Notes

103.28 On a link between Triangular and Fibonacci numbers

There are only five Fibonacci numbers, namely 1, 1, 3, 21 and 55, which are also triangular numbers. But surprisingly, there is an unexpected link between these numbers:

The product of any four consecutive Fibonacci numbers is twice a triangular number.

More precisely:

Theorem: If $T(n)$ denotes the nth triangular number and F_n denotes the nth Fibonacci number, then

$$F_nF_{n+1}F_{n+2}F_{n+3} = \begin{cases} 2T(F_{n+1}F_{n+2}), & \text{if } n \text{ is odd} \\ 2T(F_nF_{n+3}), & \text{if } n \text{ is even.} \end{cases}$$

Proof: We have the well-known relation,

$$F_nF_{n+3} - F_{n+1}F_{n+2} = (-1)^{n-1} \qquad \text{(see Appendix)}$$

$$\Rightarrow F_nF_{n+1}F_{n+2}F_{n+3} = \begin{cases} F_{n+1}F_{n+2}(F_{n+1}F_{n+2} + 1), & \text{if } n \text{ is odd} \\ F_nF_{n+3}(F_nF_{n+3} + 1), & \text{if } n \text{ is even.} \end{cases}$$

Hence the proof.

Appendix

Set
$$\begin{aligned} J_n &= F_nF_{n+3} - F_{n+1}F_{n+2} \\ &= F_n\left(F_{n+2} + F_{n+1}\right) - \left(F_n + F_{n-1}\right)F_{n+2} \\ &= F_nF_{n+1} - F_{n-1}F_{n+2} \\ &= -J_{n-1} \\ &= (-1)^2 J_{n-2} = \dots . \end{aligned}$$

This leads to

$$J_n = (-1)^{n-1}J_1.$$

But

$$J_1 = F_1F_4 - F_2F_3 = 1 \times 3 - 1 \times 2 = 1.$$

Hence

$$J_n = (-1)^{n-1}.$$

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