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Kolmogorov's dissipation number and the number of degrees of freedom for the 3D Navier–Stokes equations

Alexey Cheskidov and Mimi Dai
Department of Mathematics, Stat. and Comp. Sci., University of Illinois Chicago, Chicago, IL 60607, USA (acheskid@uic.edu; mdai@uic.edu)

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Kolmogorov's theory of turbulence predicts that only wavenumbers below some critical value, called Kolmogorov's dissipation number, are essential to describe the evolution of a three-dimensional (3D) fluid flow. A determining wavenumber, first introduced by Foias and Prodi for the 2D Navier–Stokes equations, is a mathematical analogue of Kolmogorov's number. The purpose of this paper is to prove the existence of a time-dependent determining wavenumber for the 3D Navier–Stokes equations whose time average is bounded by Kolmogorov's dissipation wavenumber for all solutions on the global attractor whose intermittency is not extreme.

Keywords: Navier–Stokes equations; determining modes; global attractor

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Secondary 37L30

1. Introduction

The Navier–Stokes equations (NSE) on a torus $\mathbb{T}^3$ are given by

$$\begin{cases} u_t + (u \cdot \nabla)u - \nu\Delta u + \nabla p = f \\ \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

where u is the velocity, p is the pressure, and f is the external force. We assume that f has zero mean, and consider zero mean solutions. We also assume that $f \in H^{-1}$ or f is translationally bounded in $L^2_{\mathrm{loc}}(\mathbb{R}, H^{-1})$.

In this paper, we investigate the number of degrees of freedom of a three-dimensional (3D) fluid flow governed by (1.1). Kolmogorov's theory of turbulence [**24**] predicts that there is a wavenumber κ_{d} above which the viscous forces dominate. This suggests that the frequencies above κ_{d} should not affect the dynamics and the number of degrees of freedom is of order κ_{d}^3. A natural question is whether this can be justified mathematically.

The notion of determining modes, which allows us to define the degrees of freedom mathematically, was introduced by Foias and Prodi in [**12**] where they showed that high modes of a solution to the 2D NSE are controlled by low modes asymptotically as time goes to infinity. Then the number of these determining modes was estimated by Foias, Manley, Temam, and Treve [**17**] and later improved by Jones and Titi

[22]. We refer the readers to [10, 11, 13–16, 18–20] and references therein for more background and related results.

In this paper, we are concerned with 3D flows governed by (1.1), for which the existence of regular solutions is one of the Millennium open questions. Therefore, we study weak solutions, whose existence was proved by Leray [25]. In [9], Cheskidov, Dai, and Kavlie proved the existence of a determining wavenumber $\Lambda_u(t)$, defined for each individual trajectory $u(t)$, whose average is uniformly bounded on the global attractor. More precisely, it was shown that two solutions $u(t)$ and $v(t)$ on the global attractor are identical, provided their projections below modes $\max\{\Lambda_u, \Lambda_v\}$ coincide. This recovered the results by Constantin, Foias, Manley, and Temam [10] in the case where $\|\nabla u(t)\|_{L^2}^2$ is uniformly bounded on the global attractor, which is known for small forces. Moreover, when the force is large and the attractor is not a fixed point, but rather a complicated object consisting of points on complete bounded trajectories that may not be regular, the determining wavenumber Λ_u from [9] still enjoys the following pointwise bound

$$\Lambda_u(t) \lesssim \frac{\|\nabla u(t)\|_{L^2}^2}{\nu^2}. \tag{1.2}$$

Note that this bound is optimal (from the physical point of view) in the case of extreme intermittency, that is, when there is only one eddy at each dyadic scale. Indeed, taking into account intermittency, Kolmogorov's dissipation wavenumber reads

$$\kappa_{\mathrm{d}} := \left(\frac{\varepsilon}{\nu^3}\right)^{1/d+1}, \quad \text{where} \quad \varepsilon := \lambda_0^d \nu \langle \|\nabla u\|_{L^2}^2 \rangle = \frac{\lambda_0^d \nu}{T} \int_t^{t+T} \|\nabla u\|_{L^2}^2 \, \mathrm{d}\tau. \tag{1.3}$$

Combined with (1.2), this gives $\langle \Lambda_u \rangle \lesssim \kappa_{\mathrm{d}}$ when $d = 0$. Here $d \in [0, 3]$ is the intermittency dimension that measures the average number of eddies at various scales. Roughly speaking, the number of eddies at the lengthscale l is proportional to l^{-d} (see [8] for precise mathematical definitions of active volumes, eddies, and their relations to intermittency). In this paper, we adopt an approach used in [5, 7, 9] and define the intermittency dimension d through the average level of saturation of Bernstein's inequality (see §3 for the precise definition).

As experimental and numerical evidence suggests, turbulent flows do not deviate much from Kolmogorov's regime where $d = 3$, that is, eddies occupy the whole region. For instance, $d \approx 2.7$ was observed in a direct numerical simulation performed by Kaneda *et al.* [23] on the Earth Simulator. In [9] it was shown that one can improve (1.2) for $d > 0$, but such an improvement was not enough to conclude that the average determining wavenumber was bounded by κ_d. For instance, in the case $d = 3$, the obtained bound was $\langle \Lambda_u \rangle \lesssim \kappa_{\mathrm{d}}^{2+}$, which suggested that the definition of Λ_u was not optimal in the physically relevant regime. In this paper, we complement the result of [9] by focussing on the region $d \in [\delta, 3]$, $\delta > 0$, and finding a different determining wavenumber Λ_u that enjoys the optimal bound $\langle \Lambda_u \rangle \lesssim \kappa_{\mathrm{d}}$ (modulo a logarithmic correction in the case $d = 3$).

We define the determining wavenumber in the following way:

$$\begin{aligned}\Lambda_u(t) := \min\{\lambda_q : (L\lambda_{p-q})^{\delta-1/2}\lambda_q^{-1}\|u_p\|_{L^\infty} \\ < c_0\nu, \ \forall p > q \text{ and } \lambda_q^{-2}\|\nabla u_{\leqslant q}\|_{L^\infty} < c_0\nu, \ q \in \mathbb{N}\},\end{aligned}$$

where $0 < \delta \leqslant 3$ is a fixed (small) parameter, and c_0 is an dimensionless constant that depends only on δ. In fact, $c_0 \to 0$ as $\delta \to 0$. Here $\lambda_q = 2^q/L$, L is the size of the torus, $u_{\leqslant q} = \sum_{p=-1}^{q} u_q$, and $u_q = \Delta_q u$ is the Littlewood–Paley projection of u (see §2).

Now, we are ready to state our main result.

THEOREM 1.1. *Let $u(t)$ and $v(t)$ be complete (ancient) bounded in L^2 Leray–Hopf solutions (i.e., solutions on the global attractor or pullback attractor). Let $\Lambda(t) := \max\{\Lambda_u(t), \Lambda_v(t)\}$ and $Q(t)$ be such that $\Lambda(t) = \lambda_{Q(t)}$. If*

$$u(t)_{\leqslant Q(t)} = v(t)_{\leqslant Q(t)}, \quad \forall t < 0, \tag{1.4}$$

then

$$u(t) = v(t), \quad \forall t \leqslant 0.$$

The dissipation wavenumber Λ_u enjoys the following bound:

$$\langle \Lambda_u \rangle - \lambda_0 \leqslant C_{\delta,d} \kappa_{\mathrm{d}} \leqslant C_{\delta,d} \kappa_0 G^{2/d+1} \left(\frac{1}{\nu T \kappa_0^2} + 1 \right)^{1/d+1},$$

for all complete bounded in L^2 Leray–Hopf solutions with $d \in [\delta, 3)$. Here $C_{\delta,d}$ is an dimensionless constant that blows up when $\delta \to 0$ or $d \to 3$. The bound is also written in terms on the dimensionless Grashof number defined as $G := \|f\|_{H^{-1}}/(\nu^2 \kappa_0^{1/2})$ in the autonomous case (see (3.8) for the nonautonomous case).

In Kolmogorov's regime where $d = 3$, we also obtain the optimal bound, but with a logarithmic correction:

$$\left\langle \frac{\Lambda_u - \lambda_0}{(\log(\Lambda_u/\lambda_0))^{1/4}} \right\rangle \leqslant \widetilde{C}_\delta \kappa_{\mathrm{d}} \leqslant \widetilde{C}_\delta \kappa_0 G^{1/2} \left(\frac{1}{\nu T \kappa_0^2} + 1 \right)^{1/4},$$

for all complete bounded in L^2 Leray–Hopf solutions with $d = 3$. Here $\widetilde{C}_\delta$ is an dimensionless constant that depends only on the parameter δ in the definition of Λ. Again, $\widetilde{C}_\delta \to \infty$ as $\delta \to 0$.

2. Preliminaries

2.1. Notation

We denote by $A \lesssim B$ an estimate of the form $A \leqslant CB$ with some absolute constant C, by $A \sim B$ an estimate of the form $C_1 B \leqslant A \leqslant C_2 B$ with some absolute constants C_1, C_2, and by $A \lesssim_r B$ an estimate of the form $A \leqslant C_r B$ with some dimensionless constant C_r that depends only on the parameter r. We write $\|\cdot\|_p = \|\cdot\|_{L^p}$, and $(\cdot,\cdot)$ stands for the L^2-inner product. We will also use $\langle\cdot\rangle$ for time averages:

$$\langle g \rangle(t) := \frac{1}{T} \int_t^{t+T} g(\tau)\, \mathrm{d}\tau,$$

for some fixed $T > 0$.

2.2. Littlewood–Paley decomposition

The techniques presented in this paper rely strongly on the Littlewood–Paley decomposition, which recall here briefly. For a more detailed description on this theory, we refer the readers to the books by Bahouri, Chemin and Danchin [1] and Grafakos [21].

We denote $\lambda_q = 2^q/L$ for integers q. A nonnegative radial function $\chi \in C_0^\infty(\mathbb{R}^3)$ is chosen such that

$$\chi(\xi) := \begin{cases} 1, & \text{for } |\xi| \leqslant \frac{3}{4} \\ 0, & \text{for } |\xi| \geqslant 1. \end{cases} \tag{2.5}$$

Let

$$\varphi(\xi) := \chi(\xi/2) - \chi(\xi)$$

and

$$\varphi_q(\xi) := \begin{cases} \varphi(2^{-q}\xi) & \text{for } q \geqslant 0, \\ \chi(\xi) & \text{for } q = -1, \end{cases}$$

so that the sequence of φ_q forms a dyadic partition of unity. Given a tempered distribution vector field u on $\mathbb{T}^3 = [0, L]^3$ and $q \geqslant -1$, an integer, the qth Littlewood–Paley projection of u is given by

$$u_q(x) := \Delta_q u(x) := \sum_{k \in \mathbb{Z}^3} \hat{u}(k) \phi_q(k) e^{i2\pi/Lk\cdot x},$$

where $\hat{u}(k)$ is the kth Fourier coefficient of u. Note that $u_{-1} = \hat{u}(0)$. Then

$$u = \sum_{q=-1}^{\infty} u_q$$

in the distributional sense. We define the H^s-norm in the following way:

$$\|u\|_{H^s} := \left(\sum_{q=-1}^{\infty} \lambda_q^{2s} \|u_q\|_2^2 \right)^{1/2},$$

for each $u \in H^s$ and $s \in \mathbb{R}$. Note that $\|u\|_{H^0} \sim \|u\|_{L^2}$. To simplify the notation, we denote

$$u_{\leqslant Q} := \sum_{q=-1}^{Q} u_q, \quad u_{(P,Q]} := \sum_{q=P+1}^{Q} u_q, \quad \tilde{u}_q := u_{q-1} + u_q + u_{q+1}.$$

2.3. Bernstein's inequality and Bony's paraproduct

Here, we recall useful properties for the dyadic blocks of the Littlewood–Paley decomposition. The first one is the following inequality:

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Lemma 2.1 (Bernstein's inequality). *Let n be the spatial dimension and $r \geqslant s \geqslant 1$. Then for all tempered distributions u,*

$$\|u_q\|_r \lesssim \lambda_q^{n(1/s-1/r)}\|u_q\|_s. \tag{2.6}$$

Secondly, we will use the following version of Bony's paraproduct formula:

$$\Delta_q(u\cdot\nabla v) = \sum_{|q-p|\leqslant 2}\Delta_q(u_{\leqslant p-2}\cdot\nabla v_p) + \sum_{|q-p|\leqslant 2}\Delta_q(u_p\cdot\nabla v_{\leqslant p-2}) + \sum_{p\geqslant q-2}\Delta_q(\tilde{u}_p\cdot\nabla v_p).$$

2.4. Weak solutions and energy inequality

A weak solution $u(t)$ of (1.1) on $[0,\infty)$ is an $L^2(\mathbb{T}^3)$ valued function in the class $u \in C([0,\infty); L^2_{\mathrm{w}}) \cap L^2_{\mathrm{loc}}(0,\infty; H^1)$ that satisfies (1.1) in the sense of distributions. A Leray–Hopf solution $u(t)$ is a weak solution satisfying the energy inequality

$$\frac{1}{2}\|u(t)\|_2^2 \leqslant \frac{1}{2}\|u(t_0)\|_2^2 - \nu\int_{t_0}^t \|\nabla u(\tau)\|_2^2\,\mathrm{d}\tau + \int_{t_0}^t (f,u)\,\mathrm{d}\tau, \tag{2.7}$$

for almost all $t_0 > 0$ and all $t > t_0$. A Leray solution $u(t)$ is a Leray–Hopf solution satisfying the above energy inequality for $t_0 = 0$ and all $t > t_0$. A complete Leray–Hopf solution $u(t)$ is an $L^2(\mathbb{T}^3)$ valued function on $(-\infty,\infty)$, such that $u(\cdot - t)|_{[0,\infty)}$ is a Leray-Hopf solution for all $t \in \mathbb{R}$.

3. Global attractor, pullback attractor, and Kolmogorov's wavenumber

In the case of a time-independent force f it can be shown that the energy inequality (2.7) implies the existence of an absorbing ball

$$B := \{u \in L^2(\mathbb{T}^3) : \|u\|_2 \leqslant R\}.$$

Here the radius R is such that

$$R > \nu\kappa_0^{-1/2}G,$$

where $\kappa_0 = 2\pi\lambda_0 = 2\pi/L$ and G is the dimensionless Grashof number

$$G := \frac{\|f\|_{H^{-1}}}{\nu^2\kappa_0^{1/2}}.$$

Note that the absorbing ball B is for all the Leray solutions, that is, the ones that satisfy the energy inequality starting from 0. More precisely, for any Leray solution $u(t)$ there exists t_0, depending only on $\|u(0)\|_2$, such that

$$u(t) \in B \qquad \forall t > t_0.$$

However, when we restrict the dynamics to the absorbing ball, we consider Leray–Hopf solutions to define the evolutionary system and the global attractor. The

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Leray–Hopf solutions are weak solutions satisfying the energy inequality starting from almost all time (but not necessarily 0). Hence, a restriction of a Leray–Hopf solution to a smaller time interval is also a Leray–Hopf solution. See [6] for a more detailed discussion.

The existence of the weak global attractor $\mathcal{A}$ was proved in [13, 18]. It has the following structure:

$$\mathcal{A} = \{u(0) : u(\cdot) \text{ is a complete bounded Leray–Hopf solution to the NSE}\}.$$

The attractor $\mathcal{A} \subset B$ is the L^2-weak omega limit of B, and it is the minimal L^2-weakly closed weakly attracting set (see [2, 4]).

In the case of a time-dependent force $f = f(t)$, a relevant object describing the long-time dynamics is a pullback attractor, whose existence was proved in [6]. In the nonautonomous case, there exists an absorbing ball for all the Leray solutions, whose radius R is such as $R > \nu\kappa_0^{-1/2}G$, just as in the autonomous case, but the Grashof number is

$$G = \frac{T^{1/2}\kappa_0^{1/2}\|f\|_{L_b^2(T)}}{\nu^{3/2}(1-e^{-\nu\kappa_0^2 T})^1/2}. \tag{3.8}$$

Here it is assumed that f is translationally bounded in $L^2_{loc}(\mathbb{R}, H^{-1})$ and

$$\|f\|^2_{L_b^2(T)} := \sup_{t\in R} \frac{1}{T}\int_t^{t+T} \|f(\tau)\|^2_{H^{-1}}\,\mathrm{d}\tau.$$

The pullback attractor is defined as the minimal weakly closed weakly pullback attracting set for all Leray–Hopf solutions in the absorbing ball. It is the weak pullback omega limit of B, and it has the following structure (see [6]):

$$\mathcal{A}(t) = \{u(t) : u(\cdot) \text{ is a complete bounded Leray–Hopf solution to the NSE}\}.$$

Let $u(t)$ be a complete bounded Leray–Hopf solution to the NSE. Then the energy inequality (2.7) implies

$$\begin{aligned} 0 \leqslant \|u(t+T)\|_2^2 &\leqslant \limsup_{\tau\to t+} \|u(\tau)\|_2^2 - 2\nu\int_t^{t+T} \|\nabla u(\tau)\|_2^2\,\mathrm{d}\tau + 2\int_t^{t+T}(f,u)\,\mathrm{d}\tau \\ &\leqslant \nu^2\kappa_0^{-1}G^2 - \nu\int_t^{t+T}\|\nabla u(\tau)\|_2^2\,\mathrm{d}\tau + \frac{1}{\nu}\int_t^{t+T}\|f\|^2_{H^{-1}}\,\mathrm{d}\tau. \end{aligned}$$

Therefore,

$$\langle\|\nabla u\|_2^2\rangle := \frac{1}{T}\int_t^{t+T}\|\nabla u(t)\|_2^2\,\mathrm{d}t \leqslant \frac{\nu G^2}{T\kappa_0} + \kappa_0\nu^2G^2. \tag{3.9}$$

We can now connect this to Kolmogorov's dissipation wavenumber defined as

$$\kappa_{\mathrm{d}} := \left(\frac{\varepsilon}{\nu^3}\right)^{1/d+1}, \qquad \varepsilon := \nu\lambda_0^d\langle\|\nabla u\|_2^2\rangle, \tag{3.10}$$

where d is the intermittency dimension and ε is average energy dissipation rate per unit active volume (i.e., the volume occupied by eddies). In order to define d, first,

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note that

$$\lambda_0^3 \lambda_q^{-1} \|u_q\|_2^2 \leqslant \lambda_q^{-1} \|u_q\|_\infty^2 \leqslant C_{\rm B} \lambda_q^2 \|u_q\|_2^2, \tag{3.11}$$

due to Bernstein's inequality. Here C_B is an absolute constant (which depends on the choice of $\chi(\xi)$ in (2.5)). The intermittency dimension d is defined as

$$d := \sup \left\{ s \in \mathbb{R} : \left\langle \sum_q \lambda_q^{-1+s} \|u_q\|_\infty^2 \right\rangle \leqslant C_{\rm B}^{3-s} \lambda_0^s \left\langle \sum_q \lambda_q^2 \|u_q\|_2^2 \right\rangle \right\}, \tag{3.12}$$

for $u \not\equiv 0$, and $d = 3$ for $u \equiv 0$ on $[t, t+T]$. Thanks to (3.11) and the fact that $\langle \sum_q \lambda_q^2 \|u_q\|_2^2 \rangle < \infty$, we have $d \in [0, 3]$ and

$$\left\langle \sum_q \lambda_q^{-1+d} \|u_q\|_\infty^2 \right\rangle = C_{\rm B}^{3-d} \lambda_0^d \left\langle \sum_q \lambda_q^2 \|u_q\|_2^2 \right\rangle.$$

The intermittency dimension d, defined in terms of a level of saturation of Bernsten's inequality (see [7, 8] for similar definitions), measures the number of eddies at various scales. The case $d = 3$ corresponds to Kolmogorov's regime where at each scale the eddies occupy the whole region. Note that $d = d(u, t)$ and $\kappa_{\rm d} = \kappa_{\rm d}(u, t)$, defined for each individual trajectory, are functions of time. We can also define their global analogues as

$$D := \inf_{u \in \mathcal{E}, t \in \mathbb{R}} d(u, t), \qquad K_{\rm d} := \sup_{u \in \mathcal{E}, t \in \mathbb{R}} \kappa_{\rm d}(u, t).$$

Here $\mathcal{E}$ is a family of all complete bounded Leray–Hopf solution to the NSE.

Finally, thanks to the bound (3.9),

$$\kappa_{\rm d} = \left\langle \frac{\lambda_0^d}{\nu^2} \|\nabla u\|_2^2 \right\rangle^{1/d+1} \leqslant (2\pi)^{-d/d+1} \kappa_0 G^{2/d+1} \left(\frac{1}{\nu T \kappa_0^2} + 1 \right)^{1/d+1}.$$

Also, taking the supremum over all $u \in \mathcal{E}$ and $t \in \mathbb{R}$, we obtain

$$K_{\rm d} \leqslant \kappa_0 G^{2/D+1} \left(\frac{1}{\nu T \kappa_0^2} + 1 \right)^{1/D+1},$$

provided $G \geqslant 1$.

4. Proof of the main result

Let $u(t)$ and $v(t)$ be completely bounded in L^2 Leray–Hopf solutions. Denote $w := u - v$, which satisfies the equation

$$w_t + u \cdot \nabla w + w \cdot \nabla v = -\nabla p' + \nu \Delta w \tag{4.13}$$

in the sense of distributions. Here p' stands for the difference of the pressures.

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Recall the definition of the determining wavenumber:

$$\Lambda_u(t) = \min\{\lambda_q : (L\lambda_{p-q})^\sigma \lambda_q^{-1}\|u_p\|_{L^\infty} < c_0\nu,$$
$$\forall p > q \text{ and } \lambda_q^{-2}\|\nabla u_{\leqslant q}\|_{L^\infty} < c_0\nu,\ q \in \mathbb{N}\},$$

where $\sigma = (\delta - 1)/2$. Let $\Lambda(t) := \max\{\Lambda_u(t), \Lambda_v(t)\}$ and $Q(t)$ be such that $\Lambda(t) = \lambda_{Q(t)}$. By our assumption, $w_{\leqslant Q(t)}(t) \equiv 0$. Recall that $0 < \delta \leqslant 3$, that is, $-1/2 < \sigma \leqslant 1$. Let

$$s = \min\left\{-\tfrac{1}{2} + \tfrac{\delta}{4}, 0\right\}.$$

Then straightforward computations give $-1 - \sigma < s < \sigma \leqslant 1$.

Multiplying (4.13) by $\lambda_q^{2s}\Delta_q^2 w$, integrating (i.e., using $\lambda_q^{2s}\Delta_q^2 w$ as a test function in the weak formulation), and adding up for all $q \geqslant -1$ yields

$$\begin{aligned}
&\frac{1}{2}\|w(t)\|_{H^s}^2 - \frac{1}{2}\|w(t_0)\|_{H^s}^2 + \nu\int_{t_0}^t \|w\|_{H^{1+s}}^2 \,\mathrm{d}\tau \\
&\leqslant \int_{t_0}^t \sum_{q\geqslant -1} \lambda_q^{2s} \left|\int_{\mathbb{T}^3} \Delta_q(w\cdot\nabla v) w_q \,\mathrm{d}x\right| \mathrm{d}\tau \\
&\quad + \int_{t_0}^t \sum_{q\geqslant -1} \lambda_q^{2s} \left|\int_{\mathbb{T}^3} \Delta_q(u\cdot\nabla w) w_q \,\mathrm{d}x\right| \mathrm{d}\tau, \\
&=: \int_{t_0}^t I \,\mathrm{d}\tau + \int_{t_0}^t J \,\mathrm{d}\tau.
\end{aligned} \tag{4.14}$$

We first decompose I using Bony's paraproduct as mentioned in §2.3,

$$\begin{aligned}
I &\leqslant \sum_{q\geqslant -1} \lambda_q^{2s} \sum_{|q-p|\leqslant 2} \left|\int_{\mathbb{T}^3} \Delta_q(w_{\leqslant p-2}\cdot\nabla v_p) w_q \,\mathrm{d}x\right| \\
&\quad + \sum_{q\geqslant -1} \lambda_q^{2s} \sum_{|q-p|\leqslant 2} \left|\int_{\mathbb{T}^3} \Delta_q(w_p\cdot\nabla v_{\leqslant p-2}) w_q \,\mathrm{d}x\right| \\
&\quad + \sum_{q\geqslant -1} \lambda_q^{2s} \sum_{p\geqslant q-2} \left|\int_{\mathbb{T}^3} \Delta_q(\tilde{w}_p\cdot\nabla v_p) w_q \,\mathrm{d}x\right| \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

It follows from Hölder's inequality that

$$\begin{aligned}
I_1 &\leqslant \sum_{q>Q} \sum_{\substack{|q-p|\leqslant 2\\ p>Q+2}} \lambda_q^{2s} \int_{\mathbb{T}^3} |\Delta_q(w_{\leqslant p-2}\cdot\nabla v_p) w_q| \,\mathrm{d}x \\
&\lesssim \sum_{q>Q} \sum_{\substack{|q-p|\leqslant 2\\ p>Q+2}} \lambda_q^{2s} \|w_{(Q,p-2]}\|_2 \lambda_p \|v_p\|_\infty \|w_q\|_2.
\end{aligned}$$

Using the definition of Λ, Young's inequality and Jensen's inequality, we obtain

$$
\begin{aligned}
I_1 &\lesssim c_0\nu \sum_{q>Q} \sum_{\substack{|q-p|\leqslant 2\\ p>Q+2}} \lambda_q^{2s}\Lambda^{1+\sigma}\lambda_p^{1-\sigma}\|w_q\|_2 \sum_{Q<p'\leqslant p-2}\|w_{p'}\|_2 \\
&\lesssim c_0\nu \sum_{q>Q}\lambda_q^{1+s}\|w_q\|_2 \left(\sum_{Q<p'\leqslant q}\lambda_{p'}^{1+s}\|w_{p'}\|_2\lambda_{p'}^{-1-s}\lambda_q^{s-\sigma}\lambda_Q^{1+\sigma}\right) \\
&\lesssim c_0\nu \sum_{q>Q}\lambda_q^{1+s}\|w_q\|_2 \left(\sum_{Q<p'\leqslant q}\lambda_{p'}^{1+s}\|w_{p'}\|_2(L\lambda_{q-p'})^{s-\sigma}\right),
\end{aligned}
$$

where we used $\sigma \geqslant -1$ and $s<\sigma$. Now using Young's inequality and Jensen's inequality, we conclude

$$
\begin{aligned}
I_1 &\lesssim c_0\nu \sum_{q>Q}\lambda_q^{2+2s}\|w_q\|_2^2 + c_0\nu\sum_{q>Q}\left(\sum_{Q<p'\leqslant q}\lambda_{p'}^{1+s}\|w_{p'}\|_2(L\lambda_{q-p'})^{s-\sigma}\right)^2 \\
&\lesssim c_0\nu \sum_{q>Q}\lambda_q^{2+2s}\|w_q\|_2^2 + c_0\nu\sum_{q>Q}\sum_{Q<p'\leqslant q}\lambda_{p'}^{2+2s}\|w_{p'}\|_2^2(L\lambda_{q-p'})^{s-\sigma} \\
&\lesssim c_0\nu \sum_{q>Q}\lambda_q^{2+2s}\|w_q\|_2^2 + c_0\nu\sum_{p'>Q}\lambda_{p'}^{2+2s}\|w_{p'}\|_2^2\sum_{q\geqslant p'}(L\lambda_{q-p'})^{s-\sigma} \\
&\lesssim c_0\nu\|\nabla^{1+s}w\|_2^2,
\end{aligned}
$$

where we needed $s<\sigma$. Note that we omit dimensionless constants that depend on δ throughout this proof. The precise bound on I_1 is

$$
I_1 \lesssim c_0\nu\|\nabla^{1+s}w\|_2^2\left(1+(1-2^{s-\sigma})^{-1}\right).
$$

Note that $(1-2^{s-\sigma})^{-1}\to\infty$ as $\delta\to 0+$ by definitions of σ and s. Because of this, we will have $c_0\to 0$ as $\delta\to 0+$ once we choose c_0 at the end of the proof. This explains why we have to avoid the case of extreme intermittency, which is covered in the companion paper [**9**].

Following a similar strategy, we have

$$
\begin{aligned}
I_2 &\leqslant \sum_{q>Q}\sum_{\substack{|q-p|\leqslant 2\\ p>Q}}\lambda_q^{2s}\int_{\mathbb{T}^3}|\Delta_q(w_p\cdot\nabla v_{\leqslant p-2})w_q|\,\mathrm{d}x \\
&\leqslant \sum_{q>Q}\sum_{\substack{|q-p|\leqslant 2\\ p>Q}}\lambda_q^{2s}\|w_p\|_2\|\nabla v_{(Q,p-2]}\|_\infty\|w_q\|_2 \\
&\quad + \sum_{q>Q}\sum_{\substack{|q-p|\leqslant 2\\ p>Q}}\lambda_q^{2s}\|w_p\|_2\|\nabla v_{\leqslant Q}\|_\infty\|w_q\|_2 \\
&\equiv I_{21}+I_{22},
\end{aligned}
$$

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where we adopt the convention that $(Q, p-2]$ is empty if $p-2 \leqslant Q$. Thus, the first part of the definition of Λ implies

$$
\begin{aligned}
I_{21} &\lesssim \sum_{p>Q} \sum_{|q-p|\leqslant 2} \lambda_q^{2s} \|w_p\|_2 \|w_q\|_2 \sum_{Q<p'\leqslant p-2} \|\nabla v_{p'}\|_\infty \\
&\lesssim \sum_{q>Q} \lambda_q^{2s} \|w_q\|_2^2 \sum_{Q<p'\leqslant q+2} \lambda_{p'} \|v_{p'}\|_\infty \\
&\lesssim c_0 \nu \sum_{q>Q} \lambda_q^{2s} \|w_q\|_2^2 \sum_{Q<p'\leqslant q+2} \lambda_{p'}^{1-\sigma} \Lambda^{1+\sigma} \\
&\lesssim c_0 \nu \sum_{q>Q} \lambda_q^{2+2s} \|w_q\|_2^2 \sum_{Q<p'\leqslant q+2} \lambda_{p'}^{1-\sigma} \Lambda^{1+\sigma} \lambda_q^{-2} \\
&\lesssim c_0 \nu \sum_{q>Q} \lambda_q^{2+2s} \|w_q\|_2^2
\end{aligned}
$$

where we need $\sigma \geqslant -1$. While the second part of the definition of Λ gives

$$
\begin{aligned}
I_{22} &\lesssim \sum_{q>Q} \sum_{\substack{|q-p|\leqslant 2 \\ p>Q}} \lambda_q^{2s} \|w_p\|_2 \|w_q\|_2 \|\nabla v_{\leqslant Q}\|_\infty \\
&\lesssim c_0 \nu \sum_{q>Q} \Lambda_v^2 \lambda_q^{2s} \|w_q\|_2^2 \\
&\lesssim c_0 \nu \sum_{q>Q} \lambda_q^{2+2s} \|w_q\|_2^2.
\end{aligned}
$$

We will now estimate I_3. It follows from integration by parts that

$$
\begin{aligned}
I_3 &= \sum_{q\geqslant -1} \lambda_q^{2s} \sum_{|q-p|\leqslant 2} \left| \int_{\mathbb{T}^3} \Delta_q (w_p \cdot \nabla v_{\leqslant p-2}) w_q \,\mathrm{d}x \right| \\
&\leqslant \sum_{q>Q} \sum_{p\geqslant q-2} \lambda_q^{2s} \int_{\mathbb{T}^3} |\Delta_q (\tilde{w}_p \otimes v_p) \nabla w_q| \,\mathrm{d}x \\
&\leqslant \sum_{p>Q} \sum_{Q<q\leqslant p+2} \lambda_q^{1+2s} \|\tilde{w}_p\|_2 \|w_q\|_2 \|v_p\|_\infty.
\end{aligned}
$$

By Hölder's inequality and definition of Λ, we have

$$
\begin{aligned}
I_3 &\lesssim \sum_{p>Q} \|\tilde{w}_p\|_2 \|v_p\|_\infty \sum_{Q<q\leqslant p+2} \lambda_q^{1+2s} \|w_q\|_2 \\
&\lesssim c_0 \nu \sum_{p>Q} \Lambda^{1+\sigma} \lambda_p^{-\sigma} \|\tilde{w}_p\|_2 \sum_{Q<q\leqslant p+2} \lambda_q^{1+2s} \|w_q\|_2 \\
&\lesssim c_0 \nu \sum_{p>Q} \lambda_p^{1+s} \|\tilde{w}_p\|_2 \sum_{Q<q\leqslant p+2} \lambda_q^{1+s} \|w_q\|_2 \lambda_Q^{1+\sigma} \lambda_p^{-1-s-\sigma} \lambda_q^{s}
\end{aligned}
$$

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Now we use Young's and Jensen's inequalities to infer

$$
\begin{aligned}
I_3 &\lesssim c_0\nu \sum_{p>Q} \lambda_p^{1+s}\|\tilde{w}_p\|_2 \sum_{Q<q\leqslant p+2} \lambda_q^{1+s}\|w_q\|_2 (L\lambda_{q-p})^{1+s+\sigma} \\
&\lesssim c_0\nu \sum_{p>Q} \lambda_p^{2+2s}\|w_p\|_2^2 + c_0\nu \sum_{p>Q} \left(\sum_{Q<q\leqslant p+2} \lambda_q^{1+s}\|w_q\|_2 (L\lambda_{q-p})^{1+s+\sigma} \right)^2 \\
&\lesssim c_0\nu \sum_{p>Q} \lambda_p^{2+2s}\|w_p\|_2^2,
\end{aligned}
$$

where we used $\sigma \geqslant -1$ and $s > -1-\sigma$.

Therefore, we have for $\sigma \geqslant -1$ and $-1-\sigma < s < \sigma$,

$$
I \lesssim c_0\nu \|\nabla^{1+s} w\|_2^2. \tag{4.15}
$$

Now applying Bony's paraproduct formula to J yields

$$
\begin{aligned}
J &= \int_{t_0}^{t} \sum_{q\geqslant -1} \lambda_q^{2s} \left| \int_{\mathbb{T}^3} \Delta_q(w\cdot\nabla v) w_q \, dx \right| \mathrm{d}\tau \\
&\leqslant \sum_{q\geqslant -1} \sum_{|q-p|\leqslant 2} \lambda_q^{2s} \left| \int_{\mathbb{T}^3} \Delta_q(u_{\leqslant p-2}\cdot\nabla w_p) w_q \, \mathrm{d}x \right| \\
&\quad + \sum_{q\geqslant -1} \sum_{|q-p|\leqslant 2} \lambda_q^{2s} \left| \int_{\mathbb{T}^3} \Delta_q(u_p\cdot\nabla w_{\leqslant p-2}) w_q \, \mathrm{d}x \right| \\
&\quad + \sum_{q\geqslant -1} \sum_{p\geqslant q-2} \sum_{|p-p'|\leqslant 1} \lambda_q^{2s} \left| \int_{\mathbb{T}^3} \Delta_q(u_p\cdot\nabla w_{p'}) w_q \, \mathrm{d}x \right| \\
&=: J_1 + J_2 + J_3.
\end{aligned}
$$

We further decompose J_1 by using a commutator form

$$
\begin{aligned}
J_1 &\leqslant \sum_{q\geqslant -1} \sum_{|q-p|\leqslant 2} \lambda_q^{2s} \left| \int_{\mathbb{R}^3} [\Delta_q, u_{\leqslant p-2}\cdot\nabla] w_p w_q \, \mathrm{d}x \right| \\
&\quad + \sum_{q\geqslant -1} \lambda_q^{2s} \left| \int_{\mathbb{R}^3} u_{\leqslant q-2}\cdot\nabla w_q w_q \, \mathrm{d}x \right| \\
&\quad + \sum_{q\geqslant -1} \sum_{|q-p|\leqslant 2} \lambda_q^{2s} \left| \int_{\mathbb{R}^3} (u_{\leqslant p-2} - u_{\leqslant q-2})\cdot\nabla\Delta_q w_p w_q \, \mathrm{d}x \right| \\
&= J_{11} + J_{12} + J_{13}.
\end{aligned}
$$

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To obtain the second term, we used $\sum_{|p-q|\leqslant 2}\Delta_q w_p = w_q$. In fact, we have $J_{12}=0$ since $\operatorname{div} u_{\leqslant q-2}=0$. In the first term, the commutator is defined as

$$[\Delta_q, u_{\leqslant p-2}\cdot\nabla]w_p := \Delta_q(u_{\leqslant p-2}\cdot\nabla w_p) - u_{\leqslant p-2}\cdot\nabla\Delta_q w_p.$$

It is easy to see (for more details, see [3]) that for any $1\leqslant r\leqslant\infty$,

$$\|[\Delta_q, u_{\leqslant p-2}\cdot\nabla]w_p\|_r \lesssim \|\nabla u_{\leqslant p-2}\|_\infty\|w_p\|_r.$$

Then J_{11} is estimated as

$$\begin{aligned}
J_{11} &\leqslant \sum_{q>Q}\sum_{\substack{|q-p|\leqslant 2\\ p>Q}}\lambda_q^{2s}\int_{\mathbb{T}^3}|[\Delta_q, u_{\leqslant p-2}\cdot\nabla]w_p w_q|\,\mathrm{d}x\\
&\leqslant \sum_{q>Q}\sum_{\substack{|q-p|\leqslant 2\\ p>Q}}\lambda_q^{2s}\|\nabla u_{(Q,p-2]}\|_\infty\|w_p\|_2\|w_q\|_2\\
&+\sum_{q>Q}\sum_{\substack{|q-p|\leqslant 2\\ p>Q}}\lambda_q^{2s}\|\nabla u_{\leqslant Q}\|_\infty\|w_p\|_2\|w_q\|_2\\
&\equiv J_{111}+J_{112}.
\end{aligned}$$

Here

$$\begin{aligned}
J_{111} &\lesssim \sum_{q>Q}\lambda_q^{2s}\|w_q\|_2^2\sum_{Q<p'\leqslant q}\lambda_{p'}\|u_{p'}\|_\infty\\
&\lesssim c_0\nu\sum_{q>Q}\lambda_q^{2s}\|w_q\|_2^2\sum_{Q<p'\leqslant q}\Lambda^{1+\sigma}\lambda_{p'}^{1-\sigma}\\
&\lesssim c_0\nu\sum_{q>Q}\lambda_q^{2+2s}\|w_q\|_2^2\sum_{Q<p'\leqslant q}\Lambda^{1+\sigma}\lambda_{p'}^{1-\sigma}\lambda_q^{-2}\\
&\lesssim c_0\nu\sum_{q>Q}\lambda_q^{2+2s}\|w_q\|_2^2,
\end{aligned}$$

where we used $\sigma\geqslant -1$. As for the second term, using the fact that $\|\nabla u_{\leqslant q}\|_\infty\leqslant c_0\nu\Lambda^2$ for $q\leqslant Q$, we obtain

$$J_{112}\lesssim c_0\nu\Lambda^2\sum_{q>Q}\lambda_q^{2s}\|w_q\|_2^2\lesssim c_0\nu\sum_{q>Q}\lambda_q^{2+2s}\|w_q\|_2^2.$$

The term J_{13} is estimated as

$$\begin{aligned}
J_{13} &\leqslant \sum_{q>Q}\sum_{\substack{|q-p|\leqslant 2\\ p>Q}}\lambda_q^{2s}\int_{\mathbb{R}^3}|(u_{\leqslant p-2}-u_{\leqslant q-2})\cdot\nabla\Delta_q w_p w_q|\,\mathrm{d}x\\
&\lesssim \sum_{q>Q}\lambda_q^{1+2s}\|u_{(q-4,q]}\|_\infty\|w_q\|_2^2
\end{aligned}$$

$$\lesssim \sum_{q>Q} \lambda_q^{1+2s} \|u_{(q-4,Q]}\|_\infty \|w_q\|_2^2 + \sum_{q>Q} \sum_{\substack{q-4<p'\leqslant q\\ p'>Q}} \lambda_q^{1+2s} \|u_{p'}\|_\infty \|w_q\|_2^2$$

$$\equiv J_{131} + J_{132}.$$

As before, we adopt the convention that $(q-4, Q]$ is empty if $q-4 \geqslant Q$. We have

$$J_{131} \lesssim c_0 \nu \Lambda \sum_{q>Q} \lambda_q^{1+2s} \|w_q\|_2^2 \lesssim c_0 \nu \sum_{q>Q} \lambda_q^{2+2s} \|w_q\|_2^2,$$

and

$$\begin{aligned} J_{132} &= \sum_{q>Q} \sum_{\substack{q-4\leqslant p'\leqslant q\\ p'>Q}} \lambda_q^{1+2s} \|u_{p'}\|_\infty \|w_q\|_2^2 \\ &\lesssim c_0 \nu \sum_{q>Q} \sum_{\substack{q-4\leqslant p'\leqslant q\\ p'>Q}} \lambda_q^{1+2s} \Lambda^{1+\sigma} \lambda_{p'}^{-\sigma} \|w_q\|_2^2 \\ &\lesssim c_0 \nu \sum_{q>Q} \lambda_q^{2+2s} \|w_q\|_2^2 (L\lambda_{Q-q})^{1+\sigma} \\ &\lesssim c_0 \nu \sum_{q>Q} \lambda_q^{2+2s} \|w_q\|_2^2, \end{aligned}$$

where we used $\sigma \geqslant -1$.

Now we continue with J_2:

$$\begin{aligned} J_2 &= \sum_{q>Q} \sum_{\substack{|q-p|\leqslant 2\\ p>Q+2}} \lambda_q^{2s} \left| \int_{\mathbb{T}^3} \Delta_q (u_p \cdot \nabla w_{\leqslant p-2}) w_q \,\mathrm{d}x \right| \\ &\leqslant \sum_{q>Q} \sum_{\substack{|q-p|\leqslant 2\\ p>Q+2}} \lambda_q^{2s} \|u_p\|_\infty \|\nabla w_{(Q,p-2]}\|_2 \|w_q\|_2. \end{aligned}$$

Using definition of Λ, Young's, and Jensen's inequalities, we obtain

$$\begin{aligned} J_2 &\lesssim c_0 \nu \sum_{q>Q} \sum_{\substack{|q-p|\leqslant 2\\ p>Q+2}} \lambda_q^{2s} \Lambda^{1+\sigma} \lambda_p^{-\sigma} \|w_q\|_2 \|\nabla w_{(Q,p-2]}\|_2 \\ &\lesssim c_0 \nu \sum_{q>Q} \Lambda^{1+\sigma} \lambda_q^{2s-\sigma} \|w_q\|_2 \|\nabla w_{(Q,q]}\|_2 \\ &\lesssim c_0 \nu \sum_{q>Q} \Lambda^{1+\sigma} \lambda_q^{2s-\sigma} \|w_q\|_2 \sum_{Q<p'\leqslant q} \lambda_{p'} \|w_{p'}\|_2 \\ &\lesssim c_0 \nu \sum_{q>Q} \lambda_q^{1+s} \|w_q\|_2 \sum_{Q<p'\leqslant q} \lambda_{p'}^{1+s} \|w_{p'}\|_2 \lambda_q^{s-\sigma-1} \lambda_{p'}^{-s} \Lambda^{1+\sigma} \\ &\lesssim c_0 \nu \sum_{q>Q} \lambda_q^{1+s} \|w_q\|_2 \left(\sum_{Q<p'\leqslant q} \lambda_{p'}^{1+s} \|w_{p'}\|_2 (L\lambda_{q-p'})^{s-\sigma-1} \right) \end{aligned}$$

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$$\lesssim c_0\nu \sum_{q>Q} \lambda_q^{2+2s}\|w_q\|_2^2 + c_0\nu \sum_{q>Q}\left(\sum_{Q<p'\leqslant q} \lambda_{p'}^{1+s}\|w_{p'}\|_2 (L\lambda_{q-p'})^{s-\sigma-1}\right)^2$$
$$\lesssim c_0\nu \sum_{q>Q} \lambda_q^{2+2s}\|w_q\|_2^2,$$

where we used $s < \sigma + 1$ and $\sigma \geqslant -1$.

Notice that the last term J_3 can be estimated in the same way as I_3. Therefore, we have for $\sigma \geqslant -1$ and $-1-\sigma < s < 1+\sigma$,

$$J \lesssim c_0\nu\|\nabla^{1+s}w\|_2^2. \tag{4.16}$$

Combining (4.15) and (4.16), we conclude that for any $\delta > 0$, there exists an dimensionless constant $C > 0$ (that depends only on δ) such that

$$I + J \leqslant Cc_0\nu\|\nabla^{1+s}w\|_2^2,$$

where $s = \min\{-1/2 + \delta/4, 0\} \leqslant 0$. Choosing $c_0 := 1/(2C)$, we infer from (4.14) that for all $t_0 \leqslant t$,

$$\|w(t)\|_{H^s}^2 \leqslant \|w(t_0)\|_{H^s}^2 - \nu\int_{t_0}^t \|\nabla^{1+s}w\|_2^2 \,\mathrm{d}\tau$$
$$\leqslant \|w(t_0)\|_{H^s}^2 - \nu\kappa_0^{2+2s}\int_{t_0}^t \|w\|_2^2 \,\mathrm{d}\tau,$$

with $\kappa_0 = 2\pi/L$. Thus

$$\|w(t)\|_{H^s}^2 \leqslant \|w(t_0)\|_{H^s}^2 e^{-\nu\kappa_0^{2+2s}(t-t_0)}, \qquad t_0 \leqslant t.$$

Recall that $s \leqslant 0$ and hence $\|w(t)\|_{H^s} \lesssim \lambda_0^s\|w(t)\|_2$, which is bounded on $\mathbb{R}$ as $w(t)$ is the difference of two complete bounded trajectories. Taking the limit as $t_0 \to -\infty$ completes the proof.

□

5. Average determining wavenumber and Kolmogorov's dissipation wavenumber

The goal of this section is to derive a uniform upper bound on the average determining wavenumber in the absorbing ball. First, recall that $\Lambda_u(t)$ is defined as

$$\Lambda_u(t) := \min\{\lambda_q : (L\lambda_{p-q})^\sigma \lambda_q^{-1}\|u_p\|_\infty < c_0\nu,$$
$$\forall p > q \text{ and } \lambda_q^{-2}\|\nabla u_{\leqslant q}\|_\infty < c_0\nu,\ q \in \mathbb{N}\},$$

where $\sigma = (\delta-1)/2$ and c_0 is an dimensionless constant that depends only on δ. Recall that $\sigma \in (-1/2, 1]$. We will drop the subscript u in Λ_u and define Q so that $\lambda_Q = \Lambda$.

LEMMA 5.1. *If $\lambda_0 \leqslant \Lambda < \infty$, then*

$$(c_0\nu)^2\Lambda^4 \lesssim \|\nabla u_{\leqslant Q-1}\|_\infty^2 + \sup_{p\geqslant Q}(L\lambda_{p-Q})^{2\sigma}\Lambda^2\|u_p\|_\infty^2. \tag{5.17}$$

If $\Lambda = \infty$, then

$$\sup_q \lambda_q^\sigma \|u_q\|_\infty = \infty.$$

Proof. First, consider the case $\Lambda = \infty$. Then for every $q \in \mathbb{N}$ either

$$\sup_{p>q}(L\lambda_{p-q})^\sigma \lambda_q^{-1}\|u_p\|_\infty \geqslant c_0\nu, \tag{5.18}$$

or

$$\lambda_q^{-2}\|\nabla u_{\leqslant q}\|_\infty \geqslant c_0\nu. \tag{5.19}$$

If (5.18) is satisfied for infinitely many $q \in \mathbb{N}$, then

$$\limsup_{q\to\infty}\sup_{p>q}\lambda_q^{-\sigma-1}(L\lambda_p)^\sigma\|u_p\|_\infty \geqslant c_0\nu.$$

Since $\sigma > -1$, this immediately implies that $\sup_q \lambda_q^\sigma\|u_q\|_\infty = \infty$.

If (5.19) is satisfied for infinitely many $q \in \mathbb{N}$, then

$$\limsup_{q\to\infty}\lambda_q^{-2}\|\nabla u_{\leqslant q}\|_\infty \geqslant c_0\nu.$$

On the contrary, since $\sigma \leqslant 1$,

$$\begin{aligned}\lambda_q^{-2}\|\nabla u_{\leqslant q}\|_\infty &\lesssim \lambda_q^{-2}\sum_{p\leqslant q}\lambda_p\|u_p\|_\infty \\ &= \lambda_q^{-\sigma-1}\sum_{p\leqslant q}(L\lambda_{q-p})^{\sigma-1}\lambda_p^\sigma\|u_p\|_\infty \\ &\leqslant \lambda_q^{-\sigma-1}\sup_{p\leqslant q}\lambda_p^\sigma\|u_p\|_\infty.\end{aligned}$$

Hence, since $-\sigma-1<0$, $\sup_q \lambda_q^\sigma\|u_q\|_\infty = \infty$.

Now if $\lambda_0 < \Lambda(t) < \infty$, then both conditions in the definition of Λ are satisfied for $q = Q$, but one of the conditions is not satisfied for $q = Q-1$, that is,

$$2^{(p-Q+1)\sigma}\lambda_{Q-1}^{-1}\|u_p\|_\infty \geqslant c_0\nu, \qquad \text{for some} \qquad p \geqslant Q, \tag{5.20}$$

or

$$\|\nabla u_{\leqslant Q-1}\|_\infty \geqslant c_0\nu\lambda_{Q-1}^2 = \tfrac{1}{4}c_0\nu\Lambda^2. \tag{5.21}$$

Thus we have

$$(c_0\nu)^2\Lambda^4 \leqslant 16(\lambda_{p-Q}L)^{2\sigma}\Lambda^2\|u_p\|_\infty^2, \qquad \text{for some} \qquad p\geqslant Q,$$

or

$$(c_0\nu)^2\Lambda^4 \leqslant 16\|\nabla u_{\leqslant Q-1}\|_\infty^2.$$

Hence, adding the right-hand sides, we obtain (5.17). □

We will now consider the average determining wavenumber

$$\langle \Lambda \rangle := \frac{1}{T}\int_t^{t+T} \Lambda(\tau)\, \mathrm{d}\tau,$$

and compare it with Kolmogorov's dissipation wavenumber defined as

$$\kappa_{\mathrm{d}} := \left(\frac{\varepsilon}{\nu^3}\right)^{1/d+1}, \qquad \varepsilon := \nu\lambda_0^d \langle \|\nabla u\|_2^2\rangle = \frac{\nu\lambda_0^d}{T}\int_t^{t+T} \|\nabla u(\tau)\|_2^2\, \mathrm{d}\tau, \tag{5.22}$$

where $d \in [0,3]$ is the intermittency dimension and ε is average energy dissipation rate per unit active volume (i.e., the volume occupied by eddies). Recall from the definition of intermittency (5.23) that

$$\left\langle \sum_{q\leqslant Q} \lambda_q^{-1+d}\|u_q\|_\infty^2 \right\rangle \lesssim \lambda_0^d \left\langle \sum_{q\leqslant Q} \lambda_q^2 \|u_q\|_2^2 \right\rangle. \tag{5.23}$$

The case $d = 3$ corresponds to Kolmogorov's regime where at each scale the eddies occupy the whole region, and $d = 0$ is the case of extreme intermittency.

Consider now a solution u for which $d \geqslant \delta$, that is, $d \geqslant 2\sigma + 1$. Then whenever $\Lambda_u(t)$ is finite, we can use (5.17) in lemma 5.1 and Jensen's inequality to get

$$\begin{aligned}
(\Lambda - \lambda_0)^{d+1} &\lesssim \frac{\Lambda^{d-3}}{(c_0\nu)^2}\left(\|\nabla u_{\leqslant Q-1}\|_\infty^2 + \sup_{q\geqslant Q} (L\lambda_{q-Q})^{2\sigma}\Lambda^2\|u_q\|_\infty^2 \right) \\
&\lesssim \frac{1}{\nu^2}\left(\sum_{q\leqslant Q-1} \lambda_q^{(d-1)/2}\|u_q\|_\infty (L\lambda_{Q-q})^{(d-3)/2} \right)^2 \\
&\quad + \frac{\Lambda^{d-1}}{\nu^2} \sup_{q\geqslant Q} (L\lambda_{q-Q})^{2\sigma}\|u_q\|_\infty^2 \\
&\lesssim_d \frac{1}{\nu^2}\sum_{q\leqslant Q-1} \lambda_q^{d-1}\|u_q\|_\infty^2 + \frac{1}{\nu^2}\sup_{q\geqslant Q} (L\lambda_{q-Q})^{2\sigma-d+1}\lambda_q^{d-1}\|u_q\|_\infty^2 \\
&\lesssim_d \frac{1}{\nu^2}\sum_q \lambda_q^{d-1}\|u_q\|_\infty^2.
\end{aligned}$$

If $\Lambda = \infty$, this inequality is also true. Indeed, in this case, lemma 5.1 implies

$$\sum_q \lambda_q^{d-1}\|u_q\|_\infty^2 \geqslant \sum_q \lambda_q^{2\sigma}\|u_q\|_\infty^2 = \infty.$$

Then thanks to Jensen's inequality,

$$\begin{aligned}
\langle \Lambda \rangle - \lambda_0 &\lesssim \langle (\Lambda - \lambda_0)^{d+1}\rangle^{1/d+1} \\
&\lesssim_d \left\langle \frac{1}{\nu^2}\sum_q \lambda_q^{d-1}\|u_q\|_\infty^2 \right\rangle^{1/d+1}.
\end{aligned}$$

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Now using (5.23), we conclude that

$$\begin{aligned}
\langle \Lambda \rangle - \lambda_0 &\lesssim_d \left\langle \frac{1}{\nu^2} \sum_{q \leqslant Q} \lambda_q^{d-1} \|u_q\|_\infty^2 \right\rangle^{1/d+1} \\
&\lesssim \left\langle \frac{\lambda_0^d}{\nu^2} \sum_{q \leqslant Q} \lambda_q^2 \|u_q\|_2^2 \right\rangle^{1/d+1} \\
&\lesssim \left\langle \frac{\nu \lambda_0^d}{\nu^3} \|\nabla u\|_2^2 \right\rangle^{1/d+1} \\
&= \kappa_{\mathrm{d}}
\end{aligned}$$

Consider now Kolmogorov's regime where $d = 3$. Then a similar computation yields

$$\begin{aligned}
\left\langle \frac{\Lambda - \lambda_0}{(\log(\Lambda/\lambda_0))^{1/4}} \right\rangle &\lesssim \left\langle \frac{(\Lambda - \lambda_0)^4}{Q} \right\rangle^{1/4} \\
&\lesssim \left\langle Q \left(\frac{1}{c_0 \nu Q} \sum_{q \leqslant Q} \|\nabla u_q\|_\infty \right)^2 + \sup_{p \geqslant Q} (L\lambda_{p-Q})^{2\sigma} \Lambda^2 \|u_p\|_\infty^2 \right\rangle^{1/4} \\
&\lesssim \left\langle \frac{1}{\nu^2} \sum_{q \leqslant Q} \lambda_q^2 \|u_q\|_\infty^2 + \frac{1}{\nu^2} \sup_{q \geqslant Q} (L\lambda_{q-Q})^{2\sigma-2} \lambda_q^2 \|u_q\|_\infty^2 \right\rangle^{1/4} \\
&\lesssim \left\langle \frac{\lambda_0^3}{\nu^2} \sum_q \lambda_q^2 \|u_q\|_2^2 \right\rangle^{1/4} \\
&\lesssim \kappa_{\mathrm{d}}.
\end{aligned}$$

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