_{3n} := 0 \text{ for } n < 0.$$

As $\rho(c) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, $\rho(c^2) = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\rho(c^3) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we have the following identities:

$$\rho(1) + \rho(c)y \cdots + \rho(c^{i-1})y^{i-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} g_{i-2} + \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} y g_{i-2} + \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} y^2 g_{i-5}, \tag{4.11}$$

$$\rho(1) + \rho(c)y \cdots + \rho(c^{j-1})y^{j-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} g_{j-1} + \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} y g_{j-4} + \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} y^2 g_{j-4}, \tag{4.12}$$

$$\rho(c^{-1})y^{-1} + \cdots + \rho(c^{-i})y^{-i} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} y^{-i+2} g_{i-5} + \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} y^{-i} g_{i-2} + \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} y^{-i+1} g_{i-2}, \tag{4.13}$$

$$\rho(c^{-1})y^{-1} + \cdots + \rho(c^{-j})y^{-j} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} y^{-j+1} g_{j-4} + \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} y^{-j+2} g_{j-4} + \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} y^{-j} g_{j-1}. \tag{4.14}$$

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Putting the identities (4.11)–(4.14) into $\Phi(\partial r/\partial c)$ in (4.1), we get

$$\begin{aligned}\Phi\left(\frac{\partial r}{\partial c}\right) = &-\begin{pmatrix}1&0\\1&1\end{pmatrix}x^{-1}y^{-j+1}g_{j-4}-\begin{pmatrix}0&1\\1&0\end{pmatrix}x^{-1}y^{-j+2}g_{j-4}-\begin{pmatrix}1&1\\0&1\end{pmatrix}x^{-1}y^{-j}g_{j-1}\\ &+\begin{pmatrix}1&1\\1&0\end{pmatrix}y^{-j}g_{j-1}+\begin{pmatrix}1&0\\0&1\end{pmatrix}y^{-j+1}g_{j-4}+\begin{pmatrix}0&1\\1&1\end{pmatrix}y^{-j+2}g_{j-4}\\ &-\begin{pmatrix}1&0\\0&1\end{pmatrix}y^{-i}g_{i-2}-\begin{pmatrix}0&1\\1&1\end{pmatrix}y^{-i+1}g_{i-2}-\begin{pmatrix}1&1\\1&0\end{pmatrix}y^{-i+2}g_{i-5}\\ &+\begin{pmatrix}1&0\\0&1\end{pmatrix}y^{-i+2}g_{i-5}+\begin{pmatrix}0&1\\1&1\end{pmatrix}y^{-i}g_{i-2}+\begin{pmatrix}1&1\\1&0\end{pmatrix}y^{-i+1}g_{i-2}.\end{aligned}$$

After changing all the coefficients to 1, we find that the matrix for $\Phi(\partial r/\partial c)$ is

$$\begin{pmatrix} \begin{matrix}y^{-i}g_{i-2}+y^{-i+1}g_{i-2}+x^{-1}y^{-j}g_{j-1}+\\ x^{-1}y^{-j+1}g_{j-4}+y^{-j}g_{j-1}+y^{-j+1}g_{j-4}\end{matrix} & \begin{matrix}y^{-i}g_{i-2}+y^{-i+2}g_{i-5}+x^{-1}y^{-j}g_{j-1}+\\ x^{-1}y^{-j+2}g_{j-4}+y^{-j}g_{j-1}+y^{-j+2}g_{j-4}\end{matrix}\\ \begin{matrix}y^{-i}g_{i-2}+y^{-i+2}g_{i-5}+x^{-1}y^{-j+1}g_{j-4}+\\ x^{-1}y^{-j+2}g_{j-4}+y^{-j}g_{j-1}+y^{-j+2}g_{j-4}\end{matrix} & \begin{matrix}y^{-i+1}g_{i-2}+y^{-i+2}g_{i-5}+x^{-1}y^{-j}g_{j-1}+\\ x^{-1}y^{-j+1}g_{j-4}+y^{-j+1}g_{j-4}+y^{-j+2}g_{j-4}\end{matrix}\end{pmatrix}.$$

Now, using column $\binom{y^{-i}}{1}$ of $\Phi(\partial r/\partial a)$, we make the second row of any other column $\binom{X}{Y}$ to be zero by replacing it with $\binom{X}{Y}+Y\binom{y^{-i}}{1}=\binom{X+y^{-i}Y}{0}$. So we bring the twisted Alexander matrix to the form

$$\begin{pmatrix}y^{-i} & 1+y^{-i}+y^{-2i} & 1+y^{-i}+y^{-j} & y^{-i-j}+y^{-i}+y^{-j} & y^{-j}C & y^{-j}D\\ 1&0&0&0&0&0\end{pmatrix},$$

where

$$\begin{aligned}C &= y^{j-2i}g_{i-2}+y^{j-2i+2}g_{i-5}+y^{j-i}g_{i-2}+y^{j-i+1}g_{i-2}+y^{-i}g_{j-1}+y^{-i+2}g_{j-4}\\ &\quad+g_{j-1}+yg_{j-4}+x^{-1}(y^{-i+1}g_{j-4}+y^{-i+2}g_{j-4}+g_{j-1}+yg_{j-4}),\\ D &= y^{j-2i+1}g_{i-2}+y^{j-2i+2}g_{i-5}+y^{j-i}g_{i-2}+y^{j-i+2}g_{i-5}+y^{-i+1}g_{j-4}+y^{-i+2}g_{j-4}\\ &\quad+g_{j-1}+y^2g_{j-4}+x^{-1}(y^{-i+1}g_{j-4}+y^{-i}g_{j-1}+g_{j-1}+y^2g_{j-4}).\end{aligned}$$

After multiplying generators of I by appropriate units, we obtain

$$I=(1+y^i+y^{2i},y^{i+j}+y^i+y^j,1+y^i+y^j,C,D).$$

As $(y^{i+j}+y^i+y^j)+(1+y^i+y^j)=1+y^{i+j}$, we have $I=(1+y^i+y^{2i},1+y^{i+j},1+y^i+y^j,C,D)$.

From Lemma 3.4, we know that $\gcd(1+y^i+y^{2i},1+y^{i+j})=1+y^d+y^{2d}$. We now show that the third generator is also divisible by $(1+y^d+y^{2d})$.

In fact, as $j\equiv 1 \bmod 3$, $i\equiv 2 \bmod 3$ we obtain $3|(2j-i)$. Moreover, as $3\not| j=ld$ we also have $\gcd(d,3)=1$. As a result, we have $3d|(2j-i)$. So we use Lemma 3.2 to get

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$(1+y^d)(1+y^d+y^{2d}) = (1+y^{3d})|(1+y^{2j-i})$. We arrive at

$$(1+y^d+y^{2d}) \mid (1+y^{2j-i}). \tag{4.15}$$

Moreover, Corollary 3.5 implies that $(1+y^d+y^{2d})|(1+y^j+y^{2j})$. Combining this with (4.15), we get that $1+y^i+y^j = (1+y^j+y^{2j}) + y^i(1+y^{2j-i})$ is also divisible by $(1+y^d+y^{2d})$.

It follows that $I = (1+y^d+y^{2d}, C, D)$.

We now proceed by simplifying the generators C and D as we did in Case 3 of Proposition 4.1. Note that adding to C or D a multiple of $(1+y^d+y^{2d})$ will not change the twisted Alexander ideal. In the proof of this Proposition, we will use the notation $X \equiv Y$ if $(X-Y)$ is a multiple of $(1+y^d+y^{2d})$. As $g_{3n} = (1+y^{3n+3})/(1+y^3)$, we simplify C and D as follows:

$$\begin{aligned}
C &= \left(y^{j-2i}\frac{1+y^{i+1}}{1+y^3} + y^{j-2i+2}\frac{1+y^{i-2}}{1+y^3} + y^{j-i}\frac{1+y^{i+1}}{1+y^3} + y^{j-i+1}\frac{1+y^{i+1}}{1+y^3}\right)\\
&\quad + (1+y^{-i-j})(y^j g_{j-1} + y^{j+2}g_{j-4}) + \left(y^j g_{j-1} + y^{j+2}g_{j-4} + g_{j-1} + yg_{j-4}\right)\\
&\quad + x^{-1}\left((1+y^{-i-j})(y^{j+1}g_{j-4} + y^{j+2}g_{j-4}) + y^{j+1}g_{j-4} + y^{j+2}g_{j-4} + g_{j-1} + yg_{j-4}\right)\\
&= \frac{(y^{j+1}+y^{j-2i}+y^{j-2i+1})(1+y)}{1+y^3} + (1+y^{-i-j})(y^j g_{j-1} + y^{j+2}g_{j-4})\\
&\quad + \left(y^j g_{j-1} + y^{j+2}g_{j-4} + g_{j-1} + yg_{j-4}\right)\\
&\quad + x^{-1}\left((1+y^{-i-j})(y^{j+1}g_{j-4} + y^{j+2}g_{j-4}) + y^{j+1}g_{j-4} + y^{j+2}g_{j-4} + g_{j-1} + yg_{j-4}\right).
\end{aligned}$$

As noted above, by Lemma 3.4, $\gcd(1+y^i+y^{2i}, 1+y^{i+j}) = 1+y^d+y^{2d}$. So $1+y^{-i-j} = y^{-i-j}(1+y^{i+j})$ is a multiple of $(1+y^d+y^{2d})$ and we may write

$$\begin{aligned}
C &\equiv \frac{(y^{j+1}+y^{j-2i}+y^{j-2i+1})(1+y)}{1+y^3} + (y^j g_{j-1} + y^{j+2}g_{j-4} + g_{j-1} + yg_{j-4})\\
&\quad + x^{-1}(y^{j+1}g_{j-4} + y^{j+2}g_{j-4} + g_{j-1} + yg_{j-4}).\\
&= \frac{(y^{j+1}+y^{j-2i}+y^{j-2i+1})}{1+y+y^2} + \left(y^j\frac{1+y^{j+2}}{1+y^3} + y^{j+2}\frac{1+y^{j-1}}{1+y^3} + \frac{1+y^{j+2}}{1+y^3} + y\frac{1+y^{j-1}}{1+y^3}\right)\\
&\quad + x^{-1}\left(y^{j+1}\frac{1+y^{j-1}}{1+y^3} + y^{j+2}\frac{1+y^{j-1}}{1+y^3} + \frac{1+y^{j+2}}{1+y^3} + y\frac{1+y^{j-1}}{1+y^3}\right)\\
&= \frac{y^{j+1}+y^{j-2i}+y^{j-2i+1}}{1+y+y^2} + \frac{(1+y^{2j+1})(1+y)}{1+y^3} + x^{-1}\frac{(1+y^j+y^{2j})(1+y)}{1+y^3}.
\end{aligned}$$

Therefore, we get

$$C \equiv \frac{y^{j+1}+y^{j-2i}+y^{j-2i+1}}{1+y+y^2} + \frac{1+y^{2j+1}}{1+y+y^2} + x^{-1}\frac{1+y^j+y^{2j}}{1+y+y^2}. \tag{4.16}$$

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We will do the same thing for D:

$$\begin{aligned}
D = &\left(y^{j-2i+1}\frac{1+y^{i+1}}{1+y^3} + y^{j-2i+2}\frac{1+y^{i-2}}{1+y^3} + y^{j-i}\frac{1+y^{i+1}}{1+y^3} + y^{j-i+2}\frac{1+y^{i-2}}{1+y^3}\right)\\
&+ (1+y^{-i-j})(y^{j+1}g_{j-4} + y^{j+2}g_{j-4}) + (y^{j+1}g_{j-4} + y^{j+2}g_{j-4} + g_{j-1} + y^2 g_{j-4})\\
&+ x^{-1}((1+y^{-i-j})(y^{j+1}g_{j-4} + y^j g_{j-1}) + y^{j+1}g_{j-4} + y^j g_{j-1} + g_{j-1} + y^2 g_{j-4})\\
= &\frac{(y^j + y^{j-2i+1})(1+y)}{1+y^3} + (1+y^{-i-j})(y^{j+1}g_{j-4} + y^{j+2}g_{j-4})\\
&+ (y^{j+1}g_{j-4} + y^{j+2}g_{j-4} + g_{j-1} + y^2 g_{j-4})\\
&+ x^{-1}((1+y^{-i-j})(y^{j+1}g_{j-4} + y^j g_{j-1}) + y^{j+1}g_{j-4} + y^j g_{j-1} + g_{j-1} + y^2 g_{j-4}).
\end{aligned}$$

Now, we discard the terms which are multiples of $(1+y^{-i-j})$ and plug in $g_{3n} = (1+y^{3n+3})/(1+y^3)$. We obtain

$$\begin{aligned}
D \equiv &\frac{(y^j + y^{j-2i+1})(1+y)}{1+y^3} + \left(y^{j+1}\frac{1+y^{j-1}}{1+y^3} + y^{j+2}\frac{1+y^{j-1}}{1+y^3} + \frac{1+y^{j+2}}{1+y^3} + y^2\frac{1+y^{j-1}}{1+y^3}\right)\\
&+ x^{-1}\left(y^{j+1}\frac{1+y^{j-1}}{1+y^3} + y^j\frac{1+y^{j+2}}{1+y^3} + \frac{1+y^{j+2}}{1+y^3} + y^2\frac{1+y^{j-1}}{1+y^3}\right)\\
= &\frac{(y^j + y^{j-2i+1})(1+y)}{1+y^3} + \frac{(1+y^j+y^{2j})(1+y)}{1+y^3} + x^{-1}\frac{(1+y^2)(1+y^j+y^{2j})}{1+y^3}.
\end{aligned}$$

We arrive at

$$D \equiv \frac{y^j + y^{j-2i+1}}{1+y+y^2} + \frac{1+y+y^{2j}}{1+y+y^2} + x^{-1}\frac{(1+y)(1+y^j+y^{2j})}{1+y+y^2}. \tag{4.17}$$

Now, we combine (4.16) and (4.17) to get

$$\begin{aligned}
D + y^2 C \equiv &\frac{(y^j + y^{j+3}) + y^{j-2i+1}(1+y+y^2) + (1+y+y^2) + (y^{2j} + y^{2j+3})}{1+y+y^2}\\
&+ x^{-1}(1+y^j+y^{2j})\\
= &\, y^j(1+y) + y^{j-2i+1} + 1 + y^{2j}(1+y) + x^{-1}(1+y^j+y^{2j})\\
= &\,(1+y)(1+y^j+y^{2j}) + y(1+y^{j-2i}) + x^{-1}(1+y^j+y^{2j}).
\end{aligned}$$

As $3\nmid j$, by Corollary 3.5, $1+y^j+y^{2j}$ is divisible by $1+y^d+y^{2d}$. It follows from the equality above that

$$D + y^2 C \equiv y(1+y^{j-2i}).$$

Similar to the proof of (4.15), we also have that $(j-2i)$ is divisible by $3d$. It follows from Lemma 3.2 that $(1+y^d)(1+y^d+y^{2d}) = (1+y^{3d}) \mid (1+y^{j-2i})$. So we obtain

$$D + y^2 C \equiv y(1+y^{j-2i}) \equiv 0.$$

So D belongs to the ideal $(1+y^d+y^{2d}, C)$. This means that $I = (1+y^d+y^{2d}, C)$.

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Now we rewrite

$$C \equiv \frac{(1+y)(1+y^{j-2i})}{1+y+y^2} + \frac{y(1+y^j+y^{2j})}{1+y+y^2} + x^{-1}\frac{1+y^j+y^{2j}}{1+y+y^2}. \tag{4.18}$$

As $3 \nmid j$, we also have $3 \nmid l$. So we divide further into three sub-cases.

Sub-case 3a: l is even. We have $(j-2i) = (l-2k)d$. So, in this case, $2 \mid (l-2k)$. Moreover, as $j \equiv 1 \mod 3$, $i \equiv 2 \mod 3$, we obtain $3|(j-2i) = (l-2k)d$. We know already that $\gcd(3,d) = 1$. So we also have $3 \mid (l-2k)$. Therefore, we obtain that $j-2i = (l-2k)d$ is divisible by $6d$. So, by Lemma 3.2,

$$(1+y^{6d}) \mid (1+y^{j-2i}). \tag{4.19}$$

On the other hand, by Lemma 3.6(i), $(1+y+y^2)(1+y^d+y^{2d})|(1+y^{2d}+y^{4d})$. So it follows that

$$(1+y+y^2)(1+y^d+y^{2d}) \mid (1+y^{2d})(1+y^{2d}+y^{4d}) = (1+y^{6d}). \tag{4.20}$$

It follows from (4.19) and (4.20) that $(1+y+y^2)(1+y^d+y^{2d}) \mid (1+y^{j-2i})$ or, equivalently, $(1+y)(1+y^{j-2i})/(1+y+y^2) \equiv 0$. Plugging this into (4.18), we obtain

$$C \equiv \frac{y(1+y^j+y^{2j})}{1+y+y^2} + x^{-1}\frac{1+y^j+y^{2j}}{1+y+y^2}.$$

Moreover, as $3 \nmid j$, by using Lemma 3.6(i) we obtain $C \equiv 0$. We deduce that I is generated by $(1+y^d+y^{2d})$.

Sub-case 3b: $l = 6t+1$. As $(l-2k)$ is odd, we have that $(j-2i) = (l-2k)d$ is not divisible by $6d$, only by $3d$. As a consequence, $6d \mid (j-2i-3d)$. So, by Lemma 3.2, we obtain $(1+y^{6d})|(1+y^{j-2i-3d})$. Combining this with (4.20), we get

$$\frac{(1+y^{j-2i})}{1+y+y^2} - \frac{(1+y^{3d})}{1+y+y^2} = \frac{y^{3d}(1+y^{j-2i-3d})}{1+y+y^2} \equiv 0.$$

We put this into (4.18) and then use Lemma 3.6(ii) to get

$$\begin{aligned} C &\equiv \frac{(1+y)(1+y^{3d})}{1+y+y^2} + \frac{y(1+y^d+y^{2d})}{1+y+y^2} + x^{-1}\frac{1+y^d+y^{2d}}{1+y+y^2} \\ &= \frac{(1+y^d+y^{d+1})}{1+y+y^2}(1+y^d+y^{2d}) + x^{-1}\frac{1+y^d+y^{2d}}{1+y+y^2}. \end{aligned}$$

Note that since $l = 6t+1$ and $j = ld = (6t+1)d \equiv 1 \mod 3$, we have $d \equiv 1 \mod 3$. So, by Lemma 3.2, we get $(1+y^3) \mid (1+y^{d-1})$. This implies that $1+y^d+y^{d+1} = (1+y+y^2) + y(1+y^{d-1}) + y^2(1+y^{d-1})$ is divisible by $(1+y+y^2)$. So we have

$$\frac{(1+y^d+y^{d+1})}{1+y+y^2}(1+y^d+y^{2d}) \equiv 0 \text{ and therefore } C \equiv x^{-1}\frac{1+y^d+y^{2d}}{1+y+y^2}.$$

So in this case I is generated by $(1+y^d+y^{2d})/(1+y+y^2)$.

Sub-case 3c: $l = 6t+5$. As in the previous case, we also have $3d \mid (j-2i)$ but $6d \nmid (j-2i)$. So by the same argument as in the previous case, using Lemma 3.6(iii), we

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obtain

$$C \equiv \frac{(1+y)(1+y^{3d})}{1+y+y^2} + \frac{y^{1-2d}(1+y^d+y^{2d})}{1+y+y^2} + x^{-1}y^{-2d}\frac{1+y^d+y^{2d}}{1+y+y^2}$$
$$= \frac{(1+y)(1+y^d)+y^{1-2d}}{1+y+y^2}(1+y^d+y^{2d}) + x^{-1}y^{-2d}\frac{1+y^d+y^{2d}}{1+y+y^2}.$$

Note that since $l = 6t+5$ and $j = ld = (6t+5)d \equiv 1 \bmod 3$, we have $d \equiv 2 \bmod 3$. So, by Lemma 3.2, both $(1+y^{d-2})$ and $(1+y^{3d})$ are divisible by $(1+y^3) = (1+y)(1+y+y^2)$. We obtain that $(1+y)(1+y^d)+y^{1-2d} = (1+y+y^2)+y^2(1+y^{d-2}) + y^{1-2d}(1+y^{3d})$ is divisible by $(1+y+y^2)$.

We deduce that $C \equiv x^{-1}y^{-2d}((1+y^d+y^{2d})(1+y+y^2))$ and therefore I is generated by $(1+y^d+y^{2d})/(1+y+y^2)$.

Case 4: $j \equiv 2 \bmod 3$, $i \equiv 1 \bmod 3$. In this case we get $\rho(a) = \left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$,

$$\Phi\left(\frac{\partial r}{\partial a}\right) = \begin{pmatrix} y^{-i}+1 & 1 \\ 1 & y^{-i}\end{pmatrix} \quad \text{and} \quad \Phi\left(\frac{\partial r}{\partial b}\right) = \begin{pmatrix} y^{-j}+1 & 1 \\ y^{-j} & y^{-j}+1\end{pmatrix}.$$

Similar to Case 3, we also use the following identities to simplify $\Phi(\partial r/\partial c)$:

$$\rho(1)+\rho(c)y\cdots+\rho(c^{i-1})y^{i-1} = \begin{pmatrix}1&0\\0&1\end{pmatrix}g_{i-1} + \begin{pmatrix}0&1\\1&1\end{pmatrix}yg_{i-4} + \begin{pmatrix}1&1\\1&0\end{pmatrix}y^2g_{i-4}, \tag{4.21}$$

$$\rho(1)+\rho(c)y\cdots+\rho(c^{j-1})y^{j-1} = \begin{pmatrix}1&0\\0&1\end{pmatrix}g_{j-2} + \begin{pmatrix}0&1\\1&1\end{pmatrix}yg_{j-2} + \begin{pmatrix}1&1\\1&0\end{pmatrix}y^2g_{j-5}, \tag{4.22}$$

$$\rho(c^{-1})y^{-1}+\cdots+\rho(c^{-i})y^{-i} = \begin{pmatrix}1&0\\0&1\end{pmatrix}y^{-i+1}g_{i-4} + \begin{pmatrix}0&1\\1&1\end{pmatrix}y^{-i+2}g_{i-4} + \begin{pmatrix}1&1\\1&0\end{pmatrix}y^{-i}g_{i-1}, \tag{4.23}$$

$$\rho(c^{-1})y^{-1}+\cdots+\rho(c^{-j})y^{-j} = \begin{pmatrix}1&0\\0&1\end{pmatrix}y^{-j+2}g_{j-5} + \begin{pmatrix}0&1\\1&1\end{pmatrix}y^{-j}g_{j-2} + \begin{pmatrix}1&1\\1&0\end{pmatrix}y^{-j+1}g_{j-2}. \tag{4.24}$$

Now we plug (4.21)–(4.24) into (4.1) to get

$$\begin{aligned}\Phi\left(\frac{\partial r}{\partial c}\right) = &-\begin{pmatrix}1&1\\0&1\end{pmatrix}x^{-1}y^{-j+2}g_{j-5} - \begin{pmatrix}1&0\\1&1\end{pmatrix}x^{-1}y^{-j}g_{j-2} - \begin{pmatrix}0&1\\1&0\end{pmatrix}x^{-1}y^{-j+1}g_{j-2}\\ &+\begin{pmatrix}0&1\\1&1\end{pmatrix}y^{-j}g_{j-2} + \begin{pmatrix}1&1\\1&0\end{pmatrix}y^{-j+1}g_{j-2} + \begin{pmatrix}1&0\\0&1\end{pmatrix}y^{-j+2}g_{j-5}\\ &-\begin{pmatrix}1&0\\0&1\end{pmatrix}y^{-i}g_{i-1} - \begin{pmatrix}0&1\\1&1\end{pmatrix}y^{-i+1}g_{i-4} - \begin{pmatrix}1&1\\1&0\end{pmatrix}y^{-i+2}g_{i-4}\\ &+\begin{pmatrix}1&0\\0&1\end{pmatrix}y^{-i+1}g_{i-4} + \begin{pmatrix}0&1\\1&1\end{pmatrix}y^{-i+2}g_{i-4} + \begin{pmatrix}1&1\\1&0\end{pmatrix}y^{-i}g_{i-1}.\end{aligned}$$

Changing all coefficients to 1, we get the following matrix for $\Phi(\partial r/\partial c)$:

$$\begin{pmatrix} y^{-i+2}g_{i-4}+y^{-i+1}g_{i-4}+x^{-1}y^{-j}g_{j-2}+ & y^{-i}g_{i-1}+y^{-i+1}g_{i-4}+x^{-1}y^{-j+1}g_{j-2}+ \\ x^{-1}y^{-j+2}g_{j-5}+y^{-j+1}g_{j-2}+y^{-j+2}g_{j-5} & x^{-1}y^{-j+2}g_{j-5}+y^{-j}g_{j-2}+y^{-j+1}g_{j-2} \\ y^{-i}g_{i-1}+y^{-i+1}g_{i-4}+x^{-1}y^{-j}g_{j-2}+ & y^{-i}g_{i-1}+y^{-i+2}g_{i-4}+x^{-1}y^{-j}g_{j-2}+ \\ x^{-1}y^{-j+1}g_{j-2}+y^{-j}g_{j-2}+y^{-j+1}g_{j-2} & x^{-1}y^{-j+2}g_{j-5}+y^{-j}g_{j-2}+y^{-j+2}g_{j-5} \end{pmatrix}.$$

As in Case 3, by using the column $\left(\begin{smallmatrix}1\\ y^{-i}\end{smallmatrix}\right)$ of $\Phi(\partial r/\partial a)$ we make the first-row entries of other columns to be zero. The twisted Alexander matrix can be brought into the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1+y^{-i}+y^{-2i} & y^{-i} & y^{-i-j}+y^{-i}+y^{-j} & 1+y^{-i}+y^{-j} & y^{-j}E & y^{-j}F \end{pmatrix},$$

where

$$\begin{aligned} E &= y^{j-2i+1}g_{i-4}+y^{j-2i+2}g_{i-4}+y^{j-i}g_{i-1}+y^{j-i+1}g_{i-4}+y^{-i+1}g_{j-2}+y^{-i+2}g_{j-5} \\ &\quad +g_{j-2}+yg_{j-2}+x^{-1}(y^{-i}g_{j-2}+y^{-i+2}g_{j-5}+g_{j-2}+yg_{j-2}), \\ F &= y^{j-2i}g_{i-1}+y^{j-2i+1}g_{i-4}+y^{j-i}g_{i-1}+y^{j-i+2}g_{i-4}+y^{-i}g_{j-2}+y^{-i+1}g_{j-2} \\ &\quad +g_{j-2}+y^2g_{j-5}+x^{-1}(y^{-i+1}g_{j-2}+y^{-i+2}g_{j-5}+g_{j-2}+y^2g_{j-5}). \end{aligned}$$

After multiplying generators of I by appropriate units, it is not hard to see that

$$\begin{aligned} I &= (1+y^i+y^{2i}, y^{i+j}+y^i+y^j, 1+y^i+y^j, E, F) \\ &= (1+y^i+y^{2i}, 1+y^{i+j}, 1+y^i+y^j, E, F). \end{aligned}$$

We argue as in Case 3. From Lemma 3.4, we know that

$$\gcd(1+y^i+y^{2i}, 1+y^{i+j}) = 1+y^d+y^{2d}.$$

We now show that the third generator is also divisible by $(1+y^d+y^{2d})$.

In this case, we also have $d|(2j-i)$, $3|(2j-i)$ and $3\nmid j=ld$. Therefore, we obtain $3d|(2j-i)$. So it follows from Lemma 3.2 that $(1+y^d)(1+y^d+y^{2d})=(1+y^{3d})|(1+y^{2j-i})$. So we have

$$(1+y^d+y^{2d}) \mid (1+y^{2j-i}). \tag{4.25}$$

Moreover, Corollary 3.5 implies that $(1+y^d+y^{2d})|(1+y^j+y^{2j})$. Combining this with (4.25), we get that $1+y^i+y^j=(1+y^j+y^{2j})+y^i(1+y^{2j-i})$ is also divisible by $(1+y^d+y^{2d})$. So we deduce that $I=(1+y^d+y^{2d}, E, F)$.

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We now proceed by simplifying the generators E, F as we did for C, D in Case 3 above. First, we substitute $g_{3n} = (1+y^{3n+3})/(1+y^3)$ into E to get

$$
\begin{aligned}
E =& \left(y^{j-2i+1}\frac{1+y^{i-1}}{1+y^3} + y^{j-2i+2}\frac{1+y^{i-1}}{1+y^3} + y^{j-i}\frac{1+y^{i+2}}{1+y^3} + y^{j-i+1}\frac{1+y^{i-1}}{1+y^3} \right) \\
&+ (1+y^{-i-j})(y^{j+1}g_{j-2} + y^{j+2}g_{j-5}) + (y^{j+1}g_{j-2} + y^{j+2}g_{j-5} + g_{j-2} + yg_{j-2}) \\
&+ x^{-1}\left((1+y^{-i-j})(y^{j}g_{j-2} + y^{j+2}g_{j-5}) + y^{j}g_{j-2} + y^{j+2}g_{j-5} + g_{j-2} + yg_{j-2} \right) \\
=& \frac{(y^j + y^{j+1} + y^{j-2i+1})(1+y)}{1+y^3} + (1+y^{-i-j})(y^{j+1}g_{j-2} + y^{j+2}g_{j-5}) + \\
&(y^{j+1}g_{j-2} + y^{j+2}g_{j-5} + g_{j-2} + yg_{j-2}) + x^{-1}(1+y^{-i-j})(y^{j}g_{j-2} + y^{j+2}g_{j-5}) \\
&+ x^{-1}(y^{j}g_{j-2} + y^{j+2}g_{j-5} + g_{j-2} + yg_{j-2}).
\end{aligned}
$$

As $1+y^{-i-j} = y^{-i-j}(1+y^{i+j})$ is a multiple of $1+y^d+y^{2d}$, we can simplify

$$
\begin{aligned}
E \equiv& \frac{(y^j + y^{j+1} + y^{j-2i+1})(1+y)}{1+y^3} + (y^{j+1}g_{j-2} + y^{j+2}g_{j-5} + g_{j-2} + yg_{j-2}) \\
&+ x^{-1}(y^{j}g_{j-2} + y^{j+2}g_{j-5} + g_{j-2} + yg_{j-2}) \\
=& \frac{(y^j + y^{j+1} + y^{j-2i+1})}{1+y+y^2} + \left(y^{j+1}\frac{1+y^{j+1}}{1+y^3} + y^{j+2}\frac{1+y^{j-2}}{1+y^3} + \frac{1+y^{j+1}}{1+y^3} + y\frac{1+y^{j+1}}{1+y^3} \right) \\
&+ x^{-1}\left(y^{j}\frac{1+y^{j+1}}{1+y^3} + y^{j+2}\frac{1+y^{j-2}}{1+y^3} + \frac{1+y^{j+1}}{1+y^3} + y\frac{1+y^{j+1}}{1+y^3} \right) \\
=& \frac{y^j + y^{j+1} + y^{j-2i+1}}{1+y+y^2} + \frac{(1+y^{2j}+y^{2j+1})(1+y)}{1+y^3} + x^{-1}\frac{(1+y^j+y^{2j})(1+y)}{1+y^3} \\
=& \frac{y^j + y^{j+1} + y^{j-2i+1}}{1+y+y^2} + \frac{1+y^{2j}+y^{2j+1}}{1+y+y^2} + x^{-1}\frac{1+y^j+y^{2j}}{1+y+y^2}.
\end{aligned}
$$

Similarly, we simplify F as follows:

$$
\begin{aligned}
F =& \left(y^{j-2i}\frac{1+y^{i+2}}{1+y^3} + y^{j-2i+1}\frac{1+y^{i-1}}{1+y^3} + y^{j-i}\frac{1+y^{i+2}}{1+y^3} + y^{j-i+2}\frac{1+y^{i-1}}{1+y^3} \right) \\
&+ (1+y^{-i-j})(y^{j}g_{j-2} + y^{j+1}g_{j-2}) + (y^{j}g_{j-2} + y^{j+1}g_{j-2} + g_{j-2} + y^2g_{j-5}) \\
&+ x^{-1}((1+y^{-i-j})(y^{j+1}g_{j-2} + y^{j+2}g_{j-5}) + y^{j+1}g_{j-2} + y^{j+2}g_{j-5} + g_{j-2} + y^2g_{j-5}) \\
=& \frac{(y^{j+1} + y^{j-2i})(1+y)}{1+y^3} + (1+y^{-i-j})(y^{j}g_{j-2} + y^{j+1}g_{j-2}) \\
&+ (y^{j}g_{j-2} + y^{j+1}g_{j-2} + g_{j-2} + y^2g_{j-5}) \\
&+ x^{-1}((1+y^{-i-j})(y^{j+1}g_{j-2} + y^{j+2}g_{j-5}) + y^{j+1}g_{j-2} + y^{j+2}g_{j-5} + g_{j-2} + y^2g_{j-5}) \\
\equiv& \frac{(y^{j+1} + y^{j-2i})(1+y)}{1+y^3} + (y^{j}g_{j-2} + y^{j+1}g_{j-2} + g_{j-2} + y^2g_{j-5}) \\
&+ x^{-1}(y^{j+1}g_{j-2} + y^{j+2}g_{j-5} + g_{j-2} + y^2g_{j-5}).
\end{aligned}
$$

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We can simplify F further:

$$
\begin{aligned}
F &\equiv \frac{y^{j+1}+y^{j-2i}}{1+y+y^2} + \left(y^j\frac{1+y^{j+1}}{1+y^3} + y^{j+1}\frac{1+y^{j+1}}{1+y^3} + \frac{1+y^{j+1}}{1+y^3} + y^2\frac{1+y^{j-2}}{1+y^3}\right) \\
&\quad + x^{-1}\left(y^{j+1}\frac{1+y^{j+1}}{1+y^3} + y^{j+2}\frac{1+y^{j-2}}{1+y^3} + \frac{1+y^{j+1}}{1+y^3} + y^2\frac{1+y^{j-2}}{1+y^3}\right) \\
&= \frac{y^{j+1}+y^{j-2i}}{1+y+y^2} + \frac{(1+y+y^{2j+1})(1+y)}{1+y^3} + x^{-1}\frac{(1+y^j+y^{2j})(1+y^2)}{1+y^3} \\
&= \frac{y^{j+1}+y^{j-2i}}{1+y+y^2} + \frac{1+y+y^{2j+1}}{1+y+y^2} + x^{-1}\frac{(1+y)(1+y^j+y^{2j})}{1+y+y^2}.
\end{aligned}
$$

From the simplification of E and F above, we have the following identity:

$$
\begin{aligned}
F+y^2E &\equiv \frac{y^{j+1}(1+y+y^2)+y^{j-2i}(1+y^3)+y^{2j+1}(1+y+y^2)+(1+y+y^2)}{1+y+y^2} \\
&\qquad + x^{-1}(1+y^j+y^{2j}) \\
&\qquad \equiv y^{j+1}+y^{j-2i}(1+y)+y^{2j+1}+1+x^{-1}(1+y^j+y^{2j}) \\
&\qquad \equiv y(1+y^j+y^{2j})+(1+y)(1+y^{j-2i})+x^{-1}(1+y^j+y^{2j}).
\end{aligned}
$$

As $3\nmid j$, by Corollary 3.5, $1+y^j+y^{2j}$ is divisible by $1+y^d+y^{2d}$. In this case, similar to the proof of (4.15), we also have that $(j-2i)$ is divisible by $3d$. It follows from Lemma 3.2 that $(1+y^d)(1+y^d+y^{2d})=(1+y^{3d})\mid(1+y^{j-2i})$. So we obtain $F+y^2E\equiv 0$ and therefore $I=(1+y^d+y^{2d},F)$.

We rewrite F as

$$
F \equiv \frac{y(1+y^j+y^{2j})}{1+y+y^2} + \frac{1+y^{j-2i}}{1+y+y^2} + x^{-1}\frac{(1+y)(1+y^j+y^{2j})}{1+y+y^2}. \tag{4.26}
$$

We obtain the twisted Alexander ideal in each of the following sub-cases.

Sub-case 4a: l is even. In this case, $j-2i=(l-2k)d$ is divisible by both $2d$ and $3d$. We deduce that $(j-2i)$ is divisible by $6d$ and therefore (4.19) and (4.20) continue to hold. It follows that

$$
F \equiv \frac{y(1+y^j+y^{2j})}{1+y+y^2} + x^{-1}\frac{(1+y)(1+y^j+y^{2j})}{1+y+y^2}.
$$

Now, by Lemma 3.6(i), we get that $F\equiv 0$ and I is generated by $(1+y^d+y^{2d})$.

Sub-case 4b: $l=6t+1$. Arguing in the same way as in sub-case 3b, we also get $3d\mid(j-2i)$ but $6d\nmid(j-2i)$. Therefore, we also get the identity

$$
\frac{(1+y^{j-2i})}{1+y+y^2} \equiv \frac{(1+y^{3d})}{1+y+y^2}.
$$

So we deduce that

$$
F \equiv \frac{y(1+y^j+y^{2j})}{1+y+y^2} + \frac{(1+y^{3d})}{1+y+y^2} + x^{-1}(1+y)\frac{1+y^j+y^{2j}}{1+y+y^2}.
$$

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By using Lemma 3.6(ii), we deduce that

$$\begin{aligned}F &\equiv \frac{y(1+y^d+y^{2d})}{1+y+y^2}+\frac{(1+y^{3d})}{1+y+y^2}+x^{-1}(1+y)\frac{1+y^d+y^{2d}}{1+y+y^2}\\ &= \frac{(1+y+y^d)}{1+y+y^2}(1+y^d+y^{2d})+x^{-1}(1+y^d+y^{2d})+x^{-1}y^2\frac{1+y^d+y^{2d}}{1+y+y^2}.\end{aligned}$$

Note that since $l=6t+1$ and $j\equiv 2 \mod 3$, we have $d\equiv 2 \mod 3$. It follows from Lemma 3.2 that $1+y+y^d=(1+y+y^2)+y^2(1+y^{d-2})$ is a multiple of $(1+y+y^2)$.

We deduce that $F\equiv x^{-1}y^2((1+y^d+y^{2d})(1+y+y^2))$ and I is generated by $(1+y^d+y^{2d})/(1+y+y^2)$.

Note that as $3\nmid d$, $(1+y^d+y^{2d})/(1+y+y^2)$ is a Laurent polynomial by Corollary 3.5.

Sub-case 4c: $l=6t+5$. In this case, we also have $3d\mid(j-2i)$ but $6d\nmid(j-2i)$. By the same arguments as in the previous case, we get

$$F\equiv \frac{y(1+y^j+y^{2j})}{1+y+y^2}+\frac{(1+y^{3d})}{1+y+y^2}+x^{-1}(1+y)\frac{1+y^j+y^{2j}}{1+y+y^2}.$$

Now, by using Lemma 3.6(iii), we deduce that

$$\begin{aligned}F &\equiv \frac{y^{1-2d}(1+y^d+y^{2d})}{1+y+y^2}+\frac{(1+y^{3d})}{1+y+y^2}+x^{-1}(1+y)y^{-2d}\frac{1+y^d+y^{2d}}{1+y+y^2}\\ &= \frac{1+y^d+y^{1-2d}}{1+y+y^2}(1+y^d+y^{2d})+x^{-1}y^{-2d}(1+y^d+y^{2d})+x^{-1}y^{2-2d}\frac{1+y^d+y^{2d}}{1+y+y^2}.\end{aligned}$$

Note that since $l=6t+5$ and $j\equiv 2 \mod 3$, we have $d\equiv 1 \mod 3$. Therefore, by Lemma 3.2, $(1+y^{2d-2})$ is divisible by $(1+y^3)$.

Since both $(1+y^{2d-2})$ and $(1+y^{3d})$ are divisible by $(1+y^3)=(1+y)(1+y+y^2)$, it follows that $1+y^d+y^{1-2d}=y^{-2d}((1+y+y^2)+y^2(1+y^{2d-2})+1+y^{3d})$ is divisible by $(1+y+y^2)$.

So in this case $F\equiv x^{-1}y^{2-2d}((1+y^d+y^{2d})/(1+y+y^2))$ and therefore I is generated by $(1+y^d+y^{2d})/(1+y+y^2)$.

The assertions (ii) and (iii) of the Proposition are proved. □

5. Applications to the isomorphism problem

In this section, we will apply the computation results obtained above to deduce that several pairs of groups $G_{i,j}$ are non-isomorphic. For each group $G_{i,j}$, we will denote by $I^1_{i,j}$, $I^2_{i,j}$ and $I^3_{i,j}$ its twisted Alexander ideals associated with representations of type 1, 2 and 3, respectively. We know that if two groups $G_{i,j}$ and $G_{i',j'}$ are isomorphic then there should be a monomial automorphism of L under which the multi-set of twisted Alexander ideals of one group is mapped to that of the other. Note that the monomial automorphism does not need to preserve the representation type, i.e. $I^1_{i,j}$ could be mapped to $I^3_{i',j'}$. We obtain the following result.

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Proposition 5.1. *Consider the following three disjoint subsets of the set of order pairs of positive integers:*

$$\begin{aligned} A &:= \{(i,j) \mid i \text{ is even or } j \text{ is even}\} \cup \{(i,j) \mid i,j \text{ are both odd}, \\ &\qquad \gcd(i,j) = 1 \text{ and } 4|(i-j)\}, \\ B &:= \{(i,j) \mid i,j \text{ are both odd}, \ \gcd(i,j) > 1 \text{ and } 4|(i-j)\}, \\ C &:= \{(i,j) \mid i,j \text{ are both odd and } 4 \not| (i-j)\}. \end{aligned}$$

If (i,j) and (i',j') do not belong to the same set among A, B and C then $G_{i,j} \not\cong G_{i',j'}$.

Proof. We put $d := \gcd(i,j)$ and $d' := \gcd(i',j')$. Suppose that $(i,j) \in A$ and $(i',j') \in B$. By Propositions 4.1 and 4.2, two twisted Alexander ideals $I^1_{i,j}$ and $I^2_{i,j}$ of $G_{i,j}$ both coincide with L. On the other hand, as (i',j') belongs to B, both twisted Alexander ideals $I^1_{i',j'}$ and $I^2_{i',j'}$ of $G_{i',j'}$ are $(f_{2(d'-1)})$. Since $d' > 1$ by the definition of B, the ideal $(f_{2(d'-1)})$ is not the whole ring. It is obvious that there exists no automorphism of L that maps a multi-set of three ideals, two of which coincide with L, to a multi-set of three ideals, two of which are proper. Therefore, we conclude that $G_{i,j} \not\cong G_{i',j'}$.

Now we consider the case where $(i,j) \in A$ and $(i',j') \in C$. We first prove the following claim.

Claim. *The ideals $(1+y^2, 1+xy)$ and $(1+y^2, 1+x)$ are neither principal nor the whole ring L.*

Proof of the claim. As $(1+y^2, 1+x)$ is the image of $(1+y^2, 1+xy)$ under the monomial automorphism (4.10), it is enough to verify the claim for $(1+y^2, 1+xy)$. Notice that the ideal $(1+y^2, 1+xy)$ does not coincide with L, because for any $f(x,y) \in (1+y^2, 1+xy)$ we have $f(1,1) \equiv 0 \mod 2$. Moreover, the ideal $(1+y^2, 1+xy)$ cannot be principal because if $(1+y^2, 1+xy) = (g)$ then $g \mid \gcd(1+y^2, 1+xy) = 1$ and this contradicts the fact that the ideal $(1+y^2, 1+xy)$ is not the whole ring. So the claim follows. □

We know from Propositions 4.1 and 4.2 that two twisted Alexander ideals of $G_{i',j'}$ are $I^1_{i',j'} = (f_{2(d'-1)})(1+y^2, 1+xy)$ and $I^2_{i',j'} = (f_{2(d'-1)})(1+y^2, 1+x)$. So, by the claim above, they are proper ideals. By the same reasoning as in the previous case, we obtain $G_{i,j} \not\cong G_{i',j'}$.

Consider the last case where $(i,j) \in B$ and $(i',j') \in C$. By Propositions 4.1 and 4.2, the twisted Alexander ideals $I^1_{i,j}$ and $I^2_{i,j}$ of $G_{i,j}$ are both principal. On the other hand, by the claim, the twisted Alexander ideals $I^1_{i',j'}$ and $I^2_{i',j'}$ of $G_{i',j'}$ are both non-principal. So we deduce that $G_{i,j} \not\cong G_{i',j'}$. □

We could not distinguish all the groups $G_{i,j}$ for (i,j) belonging to the same set B or C. However, we can find a necessary condition for two groups to be isomorphic. Before doing so, we prove a necessary condition for the existence of a monomial automorphism which maps one ideal to another.

Lemma 5.2. *Suppose that $f = 1 + y^{a_1} + \cdots + y^{a_m}$ and $g = 1 + y^{b_1} + \cdots + y^{b_n}$ are non-constant Laurent polynomials in L such that they both consist of only non-negative*

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powers of y *and have constant terms equal to* 1*. If there exists a monomial automorphism of* L *which maps the ideal* (f) *to the ideal* (g) *then* $a_m = b_n$ *and either* $f = g$ *or* $f(y^{-1})y^{a_m} = g$*.*

Proof. Suppose that the monomial automorphism is of the form $\varphi(x) = x^a y^b$ and $\varphi(y) = x^u y^v$. From the hypothesis, we get $\varphi(f) = f(x^a y^b, x^u y^v)$, and g must generate the same ideal. This means that two polynomials only differ by a unit factor of the form $x^k y^l$:

$$f(x^a y^b, x^u y^v) = 1 + x^{a_1 u} y^{a_1 v} + \cdots + x^{a_m u} y^{a_m v} = x^k y^l g = x^k y^l (1 + y^{b_1} + \cdots + y^{b_n}).$$

However, for the right-hand side containing 1, we find that k must be 0. Then the right-hand side does not depend on x. Therefore, the same must hold for the left-hand side and we get $u = 0$. From the condition of the monomial automorphism that $\left|\begin{smallmatrix} a & b \\ u & v \end{smallmatrix}\right| = av - bu = \pm 1$, we obtain $v = \pm 1$.

In the case where $v = 1$, l must be 0. So it follows that $f = g$. In the case where $v = -1$, l must be $-a_m$ and we also have $a_m = b_n$. So we deduce that $f(y^{-1})y^{a_m} = g$ as required. □

Proposition 5.3. *If* i, j, i', j' *are positive odd integers then*

$$G_{i,j} \cong G_{i',j'} \text{ implies } \gcd(i, j) = \gcd(i', j').$$

Proof. We put $d := \gcd(i, j)$ and $d' := \gcd(i', j')$. As $G_{i,j} \cong G_{i',j'}$, there exists a monomial automorphism φ of L which maps each twisted Alexander ideal of $G_{i,j}$ to a twisted Alexander ideal of $G_{i',j'}$. By Proposition 5.1, one of the following cases must happen.

Case 1: both (i, j) and (i', j') belong to A. As i, j, i', j' are positive odd integers, from the definition of A, we have $d = d' = 1$.

Case 2: both (i, j) and (i', j') belong to B. By Propositions 4.1 and 4.2, the first two twisted Alexander ideals of $G_{i,j}$ and $G_{i',j'}$ are $I^1_{i,j} = I^2_{i,j} = (f_{2(d-1)})$ and $I^1_{i',j'} = I^2_{i',j'} = (f_{2(d'-1)})$, respectively. As $G_{i,j} \cong G_{i',j'}$, there must be a monomial automorphism φ of L which maps $(f_{2(d-1)})$ to $(f_{2(d'-1)})$. It follows from Lemma 5.2 that $d = d'$.

Case 3: both (i, j) and (i', j') belong to C. In this case, the first two twisted Alexander ideals of $G_{i,j}$ and $G_{i',j'}$ are $I^1_{i,j} = (1 + y^{2d}, (1 + xy)f_{2(d-1)})$, $I^2_{i,j} = (1 + y^{2d}, (1 + x)f_{2(d-1)})$ and $I^1_{i',j'} = (1 + y^{2d'}, (1 + xy)f_{2(d'-1)})$, $I^2_{i',j'} = (1 + y^{2d'}, (1 + x)f_{2(d'-1)})$, respectively. It is obvious that one of the ideals $\varphi(I^1_{i,j}), \varphi(I^2_{i,j})$ must belong to the set $\{I^1_{i',j'}, I^2_{i',j'}\}$. We may assume, for example, that $\varphi(I^1_{i,j}) = I^2_{i',j'}$; the other case can be treated in the same way. Since the gcd of all elements of $I^1_{i,j}$ is $f_{2(d-1)}$ and that of $I^2_{i',j'}$ is $f_{2(d'-1)}$, the automorphism φ must map $(f_{2(d-1)})$ to $(f_{2(d'-1)})$. Therefore, by Lemma 5.2, we get $d = d'$. □

Using the twisted Alexander ideal of a type 3 representation, we prove the following.

Proposition 5.4. *Let* i, j, i', j' *be positive integers such that* $3|(i + j)$, $3|(i' + j')$ *and* $3 \nmid j, j'$*. Assume that* $\gcd(i, j) > 1$ *and* $\gcd(i', j') > 1$*. Then, the following holds:*

$$G_{i,j} \cong G_{i',j'} \text{ implies } \gcd(i, j) = \gcd(i', j').$$

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Proof. We put $i = kd$, $j = ld$, $i' = k'd'$, $j' = l'd'$, where $d := \gcd(i, j)$, $d' := \gcd(i', j')$. As $G_{i,j} \cong G_{i',j'}$, there exists a monomial automorphism φ of L which maps each twisted Alexander ideal of $G_{i,j}$ to a twisted Alexander ideal of $G_{i',j'}$.

We first show that l and l' must have the same parity. Suppose, for contradiction, that l is even and l' is odd. Then, by Propositions 4.1–4.3, the twisted Alexander ideals of $G_{i,j}$ are $I^1_{i,j} = I^2_{i,j} = L$ and $I^3_{i,j} = (1 + y^d + y^{2d})$.

As $3|(i' + j')$, $3\not| j'$ and l' is odd, by Proposition 4.3 we have $I^3_{i',j'} = ((1 + y^{d'} + y^{2d'})/(1 + y + y^2))$. Note that as $d' > 1$, $I^3_{i',j'} \neq L$. Since $G_{i,j} \cong G_{i',j'}$, two of the twisted Alexander ideals of $G_{i',j'}$ must coincide with L. For this to happen, we must have $I^1_{i',j'} = I^2_{i',j'} = L$.

Since the automorphism φ must map a non-trivial ideal to a Since the automorphism φ must map a non-trivial ideal to a non-trivial one, we get $\varphi(I^3_{i,j}) = I^3_{i',j'}$. As $(1 + y^{d'} + y^{2d'})/(1 + y + y^2)$ is a Laurent polynomial satisfying the hypothesis of Lemma 5.2, we get $1 + y^d + y^{2d} = (1 + y^{d'} + y^{2d'})/(1 + y + y^2)$ or, equivalently,

$$\begin{aligned} 1 + y^{d'} + y^{2d'} &= (1 + y^d + y^{2d})(1 + y + y^2) \\ &= 1 + y + y^2 + y^d + y^{d+1} + y^{d+2} + y^{2d} + y^{2d+1} + y^{2d+2}. \end{aligned}$$

As $d > 1$, the right-hand side contains more than three non-zero monomials and is different from the left-hand side. So we arrive at a contradiction.

So l and l' have the same parity. Consider the first case where l and l' are both even. By Proposition 4.3, the twisted Alexander ideals of $G_{i,j}$ and $G_{i',j'}$ are $I^1_{i,j} = I^2_{i,j} = L$, $I^3_{i,j} = (1 + y^d + y^{2d})$ and $I^1_{i',j'} = I^2_{i',j'} = L$, $I^3_{i',j'} = (1 + y^{d'} + y^{2d'})$, respectively. Therefore, the monomial automorphism φ maps $I^3_{i,j}$ to $I^3_{i',j'}$. By Lemma 5.2, we deduce that $d = d'$.

In the case where l and l' are both odd, we know that φ maps one of the twisted Alexander ideals of $G_{i,j}$ to $I^3_{i',j'} = ((1 + y^{d'} + y^{2d'})/(1 + y + y^2))$. By Propositions 4.1–4.3, $I^3_{i,j} = ((1 + y^d + y^{2d})/(1 + y + y^2))$ and $I^1_{i,j} = I^2_{i,j}$ are one of the following forms: $L, (f_{2(d-1)}), (1 + y^{2d}, f_{2(d-1)} + xyf_{2(d-1)}), (1 + y^{2d}, f_{2(d-1)} + xf_{2(d-1)})$. Note that by the claim in Proposition 5.1, the last two ideals in this list are not principal.

Since $d' > 1$, $I^3_{i',j'}$ is a proper principal ideal, so the twisted Alexander ideal of $G_{i,j}$ which is mapped to $I^3_{i',j'}$ is either $(f_{2(d-1)})$ or $((1 + y^d + y^{2d})/(1 + y + y^2))$. In either case, by Lemma 5.2, we arrive at the conclusion that $d = d'$. So the Proposition follows. □

Proof of Theorem 1.1.

(i) As $\gcd(p(2d - 1), q(2d - 1)) = 2d - 1$ and $\gcd(p(2d' - 1), q(2d' - 1)) = 2d' - 1$, part (i) is an immediate corollary of Proposition 5.3.

(ii) As $\gcd(p, q) = 1$ and $3 \mid (p + q)$, we deduce that $3\not| p$ and $3\not| q$, otherwise both p and q would be divisible by 3. So all conditions of Proposition 5.4 hold, and we obtain $d = \gcd(pd, qd) = \gcd(pd', qd') = d'$ as required.

□

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