

STANDARD BAYES LOGIC IS NOT FINITELY AXIOMATIZABLE

ZALÁN GYENIS

Department of Logic, Institute of Philosophy, Jagiellonian University
and Department of Logic, Eötvös Loránd University

Abstract. In the article [2] a hierarchy of modal logics has been defined to capture the logical features of Bayesian belief revision. Elements in that hierarchy were distinguished by the cardinality of the set of elementary propositions. By linking the modal logics in the hierarchy to the modal logics of Medvedev frames it has been shown that the modal logic of Bayesian belief revision determined by probabilities on a finite set of elementary propositions is not finitely axiomatizable. However, the infinite case remained open. In this article we prove that the modal logic of Bayesian belief revision determined by standard Borel spaces (these cover probability spaces that occur in most of the applications) is also not finitely axiomatizable.

§1. Introduction and background. Bayes logics have been introduced in the recent article [2] in order to investigate (modal) logical properties of statistical inference (Bayesian belief revision). If (X, \mathcal{B}, p) is a probability space, then elements of \mathcal{B} can be regarded as propositions or possible statements about the world, and the probability measure p describes knowledge of statistical information (or say, it represents degree of beliefs in the truth of these propositions). Learning proposition $A \in \mathcal{B}$ to be true means revising the probability measure p on the basis of the evidence A , and replacing p by some other probability q in certain ways. The transition from p to q is statistical inference: This new probability measure q can be regarded as the probability measure that one *infers* from p on the basis of the information (evidence) that A is true. A fundamental model of statistical inference is the standard Bayes model which relies on Bayes conditionalization of probabilities: given a prior probability measure p and an evidence $A \in \mathcal{B}$ the inferred measure q is defined by conditionalizing p upon A using Bayes' rule:

$$q(B) \doteq p(B \mid A) = \frac{p(B \cap A)}{p(A)} \quad (\forall B \in \mathcal{B}) \quad (1)$$

provided $p(A) \neq 0$. The article [2] aimed at studying the logical aspects of this type of inference from the perspective of modal logic.¹

It is very natural to regard the move from p to q in terms of modal logic: The core idea is to view A in the Bayes' rule (1) as a variable and say that a probability measure q can be

Received: January 12, 2018.

2010 *Mathematics Subject Classification*: 03B42, 03B45, 03A10.

Key words and phrases: modal logic, Bayesian inference, Bayes learning, Bayes logic, Medvedev frames, nonfinite axiomatizability.

¹ Bayesian inference is just a particular type of statistical inference: Various rules replacing the Bayes's rule have been considered in the literature (e.g., Jeffrey conditionalization, maxent principle; see [12] and [4]). Questions similar to what we consider here can be (and have been) asked regarding these other types of inference rules (cf. [2]), but taking the first steps we stick to the basic case of Bayesian inference here.

inferred from p if there exists an A in \mathcal{B} such that $q(\cdot) = p(\cdot \mid A)$. Equivalently, we will say in this situation that “ q can be (Bayes) learned from p .” That “it is possible to obtain/learn q from p ” is clearly a modal talk and calls for a logical modeling in terms of concepts of modal logic. For further motivation and background we suggest [2].

We will use the standard unimodal language given by the grammar

$$a \mid \perp \mid \neg\varphi \mid (\varphi \wedge \psi) \mid \diamond\varphi \quad (2)$$

defining formulas φ , where a belongs to a nonempty countable set Φ of propositional letters. As usual \Box abbreviates $\neg\diamond\neg$. We refer to the books [1, 3] concerning basic notions in modal logic and we mostly follow notation of the book [1]. We briefly recall some facts and notation in the Appendix.

Let $\langle X, \mathcal{B} \rangle$ be a measurable space and denote by $M(X, \mathcal{B})$ the set of all probability measures over $\langle X, \mathcal{B} \rangle$. Bayes accessibility relation has been defined in [2] as follows: For $\nu, w \in M(X, \mathcal{B})$ we say that w is *Bayes accessible* from ν if there is an $A \in \mathcal{B}$ such that $w(\cdot) = \nu(\cdot \mid A)$. We denote the Bayes accessibility relation on $M(X, \mathcal{B})$ by $R(X, \mathcal{B})$.

DEFINITION 1.1 (Bayes frames). *Let $\langle X, \mathcal{B} \rangle$ be a measurable space. The structure*

$$\mathcal{F}(X, \mathcal{B}) = \langle M(X, \mathcal{B}), R(X, \mathcal{B}) \rangle \quad (3)$$

is called a Bayes frame.

From the point of view of applications the most important classes of Bayes frames $\mathcal{F}(X, \mathcal{B})$ are Bayes frames determined by standard Borel spaces $\langle X, \mathcal{B} \rangle$. A measurable space $\langle X, \mathcal{B} \rangle$ is *standard Borel* if X can be endowed with a metric which makes it a complete, separable metric space in such a way that \mathcal{B} is the Borel σ -algebra (the smallest σ -algebra containing the open sets). According to the Borel isomorphism theorem (sec. 17 in [6]) a standard Borel space is always isomorphic to one of $\langle \{1, \dots, n\}, \wp(\{1, \dots, n\}) \rangle$, or $\langle \mathbb{N}, \wp(\mathbb{N}) \rangle$, or $\langle [0, 1], \mathcal{B} \rangle$, where \mathcal{B} is the set of Borel subsets of $[0, 1]$ ($[0, 1]$ endowed with the Euclidean topology). If $\langle X, \mathcal{B} \rangle$ is a standard Borel space and w is a probability measure on \mathcal{B} , then the completion of $\langle X, \mathcal{B}, w \rangle$ is a standard probability space (see sec. 17.F in [6]). Thus any standard probability space is isomorphic (modulo zero) to either $\langle [0, 1], \mathcal{L}, \lambda \rangle$, where \mathcal{L} is the Lebesgue σ -algebra and λ is the Lebesgue measure, or to an atomic probability space with countable (possibly finitely) many atoms, or the combination (disjoint union) of both. Let $\langle X, \mathcal{A}, p \rangle$ and $\langle Y, \mathcal{B}, q \rangle$ be two probability spaces and $r \in (0, 1)$. Denote the disjoint union of the two spaces by $\langle X + Y, \mathcal{A} + \mathcal{B}, p + q \rangle$, where $X + Y$ is the disjoint union of X and Y ; $\mathcal{A} + \mathcal{B}$ is the σ -algebra generated by $\mathcal{A} \cup \mathcal{B}$ and $p + q$ is the probability measure $r \cdot p + (1 - r) \cdot q$. (If we would like to make r explicit, we write $p +_r q$.) It follows that any standard probability space is isomorphic (modulo zero) to $\langle \mathbb{N} + [0, 1], \wp(\mathbb{N}) + \mathcal{L}, p + \lambda \rangle$, where $\langle \mathbb{N}, \wp(\mathbb{N}), p \rangle$ is an arbitrary probability space (as p might not be faithful, this covers the finite case as well).

Let us introduce the following notation

$$\mathcal{F}_n = \langle M(n, \wp(n)), R(n, \wp(n)) \rangle, \quad \mathcal{F}_\omega = \langle M(\mathbb{N}, \wp(\mathbb{N})), R(\mathbb{N}, \wp(\mathbb{N})) \rangle. \quad (4)$$

Note that if the measurable space $\langle X, \mathcal{B} \rangle$ is finite or countable, then \mathcal{B} is the full powerset algebra $\wp(X)$ (we rely on the convention that elementary events $\{x\}$ for $x \in X$ always belong to the algebra \mathcal{B}).

DEFINITION 1.2 (Bayes logics). *A family of normal modal logics have been defined in [2] based on finite or countable or countably infinite or all Bayes frames as follows.*

$$\mathbf{BL}_n = \{\phi : \mathcal{F}_n \Vdash \phi\} \tag{5}$$

$$\mathbf{BL}_{<\omega} = \{\phi : (\forall n \in \mathbb{N}) \mathcal{F}_n \Vdash \phi\} \tag{6}$$

$$\mathbf{BL}_\omega = \{\phi : \mathcal{F}_\omega \Vdash \phi\} \tag{7}$$

$$\mathbf{BL}_{\leq\omega} = \mathbf{BL}_{<\omega} \cap \mathbf{BL}_\omega \tag{8}$$

$$\mathbf{BL} = \{\phi : (\forall \text{ Bayes frames } \mathcal{F}) \mathcal{F} \Vdash \phi\}. \tag{9}$$

We call $\mathbf{BL}_{<\omega}$ (resp. $\mathbf{BL}_{\leq\omega}$) the logic of finite (resp. countable) Bayes frames; however, observe that the set of possible worlds $M(X, \mathcal{B})$ of a Bayes frame $\mathcal{F}(X, \mathcal{B})$ is finite if and only if X is a one-element set, otherwise it is at least of cardinality continuum.

$\mathbf{BL}_{<\omega}$ is the set of general laws of Bayesian learning based on all finite Bayes frames. The general laws of Bayesian learning independent of the particular representation $\langle X, \mathcal{B} \rangle$ of the events is then the modal logic \mathbf{BL} .

The following theorem has been proved in [2] (see Proposition 3.1 and Theorem 4.1 therein).

THEOREM 1.3. *The following propositions hold.*

- $\mathbf{S4} \subseteq \mathbf{BL} \subseteq \mathbf{BL}_\omega = \mathbf{BL}_{\leq\omega} \subsetneq \mathbf{BL}_{<\omega}$,
- $\mathbf{S4.1} \subsetneq \mathbf{BL}_\omega$,
- $\mathbf{S4.1} + \mathbf{Grz} \subsetneq \mathbf{BL}_{<\omega}$.

The finite Bayes frame case has been completely described in [2] and, in particular, it has been shown that $\mathbf{BL}_{<\omega}$ has the finite frame property and is *not* finitely axiomatizable (see Propositions 5.8 and 5.9 in [2]), but the infinite case remained almost completely open.

In this article we deal with Bayes frames $\mathcal{F}(X, \mathcal{B})$, where $\langle X, \mathcal{B} \rangle$ is a standard Borel space (such frames we will refer to as standard Borel Bayes frames). This covers the finite and countably infinite cases (e.g., $\mathbf{BL}_{<\omega}$ and \mathbf{BL}_ω) but is more general because the uncountably infinite space $\langle [0, 1], \mathcal{B}, \lambda \rangle$ is also considered. Let us write \mathbf{BL}_{st} for the standard Bayes logic

$$\mathbf{BL}_{st} = \{\phi : (\forall \text{ Standard Borel Bayes frames } \mathcal{F}) \mathcal{F} \Vdash \phi\}.$$

By relating standard Borel Bayes frames and standard Bayes logics to generalized Medvedev frames (the definitions will be recalled later on) and continuing Shehtman’s [10] analysis we will prove the following two main theorems.

Theorem 3.3. Let $\langle X, \mathcal{B} \rangle$ be a standard Borel space and $\mathcal{F}(X, \mathcal{B})$ be the corresponding Bayes frame. The logic $\Lambda(\mathcal{F}(X, \mathcal{B}))$ is not finitely axiomatizable.

Theorem 3.4. \mathbf{BL}_{st} is not finitely axiomatizable.

Similar problems regarding the general case \mathbf{BL} remain open, see Problems 3.5, 3.6, and 3.7 at the open problems section.

Structure of the article. To obtain nonfinite axiomatizability results we employ the techniques presented in Shehtman [10] which we recall in §2. In §2.1 and §2.2 we prove that the logics corresponding to finitary Boolean frames and measure algebra frames are not finitely axiomatizable. These are the key preliminary theorems that we will apply in §3, where we prove that standard Bayes logic \mathbf{BL}_{st} is not finitely axiomatizable (Theorem 3.4).

§2. A method for nonfinite axiomatizability. We start by recalling definitions and theorems from [10] and [2] and the notion of Medvedev frames. Such frames originate from intuitionistic logic, for an overview about the history we refer to the book [3] and to Shehtman [10]; for further works on Medvedev’s logic of finite problems or Medvedev frames we refer to [5, 7–11]).

DEFINITION 2.1 (Medvedev frame). A Medvedev frame is a frame that is isomorphic (as a directed graph) to $\mathcal{P}^0(X) = \langle \wp(X) \setminus \{\emptyset\}, \supseteq \rangle$ for a nonempty finite set X .

For convenience, as a slight abuse of notation, we will call every frame of the form $\langle \wp(X) \setminus \{\emptyset\}, \supseteq \rangle$ (X being finite or infinite) a Medvedev frame and we will use the notation

$$\mathcal{P}^0(X) = \langle \wp(X) \setminus \{\emptyset\}, \supseteq \rangle. \tag{10}$$

A hierarchy of normal modal logics that correspond to the frames $\mathcal{P}^0(X)$ can be defined as follows.

$$\mathbf{ML}_n = \{ \phi : \mathcal{P}^0(n) \Vdash \phi \} \tag{11}$$

$$\mathbf{ML}_{<\omega} = \{ \phi : (\forall n \in \mathbb{N}) \mathcal{P}^0(n) \Vdash \phi \} \tag{12}$$

$$\mathbf{ML}_\omega = \{ \phi : \mathcal{P}^0(\omega) \Vdash \phi \} \tag{13}$$

$$\mathbf{ML}_{\leq\omega} = \mathbf{ML}_{<\omega} \cap \mathbf{ML}_\omega \tag{14}$$

$$\mathbf{ML} = \bigcap_\alpha \mathbf{ML}_\alpha. \tag{15}$$

Observe that for $\alpha < \beta$ the frame $\mathcal{P}^0(\alpha)$ is a generated subframe of $\mathcal{P}^0(\beta)$, consequently $\mathbf{ML}_\beta \subseteq \mathbf{ML}_\alpha$. Since there are countably many modal formulas and proper class many cardinals, there must exist a cardinal α_0 such that the sequence \mathbf{ML}_α stabilizes, that is $\mathbf{ML} = \mathbf{ML}_{\alpha_0}$. It is a difficult open problem to determine the value α_0 .

PROBLEM 2.2. What is the exact value of α_0 ?

One of the main results in [2] is Theorem 5.2:

THEOREM 2.3. Countable Bayes logics and the modal logics of countable Medvedev frames coincide.

$$\begin{array}{ccccccc} \mathbf{ML} & \subseteq & \mathbf{ML}_\omega & = & \mathbf{ML}_{\leq\omega} & \subsetneq & \mathbf{ML}_{<\omega} & \subsetneq & \mathbf{ML}_n \\ \cup\! \! \! \cup & & \parallel & & \parallel & & \parallel & & \parallel \\ \mathbf{BL} & \subsetneq & \mathbf{BL}_\omega & = & \mathbf{BL}_{\leq\omega} & \subsetneq & \mathbf{BL}_{<\omega} & \subsetneq & \mathbf{BL}_n. \end{array} \tag{16}$$

To gain nonfinite axiomatizability results we follow the method presented in Shehtman [10] and we recall the most important lemmas that we make use of.

LEMMA 2.4 (cf. Proposition 4 in [10]). Let \mathcal{F} be a generated finite **S4**-frame. Then there is a modal formula $\chi(\mathcal{F})$ with the following properties:

- (A) For any **S4**-frame \mathcal{G} we have $\mathcal{G} \not\Vdash \chi(\mathcal{F})$ if and only if $\exists u \mathcal{G}^u \twoheadrightarrow \mathcal{F}$.
- (B) For any logic $\mathbf{L} \supseteq \mathbf{S4}$ we have $\mathbf{L} \subseteq \Lambda(\mathcal{F})$ if and only if $\chi(\mathcal{F}) \notin \mathbf{L}$.

DEFINITION 2.5 ([10]). For $m > 0$ and $k > 2$ the Chinese lantern is the **S4**-frame $\mathcal{C}(k, m)$ formed by the set

$$\{(i, j) : (1 \leq i \leq k - 2, 0 \leq j \leq 1) \text{ OR } (i = k - 1, 1 \leq j \leq m) \text{ OR } (i = k, j = 0)\}$$

with the accessibility relation being an ordering:

$$(i, j) \leq (i', j') \text{ iff } (i, j) = (i', j') \text{ OR } i > i'$$

$\mathcal{C}(m, k)$ is illustrated on p. 373 in [10]; however, we will not need any particular information about \mathcal{C} apart from two lemmas that we recall below.

LEMMA 2.6 (Lemma 22 in [10]). *Let ϕ be a modal formula using l variables and let $m > 2^l$. Then $\mathcal{C}(k, m) \not\models \phi$ implies $\mathcal{C}(k, 2^l) \not\models \phi$.*

LEMMA 2.7 (Lemma 24 in [10]). *For any $n > 1$ we have $\mathcal{C}(2^n, 2^n) \Vdash \mathbf{ML}_{<\omega}$.*

For a natural number l a logic \mathbf{L} is l -axiomatizable if it has an axiomatization using only formulas whose propositional variables are among p_1, \dots, p_l . Every finitely axiomatizable logic is l -axiomatizable for a suitable l : take l to be the maximal number of variables the finitely many axioms in question use.

THEOREM 2.8. *Let \mathbf{L} be a normal modal logic with $\mathbf{S4} \subseteq \mathbf{L} \subseteq \mathbf{ML}_{<\omega}$. Suppose that for every $l \geq 1$ and $k > l$ there is $n \geq k$ such that $\chi(\mathcal{C}(k, 2^n)) \in \mathbf{L}$. Then \mathbf{L} is not l -axiomatizable for any number l .*

Proof. By way of contradiction suppose \mathbf{L} is l -axiomatizable, that is, $\mathbf{L} = \mathbf{S4} + \Sigma$ where Σ is a set of formulas that can use only the first l propositional variables. Let $k = 2^l$. By assumption there is $n \geq k$ so that $\chi(\mathcal{C}(k, 2^n)) \in \mathbf{L}$. That Σ axiomatizes \mathbf{L} means that every formula in \mathbf{L} can be derived (in the normal modal calculus) from a finite set of axioms from Σ . Therefore there is an l -formula $\phi \in \mathbf{L}$ such that $\chi(\mathcal{C}(k, 2^n)) \in \mathbf{S4} + \phi$. This implies, by Lemma 2.4(B), that $\mathcal{C}(k, 2^n) \not\models \phi$. As $n \geq k = 2^l > l$, Lemma 2.6 ensures $\mathcal{C}(k, 2^l) \not\models \phi$. In particular, $\mathcal{C}(k, 2^l) \not\models \mathbf{L}$.

On the other hand Lemma 2.7 implies (as $k = 2^l$) that $\mathcal{C}(k, 2^l) \Vdash \mathbf{ML}_{<\omega}$. By assumption $\mathbf{L} \subseteq \mathbf{ML}_{<\omega}$ so it follows that $\mathcal{C}(k, 2^l) \Vdash \mathbf{L}$ which is a contradiction. □

COROLLARY 2.9. *Let \mathcal{F} be a frame, $\mathbf{L} = \Lambda(\mathcal{F})$ and assume $\mathbf{S4} \subseteq \mathbf{L} \subseteq \mathbf{ML}_{<\omega}$. Suppose for any $k \geq 1$ there is $n \geq k$ such that for all $u \in \mathcal{F}$ we have $\mathcal{F}^u \not\models \mathcal{C}(k, 2^n)$. Then \mathbf{L} is not l -axiomatizable for any finite number l .*

Proof. Under the given assumptions Lemma 2.4(A) implies that for all $k \geq 1$ there is $n \geq k$ so that $\chi(\mathcal{C}(k, 2^n)) \in \mathbf{L}$. Then Theorem 2.8 applies. □

2.1. The modal logic of finitary Boolean frames is not finitely axiomatizable. Let $\mathcal{F} = \langle W, \leq \rangle$ be a finite ordering (partially ordered set) and pick $x \in W$. y is an immediate successor of x if $x < y$ and there is no $x < z < y$. (As usual $<$ means $\leq \cap \neq$). The branching index $b_{\mathcal{F}}(x)$ is the cardinality of the set of immediate successors of x , and the depth $d_{\mathcal{F}}(x)$ is the least upper bound of cardinalities of chains in \mathcal{F} whose least element is x . Thus, $d_{\mathcal{F}}(x) = 1$ means that x has no immediate successors. We adopt Lemma 17 in [10] in a slightly more general setting.

DEFINITION 2.10. *Let $\mathcal{B} \subseteq \wp(X)$ be a Boolean algebra of subsets of X such that all finite subsets of X are contained in \mathcal{B} . A frame isomorphic to $\mathcal{B}^0 = \langle \mathcal{B} \setminus \{\emptyset\}, \supseteq \rangle$ is called a finitary Boolean frame.*

The connection with Medvedev frames is transparent: for any set X the frames $\mathcal{P}^0(X) = \langle \wp(X) \setminus \{\emptyset\}, \supseteq \rangle$ are finitary Boolean frames. Note that any point-generated subframe of \mathcal{B}^0 is also a finitary Boolean frame.

LEMMA 2.11. Let \mathcal{B}^0 be a finitary Boolean frame and let \mathcal{F} be a finite, point-generated ordering (frame) such that $b_{\mathcal{F}}(x) \neq 1$ for every $x \in \mathcal{F}$. If $\mathcal{B}^0 \rightarrow \mathcal{F}$, then $b_{\mathcal{F}}(x) < 2^{d_{\mathcal{F}}(x)}$ for all $x \in \mathcal{F}$.

An upper semilattice S is an ordered set in which every two elements x, y have a supremum $x \vee y$. S is atomic if, except for the least element 0_S (if it exists), below every element there is an atom (the set of atoms is coinital). Note that every Boolean algebra can be regarded as an atomic semilattice. The anonymous referee pointed out that Lemma 2.11 can be proved in the context of atomic upper semilattices: Lemma 2.11 is a direct consequence of our next Lemma 2.12.

LEMMA 2.12. For an atomic upper semilattice (S, \leq) consider the frame $\mathcal{S} = \langle S \setminus \{0_S\}, \geq \rangle$. Let \mathcal{F} be a finite, point-generated ordering such that $b_{\mathcal{F}}(x) \neq 1$ for every $x \in \mathcal{F}$. If $\mathcal{S} \rightarrow \mathcal{F}$, then $b_{\mathcal{F}}(x) < 2^{d_{\mathcal{F}}(x)}$ for all $x \in \mathcal{F}$.

Proof. For elements $A \in \mathcal{S}$ let us define $|A|$ as the cardinality of the set of atoms $B \in \mathcal{S}$ such that $B \leq A$. Suppose $h : \mathcal{S} \rightarrow \mathcal{F}$ is a surjective bounded morphism. We show first that for all $x \in \mathcal{F}$ there is a set $A_x \in \mathcal{S}$ such that $h(A_x) = x$ and $|A_x| < 2^{d_{\mathcal{F}}(x)}$. We proceed by induction. The case $d(x) = 1$ is straightforward: For some $A \in \mathcal{S}$ we have $h(A) = x$. As $d(x) = 1$, x has no proper successors, therefore for all $B \leq A$, $B \in \mathcal{S}$ we have $h(B) = x$. Pick any atom $A_x \leq A$. Then $h(A_x) = x$ and $|A_x| \leq 1 < 2^1$.

Suppose now (inductive hypothesis) that we know the statement for all $x \in \mathcal{F}$ with $d(x) = n$ and consider the case $d(x) = n + 1$. There is a set $A \in \mathcal{S}$ such that $h(A) = x$. By assumption $b(x) \neq 1$; therefore x has at least two immediate successors x_1 and x_2 . Then $d(x_i) = n$; therefore, by induction, there are elements $A_1, A_2 \leq A$ such that $h(A_i) = x_i$ and $|A_i| < 2^{d(x_i)}$. Let $A_x = A_1 \vee A_2$ be the supremum. It is clear that $|A| < 2^{d(x_1)} + 2^{d(x_2)} \leq 2^{n+1}$, so we need to show $h(A_x) = x$. Since $A \geq A_x$ and h is a homomorphism, it follows that $x \leq h(A_x)$. Similarly, $A_x \geq A_i$, therefore $h(A_x) \leq h(A_i) = x_i$. As x_1 and x_2 are immediate successors, the only element $h(A_x)$ that can satisfy the equations $x \leq h(A_x) \leq x_i$ for $i = 1, 2$ is x .

To complete the proof pick an arbitrary $x \in \mathcal{F}$ and a set $A \in \mathcal{S}$ such that $h(A) = x$ and $|A| < 2^{d(x)}$. As $|A|$ is finite we can assume that $|A|$ is as small as possible. The branching index of A is $b(A) \leq |A| < 2^{d(x)}$ and since h is a bounded homomorphism we also have $b(h(A)) \leq b(A)$. This completes the proof. \square

COROLLARY 2.13. Let \mathcal{S} be as in Lemma 2.12. Then there is no surjective bounded morphism $\mathcal{S} \rightarrow \mathcal{C}(k, 2^k)$.

Proof. The point $x = (k, 0)$ in $\mathcal{C}(k, 2^k)$ has depth $d(x) = k$ and branch index $b(x) = 2^k$. If there is $\mathcal{S} \rightarrow \mathcal{F}$, then Lemma 2.12 would apply and we would have $b(x) < 2^{d(x)}$ which is impossible. \square

COROLLARY 2.14. Let \mathcal{S} be as in Lemma 2.12 and suppose \mathcal{S} has infinitely many atoms. Then $\mathbf{L} = \Lambda(\mathcal{S})$ is not finitely axiomatizable.

Proof. We intend to apply Corollary 2.9. It is straightforward that the reverse ordering \geq of the semilattice S is transitive and reflexive, therefore \mathcal{S} is an $\mathbf{S4}$ -frame (i.e., $\mathbf{S4} \subseteq \mathbf{L}$).

Next, let us verify that $\mathbf{L} \subseteq \mathbf{ML}_{<\omega}$. To this end it is enough to prove that for each finite set D we have $\mathcal{S} \rightarrow \mathcal{P}^0(D)$. Let us denote by A the set of atoms of \mathcal{S} and fix an arbitrary finite set $D = \{d_1, \dots, d_n\}$. Take $A' \subseteq A$ such that $|A'| = |D|$ (such set A' exists as \mathcal{S} has infinitely many atoms), let us say $A' = \{a_1, \dots, a_n\}$. Take the mapping $g : A \rightarrow \mathcal{P}^0(D)$

defined as

$$g(a) = \begin{cases} d_i & \text{if } a = a_i \\ d_1 & \text{otherwise.} \end{cases}$$

The idea is to extend the mapping g to a bounded morphism. Define $f : \mathcal{S} \rightarrow \mathcal{P}^0(D)$ as

$$f(s) = \{f(a) : a \leq s, a \in A\} \neq \emptyset.$$

Checking that f is a surjective bounded morphism is easy and is left to the reader.

Finally, to fulfill all requirements of Corollary 2.9 we show that for all $u \in \mathcal{S}$ we have $(\mathcal{S})^u \not\cong \mathcal{C}(k, 2^k)$. As point-generated subframes of \mathcal{S} are also atomic upper semilattices the result follows from Corollary 2.13. □

The next two corollaries are immediate consequences of Corollaries 2.13 and 2.14.

COROLLARY 2.15. *Let \mathcal{B}^0 be a finitary Boolean frame. Then there is no surjective bounded morphism $\mathcal{B}^0 \rightarrow \mathcal{C}(k, 2^k)$.*

COROLLARY 2.16. *Let \mathcal{B}^0 be an infinite finitary Boolean frame. Then $\mathbf{L} = \Lambda(\mathcal{B}^0)$ is not finitely axiomatizable.*

We close this subsection by mentioning the following hard open problem (see Problem 2.2).

PROBLEM 2.17. *Is it true that the modal logic of all finitary Boolean frames coincides with \mathbf{ML}_ω ?*

2.2. On the modal logic of certain measure algebra frames. Suppose $\langle X, \mathcal{B}, w \rangle$ is a probability space. Two measurable sets A and B are said to be w -equivalent, $A \sim_w B$ in symbols (when it is clear we omit w from the subscript), if the w -measure of their symmetric difference is 0. \sim is a congruence on the Boolean σ -algebra \mathcal{B} and thus we can consider the quotient structure \mathcal{B}/\sim which also is a σ -complete Boolean algebra with the quotient operations. E.g., $A/\sim \leq B/\sim$ if and only if $A \subseteq B$ modulo w -measure zero for any representatives A and B respectively of A/\sim and B/\sim . The measure w can be pushed down to \mathcal{B}/\sim by the definition $\bar{w}(A/\sim) = w(A)$ for any $A \in \mathcal{B}$. For every $A \in \mathcal{B}$ with $w(A) = 0$ we have $A/\sim = \emptyset/\sim$, therefore \emptyset/\sim is the only element in \mathcal{B}/\sim which has \bar{w} -measure zero. A similar argument shows that only X/\sim has \bar{w} -measure 1 in \mathcal{B}/\sim . $\langle \mathcal{B}/\sim, \bar{w} \rangle$ is called a measure algebra.

DEFINITION 2.18. *A frame isomorphic to $\mathcal{M}(X, \mathcal{B}, w) = \langle \mathcal{B}/\sim_w \setminus \{\emptyset/\sim_w\}, \geq \rangle$ is called a measure algebra frame. Here $\langle X, \mathcal{B}, w \rangle$ is a probability space and \geq is the converse of the Boolean ordering of \mathcal{B}/\sim_w .*

There is a straightforward connection with Medvedev frames: the set of possible worlds of a Medvedev frame is $\wp(X) \setminus \{\emptyset\}$ for some (finite) set X . $\wp(X)$ is a Boolean algebra, thus we can think of the possible worlds of a Medvedev frame as the structure that results when we cut out the least element of a Boolean algebra. We did the same thing in case of finitary Boolean frames (Definition 2.10) and we do something similar here: we take the Boolean algebra \mathcal{B}/\sim and cut out its least element. If X is a finite set, then every subset of X is measurable, thus $\mathcal{B} = \wp(X)$. Take now a faithful probability measure w over \mathcal{B} . Then no proper subsets of X are \sim_w equivalent, therefore \mathcal{B}/\sim is (isomorphic to) $\wp(X)$. Consequently, the Medvedev frame $\mathcal{P}^0(X)$ and the measure algebra frame $\mathcal{M}(X, \mathcal{B}, w)$ are isomorphic when X is finite. For infinite X the connection is more subtle: the Lebesgue measure algebra (where $X = [0, 1]$, $\mathcal{B} =$ all Borel sets of $[0, 1]$, $w =$ Lebesgue measure)

has no σ -complete set representation (it is not realized as a σ -complete Boolean algebra of subsets of some set), therefore it is not isomorphic to any (finitary) Boolean frame.

From now on, in this subsection, \mathcal{B} denotes the Borel σ -algebra over the unit interval $[0, 1]$, \mathcal{L} the Lebesgue σ -algebra over $[0, 1]$, and λ the Lebesgue measure. \sim means \sim_λ . As \mathcal{L} is the completion of \mathcal{B} , it follows that the measure algebras \mathcal{L}/\sim and \mathcal{B}/\sim are isomorphic. Therefore $\mathcal{M}([0, 1], \mathcal{L}, \lambda)$ and $\mathcal{M}([0, 1], \mathcal{B}, \lambda)$ are also isomorphic (as frames). Note that both $\mathcal{B} \setminus \{\emptyset\}$ and $\mathcal{L} \setminus \{\emptyset\}$ are infinite finitary Boolean frames (in the sense of Definition 2.10), thus their modal logics are not finitely axiomatizable (Corollary 2.16).

LEMMA 2.19. *If \mathcal{M} is a measure algebra frame and $a \in \mathcal{M}$, then the generated subframe \mathcal{M}^a is also a measure algebra frame. If $\mathcal{M} = \mathcal{M}([0, 1], \mathcal{L}, \lambda)$, then \mathcal{M}^a is isomorphic to \mathcal{M} .*

Proof. Pick any $\mathcal{M} = \mathcal{M}(X, \mathcal{B}, w)$ and an element $a \in \mathcal{M}$. There is $A \in \mathcal{B}$ such that $A/\sim_w = a$. Measurable subsets of A are exactly the sets in $\mathcal{B} \upharpoonright A = \{A \cap B : B \in \mathcal{B}\}$ and $v(\cdot) = w(\cdot \mid A)$ is a probability measure on $\mathcal{B} \upharpoonright A$. It is not hard to see that the generated subframe \mathcal{M}^a is isomorphic to $\mathcal{M}(A, \mathcal{B} \upharpoonright A, v)$.

For the second statement we only have to note that for a Lebesgue measurable $A \subseteq [0, 1]$ we have $\mathcal{M}(A, \mathcal{L} \upharpoonright A, \lambda) \cong \mathcal{M}([0, 1], \mathcal{L}, \lambda)$. □

THEOREM 2.20. *The modal logic of $\mathcal{M} = \mathcal{M}([0, 1], \mathcal{L}, \lambda)$ is not finitely axiomatizable.*

Proof. Write $\mathbf{L} = \Lambda(\mathcal{M})$. We intend to apply Corollary 2.9; the first part of the proof is almost identical to that of Corollary 2.16. It is straightforward that the reverse-Boolean ordering \geq is transitive and reflexive, therefore \mathcal{M} is an **S4**-frame (i.e., $\mathbf{S4} \subseteq \mathbf{L}$).

Next, we verify $\mathbf{L} \subseteq \mathbf{ML}_{<\omega}$. It is enough to prove $\mathcal{M} \twoheadrightarrow \mathcal{P}^0(\mathbb{N})$ as in this case $\mathbf{L} \subseteq \Lambda(\mathcal{P}^0(\mathbb{N})) = \mathbf{ML}_\omega \subseteq \mathbf{ML}_{<\omega}$. As \mathcal{M} is infinite, there are countably infinite many pairwise disjoint elements $a_i \in \mathcal{M}$ ($i \in \mathbb{N}$) such that $\bigvee_{i \in \mathbb{N}} a_i = [0, 1]/\sim$. (Note that each a_i has a representative $A_i \in \mathcal{B}$ so that $a_i = A_i/\sim$. Then $A_i \cap A_j$ is λ -measure zero for all $i \neq j$ and $\bigcup_{i \in \mathbb{N}} A_i = [0, 1]$ (modulo λ -measure zero)). Define $f : \mathcal{M} \rightarrow \mathcal{P}^0(\mathbb{N})$ as

$$f(a) = \{i \in \mathbb{N} : a \wedge a_i \neq 0\} \neq \emptyset.$$

Then $f([0, 1]/\sim) = \mathbb{N}$ and $f(a_i) = \{i\}$. We claim that f is a surjective bounded morphism.

Surjectivity: For a nonempty $A \subseteq \mathbb{N}$ we have $f(\bigvee_{i \in A} a_i) = A$.

Homomorphism: Suppose for $a, b \in \mathcal{M}$ we have $a \geq b$. Then whenever $b \wedge a_i \neq 0$ we also have $a \wedge a_i \neq 0$, therefore

$$\{i \in \mathbb{N} : a \wedge a_i \neq 0\} \supseteq \{i \in \mathbb{N} : b \wedge a_i \neq 0\}$$

meaning that $f(a) \supseteq f(b)$.

Zig-zag property: Suppose $f(a) = A$ and $A \supseteq B \neq \emptyset$. We need to find a b with $a \geq b$ and $f(b) = B$. Take $b = a - \bigvee_{i \in A \setminus B} a_i$. Then $a \geq b$ and $f(b) = \{i : b \wedge a_i \neq 0\} = B \setminus A = B$.

Finally, to fulfill all requirements of Corollary 2.9 we prove that for all $u \in \mathcal{M}$ we have $\mathcal{M}^u \not\approx \mathcal{C}(k, 2^k)$. As generated subframes of measure algebra frames are measure algebra frames (Lemma 2.19) it is enough to prove that $\mathcal{M} \not\approx \mathcal{C}(k, 2^k)$.

Suppose, seeking a contradiction, that there is $f : \mathcal{M} \twoheadrightarrow \mathcal{C}(k, 2^k)$. Write $\mathcal{L}^+ = \{B \in \mathcal{L} : \lambda(B) > 0\}$ and let \mathcal{N} be the frame $\mathcal{N} = \langle \mathcal{L}^+, \supseteq \rangle$. The mapping f can be lifted up to a surjective bounded morphism $f^+ : \mathcal{N} \twoheadrightarrow \mathcal{C}(k, 2^k)$ by letting $f^+(B) = f(B/\sim)$. Clearly, if $A \sim B$, then $f^+(A) = f^+(B)$. To complete the proof we would like to apply Corollary 2.15, but the problem is that \mathcal{N} is not a finitary Boolean frame as finite subsets

of $[0, 1]$ have λ -measure zero. Therefore, we need to further extend f^+ as follows. Let A be a nonempty Lebesgue measurable set and suppose $\{A_i\}_{i \in \mathbb{N}}$ is a sequence of positive measure sets ($A_i \in \mathcal{L}^+$) such that $A_i \supseteq A_{i+1}$ modulo measure zero and $A = \bigcap_{i \in \mathbb{N}} A_i$. Then define $F(A) = \sup_{\{A_i\}_{i \in \mathbb{N}}} \lim_i f^+(A_i)$. For every such sequence $\{A_i\}$ the limit exists as $\mathcal{C}(k, 2^k)$ is finite and $f^+(A_i) \leq f^+(A_{i+1})$. If two sequences $\{A_i\}$ and $\{B_i\}$ give a different limit, then $\{A_i \cap B_i\}$ yields a greater (or equal) limit as $f^+(A_i \cap B_i) \geq f^+(A_i), f^+(B_i)$. It follows that the supremum always exists. Denote the frame extended with elements of such form by $\mathcal{N}^* = \langle \mathcal{L}^*, \supseteq \rangle$, where

$$\mathcal{L}^* = \{A : A \in \mathcal{L}, \exists \{A_i\} \subseteq \mathcal{L}^+ : A = \bigcap_i A_i, A_i \supseteq A_{i+1}\}.$$

It is straightforward to verify that $F : \mathcal{N}^* \rightarrow \mathcal{C}(k, 2^k)$ is a surjective bounded morphism.

To complete the proof observe that each finite subset can be obtained in the form $\bigcap_{i \in \mathbb{N}} A_i$: if Z is a finite subset of $[0, 1]$, then $Z = \bigcap_{i \in \mathbb{N}} (Z \cup (0, \frac{1}{i}))$. Therefore the frame \mathcal{N}^* is a finitary Boolean frame, thus Corollary 2.15 implies that there is no bounded morphism $\mathcal{N}^* \rightarrow \mathcal{C}(k, 2^k)$. This contradicts to F being such a bounded morphism. \square

Suppose X is a finite or countably infinite set and consider the probability space $\langle X, \wp(X), w \rangle$ for some probability measure w . Without loss of generality we can assume that w is faithful: Otherwise restrict X to the support $\text{supp}(w)$ of w (let $Y = X \cap \text{supp}(w)$) and consider the probability space $\langle Y, \wp(Y), w \upharpoonright Y \rangle$. Then the two frames $\mathcal{M}(X, \wp(X), w)$ and $\mathcal{M}(Y, \wp(Y), w \upharpoonright Y)$ are isomorphic. As noted above $\mathcal{M}(X, \wp(X), w)$ is isomorphic to the Medvedev frame $\mathcal{P}^0(X)$. Consider now the probability space whose underlying set is $[0, 1] \cup X$, its σ -algebra is the algebra generated by $\mathcal{L} \cup \wp(X)$ and the probability measure is $\nu = r \cdot \lambda + (1 - r) \cdot w$ for some $r \in (0, 1)$. Denote the corresponding measure algebra frame by $\mathcal{M}([0, 1] \cup X, \mathcal{L} + \wp(X), \nu)$. (By replacing \mathcal{L} with \mathcal{B} we obtain the same measure algebra frame).

THEOREM 2.21. *The modal logic of $\mathcal{M} = \mathcal{M}([0, 1] \cup X, \mathcal{L} + \wp(X), \nu)$ is not finitely axiomatizable.*

Proof. The same proof as that of Theorem 2.20 works with the obvious modifications. The proof for $\mathbf{S4} \subseteq \mathbf{L} \subseteq \mathbf{ML}_{<\omega}$ is identical. To show that for all $u \in \mathcal{M}$ we have $\mathcal{M}^u \not\cong \mathcal{C}(k, 2^k)$ we need two cases: if $u \leq X/\sim$, then \mathcal{M}^u is isomorphic to a subframe of $\mathcal{P}^0(X)$ which is a finitary Boolean frame, thus the proof completes by applying Corollary 2.15; if u intersects $[0, 1]/\sim$, then the proof continues identically. \square

§3. Standard Bayes logic is not finitely axiomatizable. Recall (from the Introduction) that every standard probability space is isomorphic (modulo zero) to $\langle \mathbb{N} + [0, 1], \wp(\mathbb{N}) + \mathcal{L}, p + \lambda \rangle$, where $\langle \mathbb{N}, \wp(\mathbb{N}), p \rangle$ is an arbitrary probability space.

LEMMA 3.1. *There is no surjective bounded morphism $\mathcal{M}(\mathbb{N} + [0, 1], \wp(\mathbb{N}) + \mathcal{L}, p + \lambda) \rightarrow \mathcal{C}(k, 2^k)$.*

Proof. The proof of this statement is part of the proof of Theorem 2.21 (cf. the proof of Theorem 2.20). \square

LEMMA 3.2. *Let $\mathcal{F} = \mathcal{F}(X, \mathcal{B})$ be a Bayes frame. For any possible world (probability measure) $w \in \mathcal{F}$ the generated subframe \mathcal{F}^w is isomorphic to $\mathcal{M}(X, \mathcal{B}, w)$.*

Proof. As \mathcal{F}^w is generated by w , for any $u \in \mathcal{F}^w$ there is a non w -measure zero set A_u such that $u(\cdot) = w(\cdot \mid A_u)$. This A_u is unique up to w -measure zero equivalence: if

$A_u \sim_w A'_u$, then $w(\cdot \mid A_u) = w(\cdot \mid A'_u)$. Therefore u can be identified with the element $A_u/\sim \in \mathcal{B}/\sim$. Indeed, let f be the mapping $f : \mathcal{F}^w \rightarrow \mathcal{B}/\sim \setminus \{\emptyset/\sim\}$ defined by $f(u) = A_u/\sim$. (In particular $f(w) = X/\sim$ which is the top element of the Boolean algebra \mathcal{B}/\sim , i.e., the element which generates \mathcal{M} .) It is fairly easy to check that f is an isomorphism between the frames \mathcal{F} and \mathcal{M} . □

THEOREM 3.3. *Let $\langle X, \mathcal{B} \rangle$ be a standard Borel space and $\mathcal{F}(X, \mathcal{B})$ be the corresponding Bayes frame. The logic $\Lambda(\mathcal{F}(X, \mathcal{B}))$ is not finitely axiomatizable.*

Proof. Let $\mathbf{L} = \Lambda(\mathcal{F}(X, \mathcal{B}))$. We intend to apply Corollary 2.9. We already know that $\mathbf{S4} \subseteq \mathbf{L} \subseteq \mathbf{ML}_{<\omega}$ (cf. Theorem 1.3) thus we only need to see that for any k and every $u \in \mathcal{F}$ we have $\mathcal{F}^u \not\rightarrow \mathcal{C}(k, 2^k)$. By Lemma 3.2 we know that \mathcal{F}^u is isomorphic to $\mathcal{M}(X, \mathcal{B}, u)$. But as $\langle X, \mathcal{B} \rangle$ is a standard Borel space, the probability space $\langle X, \mathcal{B}, u \rangle$ is also standard (more precisely, the completion of it is standard), therefore it is isomorphic modulo zero to $\langle \mathbb{N} + [0, 1], \wp(\mathbb{N}) + \mathcal{L}, p + \lambda \rangle$ for some p . Lemma 3.1 states there is no bounded morphism $\mathcal{M}(\mathbb{N} + [0, 1], \wp(\mathbb{N}) + \mathcal{L}, p + \lambda) \rightarrow \mathcal{C}(k, 2^k)$, which completes the proof. □

THEOREM 3.4. *The standard Bayes logic*

$$\mathbf{BL}_{st} = \{ \phi : (\forall \text{Standard Borel Bayes frames } \mathcal{F}) \mathcal{F} \Vdash \phi \}$$

is not finitely axiomatizable.

Proof. Take $\mathcal{F} = \mathcal{F}(\mathbb{N} + [0, 1], \wp(\mathbb{N}) + \mathcal{L}, p + \lambda)$, where p is any faithful measure on $\wp(\mathbb{N})$. Then every standard Borel Bayes frame is a generated subframe of \mathcal{F} , thus $\Lambda(\mathcal{F}) = \mathbf{BL}_{st}$. But then Theorem 3.3 applies and completes the proof. □

Open problems. We have seen that none of the logics $\mathbf{BL}_{<\omega}$, \mathbf{BL}_ω and \mathbf{BL}_{st} are finitely axiomatizable (moreover, none of them can be axiomatized by formulas using finitely many propositional variables only). We know that the following inclusions hold:

$$\mathbf{S4} \subseteq \mathbf{BL} \subseteq \mathbf{BL}_{st} \subseteq \mathbf{BL}_\omega = \mathbf{BL}_{<\omega} \subsetneq \mathbf{BL}_{<\omega}.$$

We raise the following open problems.

PROBLEM 3.5. *Is $\mathbf{BL} = \mathbf{BL}_{st}$?*

PROBLEM 3.6. *Is $\mathbf{BL} = \mathbf{S4}$?*

PROBLEM 3.7. *If $\mathbf{S4} \neq \mathbf{BL} \neq \mathbf{BL}_{st}$, then is \mathbf{BL} finitely axiomatizable?*

It is a longstanding hard open problem whether there are recursive axiomatizations for any of the logics \mathbf{ML}_α (α infinite). In light of this we raise the next problem.

PROBLEM 3.8. *Are $\mathbf{BL}_{<\omega}$, \mathbf{BL}_{st} or \mathbf{BL} recursively axiomatizable?*

Appendix. ω is the least infinite cardinal (that is, the set of natural numbers). By a frame we always understand a Kripke frame, that is, a structure of the form $\mathcal{F} = \langle W, R \rangle$, where W is a nonempty set (of possible worlds) and $R \subseteq W \times W$ a binary relation (accessibility). Validity of a formula ϕ ($\mathcal{F} \Vdash \phi$ in symbols) is defined in the standard way (see Definition 1.28 in [1]). For a class \mathbf{C} of frames we let

$$\Lambda(\mathbf{C}) = \{ \phi : (\forall \mathcal{F} \in \mathbf{C}) \mathcal{F} \Vdash \phi \} \tag{17}$$

$\Lambda(\mathbf{C})$ is always a normal modal logic.

Let us recall the most standard list of modal axioms (frame properties) that are often considered in the literature (cf. [1] and [3]). Such axioms are

- K** $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$
T $\Box\phi \rightarrow \phi$
4 $\Box\phi \rightarrow \Box\Box\phi$
M $\Box\Diamond\phi \rightarrow \Diamond\Box\phi$
Grz $\Box(\Box(\phi \rightarrow \Box\phi) \rightarrow \phi) \rightarrow \phi$.

Let us recall some of the standard frame properties corresponding to these axioms (cf. [1] and [3]).

Logic	Axioms	Adequate frames
K	K	All frames
T	K+T	Reflexive frames
4	K+4	Transitive frames
S4	K+T+4	Preorders
S4.1	K+T+4+M	Preorders in which every point sees an endpoint
S4.Grz	K+T+4+Grz	Preorders without infinite chains

For two frames $\mathcal{F} = \langle W, R \rangle$ and $\mathcal{G} = \langle W', R' \rangle$ we write $\mathcal{F} \trianglelefteq \mathcal{G}$ if \mathcal{F} is (isomorphic as a frame to) a generated subframe of \mathcal{G} . We recall that if $\mathcal{F} \trianglelefteq \mathcal{G}$, then $\mathcal{G} \Vdash \phi$ implies $\mathcal{F} \Vdash \phi$, whence $\Lambda(\mathcal{G}) \subseteq \Lambda(\mathcal{F})$ (see Theorem 3.14 in [1] where the symbol \succrightarrow was used instead of \trianglelefteq). If $w \in W$, then we write \mathcal{F}^w to denote the subframe of \mathcal{F} generated by w , and we call such subframes point-generated subframes. Further, let $\mathcal{F} \twoheadrightarrow \mathcal{G}$ denote a surjective, bounded morphism (sometimes called p-morphisms). Such morphisms preserve the accessibility relation and have the zig-zag property (see [1]). Recall that if $\mathcal{F} \twoheadrightarrow \mathcal{G}$, then $\mathcal{F} \Vdash \phi$ implies $\mathcal{G} \Vdash \phi$, hence $\Lambda(\mathcal{F}) \subseteq \Lambda(\mathcal{G})$ (see Theorem 3.14 in [1]).

Acknowledgments. We are truly grateful to the anonymous referee whose conscientious work and whose suggestions improved the article substantially. Project was supported by the Premium Postdoctoral Grant of the Hungarian Academy of Sciences and in part by the Hungarian Scientific Research Found (OTKA), contract numbers: K100715, K115593.

BIBLIOGRAPHY

- [1] Blackburn, P., de Rijke, M., & Venema, Y. (2002). *Modal Logic*. Cambridge: Cambridge University Press.
- [2] Brown, W., Gyenis, Z., & Rédei, M. (2018). The modal logic of Bayesian belief revision. *Journal of Philosophical Logic*, doi:10.1007/s10992-018-9495-9.
- [3] Chagrov, A. & Zakharyashev, M. (1997). *Modal Logic*. Oxford: Clarendon Press.
- [4] Diaconis, P. & Zabell, S. (1983). Some alternatives to Bayes' rule. Technical Report 205, Stanford University.
- [5] Holliday, W. H. (2016). On the modal logic of subset and superset: Tense logic over Medvedev frames. *Studia Logica*, **105**, 13–35.
- [6] Kechris, A. S. (1995). *Classical Descriptive Set Theory*. New York: Springer-Verlag.
- [7] Łazarz, M. (2013). Characterization of Medvedev's logic by means of Kubiński's frames. *Bulletin of the Section of Logic*, **42**, 83–90.

- [8] Medvedev, Y. T. (1996). On the interpretation of the logical formulas by means of finite problems. *Doklady Akademii Nauk SSSR*, **169**, 20–24 (in Russian).
- [9] Prucnal, T. (1976). Structural completeness of Medvedev's propositional calculus. *Reports on Mathematical Logic*, **6**, 103–105.
- [10] Shehtman, V. (1990). Modal counterparts of Medvedev logic of finite problems are not finitely axiomatizable. *Studia Logica*, **49**, 365–385.
- [11] Skvortsov, D. (1979). The logic of infinite problems and the Kripke models on atomic semilattices of sets. *Doklady Akademii Nauk SSSR*, **245**, 798–801 (in Russian).
- [12] Williamson, J. (2010). *In Defence of Objective Bayesianism*. Oxford: Oxford University Press.

DEPARTMENT OF LOGIC
INSTITUTE OF PHILOSOPHY
JAGIELLONIAN UNIVERSITY
KRAKÓW, POLAND

and

DEPARTMENT OF LOGIC
EÖTVÖS LORÁND UNIVERSITY
BUDAPEST, HUNGARY

E-mail: zalan.gyenis@gmail.com