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The Alfvén limit revisited and its relevance to laser-plasma interactions

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Abstract

Alfvén's derivation of his current limit is given. It demonstrates that it does not give the maximum possible current of a beam, but the maximum current that can propagate for an indefinite distance and time, from a source, in a charge neutral beam. Furthermore, the value Alfvén obtained applies to a uniform current density and to particles initially moving in the direction of the beam. It is also shown that Alfvén predicted that beams which exceed the limit will filament as a result of the particles that are turned back by the magnetic field. His work is extended to beams with particles that have transverse momentum, to beams with non-uniform current densities, to beams that are not charge neutral and to the time dependent case. These extensions of Alfvén's work are found to require numerical calculations in most cases and to give ambiguous results in some cases. A general formula for the current limit is given based on the conservation of energy. It is calculated for the cases considered previously and found to confirm the accuracy of Alfvén's original estimate. The relevance of the current limit to high intensity laser-solid interactions and fast ignition is then discussed.

Keywords: Alfvén limit; Fast ignition; Laser-plasma

1. THE ALFVÉN LIMIT REVISITED

1.1. Introduction

The Alfvén limit is frequently quoted as giving the maximum current that can be carried by a beam of charged particles, and its validity as such is questioned almost as frequently. In this article, I return to Alfvén's original paper, and demonstrate that this interpretation of his result is inaccurate and that a more accurate consideration answers most of the many questions regarding the limit and reveals an overlooked aspect of his work: the filamentation of beams that exceed the limit.

Alfvén (1939) wrote his paper on the motion of cosmic rays in interstellar space. As a simple model, he considered identical, charged particles with identical momenta being emitted from a circular region of an infinite plane, and forming a beam with a uniform net current density and no net charge density. He justified the assumption of charge neutrality by stating that interstellar space is a good conductor with a charge density much greater than that of the cosmic rays. He then calculated the trajectories of the particles in their steady state, self-generated, azimuthal magnetic field, in his words, the situation when all transients have died down. I will start with a slightly more general model, calculating the trajectory of a charged particle with an arbitrary initial momentum in the azimuthal magnetic field B generated by an arbitrary current J, given by $\mu_0 J/2\pi r$, that depends only on radius. I will use cylindrical coordinates (r, z, θ), defining $z = 0$ to be the source and the positive z direction to be the direction of beam propagation. The axial equation of motion can be solved by dividing by the radial velocity and integrating with respect to radius, giving

$$p_z = p_{z0} + \frac{q\mu_0}{2\pi} \int_{r_0}^{r} \frac{J}{r}\, dr, \tag{1}$$

where q is the particle charge and the subscript 0 indicates initial values. This can also be expressed as the conservation of the canonical momentum $p_z + qA_z$, where A_z is the axial vector potential given by $B = -\partial A_z/\partial r$. The azimuthal momentum is given by the conservation of angular momentum,

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$$p_\theta = \frac{r_0}{r} p_{\theta 0}. \quad (2)$$

The radial momentum can then be determined from

$$p_r^2 = p^2 - p_z^2 - p_\theta^2. \quad (3)$$

The particle trajectory is then given by integrating the equation

$$\frac{dz}{dr} = \frac{p_z}{p_r}. \quad (4)$$

For a uniform current density j_0 and a particle initially moving in the axial direction, as Alfvén considered, Eq. (4) gives

$$z = \pm \int_{r_0}^{r} \frac{4p + q\mu_0 j_0 (r^2 - r_0^2)}{\sqrt{(r^2 - r_0^2)(q^2\mu_0^2 j_0^2 (r_0^2 - r^2) - 8q\mu_0 j_0 p)}} dr, \quad (5)$$

where the sign is negative as the particle moves from r_0, the maximum radius, to 0, the minimum radius, and vice versa. Eq. (5) can be expressed in terms of elliptic integrals. Alfvén must have sat down with a table of elliptic integrals and drawn some sample trajectories by hand (see his Fig. 2). Nowadays this can be carried out almost instantly on a computer, which is how Figure 1 was generated. It shows that all particles beyond a certain radius move backwards, so the beam that actually propagates forward, the direct beam as Alfvén called it, has a maximum radius, which defines a maximum current. In general, this current is enclosed within the initial radius of the particle trajectory that satisfies

$$\int_{r_{\min}}^{r_{\max}} \frac{p_z}{\sqrt{p_r^2}} dr \leq 0. \quad (6)$$

For Alfvén's case, Eq. (6) gives a complete elliptic integral, and he gave the solution as

$$J_{A-U} = 1.65 \frac{4\pi}{q\mu_0} p \equiv 1.65 J_0. \quad (7)$$

I will refer to Eq. (7) as the Alfvén limit for a uniform current density, hence the subscript.

It can be seen from the derivation of the Alfvén limit that it does not give the maximum possible current of a beam, but the maximum current that can propagate for an indefinite distance and time, from a source, in a charge neutral beam, and that the value Alfvén calculated applies to a beam with a uniform current density where all the particles are initially moving in the direction of the beam. The Alfvén limit only applies after a certain distance because particles outside the direct beam propagate a limited distance. In Figure 1, the current is only limited to that in the direct beam beyond a distance of $0.84\sqrt{p/q\mu_0 j_0}$, which is about one-third of the radius of the direct beam. The Alfvén limit only applies after

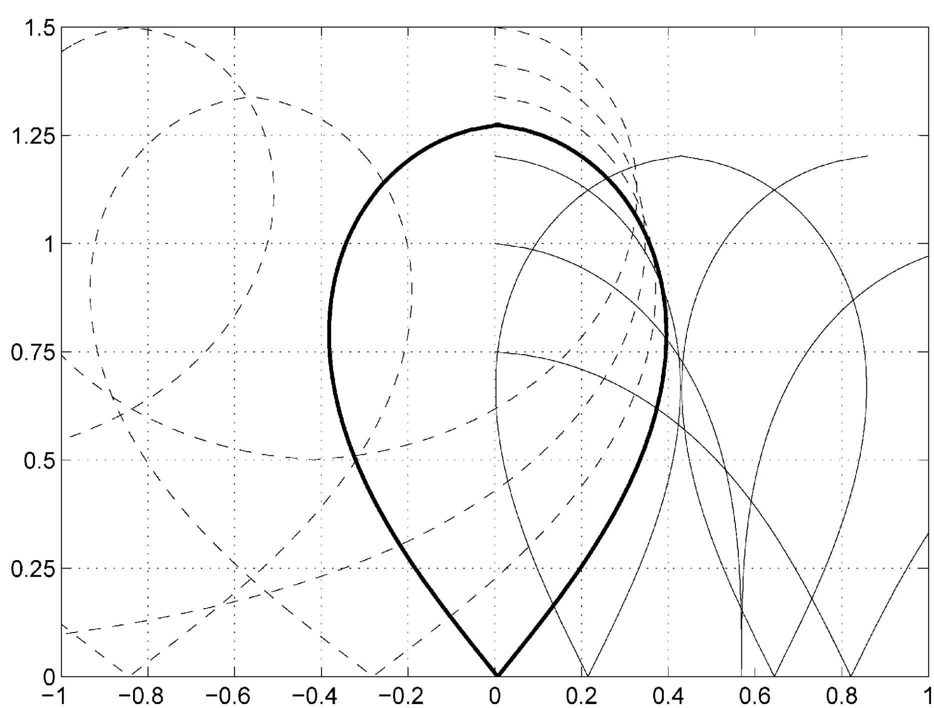

Fig. 1. Trajectories of particles with no azimuthal momentum in the magnetic field generated by a uniform axial current density. Distances in units of $\sqrt{4p/q\mu_0 j_0}$. Trajectories with a net backwards motion are indicated by a dashed line. The trajectory that defines the Alfvén limit is indicated by a thick line.

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a certain time because the steady state magnetic field was used. A secondary point is that the Alfvén limit is not self-consistent, because it indicates that the current profile used to obtain the magnetic will be modified by the magnetic field. Alfvén paid particular attention to this point. He argued that since the particles outside the direct beam are moving backward, the magnetic field would actually fall with radius, allowing particles further out to move forward, until the magnetic field increased again and the process repeated itself. The maximum forward current that could flow would thus be much higher than the current in the direct beam. This argument basically states that a current higher than the limit for a single beam can be carried in a series of separate beams, to which separate limits can be applied. Here separate means beams whose currents are isolated from one another by equal return currents that do not overlap one another. If this were not the case then the magnetic field from the surrounding beams would have to be considered in calculating the limit for each beam, a situation that could be better described as a single beam with an irregular current profile. The beams would also attract one another. In the situation Alfvén described, the separate beams are concentric, hollow cylinders, which is the form filamentation takes in rotational symmetry. In other words, what Alfvén suggested is that a beam which exceeds the limit will filament as a result of particles turned back by the magnetic field. This could also be considered as a form of two-stream instability. A notable feature of this Alfvén filamentation is that it does not require any initial perturbation in the current profile, as is required by other filamentation instabilities, it merely requires the beam current to exceed a certain value. This value could be lower than Eq. (7), since particles start to move back through the beam at a current of J_0 and are returned to the source at a current of $1.46J_0$. Indeed, the current limit may be given by the current at which particles return to the source, since the assumed magnetic field may not exist behind the source. Another feature of Alfvén filamentation that can be predicted from Figure 1 is that it will be localized near the source of the beam, since this is where the returning beam particles are localized. This indicates that it will not actually increase the Alfvén limit, that is, the current that can propagate for an indefinite distance.

I will now extend Alfvén's calculation to include transverse momentum, to other current profiles, to beams that are not charge neutral and to the time dependent case. Then I will consider a different current limit based on the conservation of energy and compare this energy limit to the Alfvén limit. Finally, I will demonstrate the relevance of the Alfvén limit to high intensity laser-solid interactions and fast ignition.

1.2. Transverse momentum

To illustrate the effect of transverse momentum, I will calculate the Alfvén limit for particles with identical, absolute values of either radial or azimuthal momentum for a uniform, still purely axial, current density.

The calculation for radial momentum is straightforward, giving

$$J_{A-U} = \left(\frac{p_{z0}}{p} + 0.65\right) J_0, \tag{8}$$

which shows that the inclusion of radial momentum lowers the Alfvén limit by a factor of up to 2.54, justifying its neglect in estimating an upper limit. A notable feature of Eq. (8) is that it does not tend to zero as the axial momentum tend to zero. Of course, if the axial momentum were zero then there would be no magnetic field so Eq. (8) would not apply. This occurs because the magnetic field increases the average axial momentum of particles with sufficient transverse momentum, so it acts to enhance rather than to inhibit the beam propagation. The magnetic field can be considered to rapidly impose a certain distribution of radial and axial momentum within the direct beam, which is why the limit is relatively insensitive to the radial momentum.

The calculation for azimuthal momentum has to be carried out numerically. However, it is immediately obvious that azimuthal momentum will increase the Alfvén limit. If a particle has an azimuthal momentum such that $p_\theta^2 = q\mu_0 J p_z$ (note that if q is negative then so is J), then there will be no net radial force on the particle. It is possible to construct an equilibrium where no net force acts on any of the particles, and which has no Alfvén limit. Lai (1980) discusses such equilibria in more detail. The numerical solution for particles with a fixed azimuthal momentum, given in Figure 2, also shows that the Alfvén limit is removed when $p_{\theta 0} \geq p/2$, a condition that can be obtained analytically. This indicates that Alfvén's result can be seriously in error for beams with sufficient azimuthal momentum. The limit at which particles return to the source, which is also given in Figure 2, is never removed, but it still greatly exceeds Alfvén's result when the azimuthal momentum is the largest single component of the momentum.

A related question is what occurs to the Alfvén limit when the particles does not have identical momenta. In this case, a range of values of the Alfvén limit could be calculated using the results for particles with identical momenta. It is clear that particles will start to move back at the lowest value, leading to a fall in the beam current and possibly to Alfvén filamentation, but the maximum possible current that can travel an indefinite distance and time will be given by the highest value.

1.3. Current profile

To illustrate the effect of varying the current profile, I will calculate the Alfvén limit for different current profiles, assuming that all particles start with only axial momentum.

A general idea of how the Alfvén limit will vary with current profile can be obtained by determining lower and

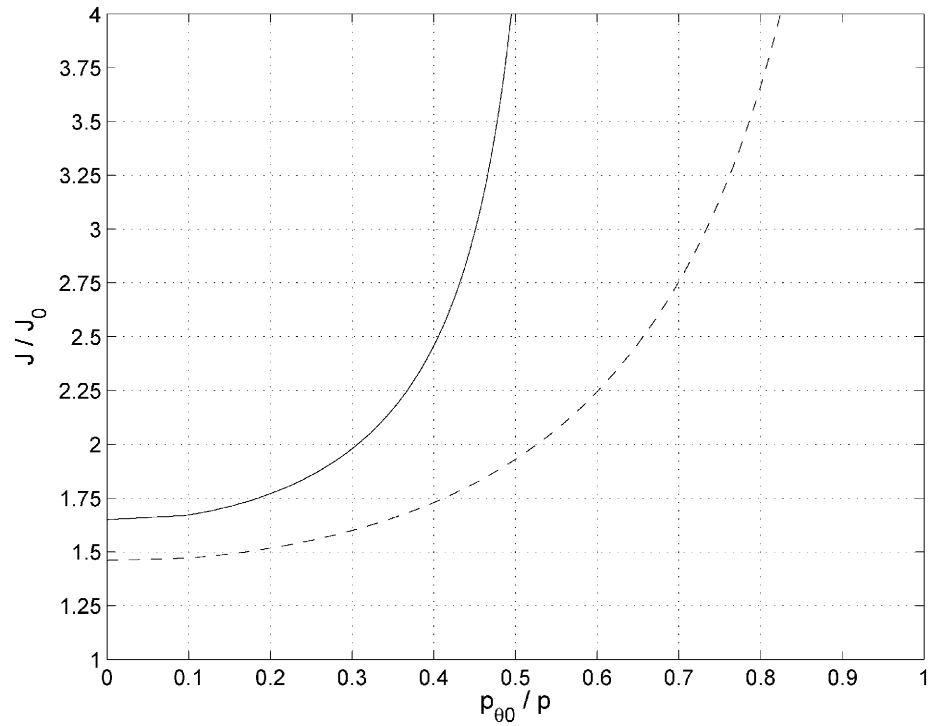

Fig. 2. The Alfvén limit (solid line) and the limit at which particles return to the source (dashed line) for a beam with a uniform current density as a function of the initial azimuthal momentum of the particles.

upper values using Eq. (1). If a particle has $p_z \geq 0$ at $r = 0$ it will never move backward, whereas if it has $p_z = -p$ it will continue to move backward along the axis. The Alfvén limit for a current profile $J(r)$ at a radius r_b, and the limit at which particles return to the source, therefore satisfies

$$J_A = (0.5 - 1)\, \frac{J(r_b)}{\displaystyle\int_0^{r_b} J(r)/r\, dr}\, J_0. \tag{9}$$

Another lower value on the Alfvén limit is given by the current within a radius r_0 that satisfies

$$\int_0^{r_0} p_z\, dr \leq 0, \tag{10}$$

provided that $p_r \neq 0$ at $r = 0$, so that the Alfvén limit is defined by a particle that forms a closed-loop. If this is not the case, then the Alfvén limit is defined by the particle that moves back along the axis and is equal to the upper value given by Eq. (9). It is clear from Eq. (9), and indirectly from Eq. (10), that the Alfvén limit will be lower than that for a uniform current density if the current density peaks on-axis and higher if it peaks off-axis, since it is determined by the radially integrated value of J/r. I will now consider some specific examples.

A calculation of the Alfvén limit for a current density that peaks on-axis is given by its most commonly quoted derivation, which states that it is given by the current at which a particle at the edge of a beam has a Larmor radius less than half the beam radius, giving

$$J_{A-1/r} = J_0. \tag{11}$$

At first sight this derivation indicates that the Alfvén limit is independent of the current profile, which is a common misconception, but on closer inspection it is clear that it will only be an accurate calculation if the magnetic field is uniform, which requires a current density proportional to $1/r$, hence the subscript. In this case, the Alfvén limit is defined by the particle that moves back along the axis, so equals the upper value given by Eq. (9), and it is equal to the limit at which particles return to the source. This infinite peak in the current density on-axis leads to a relatively small reduction in the Alfvén limit, indicating that it is quite an accurate estimate.

Although this is not Alfvén's derivation of the limit, Eq. (11) is the value he gave as his final result, because he dropped the factor of 1.65 in Eq. (7) when calculating the order of magnitude of the current limit. The derivation is often credited to Lawson (1957) rather than to Alfvén, but this is also incorrect. Lawson derived a condition for the radial velocity to be much less than the axial velocity, not an explicit current limit, using the same assumptions as Alfvén, but approximating the equation of motion. He then noted that this condition was reproduced by requiring the Larmor radius of a particle at the edge of the beam to be much less than the beam radius, not less than half the beam radius. Alfvén's choice of Eq. (11) as his final result rather than Eq. (7) has also led to the misconception that the limit is defined by the particle whose axial momentum goes to zero (the lower value of Eq. (9)) rather than by the particle whose average axial momentum is less than or equal to zero.

Another calculation of the Alfvén limit for a current density that peaks on-axis has been given by Honda (2000), who calculated the limit for the Bennet profile (Bennett, 1933, 1955)

$$j_B = \frac{j_0}{(1 + r^2/R^2)^2}, \tag{12}$$

for which the current is given by $\pi r^2 j_0/(1 + r^2/R^2)$. This is a particular, equilibrium solution for a beam with uniform, constant transverse temperature kT, and propagation velocity v, charge neutralized by a similar beam. It differs from the other cases in that it has a finite total current, given by $\pi R^2 j_0$, so the radius of the direct beam depends on the total current and it does not define a unique current. Honda obtained the current in the direct beam as a function of its radius r_b from a numerical solution of Eq. (6), since it cannot be solved analytically in this case. However, from Eqs. (9) and (10) it is possible to determine that

$$J_{A-B} > \frac{r_b^2/R^2}{1 + r_b^2/R^2} \frac{r_b/R}{2(r_b/R - \tan^{-1} r_b/R)} J_0,\ r_b/R < 3.18$$

$$= \frac{r_b^2/R^2}{1 + r_b^2/R^2} \frac{2}{\ln(1 + r_b^2/R^2)} J_0,\ r_b/R \geq 3.18, \tag{13}$$

because for $r_b/R \geq 3.18$, a transition that I determined numerically, the Alfvén limit is determined by the particle that moves back along the axis. It is also possible to determine the exact result for $r_b/R \to 0$ because Eq. (12) then tend to a uniform current density, for which the Alfvén limit is $1.65J_0$. Eq. (13) then gives a lower value of $1.5J_0$, so it is a good approximation for $r_b/R < 3.18$. This indicates that the Alfvén limit of a beam can be much lower than the value he gave. However, if we draw the particle trajectories we find that all particles at large enough radii move forward and that when the limit is defined by the particle that moves back along the axis this is the only particle that moves backward. This case is illustrated in Figure 3. It could be argued that the particles moving backward are inconsistent with the assumed current density profile beyond the radius of the direct beam so that Eq. (13) does give the current limit, but it could also be argued that all of the particles that move forward should be included in the direct beam. In the latter case, when only one particle moves backward the current in the direct beam

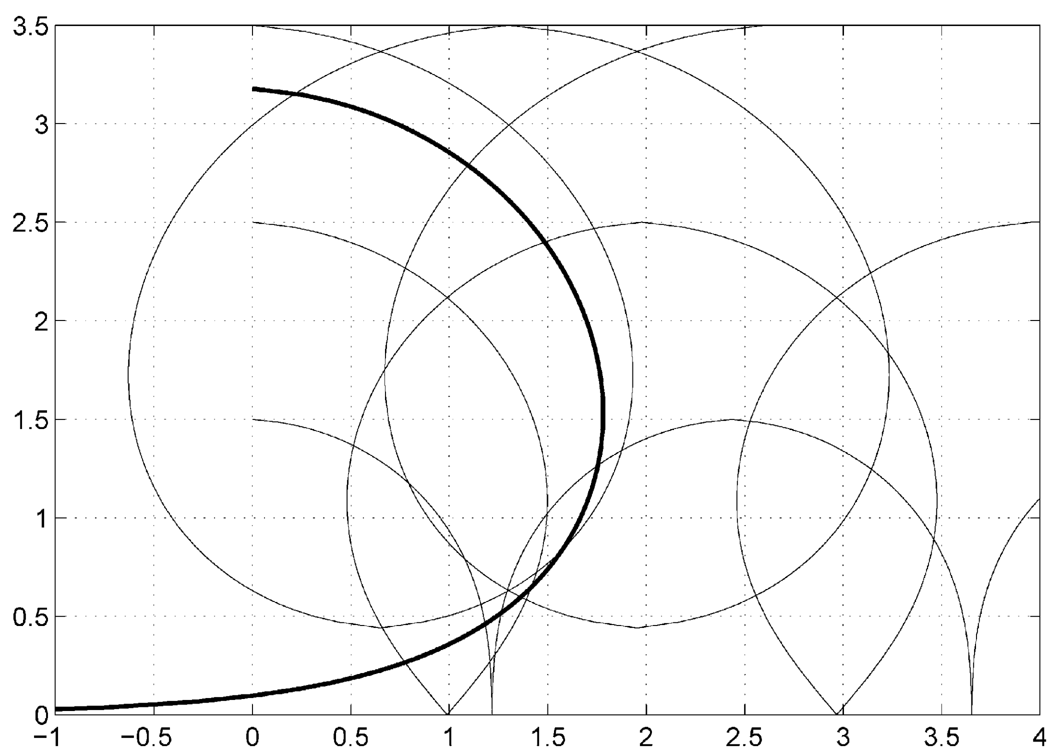

Fig. 3. Trajectories of particles with no azimuthal momentum in the magnetic field generated by the Bennett profile with a total current of $0.831J_0$. Distances in units of R. The trajectory that defines the Alfvén limit is indicated by a thick line.

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will equal the total beam current, so the current limit for the Bennet profile could be defined as

$$J_{A-B} = 0.831 J_0. \qquad (14)$$

It is not clear which interpretation is correct. To further confuse the issue, it also frequently argued that since this is a self-consistent, equilibrium solution, the equilibrium current must be able to propagate. For a beam charge neutralized by a cold, stationary background of particles the Bennett current is

$$J_{\mathrm{eq}} = \frac{8\pi}{q\mu_0}\frac{kT}{v}. \qquad (15)$$

This equilibrium current exceeds the Alfvén limit given by Eq. (13), which tend to zero for an infinitely wide beam, and exceeds that given by Eq. (14) when the mean transverse velocity exceeds the propagation velocity. There appears to be a fundamental contradiction here. The resolution to this paradox is that the Alfvén limit is calculated for a beam propagating from a source, whereas the equilibrium was calculated for an infinitely long beam. It can be seen from Figure 3 that there is no contradiction between the particle trajectories and the assumed current density if there are particles starting at every point in space, because every point is then crossed by particles with a net forward motion. Indeed, this ambiguity does not arise for the limit at which particles return to the source, which is slightly lower than that given by Eq. (13). Nonetheless, this still leaves a choice of two Alfvén limits.

To give an example of a current density profile that peaks off-axis, I will consider the extreme case of a uniform ring situated at a radius r_b with a width $R \ll r_b$. The equation of motion of a particle within the ring will then be approximately the same as that for a uniform current density with the magnetic field lowered by a factor of r_b/R. The Alfvén limit will be determined by the particle that reaches the inside of the ring with no axial momentum, since a particle moving backward inside the ring will travel far further backward than it can travel forward once it returns to the ring, giving

$$J_{A-R} \approx \frac{r_b}{R} J_0, \; r_b \gg R. \qquad (16)$$

This demonstrates that the Alfvén limit can be much greater than the value given by Alfvén, if the current is concentrated at the edge of the beam. Such current profiles do occur naturally: the ring profile could also be described as a skin current and filamentation in rotational symmetry leads to hollow beams. It should be emphasized that this is only a ring in current, not density. The magnetic field means that particles will cross the axis, unless they have sufficient azimuthal momentum, in which case there is no Alfvén limit (Section 1.2).

Other authors, such as Hammer and Rostoker (1970) and Gratreau (1978), have stated that current density profiles that peak off-axis allow currents higher than Alfvén's estimate of the current limit to propagate, but these statements were based on the fact that such current profiles have higher equilibrium currents, not on calculations of the Alfvén limit. This has led to the misconception that the Alfvén limit does not apply to certain current profiles.

1.4. Charge neutrality

If a beam is not charge neutral then its self-generated electric field must be included when calculating the Alfvén limit. An Alfvén limit for a beam with a charge neutralization factor f_E is often quoted, where $f_E = 1$ corresponds to a charge neutral beam and $f_E = 0$ to a beam with no charge neutralization. A similar current neutralization factor f_M is also often included, giving

$$J_{AL} = \frac{J_0}{|1 - f_M - (1 - f_E)c^2/v^2|}, \qquad (17)$$

which is known as the Alfvén-Lawson limit, where v is both the particle and beam propagation velocity. For a charge neutral beam with no current neutralization ($f_E = 1, f_M = 0$) it agrees with Eq. (11), which is the result generally known as the Alfvén limit. The effect of current neutralization on the Alfvén-Lawson limit for a charge neutral beam ($f_E = 1$) is intuitively obvious, since it is clearly the net current $(1 - f_M)J$ that is limited, not the beam current J. As the charge neutralization factor is lowered, the Alfvén-Lawson limit increases until $f_E = 1 - (v/c)^2(1 - f_M)$, when it tend to infinity, which implies that there is no limit. For lower values of the charge neutralization factor, there is current limitation due to the electric field, which exists even when there is current neutrality ($f_M = 1$). This is known as the space charge limit, which is better expressed as a limit on the charge per unit length (J/v), since it exists in the absence of a current. The origin of the Alfvén-Lawson limit is not clear. Lawson (1958) obtained an approximate solution to the equation of motion of a particle at the edge of a partially charge neutral, infinitely long, cylindrical beam by assuming that the particle's transverse momentum was negligible, and that it remained at the edge of the beam. He then stated that for a beam to exist $|rd^2r/dz^2| \ll 1$, which is basically a requirement that the beam propagates faster than it contracts or expands. This condition requires the current to be much less than half that given by Eq. (17) with $f_M = 0$. It is not clear who modified it to agree with Alfvén's result, nor who included the current neutralization factor. Unfortunately, Lawson's result cannot be extended to the case Alfvén considered because the transverse momentum exceeds the axial momentum and particle trajectories cross one another. However, it is possible to calculate the effect of the self-generated electric field of a partially charge neutral, infinitely long beam on the Alfvén limit, if the net charge and current densities have the same dependence on radius. In this case the radial electric field E and the azimuthal magnetic field B are related by

$$\frac{E}{B} = \frac{1-f_E}{1-f_M}\frac{c^2}{v} = \text{constant.} \tag{18}$$

This means that we can transform to a frame moving at a speed E/B in the direction of the beam where there is no electric field, calculate the Alfvén limit on the net current in the normal manner, transform back again and divided by $1 - f_M$ to obtain the limit on the beam current

$$J_{AE} = \frac{1-f_M-(1-f_E)c^2/v^2}{(1-f_M)^2-(1-f_E)^2c^2/v^2}\,J_A, \tag{19}$$

where J_A is the conventional Alfvén limit ($f_E = 1, f_M = 0$) for the current density profile being considered. On the contrary to the Alfvén-Lawson limit, lowering the charge neutralization factor lowers this limit, until $f_E \leq 1 - (v/c)^2(1 - f_M)$, when there is no limit. The two limits only agree in the exact point at which the limit is removed. The physical reason for this change to the Alfvén limit is that the radial electric field lowers the momentum of the particles that are turned inward by the magnetic field, lowering the limit, until $E \geq vB$ when the particles move outward under the influence of the electric field and no particle has a net backward motion, so there is no limit. Indeed, in the presence of a radial electric field, all particles that do not cross the axis may have a net forward motion, which leads to a similar problem in defining the Alfvén limit to that encountered with the Bennet profile (Section 1.3). Again, this ambiguity does not arise with the limit at which particles return to the source.

In general, the net charge density will not be proportional to the net current density, the charge and current neutralization factors will be different functions of space and time, and we should consider the axial electric field generated by a beam propagating from a source, so this can only be considered as a qualitative treatment. As such, it indicates that the self-generated electric field of a partially charge neutral beam will not increase the Alfvén limit, unless the net charge of the beam is sufficient for the force from the electric field to exceed that from magnetic field, when there is no Alfvén limit. It is notable that this is always the case for a non-neutral beam.

1.5. Time dependence

The consideration of the time dependence of the Alfvén limit before a steady state is achieved amounts to the consideration of current neutralization by the return current that must be present. The return current can be provided by a current of charged particles or by the displacement current. A return current of charged particles could come from the conductor that the beam has been assumed to be propagating in or from the beam itself, as occurs when the Alfvén limit is exceeded. This problem is clearly a complex one, so I will only obtain order of magnitude estimates for the time at which the Alfvén limit will apply for some greatly simplified cases.

Hammer and Rostoker (1970) considered the degree of current neutralization that can be provided by a conductor. They obtained an approximate solution for a cylindrical electron beam with a constant, uniform charge density moving at a constant velocity through a cold electron fluid, charge neutralized by fixed ions, assuming that the charge density of the beam was much less than that of the conductor, a situation very similar to that considered by Alfvén. The solution of even this greatly simplified model is complex, but its central results can be illustrated with simple physical arguments. The inertia of the conduction electrons leads to a delay in them establishing a return current given by the reciprocal of their plasma frequency ω_p. During this time the conductor behaves like a vacuum and the return current is provided by the displacement current, which separates from the beam at the speed of light. Thus, by the time the return current is provided by the conductor it has separated from the beam current at a distance c/ω_p, which is known as the magnetic or collisionless skin depth l. This means that a significant degree of current neutralization will only be provided by the conductor for beams with a radial extent $R \gg l$. This can be expressed in terms of a current neutralization factor (Section 1.4)

$$f_M \approx 1 - \frac{2ct}{R},\ ct \ll R,\ ct \leq l. \tag{20}$$

Following this initial, rapid decay in the current neutralization there will be a subsequent, slower decay arising from the mutual repulsion of the beam and conduction currents, and the decay of the conduction current due to collisions. This was not considered by Hammer and Rostoker. Alfvén excluded the possibility of current neutralization by the conductor on the grounds that this situation was unstable as a result of the mutual repulsion of the currents. Here I will estimate the subsequent decay time of the conduction current due to collisions, which will tend to dominate when the density of the conductor is much greater than that of the beam, because the conduction current then consists of electrons drifting slowly through a dense medium. Since the conduction electrons are now in a quasi-steady state, I will use the basic Ohm's law $\mathbf{E} = \eta\mathbf{j}_c$ for the current density of the conductor $\mathbf{j}_c$, where η its resistivity. The displacement current is now negligible, so the evolution of the net current density $\mathbf{j}_n$ is given by

$$\frac{\partial \mathbf{j}_n}{\partial t} + \nabla\times\nabla\times\frac{\eta}{\mu_0}\mathbf{j}_n = \nabla\times\nabla\times\frac{\eta}{\mu_0}\mathbf{j}, \tag{21}$$

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where **j** is the beam current density. In general, Eq. (21) cannot be solved analytically. However, it is clear that its steady state solution is indeed $\mathbf{j}_n = \mathbf{j}$, and from a dimensional analysis it can be seen that the time scale for this steady state to be established is given by

$$t_D = \frac{\mu_0 R^2}{\eta}, \quad (22)$$

which is known as the magnetic diffusion time, where R is the appropriate radial scale length. The Alfvén limit will be valid for times greater than this. The initial decay of the conduction current can be calculated from Eq. (21) by assuming that $j_n \ll j$, provided that the resistivity and the beam current density are known, well-behaved, functions. The only current profile that we have considered to which this approximation can be applied is the Bennett profile (Eq. (12)), which gives

$$j_n = 8\frac{t}{t_D}\frac{1-2r^2/R^2}{(1+r^2/R^2)^4}j_0, t \ll t_D, R \gg l, \quad (23)$$

for a constant resistivity. The corresponding current neutralization factor is

$$f_M = 1 - \frac{8t/t_D}{(1+r^2/R^2)^2}. \quad (24)$$

This confirms the statement made in Section 1.4 that the current neutralization factor is, in general, time and space dependent. Calculating the Alfvén limit for a time dependent current density requires a numerical solution, so I will estimate it from the lower value given by Eq. (9) at a fixed time. This gives the time at which the Alfvén limit is exceeded to be

$$t_I \sim \frac{t_D}{3.64J/J_0}, J \gg J_0, \quad (25)$$

which I have called the magnetic inhibition time (Davies, 2003, 2004), where J is the total beam current. Similar estimates can be obtained for other current density profiles. An important exception is the ring profile (Section 1.3), which was found to increase the Alfvén limit (Eq. (16)). In this case, the magnetic inhibition time will be increased by a factor of r_b/R, but since it is the width of the ring R that appears in the magnetic diffusion time, the magnetic inhibition time will actually be lowered by a factor of r_b/R, unless the Alfvén limit exceeds the beam current. The magnetic inhibition time is also not increased if the current is divided up into a series of separate filaments. Furthermore, the decay of the return current around each filament means that they will not remain separate, so even if each filament carries a current less than its Alfvén limit, the magnetic inhibition time of the beam as a whole will not be significantly increased. The only apparent means of significantly increasing the time for which a current greater than the Alfvén limit can propagate, is if it propagates in randomly fluctuating filaments, in which case a steady state may never be reached.

This calculation also gives an axial electric field, which will lower the Alfvén limit since it decelerates the beam particles. As was seen in Section 1.4, the importance of the electric field can be quantified by the ratio of the force from the electric field to that from the magnetic field. Estimating the initial growth of the magnetic field from $\eta jt/R$ gives

$$\frac{E}{vB} \sim \frac{R/v}{t}, t \ll t_D, R \gg l, \quad (26)$$

so the electric field will be negligible after a light transit time across the beam radius, which is much less than the magnetic diffusion time. This clearly justifies Alfvén's neglect of the electric field when the charge density of the conductor is much greater than that of the beam.

1.6. Comparison to the energy limit

A limit on the current that can propagate for an indefinite distance and time from a source can also be obtained by equating the energy per unit length in the steady state magnetic field to the kinetic energy per unit length in the particles that must generate it as they propagate, giving

$$J = \frac{J^2(r_b)}{\int_0^{r_b} J^2(r)/r\,dr}\frac{4\pi}{q\mu_0}\frac{\langle K\rangle}{v} \equiv \frac{J^2(r_b)}{\int_0^{r_b} J^2(r)/r\,dr}J_0', \quad (27)$$

where $J(r)$ is the current profile of the beam, $\langle K\rangle$ is the mean kinetic energy of the particles, and v is the propagation velocity (Davies, 2003).

Eq. (27) can be easily evaluated for the current profiles considered so far, giving

$$J_U = 4J_0', \quad (28)$$

$$J_{1/r} = 2J_0', \quad (29)$$

$$J_B = \frac{r_b^4/R^4}{(1+r_b^2/R^2)^2}\frac{2}{\ln(1+r_b^2/R^2) - (r_b^2/R^2)/(1+r_b^2/R^2)}J_0', \quad (30)$$

$$J_R \approx 3\frac{r_b}{R}J_0', r_b \gg R. \quad (31)$$

To compare these results to the corresponding Alfvén limits we must make the same assumptions, namely that the particles have the same momentum and are moving in the same direction, giving $J_0' = J_0/(1+1/\gamma)$. The value of J_0' varies from $J_0/2$ in the non-relativistic limit to J_0 in the strongly

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relativistic limit. The ratio of the energy limit to the Alfvén limit is then 1.21–2.42 for a uniform current density, 1–2 for $j \propto 1/r$ and 1.5–3 for the ring profile. For the Bennett profile, it shows that it is correct to assume that a single particle trajectory with a net backward motion represents a limit on the current. The ratio of the energy limit to the Alfvén limit (Eq. (13)) falls with radius, from the result for a uniform current density of 1.21–2.42 as the radius tend to zero to 0.5–1 as the radius tend to infinity. More generally, it can be seen from Eqs. (9) and (27) that the Alfvén and energy limits will have a similar dependence on current profile, with the energy limit showing a slightly stronger dependence, provided that all of the particles are initially moving in the same direction.

Transverse momentum, whether radial or azimuthal, increases the energy limit since it increases the mean energy without a corresponding increase in the propagation velocity. This dependence on transverse momentum is quite different to that of the Alfvén limit (Section 1.2). The most significant difference is that the energy limit is never removed by azimuthal momentum. However, to achieve this increase in the energy limit, the transverse momentum must be converted into axial momentum at a specific rate as the beam propagates, because the energy in the azimuthal magnetic field can only come from the axial momentum. It would appear to be more accurate to define the energy limit in terms of the axial energy. The problem with this is that, as can be seen in Figures 1 and 3, the magnetic field rapidly dominates the momentum distribution of the particles, so to give an accurate value of the current limit we cannot take any given value of $\langle K\rangle/v$. Although the energy limit has the advantage that it is directly applicable to an arbitrary momentum distribution, it is not self-consistent in its treatment of the momentum distribution. A reasonable estimate of the current limit should be given by assuming that all of the particles start to move along the axis, so that $\langle K\rangle/v$ is replaced by $\langle K\rangle/\langle v\rangle$.

The self-generated, radial electric field of a beam that is not charge neutral considered in Section 1.4 would not be expected to change the energy limit, because the energy in the electric field would only become available to generate the magnetic field if the radial separation of the particles tended to infinity and the radial momentum gained were converted to axial momentum. The most significant difference between this behavior and that of the Alfvén limit (Eq. (19)) is that the energy limit is not removed when $E \geq vB$. This assumes that we start with a beam that is not charge neutral. If the lack of charge neutrality resulted from the propagation of the beam then particle energy would be lost in generating the electric field and the energy limit would be reduced.

The discussion of the time dependence of the Alfvén limit (Section 1.5) applies equally to the energy limit. The calculation of the energy limit for a time dependent current density such as Eq. (23) is far simpler than that of the Alfvén limit, the only difficulty arises in defining the kinetic energy available to generate the magnetic field. This could be calculated from the total beam current or from the net current, and the net current varies with radius, tending to zero as the radius tend to infinity. The best choice would appear to be the maximum net current (Davies, 2004). The magnetic inhibition time for the Bennet profile then becomes

$$t_I \approx \frac{t_D}{1.35J/J_0'},\ J \gg J_0', \tag{32}$$

which is similar to Eq. (25).

From this we can conclude that the energy limit is practically the same as the Alfvén limit. The only exceptions are the situations in which the Alfvén limit does not exist, when the energy limit remains. This begs the question as to what the mechanism of current limitation is if not particles moving backward. In most cases it can be ascribed to particles being returned to the source, but not all. What the energy limit shows is that a beam with given values of $J(r)$ and $\langle K\rangle/v$ could not be established by emitting particles from a source because the induced electric field would stop the beam propagating.

1.7. Conclusions

The Alfvén limit is the current at which particles start to have a net backward motion in the self-generated magnetic field of a beam. It gives the maximum current that can propagate for an indefinite distance and time, from a source, in a charge neutral beam. Alfvén estimated the order of magnitude of this current by calculating it for a beam with a uniform current density, where all the particles have the same momentum and are initially moving in the same direction. This gives Eq. (7), but as Alfvén was only estimating an order of magnitude, he dropped the factor of 1.65 to give Eq. (11). Alfvén also predicted that the particles turned back by the magnetic field would lead to filamentation near the source of the beam, where the limit is exceeded.

The extension of Alfvén's work presented here has largely confirmed this estimate. The only exception occurs when the current of a beam is concentrated at its edge, when the limit can be much higher, as shown by Eq. (16). However, the Alfvén limit cannot be generalized: only an implicit expression for the limit can be given (Eq. (6)) and it can give ambiguous, even incorrect, results. A general expression for the current limit can be obtained from the conservation of energy, giving Eq. (27). It is this result that confirms the accuracy of Alfvén's original estimate. A better definition of the current limit based on particle trajectories would be the current at which particles return to the source. The other key result of this work is the estimate of the time at which the Alfvén limit applies (Eqs. (20) and (25)).

One of the most important conclusions that can be drawn from this is something that was clearly expressed by Alfvén at the beginning of his article:

"It is clearly understood that there may be some danger in drawing too far-reaching general conclusions from such a special case. But on the other hand . . . the treatment of a special case is very likely to give the right order of magnitude. . .".

2. RELEVANCE TO LASER-PLASMA INTERACTIONS

2.1. Laser-solid interactions

When a laser with a value of intensity times wavelength squared ($I\lambda^2$) greater than roughly 10^7 W interacts with a solid it rapidly forms a plasma, and the interaction of the laser with this plasma has been found to transfer a significant fraction f_L of the laser's energy into electrons entering the target. The current J of these electrons can be written as

$$\frac{J}{J_0'} = 10^{-7}\,\frac{f_L P v}{(\langle K\rangle/e)^2}, \tag{33}$$

where P is the laser power and J_0' has been chosen as a reference value since the energy limit is more clearly defined for a beam that is not mono-energetic. Values of the fraction of the laser energy transferred to the electrons and their mean energy have been determined by numerous experiments and theoretical models. The mean energy has been found to scale as $(I\lambda^2)^\alpha$ with $\alpha \approx 0.3$ for values of $I\lambda^2$ up to about 10^{10} W and $\alpha \approx 0.5$ at much higher values. This transition corresponds to the point at which electron motion in the laser fields starts to become relativistic. This does not appear to be a coincidence, since $\alpha = 0.5$ ensures that the number density of the electrons will never exceed the relativistic critical density, and the laser is absorbed at or below this density since a plasma is rapidly formed on the surface of the solid. Using the mean energy $\langle K\rangle/e = 100(I\lambda^2)^{1/3}$ eV determined by Beg *et al.* (1996) for the non-relativistic case gives

$$\frac{J}{J_0'} = f_L\left(\frac{P}{9.07\times 10^7}\right)^{1/2}\frac{R}{\lambda}, \tag{34}$$

where R is the laser spot radius. The fraction of the laser energy transferred to the electrons has been found to be from 0.1 to 0.5 and the spot radius must exceed the laser wavelength, so Eq. (34) will typically be greater than one for all laser powers of interest. Using the ponderomotive potential for the mean energy in the strongly relativistic limit ($\langle K\rangle/e \approx 4.77(I\lambda^2)^{0.5}$ eV, $I\lambda^2 \gg 10^{10}$ W) gives

$$\frac{J}{J_0'} = 4.14 f_L\left(\frac{R}{\lambda}\right)^2. \tag{35}$$

This indicates that the ratio J/J_0' will eventually saturate with increasing laser power. The most notable point of both Eqs. (34) and (35) is the strong dependence on the laser focussing. Indeed, the current limit may not be exceeded when the laser is tightly focussed ($R \approx \lambda$). Even when the current limit is exceeded it may not be a relevant issue for the electron propagation because of current neutralization (Section 1.4). The initial decay of the return current described by Eq. (20) occurs over a time certainly less than the laser period, so the beam current may exceed the current limit by a factor of at most $R/2l$. The skin depth l in the solid will be much less than the laser wavelength so the current limit will only be exceeded initially if the spot radius is very much greater than the laser wavelength. This will be true even in an insulator because the currents being considered are very much greater than the breakdown threshold of any insulator. The magnetic inhibition time given by Eq. (32) for the strongly relativistic case of Eq. (35) is

$$t_I = 2.25\times 10^{-7}\,\frac{\lambda^2}{f\eta}. \tag{36}$$

For a laser wavelength of 1 μm, the maximum resistivity of a solid of ~2 $\mu\Omega$m and the upper value for the fraction of the laser energy transferred to the electrons of 0.5, Eq. (36) gives a lower value on the magnetic inhibition time of 225 fs. Current limitation will occur if the laser pulse duration exceeds this value. The actual value is likely to be somewhat lower because the magnetic field can focus the beam, reducing the radius, and hence the magnetic inhibition time, as described by Bennett (1933, 1955). This will occur when the net current exceeds the Bennett current (Eq. (15)), which is typically lower than the current limit.

These estimates apply in the solid, but the electrons are generated at or below the critical density, where the skin depth equals the laser wavelength, which would not give sufficient current neutralization. However, the plasma density will increase toward the solid, so current limitation will only occur before the beam enters the solid if the density gradient scale length beyond the critical density is large enough. Eqs. (34) and (35) indicate that the radius that contains a current equal to the current limit will be of the order of the laser wavelength, and this determines the distance over which current limitation occurs. The exact value, however, will depend on the current profile, which will be modified by the magnetic field. Therefore, it can only be concluded that beyond some undetermined pre-pulse level or pulse duration the current entering the solid will be limited.

When current limitation occurs it will lead to a significant fraction of the electrons being trapped within a distance less than the spot radius, where Alfvén filamentation would be expected to occur. A certain fraction of the beam will always be able to propagate, since the current limit only applies after a certain time, and when current limitation does set in the high energy component of the typically broad energy spectra produced in laser-solid interactions will also be able to propagate. This will have important consequences for

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numerous aspects of interest in laser-solid interactions, such as target heating, Kα emission, bremsstrahlung emission, and proton acceleration. Current limitation could also have important consequences for the laser interaction, since electrons being returned to the source implies electrons re-interacting with the laser.

2.2. Fast ignition

The fast ignition proposal of Tabak *et al.* (1994) requires a minimum deposited power and a maximum electron energy so that the electrons stop within the core, so there exists a minimum ignition current J_{ig}. Atzeni (1999) gives the ignition power to be $2.6\bar{\rho}^{-1}$ PW, where $\bar{\rho}$ is the Deuterium-Tritium (DT) fuel density in units of 100 gcm^{-3}, which is 10^5 kgm^{-3} in SI units. Tabak *et al.* (1994) estimated the maximum electron energy to be 1 MeV, so I will define $\langle \bar{K} \rangle \equiv \langle K \rangle / 10^6 e$. For these values Eq. (33) gives

$$\frac{J_{ig}}{J_0'} = 7.8 \times 10^4 \frac{1}{f_H} \frac{1}{\bar{\rho}} \frac{v}{c} \frac{1}{\langle \bar{K} \rangle^2}, \tag{37}$$

where f_H is the fraction of the electron energy that provides useful heating. From Eq. (37) it can be seen that the ignition current is certainly much greater than the current limit. The maximum initial current neutralization provided by the plasma (Eq. (20)) gives a minimum plasma electron density for the net current to be less than J_0' of

$$n_{min} \approx 5.0 \times 10^{31} \frac{1}{f_H^2} \frac{v^2}{c^2} \frac{1}{\langle \bar{K} \rangle^4}. \tag{38}$$

An electron number density of 5×10^{31} m^{-3} corresponds to a DT density of $\bar{\rho} = 2.1$, so the beam cannot propagate through the corona.

In Davies (2004), I showed that the subsequent, slower, decay of the current neutralization due to collisions of the plasma electrons (Eq. (32)) will also be less than the required pulse duration for fast ignition, and discussed means of avoiding current limitation. The only practical means appear to be spherical irradiation or increasing the mean energy at the cost of a reduction in efficiency (f_H). The obvious solution of raising the current limit by using the ring profile (Eqs. (16) and (31)) is not practical because the width of the ring would have to be less than $2.25 f_H \langle \bar{K} \rangle^2 c/v$ nm, which is much less than practical laser wavelengths. It can also be seen from Eqs. (34) and (35) that increasing the current limit by a factor of R/λ would be insufficient to avoid the current limit for any parameters of interest.

Alternatively, current limitation could be used to achieve ignition, taking advantage of the increase in energy deposition that will occur in the region where electrons are being returned to the source. This energy deposition could occur due to collisions or due to the excitation of any of a wide variety of plasma instabilities, the exact mechanism is irrelevant. This region will have an extent somewhat less than the beam radius, so the laser would have to be absorbed at the edge of the region to be heated. It has been demonstrated that relying on the laser ponderomotive force to bore a hole to the core will not work. To give a critical density equal to that of the core requires a laser wavelength of $65\bar{\rho}^{-1}$ nm in the non-relativistic case, which would certainly apply for the intensities required for fast ignition at this wavelength. Unfortunately, such a short laser wavelength is not currently practical. The laser might be brought sufficiently close to the core in schemes using cones, such as those discussed by Norreys *et al.* (2000), depending on the position of the core after compression and the thickness of the cone required. However, this may not be necessary, because Hain and Mulser (2001) found that ignition can be achieved when a total energy somewhat less than that given by Atzeni is deposited in the corona, provided that the energy does not go into an electron beam. It has been concluded that this scheme would require some form of anomalous electron stopping, which is precisely what current limitation provides.

I will now try to estimate how efficient the use of current limitation could be, in other words, the value of f_H, which is the crucial parameter for any ignition scheme. I will ignore the energy of the electrons that propagate before current limitation occurs and of those in the direct beam, and consider only the effect of the electron energy distribution. In the strongly relativistic case, the energy limit is typically twice the Alfvén limit (Section 1.6), which indicates that if the energy limit is reached, electrons with an energy less than twice the mean energy will be turned back. For an exponential energy distribution, which in the strongly relativistic limit is a one-dimensional Maxwellian, this would be 86% of the total electron energy. For a strongly relativistic, three-dimensional Maxwellian it would be 60%. The energy deposition would be expected to fall sharply away from the laser, but ignition requires a minimum volume to be heated to a minimum temperature, so this will lower the efficiency. Assuming that this reduction amounts to a factor of two gives $f_H \sim 0.4$. Combining this with $f_L = 0.5$, Atzeni's energy, power and intensity ignition thresholds become laser values of $0.7\bar{\rho}^{-1.85}$ MJ, $13\bar{\rho}^{-1}$ PW, and $1.2 \times 10^{24} \bar{\rho}^{0.95}$ Wm^{-2}, respectively. The laser wavelength would merely have to be much less than the spot radius, which is given by $58.7\bar{\rho}^{-0.975}$ μm according to Atzeni's criteria. For the commonly quoted fuel density of $\bar{\rho} = 3$ this gives a laser energy of 92 kJ, a power of 4.3 PW and an intensity of 3.4×10^{24} Wm^{-2}, which corresponds to a pulse duration of 21 ps and a spot radius of 20 μm, so a wavelength of 1 μm should be adequate, although a shorter wavelength would be better. These values are within reach of current technology.

2.3. Conclusions

The interaction of even relatively low powered lasers with an over dense plasma, which occurs in laser-solid interactions and in the proposed fast ignition scheme, generates an electron beam with a current that greatly exceeds the Alfvén limit. If the electron beam does not rapidly enter a plasma with a density much greater than the critical density or if the pulse duration is long enough then the net current will exceed the Alfvén limit, and current limitation will occur. This is certainly the case in the fast ignition proposal, requiring the scheme to be rethought. One alternative is to use current limitation to ensure sufficient energy deposition. In this scheme the laser must be absorbed at the edge of the region to be heated, which could be achieved by the use of a cone or simply as a result of coronal ignition.

Laser-solid interactions provide one of the few means of generating electron beams with currents that greatly exceed the Alfvén limit, and thus of confirming Alfvén's predictions. If the interaction is considered in terms of the conversion of photons into electrons then the reason for this is obvious: there is no limit on the current or density of photons because they have no charge.

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