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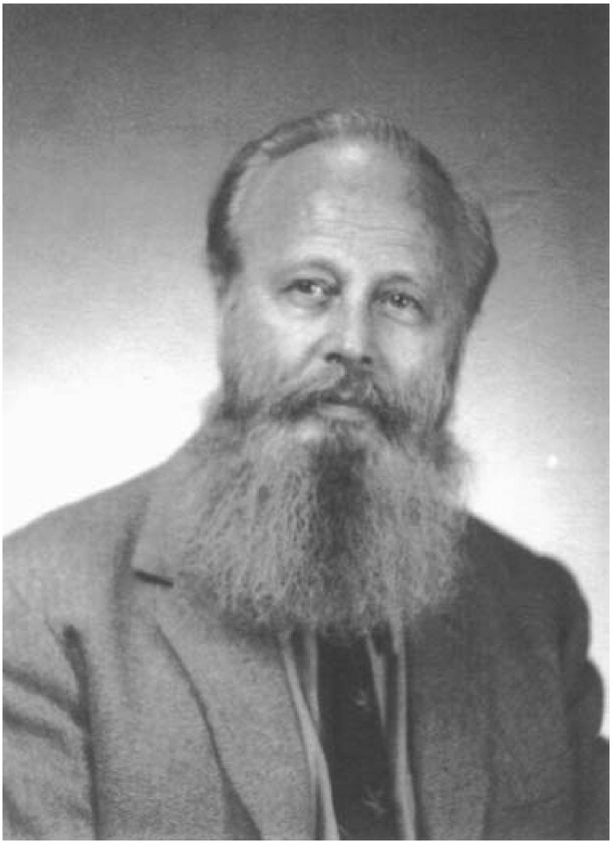
OBITUARY

LAURENCE CHISHOLM YOUNG (1905–2000)



Laurence Chisholm Young, a major contributor to mathematical analysis, died peacefully at home in Madison, Wisconsin, USA, on Sunday, 24 December 2000, aged 95. Young is known especially for his work on the calculus of variations, particularly for his approach based on what he called *generalized* curves and surfaces; this theory is often referred to as *Young measure*.

Young's mathematical work is outlined in the second part of this article. First we discuss his life, family and interests from a personal perspective.

Born in Göttingen, Germany, on 14 July 1905, Laurence, often called 'Laurie' or just 'L. C.', was the fifth of six children of the British mathematicians William Henry Young ('Will') and Grace Chisholm Young. Grace, in 1895, was the first woman to receive an official PhD in any field in Germany. (Grace was the first who was granted a PhD by the usual procedure (including examinations); Sonya Kovalevsky's PhD about 20 years earlier was awarded 'in absentia' in recognition of her work.) Although Grace was the primary caregiver and household manager, she still contributed significantly to Will's papers when the children were young; her major work came later in her part of a well-known theorem on the derivatives of a function, now called the Denjoy–Saks–Young theorem ⟨**1**, **4**⟩. Will, who started his career as a tutor at Cambridge University, later became an eminent analyst associated with the Universities of Liverpool, Aberystwyth and Calcutta. Will was President of the London Mathematical Society from 1922 to 1924, and in 1929 became the second President of the International Mathematical Union ⟨**3**⟩. The senior Youngs together wrote over 220 articles and several books, mostly in analysis and mostly signed by Will. They traveled widely and communicated with many of the leading mathematicians of their times. (The archives of the University of Liverpool contain over 6000 letters of their correspondence with each other and with other noteworthy mathematicians, as well as photographs and notes describing their daily lives.)

Laurence was considerably influenced by his mathematician parents, their contacts with other mathematicians and their lifestyle of travel, music, and languages. He was taught to be resourceful and independent. Like his parents, Laurence liked to travel. This began very early; his first solo international trip was at age two! Grace and Will had left him in Germany with a nanny while they made a professional visit to Cambridge University. He decided to follow them; on his own, he walked down to the train station and got onto a train going from Germany to France, while declaring proudly to everyone within earshot that he was going to England. Even at the frontier, it was assumed he was with someone – he was so full of confidence and conviction. (He was caught by the Dutch police at the second border; they entertained him until he was picked up.)

According to Laurence's reminiscences of growing up in Göttingen and later Lausanne, Switzerland, the family with six children lived on a shoestring budget and grew most of their food themselves; he was in charge of the chickens and rabbits and, as did each of the children, he cooked the family's meals one day each week. Grace taught the children languages and music, and encouraged their intellectual pursuits. Laurence finished Gymnasium two years early; then he was somewhat held back by scarlet fever, but he read mathematics and physics books extensively while ill, and felt well prepared for further study in these areas.

In 1924, Laurence went to Munich to study with Carathéodory; he organized the seminar there and assisted other students with their presentations. He often returned there in the summers while he was a student at Trinity College, Cambridge University. He wrote his first book, a Cambridge tract, at the age of 20; he was named the Isaac Newton Student at Cambridge in 1930. Studying under Fowler and Littlewood, Laurence became a Fellow in 1931 and was awarded an ScD in 1939. During his Cambridge years, Laurence became friends with H. S. M. Coxeter; together they started the 'Group group', to study group theory.

Laurence met Joan Dunnett ('Elizabeth') at a tea with Geoffrey Miller, her cousin and his friend, who was later a Professor of Mathematics at Oxford. After seven years of courtship, Laurence and Elizabeth married in 1934. They first lived in Cambridge, where the first two children, Francis and Rosalind ('Elizabeth'), were born. Laurence served as Professor and the first Head of the Mathematics Department at the University of Cape Town, South Africa, from 1938 to 1948, a time of terrific growth for the department. The family grew also, to include four more children: David, Sylvia, Angela and Beatrice.

From 1949 to 1975, Laurence was Professor of Mathematics at the University of Wisconsin–Madison. (Before going to Madison, he visited Ohio State University for part of a year.) Chair of the Department from 1962 to 1964, he started the Wisconsin High School Mathematics Talent Search in 1963, an event that continues to thrive and that has identified and nurtured many talented young mathematicians. During these two years also, the Department's graduate program and faculty increased substantially in size. According to longtime faculty members, this happened because Laurence persuaded the dean to let him make extra offers on the grounds that at most half of the graduate students and faculty would accept offers. However, Laurence was so persuasive that most of the candidates accepted the offers – and then finding offices for all those new people was a real challenge! The dean, after becoming resigned to having them all, was ecstatic about the qualifications of the new people and the esteem they brought to the department. Laurence was not an easy professor: he was said to begin the first day of class by saying, 'Here is the recommended reading for the course,' (pointing to a huge pile of books in several different languages). 'Don't worry, the mathematics is *much* harder than the language.' An often-told story is that once during a colloquium Laurence, who was apparently napping (which was not unusual), suddenly sat upright in the middle of the talk and startled everyone by proclaiming, 'That's Mummy's theorem!' Laurence was named Distinguished Research Professor at Wisconsin in 1968. After retiring in 1975, Laurence continued to lecture and mentor graduate students and postdoctoral fellows in Brazil for many years, at the University of Campinas in Campinas, São Paulo, Brazil and at the Institute for Pure and Applied Mathematics in Rio de Janeiro. During this time he worked in Brazil and in the US with Pedro Nowasad on problems arising from physics.

A champion chess player, Laurence won the Heart of America Competition in 1955. He acted in plays in South Africa, and then on Wisconsin Public Television in the 1950s. He knew twelve languages, and was fluent in German, Italian, and French (and English); he loved classical languages and he translated many Russian mathematical articles into English. He often celebrated Bastille Day (his birthday) at the French House in Madison.

Laurence received an honorary degree from the Université de Paris–Dauphine in 1984. The ceremony, which included Laurence's hour-long acceptance speech in French, was televised across France. Laurence became a member of the London Mathematical Society in 1933. He was a longstanding member of the Royal Astronomical Society, and of the American Mathematical Society.

Throughout his life Laurence was a brilliant and vigorous man. He felt that surviving scarlet fever when he was young had made him especially tough and thus had kept him healthy through his eighties and nineties. At age 92, Young gave an expository address on Young measures, and told how he had come to discover generalized curves as a younger man – the notion was inspired by bicycling and pedalling in a zig-zag fashion up a hill, and by sailing and tacking against the wind; he realized that other wandering, wavy paths might be the most efficient. His interest in active pursuits thus fuelled his mathematics. As a young man, he also rowed for his College, played tennis and took fifty-mile walks. The Youngs' home in Madison was on the shore of Lake Mendota; in the 1950s, when winter conditions permitted, it was his habit to skate to class. In his nineties he still occasionally walked the five miles to the University, he never missed a talent search presentation (held at the Mathematics Department every May), and he gardened for many hours each day at his home.

During Laurence's nineties, he worked intensely on several intellectual projects, including:

(1) a lengthy philosophical book in four separate versions: English, French, Italian, and German (Not technically about mathematics, it extolled the beauty of mathematics; he completed the French and English volumes, but they have not been published.);

(2) a volume of his parents' collected works (This volume has appeared ⟨**2**⟩. Laurence wrote an interesting preface for the volume, but due to the cost, neither his nor other solicited articles concerning the work of the senior Youngs could be included in the book. It is hoped that they will be published separately.);

(3) his mother's historical novel concerning a 17th-century queen of England (She had not completed the book; he made some progress, but it is still incomplete.).

Like many others, he still dreamed of solving the Riemann hypothesis (whatever ideas he had for this are apparently gone and unknown to us). Laurence maintained his skill at chess, and could still easily beat competent players who called out their moves while they and the chessboard were out of his sight in another room.

In September 2000, when Laurence heard that he had cancer and the doctor predicted one more month to live, he took it as a challenge; he had 'defied death before' (he did survive four more months). The night before he died, he expressed his appreciation to each of the family members gathered around his bed. It was a reasoned and fitting exit for a rational man to take after a full and productive life; he went out in style.

Laurence's son David died in 1964 and his wife Elizabeth in 1995. Laurence is survived by the remaining five children: Frank Young, Elizabeth Young, Sylvia Wiegand, Angela Young, and Beatrice Nearey; two sons-in-law: Roger Wiegand and Terry Nearey; five grandchildren: David and Andrea Wiegand, and Siobhan Laura, Brendan and Kenneth Nearey; as well as granddaughter-in-law Chris (Carmichael) Wiegand and great-granddaughter Samantha Wiegand. Sylvia and Roger are mathematicians at the University of Nebraska.

Laurence Young had one PhD student at Cambridge University: E. R. Love (1938); and twelve at Wisconsin: Wendell Fleming and Kennan T. Smith (both in 1951), Kermit H. Carlson (1954), Raymond Rishel and Daniel Sokolowsky (both 1959), Dattatraya J. Patil and Mahavirendra Vasavada (both 1968), Gerald M. Armstrong and Edgar E. Escultura (both 1970), Graham Donald Allen (1971), Dar-Biau Liu (1972), and Yuan Chia Wang (1975).

A memorial service was held in Madison, Wisconsin in August 2001.

Mathematical contributions

L. C. Young's published work spans a long period, from 1927 to 1983. His imagination and vision were instrumental in bringing calculus of variations and related fields to their present form. Young's work was ahead of its time in many ways, and the breadth of its applications has been increasingly recognized in recent years. For instance, Young measure leads to a new concept of weak solution for

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nonlinear partial differential equations which is relevant to contemporary studies of phase transitions and microstructures in crystalline and composite materials.

Starting in 1933, L. C. Young introduced the radically new idea of generalized curves and surfaces. This provided a setting in which variational problems with nonconvex integrands have solutions. In control theory, these ideas reappeared in the 1960s with the name *chattering* (or *relaxed*) controls. The use of relaxed controls remains an indispensable tool in both deterministic and stochastic control theory. Recent studies of nonconvex variational integrands in material science are strongly influenced by Young's generalized surfaces.

In the 1950s, Young recast the theory of generalized curves and surfaces in a conceptually appealing framework, in which methods of functional analysis could be applied. This work set the stage for present-day geometric measure theory, a powerful tool in several areas of mathematical analysis (including partial differential equations and complex variables) and in differential geometry.

L. C. Young is the author of two books. The first, *Lectures on the calculus of variations and optimal control theory* [**57**], written in Young's highly personal style, is a highly readable introduction to the topics of the title. The book is illuminated and enlivened by many analogies drawn from history, literature and everyday life. The second book, *Mathematicians and their times* [**66**], is a remarkable collection of reflections on the development of mathematics since antiquity, and on the lives of great mathematicians who created it. The book begins with a lengthy introduction, concerned with mathematics from antiquity through the 18th century. Then Young turns to what he calls 'the Romantic Period' in mathematics, from around the time of the French Revolution until the end of the 19th century. The final part gives a fine perspective on the development of mathematics in the early 20th century, including remembrances of the rich mathematical life at Cambridge, Göttingen and other leading European centers. Additional personal reminiscences appear in Young's last published paper [**69**].

Young translated into English the book *Theory of the integral* by S. Saks. The translation appeared in 1937. This was a great service to the mathematical community, since Saks' book was for many years a basic reference for measure and integration.

We now give a concise overview of Young's research contributions, organized according to topics and time periods.

1. *Measure, integration, Fourier series* (1927–1943) [**1–4**, **6–14**, **16–18**, **20**, **23**, **24**]

L. C. Young had a long-standing interest in measure and integration theory. His earliest publication [**1**] is a concise introduction to Lebesgue–Stieltjes integrals, using methods of monotone sequences of functions. A series of papers on Stieltjes integrals and applications to Fourier series followed (see the list above).

It is elementary that the Stieltjes integral $\int f dg$ over a compact interval I of the real line exists, provided that one of the functions f, g is continuous and the other is of bounded variation. Young replaced this asymmetric condition of f, g by more symmetric conditions which also guarantee that the Riemann–Stieltjes integral exists. For instance, f can have bounded pth power vairation and g bounded qth power variation, where $p^{-1} + q^{-1} > 1$. This condition is close to optimal. For example, if $p = q = 2$, then f and g have bounded quadratic variation, and the Riemann–Stieltjes integral need not exist. This situation is frequently encountered in the Itô stochastic calculus. This framework led to a number of new results about convergence of $\int f_n\,dg_n$ to $\int f\,dg$ when f_n, g_n tend to f, g (in suitable senses) as $n \to \infty$. It also led to new results about convergence of Fourier series, including a nice generalization of a convergence criterion due to W. H. Young (see [**16**]). Also related are L. C. Young's results with Love on fractional integration by parts.

2. *Generalized curves and surfaces* (1933–1942) [**5**, **15**, **19**, **21**, **22**]

In connection with his proof of the Dirichlet principle, Hilbert stated. 'Every problem in the calculus of variations has a solution, provided the word "solution" is suitably interpreted'; see [**57**, p. 123; **66**, p. 241]. In traditional formulations of calculus of variations, this guiding principle turns out to be correct only if an additional convexity condition is imposed. To illustrate the difficulty, consider the following simple problem. For a given finite interval I and real-valued function f of three variables, find a real-valued function $x(\cdot)$ having prescribed values at the endpoints of I, which minimizes

$$J(x(\cdot)) = \int_I f(t, x(t), \dot{x}(t))\,dt. \tag{1}$$

The customary method to prove existence of a minimum is to find a minimizing sequence $x_n(\cdot)$ converging uniformly on I to a limit $x^*(\cdot)$, and to invoke lower semicontinuity of J under uniform convergence to conclude that $x^*(\cdot)$ is minimizing. However, semicontinuity requires convexity in $\dot{x}$ of $f(t, x, \dot{x})$, and examples show that there may be no $x^*(\cdot)$ that minimizes $J(x(\cdot))$ if this convexity assumption does not hold.

L. C. Young resolved this difficulty by recasting the problem in an altogether different way. This was done by enlarging the space of ordinary curves to include entities that he called 'generalized curves', and to replace uniform convergence of curves $x_n(\cdot)$ on I by weak limits of associated linear functionals.

A generalized curve $\mathcal{C}$ can be described by a pair of functions $x(\cdot)$, $M(\cdot)$ on I. For the simple problem mentioned above, $x(\cdot)$ is real-valued and $M(t)$ is a probability measure on the real line $\mathbb{R}^1$ with mean $\dot{x}(t)$. Let

$$J(\mathcal{C}) = \int_I \int_{\mathbb{R}^1} f(t, x(t), \dot{x}) M(t, d\dot{x})\,dt. \tag{2}$$

Nowadays, $M(t)$ is often called a *Young measure*.

Every generalized curve $\mathcal{C}$ can be approximated by a sequence $x_n(\cdot)$ of ordinary curves such that $x_n(\cdot)$ tends to $x(\cdot)$ on I and $J(x_n(\cdot))$ tends to $J(\mathcal{C})$ as $n \to \infty$. The derivatives $\dot{x}_n(\cdot)$ may oscillate rapidly for large n. The probability distribution $M(t, d\dot{x})$ can be obtained as a limit of empirical distributions of $\dot{x}_n(s)$ for large n and s in a small neighborhood of t.

This definition of generalized curve appears in [**15**], and in a preliminary form in [**5**]. Young obtained general theorems insuring the existence of a generalized curve $\mathcal{C}$ that minimizes $J(\mathcal{C})$, in accord with the principle that Hilbert stated. If $M(t, \cdot)$ is a Dirac measure concentrated at $\dot{x}(t)$, then $\mathcal{C}$ corresponds to the ordinary curve represented by $x(\cdot)$ and (2) reduces to (1). If the function f from (1) is convex in $\dot{x}$, the existence of an ordinary minimizing curve is then immediate from Jensen's inequality.

Once the existence of a minimizing generalized curve is known, the next step is to obtain necessary conditions for a minimum corresponding to the classical conditions of Euler and Weierstrass. This was addressed by Young in [**19**].

In [**21**] and [**22**], Young extended these ideas from single integral to double integral problems of the calculus of variations. He defined 'generalized surfaces' described by a pair $x(\cdot,\cdot), M(\cdot,\cdot)$, where $x(\cdot,\cdot)$ is a Lipschitz continuous function of two variables (t_1, t_2) and $M(t_1, t_2)$ is a probability measure on the space $\mathbb{R}^2$ of possible gradient vectors. Young's methods later provided the basis for study of minimum energy configurations in solid mechanics with nonconvex energy integrands. In addition, Tartar and Murat recognized the role of Young measures and developed the method of compensated compactness. Subsequently Di Perna, Ball and others applied Young measures to problems of hyperbolic partial differential equations, microstructures and phase transitions.

3. *Area theory* (1944–1955) [**25–29**, **32**, **33**, **36**]

The *Plateau problem* is to show that there is a surface S in 3-dimensional Euclidean space $\mathbb{R}^3$ of minimum area, with given boundary C. Douglas and Rado independently solved the Plateau problem in the early 1930s, for the class of surfaces described by parametric representations $\mathbf{x} = \mathbf{x}(\cdot,\cdot)$ from a circular disc D into $\mathbb{R}^3$, such that the curve C is represented by the restriction of $\mathbf{x}$ to ∂D. During the following 25 years, the field of surface area theory evolved. One goal of area theory was to provide a framework in which the existence of minima could be proved for a wide class of geometric problems in the calculus of variations. While this effort proved successful, area theory turned out to involve difficult technical complexities. Moreover, the methods were limited to two-dimensional surfaces represented by mappings from a domain D of given topological type into $\mathbb{R}^n, n \geqslant 3$. If D is simply connected, by conformal mapping one can take D to be a circular disc.

Several definitions of 'area' for a continuous mapping $\mathbf{x}$ from a disc D into $\mathbb{R}^3$ were given. If $\mathbf{x}$ is Lipschitz continuous, each of these areas agrees with the classical area integral formula $A(\mathbf{x})$, but this is not true under weaker assumptions that ensure that $A(\mathbf{x})$ exists. The Lebesgue definition of area $A_{\mathrm{L}}(\mathbf{x})$ has the attractive feature that $A_{\mathrm{L}}(\cdot)$ is lower semicontinuous under uniform convergence. Lebesgue area theory was extensively developed by Cesari, Rado, Federer, and others. Among its drawbacks is that finiteness of $A_{\mathrm{L}}(\mathbf{x})$ does not exclude various pathologies. For instance, the compact set $\mathbf{x}(\mathbf{D})$ may have positive 3-dimensional Lebesgue measure even if $A_{\mathrm{L}}(\mathbf{x})$ is finite.

In [**28**], Young introduced an alternative definition $A_{\mathrm{I}}(\mathbf{x})$ for area of a continuous mapping $\mathbf{x}$, called *intrinsic area*. This definition avoids some objectionable features of Lebesgue area. Besicovitch independently gave a definition of area similar to Young's $A_{\mathrm{I}}(\mathbf{x})$. Intrinsic area played an important role in Young's fundamental 1948 paper [**29**]. That paper also made use of his results on harmonic interpolation [**26**, **28**]. Bounds on the area of a family of surfaces do not imply any compactness property in the uniform topology. This is easily seen from the fact that surfaces may include arbitrarily shaped thin protrusions of arbitrarily small area. However, in [**29**] Young proved results that involve a weaker property called quasicompactness. Roughly speaking, a family of surfaces is quasicompact if, given $\epsilon > 0$, it is transformed into a compact family by excising certain portions of each surface which are then replaced by new portions of total area less than ϵ. The results of [**29**] provided a foundation for Young's 1955 memoir [**36**], which characterizes generalized parametric surfaces of finite topological type.

4. *Generalized surfaces and geometric measure theory* (1951–1965) [**30**, **34**, **35**, **37**–**42**, **44**–**56**]

Young's seminal 1951 paper [**30**] had a profound influence on what later came to be known as geometric measure theory. Geometric measure theory provides a framework for measure and integration over broadly defined classes of 'surfaces' in n-dimensional $\mathbb{R}^n$, of arbitrary dimension $k < n$. The concept of a surface should include oriented k-dimensional manifolds with lower-dimensional singularities. However, for geometric problems of the calculus of variations, a much broader class of surfaces is needed. An example is the problem of minimizing k-dimensional area, among all k-dimensional surfaces with given $(k-1)$-dimensional boundary (a higher-dimensional Plateau problem). To prove that a minimum exists, closure and compactness theorems in a suitable topology on the space of admissible surfaces are needed. By 'surface' let us mean a generalized surface, as defined in [**30**]. As in [**30**], let us take for simplicity $k = 2$. Let $\mathbf{x} \in \mathbb{R}^n$ and let $\boldsymbol{\alpha} = \mathbf{v}_1 \wedge \mathbf{v}_2$ denote the exterior product of $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$. Let E denote the space of all continuous real-valued functions $f(\mathbf{x}, \boldsymbol{\alpha})$, such that $f(\mathbf{x}, k\boldsymbol{\alpha}) = kf(\mathbf{x}, \boldsymbol{\alpha})$ when $k \geqslant 0$. A generalized surface L is any non-negative linear functional on E. To any ordinary surface S, represented parametrically by a Lipschitz continuous mapping $\mathbf{x}$ from a bounded region $D \subset \mathbb{R}^2$ into $\mathbb{R}^n$, corresponds the generalized surface L_{S} such that for all $f \in E$

$$L_{\mathrm{S}}(f) = \int_D \int f[\mathbf{x}(u, v), \mathbf{x}_u(u, v) \wedge \mathbf{x}_v(u, v)]\, du\, dv. \tag{3}$$

It turns out that every generalized surface is the weak limit of sequences of convex combinations of ordinary surfaces L_{S}. By adopting this broad notion of 'surface' in a topological vector space setting, Young opened the way to using functional analysis methods in geometric measure theory and the calculus of variations.

A function $f \in E$ is called *exact* if f corresponds to the exterior differential $d\omega$ of a smooth differential form ω of degree 1. The boundary ∂L of a generalized surface L is the restriction of L to the subspace of exact f. In particular, we write $\partial L = C$ where C is an oriented compact 1-manifold in $\mathbb{R}^n$ of finite length, if Stokes' formula holds for all such ω:

$$\int_C \omega = L(d\omega). \tag{4}$$

Let Γ_C consist of all L with $\partial L = C$.

In Young's framework, geometric problems in the calculus of variations are formulated as follows. Given $f_0 \in E$, find L^* which minimizes $L(f_0)$ among all $L \in \Gamma_C$. A sufficient condition for L^* to be minimizing is that there exists an exact φ_0 such that $f_0 + \varphi_0 \geqslant 0$ and $L^*(f_0 + \varphi_0) = 0$. The existence of such an exact φ_0 was obtained in [**34**], using a separation principle from functional analysis. Young called this result a homology principle [**35**, **39**]. For the simpler case $k = 1$, this homology principle corresponds to an equivalence principle of Carathéodory [**40**].

If $\partial L = C$, let Γ_C^0 be the set of all 'ordinary surfaces' L_S of finite topological type with boundary C. In the spirit of Young's earlier work, one can seek to minimize $L(f_0)$ on the weak closure $\bar{\Gamma}_C^0$ of Γ_C^0. Each $L \in \bar{\Gamma}_C^0$ has a representation that is a kind of extension of Young's representation (2) for generalized curves; see [**30**] and [**38**] for the case $n = 3$. For f_0 that is positive and convex in α, this implies that any L^* that minimizes $L(f_0)$ on $\bar{\Gamma}_C^0$ differs in arbitarily small area from an ordinary surface L_S represented as in (3). In particular, this is true for the area integrand $f_0(\boldsymbol{\alpha}) = |\boldsymbol{\alpha}|$. (Later, much stronger regularity results were obtained by De Giorgi, Almgren, and others for multidimensional geometric variational problems, including the minimum area problem.) In dimension $n = 3$, the convex closure of Γ_C^0 turns out to be Γ_C. This implies that the minimum of $L(f_0)$ on $\bar{\Gamma}_C^0$ is the same as on Γ_C, and the homology principle applies. Unfortunately, Γ_C is not the convex closure of Γ_C^0 in dimension $n \geqslant 4$ [**38**]. Moreover, an example shows that the minimum areas in the classes Γ_C and $\bar{\Gamma}_C^0$ can be different when $n \geqslant 4$ [**52**].

There are similarities between De Rham's theory of currents and Young's generalized surfaces. In the definition of a current, f is linear in $\boldsymbol{\alpha}$: $f(\mathbf{x}, \boldsymbol{\alpha}) = \omega(\mathbf{x}) \cdot \boldsymbol{\alpha}$, and positivity of the linear functionals that are currents is not required. In 1960, Federer and Fleming introduced normal and integral currents. For surface dimension $k = 2$ and boundary C, normal currents correspond roughly to generalized surfaces in Γ_C, and integral currents to generalized surfaces in $\bar{\Gamma}_C^0$. In some respects the normal and integral current framework is more convenient. However, to obtain a Morse theory, Almgren and Allard found a generalized surface formulation more convenient. Instead of the name 'generalized surface', Almgren used the catchier term 'varifold'.

In a series of papers during the late 1950s and 1960s [**44**–**56**], Young significantly expanded his theory. He called the higher-dimensional analogues of generalized curves and surfaces *k-dimensional hypersurfaces* (or *generalized k-varieties*) in $\mathbb{R}^n$, for $2 < k < n$. In [**44**], Young introduced a tool that is very useful in geometric measure theory. He called it a *partial area formula*. Independently, Federer discovered an equivalent result, which he called a *coarea formula*; a special case of the partial area/coarea formula was used earlier by De Giorgi. In [**45**] and [**46**], the partial area formula was used to define *slicing* of a generalized k-variety by the family of level sets of a function g from $\mathbb{R}^n$ into $\mathbb{R}^m$, where $m < n$. For instance, if $g(\mathbf{x}) = |\mathbf{x} - \mathbf{x}_0|$, the level sets are $(n-1)$-dimensional spheres with center x_0. In [**45**], slicing by $(n-1)$-spheres and construction of comparison cones were used to obtain necessary conditions that $L^*(f_0)$ is extremal in comparison with $L(f_0)$, among generalized k-varieties L with $\partial L = \partial L^*$. In [**47**], slicing methods were used in obtaining generalized hypersurface versions of an isoperimetric inequality and the Schwarz symmetrization principle for least area problems.

In [**50**] and [**51**], Young considered the following difficult long-standing question. Consider the problem of minimizing $L(f_0)$ when ∂L is given. What conditions on f_0 insure that probability distributions over possible tangent k-vectors $\boldsymbol{\alpha}$ at a point $\mathbf{x}$ (as in (2)) can be avoided, in seeking an L^* that gives the minimum? For $k = n - 1$, convexity in $\boldsymbol{\alpha}$ of f_0 suffices, by Jensen's inequality. No such simple criterion is available if $k < n - 1$. Young proposed a criterion called *restricted convexity*, which is defined only indirectly. A similar difficulty occurs for multidimensional nonparametric problems of calculus of variations, such as energy minimization in solid mechanics. For such problems, the condition of quasiconvexity has no simple explicit characterization.

5. *Other topics*

(a) *Prime ends.* Carathéodory's theory of prime ends arose from conformal mapping of a simply connected region G with 'irregular' boundary ∂G from within G. Ursell and Young [**31**] considered the approach from the exterior $\mathbb{R}^2 - \bar{G}$, and studied the intrinsic nature of ∂G.

(b) *Control theory.* The field of control theory flowered during the 1960s and 1970s. A broad class of deterministic optimal control problems were formulated as calculus of variations type problems, with additional conditions in the form of differential equations and inequality constraints on the control variables. In many examples taken from engineering applications there is no optimum among ordinary controls. To obtain a minimum, chattering (or relaxed) controls were introduced. These are direct analogues of Young's generalized curves. The second half of Young's book [**57**] is an introduction to control theory, including Pontryagin's necessary conditions for an optimum and relaxed controls. Further developments of Young's approach to control theory appear in [**59**, **62**].

(c) *Stochastic integrals.* In a series of papers [**58**, **60**, **61**, **64**], Young developed his own approach to stochastic integration, and to related questions about Stieltjes integrals. In [**58**], a stochastic integral $\int y(t)\,dx(t,\omega)$ is defined when $y(\cdot)$ is deterministic and $x(t,\omega)$ is a stochastic process with sample paths $x(\cdot,\omega)$ in some Hardy–Littlewood class. In [**61**], $y(t)$ is stochastic but $x(t,\omega)$ must be 'almost a martingale'. The results involve estimates for higher-order derivatives of functions represented in terms of multiple Fourier kernels. Reference [**64**] establishes via counterexamples the best possible character of these results.

(d) *Connections with particle physics.* If a generalized curve $\mathcal{C}$ is regarded as the motion of a physical particle, then at time t the position $\mathbf{x}(t)$ is observed, but, only a probability distribution $M(t)$ of possible velocities is known. See equation (2) above in the scalar case. From his earliest paper [**5**] on generalized curves, Young was interested in possible connections with quantum mechanics. He returned to this much later in joint work with Nowosad [**67**, **68**]. In this approach, the apparent velocity $\mathbf{v}$ of a particle is the weighted average

$$\mathbf{v} = p\mathbf{v}_1 + (1-p)\mathbf{v}_2$$

of velocities $\mathbf{v}_1$ and $\mathbf{v}_2$. For instance, $\mathbf{v}_1$ and $\mathbf{v}_2$ may be the velocities of emitted and absorbed photons.

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References

⟨**1**⟩ M. L. Cartwright, 'Grace Chisholm Young', *J. London Math. Soc.* 19 (1944) 185–192.
⟨**2**⟩ S. D. Chatterji and H. Wefelscheid (eds), *Selected papers of G. C. Young and W. H. Young* (Presses Polytechniques et Universitaires Romandes, Lausanne, 2000).
⟨**3**⟩ G. H. Hardy, 'William Henry Young', *J. London Math. Soc.* 17 (1942) 218–237.
⟨**4**⟩ F. Riesz and B. Nagy, *Functional analysis* (Ungar, New York, 1955).

Publications of L. C. Young

1. *The theory of integration*, Cambridge Tracts in Mathematics and Mathematical Physics 21 (Cambridge University Press, 1927). Reprinted by Stechert-Hafner, New York, 1964.
2. 'Changement de variables dans les integrales de Stieltjes', *Comptes Rendus Acad. Sci. Paris* 187 (1928) 704–706.
3. 'Note on the theory of measure', *Proc. Cambridge Philosophical Society* 26 (1930) 88–93.
4. 'On the combinatory scheme of analysis', *Proc. Cambridge Philosophical Society* 27 (1931) 232–239.
5. 'Approximation by polygons in the calculus of variations', *Proc. Roy. Soc. London Sec.* A 141 (1933) 325–341.
6. 'Limits of Stieltjes integrals', *J. London Math. Soc.* 9 (1934) 119–126.
7. 'Remarks on Fourier series and Parseval's equation', *Proc. Internat. Congress Math.*, Oslo (1936).
8. 'An inequality of the Hölder type connected with Stieltjes integration', *Proc. Internat. Congress Math.*, Oslo (1936).
9. 'An inequality of the Hölder type connected with Stieltjes integration', *Acta Math.* 67 (1936) 251–282.
10. 'Sur une généralization de la notion de variation de puissance p-ième bornée au sens de M. Wiener, et sur la convergence des séries de Fourier', *Comptes Rendus Acad. Sci. Paris* 204 (1937) 470–473.
11. 'Integrales généralisées...de Stieltjes...', *Conference de la Reunion Internationale des Mathematiciens, Paris* (1937).
12. 'Inequalities connected with binary derivation...', *Math. Z.* 43 (1937) 255–270.
13. 'Inequalities connected with p-th power variation in the Wiener sense and with integrated Lipschitz conditions...', *Proc. London Math. Soc.* (2) 43 (1937) 449–467.
14. 'General inequalities for Stieltjes integrals', *Math. Ann.* 115 (1938) 581–612.
15. 'Generalized curves and the existence of an attained absolute minimum...', *Comptes Rendus, Soc. Sciences & Lettres, Warsaw, cl. III* 30 (1937) 212–234.
16. 'A convergence criterion for Fourier series', *Oxford Quarterly Journal of Mathematics* 9 (1938) 115–120.
17. (With E. R. Love) 'Sur une classe de fonctionnelles linéaires', *Fund. Math.* 28 (1937) 243–257.
18. (With E. R. Love) 'On fractional integration by parts', *Proc. London Math. Soc.* (2) 44 (1938) 1–35.
19. 'Necessary conditions in the calculus of variations', *Acta Math.* 69 (1938) 229–258.
20. 'On an inequality of Marcel Riesz', *Ann. of Math.* 40 (1938) 567–574.
21. 'Generalized surfaces in the calculus of variations', *Ann. of Math.* 43 (1942) 84–103.
22. 'Generalized surfaces in the calculus of variations II', *Ann. of Math.* 43 (1942) 530–544.
23. 'On the convergence of Fourier–Bessel series', *Proc. London Math. Soc.* (2) 47 (1942) 290–307.
24. 'A further inequality for Stieltjes integrals', *J. London Math. Soc.* 18 (1943) 78–82.
25. 'An expression connected with the area of a surface $z = F(x, y)$', *Duke Math. J.* 11 (1944) 43–57.
26. 'A lemma in the theory of surfaces', *J. London Math. Soc.* 19 (1944) 209–212.
27. 'On the isoperimetric ratio for a harmonic surface', *Proc. London Math. Soc.* (2) 49 (1947) 396–408.
28. 'On area and length', *Fund. Math.* 35 (1948) 275–302.
29. 'Some applications of the Dirichlet integral to the theory of surfaces', *Trans. Amer. Math. Soc.* 64 (1948) 317–335.
30. 'Surfaces paramétriques généralisées', *Bull. Soc. Math. France* 79 (1951) 59–84.
31. (With H. Ursell) 'Remarks on the theory of prime ends', *Mem. Amer. Math. Soc.* 3 (1951).
32. (With K. H. Carlson) 'Continuity of area for harmonic surfaces with boundaries of uniformly bounded length', *Proc. Amer. Math. Soc.* 3 (1952) 88–91.
33. 'On the compactness and closure of surfaces of finite area, continuous or otherwise and on generalized surfaces', *Rend. Circ. Mat. Palermo* 2 (1953) 106–118.
34. (With W. H. Fleming). 'A generalized notion of boundary', *Trans. Amer. Math. Soc.* 76 (1954) 457–484.
35. 'A variational algorithm', *Riv. Mat. Univ. Parma* 5 (1954) 255–268.
36. 'On generalized surfaces of finite topological types', *Mem. Amer. Math. Soc.* 17 (1955).
37. (With W. H. Fleming) 'Representations of generalized surfaces as mixtures', *Rend. Circ. Mat. Palermo* 5 (1956) 117–144.
38. (With W. H. Fleming) 'Generalized surfaces with prescribed elementary boundary', *Rend. Circ. Mat. Palermo* 5 (1956) 320–340.
39. 'Some new methods in two-dimensional variational problems with special reference to minimal surfaces', *Comm. Pure Appl. Math.* 9 (1956) 625–632.

40. 'Champs vectoriels attachés a une mesure plane', *J. Math. Pures Appl.* 35 (1956) 345–358.
41. 'Caractere local élémentaire d'une surface qui rend minima une integrale double', *Comptes Rendus Acad. Sci. Paris* 248 (1959) 1110–1112.
42. 'Existence du plan tangent d'une surface qui rend minima une integrale double', *Comptes Rendus Acad. Sci. Paris* 248 (1959) 916–919.
43. 'Remarks on a chapter of the integral', *Rend. Circ. Mat. Palermo* 7 (1958) 48–54.
44. 'Partial area, I', *Riv. Mat. Univ. Parma* 10 (1959) 103–113.
45. 'Contours on generalized and extremal varieties', *J. Math. & Mech.* 11 (1962) 615–646.
46. 'Partial area, II', *Riv. Mat. Univ. Parma* 10 (1959) 117–182.
47. 'Partial area, III', *Riv. Mat. Univ. Parma* 10 (1959) 257–263.
48. 'Some extremal questions for simplicial complexes, I', *Rend. Circ. Mat. Palermo* (2) 11 (1962) 178–184.
49. 'Some extremal questions for simplicial complexes, II', *Rend. Circ. Mat. Palermo* (2) 11 (1962) 271–279.
50. 'Some extremal questions for simplicial complexes, III', *Rend. Circ. Mat. Palermo* (2) 12 (1963) 41–58.
51. (With W. H. Fleming) 'Some extremal questions for simplicial complexes, IV', *Rend. Circ. Mat. Palermo* (2) 12 (1963) 200–210.
52. 'Some extremal questions for simplicial complexes, V', *Rend. Circ. Mat. Palermo* (2) 12 (1963) 257–274.
53. 'Constructions élémentaires et methodes abstraites dans le calcul des variations géneralisé', *Atti del Simposio Lagrangiano Ac. d. Sc. Torino* (1963) 120–137.
54. 'Generalized varieties as limits', *J. Math. & Mech.* 18 (1964) 673–692.
55. 'Remarques concernant un problème de représentation des variétés généralisées, et son rapport au mouvement stationnaire d'un fluide', *Enseign. Math. T.* (1965) 11, fasc. 2–3, 203–239.
56. 'A theory of boundary values', *Proc. London Math. Soc.* (3) 14A (1965) 300–314.
57. *Lectures on the calculus of variations and optimal control theory* (W. B. Saunders, Philadelphia, 1969).
58. 'Some new stochastic integrals and Stieltjes integrals I, Analogues of Hardy–Littlewood classes', *Advances in probability and related topics*, Vol. 2 (Dekker, New York, 1970) 161–240.
59. 'Strengthening Carathéodory's method to apply to control problems', *Control theory and calculus of variations* (ed. A. V. Balakrishnan, Academic Press, New York, 1969) 133–151.
60. 'An approximation algorithm for nth derivatives and its applications to analysis and to stochastic integrals', *Techniques of optimization*, Fourth IFIP Colloq. Optimization Techniques (ed. A. V. Balakrishnan, Academic Press, New York, 1972) 129–133.
61. 'Some new stochastic integrals and Stieltjes integrals II, Nigh-martingales', *Advances in probability and related topics*, Vol. 3 (Dekker, New York, 1974) 101–178.
62. 'Remarks on some basic concepts of optimal control', *Optimal control theory and its applications*, Part II, Lecture Notes in Econom. and Math. Systems 106 (ed. B. J. Kirby, Springer, 1974) 385–398.
63. *On D. J. Patil's remarkable generalization of Cauchy's formula*, London Mathematical Society Lecture Note Series 12 (Cambridge University Press, London, 1974) 137–138.
64. 'A converse to some inequalities and approximations in the theory of Stieltjes and stochastic integrals, and for nth derivatives', *Studia Math.* 55 (1976) 215–223.
65. 'Generalized curves, generalized varieties, optimal control', *Proc. Ninth National Math. Conf., Isfahan, 1978* (Univ. Isfahan, Isfahan, 1978) 364–374.
66. Mathematicians and their times. History of mathematics and mathematics of history, North-Holland Math. Stud. 48 (North-Holland, Amsterdam, 1981).
67. 'New applications of generalized curves, particularly in physics', *Convex analysis and optimization*, Res. Notes in Math. 57 (ed. J. P. Aubin and R. Vinter, Pitman, Boston, 1982) 181–210.
68. (With P. Nowosad) 'Generalized curve approach to elementary particles', *J. Optim. Theory Appl.* 41 (1983) 261–278.
69. 'Remarks and personal reminiscences', *Modern optimal control*, Lecture Notes in Pure and Applied Mathematics 119 (ed. E. O. Roxin, Dekker, New York, 1989) 421–433.

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